



# Loci in strata of meromorphic differentials with fully degenerate Lyapunov spectrum

Julien Grivaux, Pascal Hubert

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# LOCI IN STRATA OF MEROMORPHIC QUADRATIC DIFFERENTIALS WITH FULLY DEGENERATE LYAPUNOV SPECTRUM

J. GRIVAUX AND P. HUBERT

ABSTRACT. We construct explicit closed  $GL(2; \mathbb{R})$ -invariant loci in strata of meromorphic quadratic differentials of arbitrary large dimension with fully degenerate Lyapunov spectrum. This answers a question of Forni-Matheus-Zorich.

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## 1. INTRODUCTION

Lyapunov exponents of the Teichmüller flow have been studied a lot since the work of Zorich ([Zor97], [Zor99]) and Forni [For02]. Their understanding is important for applications to the dynamics of interval exchange transformations and polygonal billiards. A big breakthrough is the Eskin-Kontsevich-Zorich formula for the sum of positive Lyapunov exponents [EKZ11b]. Given a  $SL(2; \mathbb{R})$ -invariant suborbifold of a stratum of quadratic differentials, they relate the sum  $\lambda_1 + \dots + \lambda_g$  to the Siegel-Veech constant of the invariant locus<sup>1</sup>.

By a theorem of Kontsevich and Forni, the sum  $\lambda_1 + \dots + \lambda_g$  is also the integral over the invariant locus of the curvature of the Hodge bundle along Teichmüller

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<sup>1</sup>For quadratic differentials, two bundles can be considered. In this article, we will only be interested in the bundle with fiber  $H^1(X, \mathbb{R})$  over a Riemann surface  $X$ . The Lyapunov exponents of this bundle are often denoted by  $\lambda_1^+, \dots, \lambda_g^+$ .

discs ([For02], [EKZ11b]). Using this interpretation, every Lyapunov exponent is computed for cyclic covers of the sphere branched over 4 points ([EKZ11a], [FMZ11], see also [BM10], and [Wri12] for abelian covers). For some cyclic covers, Forni-Matheus-Zorich remarked that the sum  $\lambda_1 + \dots + \lambda_g$  is equal to zero [FMZ11, Thm. 35]. This surprising fact means that the complex structure of the underlying Riemann surface is constant along the Teichmüller disc. Forni-Matheus-Zorich ask whether it is possible to construct other invariant loci with this property (see [FMZ11, p. 312]). The content of this article is to give a simple explanation of the phenomenon discovered by Forni-Matheus-Zorich.

**Theorem 1.** *There exist closed  $GL(2; \mathbb{R})$ -invariant loci of quadratic differentials of arbitrarily large dimension with zero Lyapunov exponents.*

This result can be interpreted in the following way: the projection of such a locus to the moduli space of *compact* Riemann surfaces is a point. Remark that the situation for strata of *abelian* differentials is completely different: there are finitely many invariant suborbifolds with fully degenerate Lyapunov spectrum (meaning in this setting that all exponents are zero except  $\lambda_1$  which is 1), and they are arithmetic Teichmüller curves (see [Möl11], [For06], [FMZ11] and [Aul13]).

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## 2. BACKGROUND MATERIAL

### 2.1. The Teichmüller flow for translation surfaces.

A translation surface is a pair  $(X, \omega)$  where  $X$  is a compact Riemann surface and  $\omega$  is a holomorphic one-form on  $X$ . If  $S(\omega)$  is the set of the zeroes of  $\omega$ , there exists an open covering of  $\tilde{X} = X \setminus S(\omega)$  and holomorphic charts  $\varphi_i: U_i \rightarrow X$  such that  $\varphi_i^* \omega = dz$  for all  $i$ . For such an atlas, the transition functions are translations. The form  $\omega$  induces a flat metric  $|\omega|^2$  on the open surface  $\tilde{X}$ , whose area is the integral  $\frac{i}{2} \int_{\tilde{X}} \omega \wedge \bar{\omega}$ . We could have taken meromorphic 1-forms instead of holomorphic ones, but in that case the area of the surface is never finite.

There is a natural action of  $GL(2, \mathbb{R})$  on translation surfaces given as follows: first we pick an atlas of charts of  $\tilde{X}$  where all transitions functions are translations by some complex vectors  $v_{ij}$  which we will consider as vectors in  $\mathbb{R}^2$ . Then, for any  $M$  in  $GL(2, \mathbb{R})$ , we get an open surface  $\tilde{X}_M$  defined by an atlas whose transition functions are translations by  $M.v_{ij}$ . This surface is diffeomorphic to  $\tilde{X}_M$ . Therefore, we can fill the holes and extend the complex structure in a unique way: the result is a compact Riemann surface  $X_M$  diffeomorphic to  $X$  endowed with a meromorphic differential  $\omega_M$  of finite volume, hence holomorphic. The action of  $GL(2; \mathbb{R})$  is defined by the formula  $M.(X, \omega) = (X_M, \omega_M)$ . The action of  $SL(2; \mathbb{R})$  preserves the volume of translation surfaces.

The subgroup of  $\mathrm{SL}(2; \mathbb{R})$  of matrices  $M$  such that  $M.(X, \omega) = (X, \omega)$  up to diffeomorphism is called the Veech group of  $(X, \omega)$ . If  $M_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$  then the curve  $(X_t, \omega_t) := M_t.(X, \omega)$  is called the orbit under the Teichmüller flow of  $(X, \omega)$ . If  $M_\theta$  is a rotation in  $\mathrm{SO}(2, \mathbb{R})$ , then  $M_\theta.(X, \omega) = (X, e^{i\theta}\omega)$ .

If we fix multiplicities  $(m_1, \dots, m_r)$  such that  $\sum_{i=1}^r m_i = 2g - 2$ , the stratum of translation surfaces  $\mathcal{H}(m_1, \dots, m_r)$  is the set of translations surfaces  $(X, \omega)$  where  $\omega$  has  $r$  distinct zeroes of multiplicities  $m_1, \dots, m_r$  modulo diffeomorphism. The normalized stratum  $\mathcal{H}_1(m_1, \dots, m_r)$  is the locus of flat surfaces with unit area in  $\mathcal{H}(m_1, \dots, m_r)$ , and the projective stratum  $\mathrm{PH}(m_1, \dots, m_r)$  is obtained by taking the quotient of  $\mathcal{H}(m_1, \dots, m_r)$  under the natural  $\mathbb{C}^\times$ -action on forms. Strata and projective strata are complex orbifolds of respective dimensions  $2g + r - 1$  and  $2g + r - 2$  if  $g \geq 2$ .

There are standard coordinates on the stratum  $\mathcal{H}(m_1, \dots, m_r)$ , called period coordinates. Fix  $(X, \omega)$  in this stratum, and let  $A_1, \dots, A_g, B_1, \dots, B_g$  be a symplectic basis of  $H_1(X, \mathbb{Z})$  and  $C_1, \dots, C_{r-1}$  be  $r - 1$  paths joining a zero of  $\omega$  to all the  $r - 1$  other zeroes. The map

$$(X, \omega) \rightarrow \left( \int_{A_1} \omega, \dots, \int_{A_g} \omega, \int_{B_1} \omega, \dots, \int_{B_g} \omega, \int_{C_1} \omega, \dots, \int_{C_{r-1}} \omega \right)$$

yields an orbifold chart on  $\mathcal{H}(m_1, \dots, m_r)$ . These charts allow to define a canonical volume element on  $\mathcal{H}(m_1, \dots, m_r)$ ,  $\mathcal{H}_1(m_1, \dots, m_r)$ , and  $\mathrm{PH}(m_1, \dots, m_r)$ . By classical results of Masur and Veech, connected components of projective strata have finite volume.

Let  $\mathbb{H} = \mathrm{SL}(2; \mathbb{R})/\mathrm{SO}(2)$  denote the Poincaré upper-half plane. For any  $(X, \omega)$  in a projective stratum, the  $\mathrm{SL}(2; \mathbb{R})$ -action factorizes to a holomorphic map

$$\mathbb{H} \rightarrow \mathrm{PH}(m_1, \dots, m_r)$$

which is an immersion. The image of this map is called a Teichmüller disc, it is stable under the Teichmüller flow. Besides, Teichmüller discs induce a smooth foliation with holomorphic leaves on  $\mathrm{PH}(m_1, \dots, m_r)$ .

Assume that the Veech group  $\Gamma$  of  $(X, \omega)$  is a lattice in  $\mathrm{SL}(2; \mathbb{R})$ . Then the image  $\mathbb{H}/\Gamma$  of the corresponding Teichmüller disc in the projective stratum is called a Teichmüller curve.

All these considerations generalize to the so-called half-translation surfaces, which are pairs  $(X, q)$  where  $q$  is a quadratic holomorphic (for the time being) differential on  $X$ . The transition functions of a well-chosen atlas of charts on the open surface are half translations, that is either translations or flips. The area of the flat metric on  $\tilde{X}$  is  $\frac{1}{2} \int_X |q|$ , and we still have an action of  $\mathrm{GL}(2, \mathbb{R})$  as well as a Teichmüller flow. The period coordinates on strata of quadratic differentials are obtained as follows: for any  $(X, q)$  in a stratum, we take the twofold branched covering  $\hat{p}: \tilde{X} \rightarrow X$  given by the holonomy representation of  $q$ , which is given by a morphism from  $\pi_1(X)$  to  $\mathbb{Z}/2\mathbb{Z}$ . Let  $j$  be the corresponding involution acting on

$\widehat{X}$ . The quadratic differential  $p^*q$  is the square of an abelian differential  $\omega$ . The period coordinates of  $(X, q)$  are obtained by taking  $j$ -anti-invariant absolute and relative periods of  $(X, q)$ .

However, a major difference happens for quadratic differentials: it is possible to take meromorphic quadratic differentials and still get finite volume for the corresponding flat surface. More precisely,  $(X, q)$  has finite volume if and only if  $q$  has poles of order at most one. Therefore we have strata, normalized strata and projective strata  $\mathcal{Q}(m_1, \dots, m_r)$ ,  $\mathcal{Q}_1(m_1, \dots, m_r)$  and  $\text{PQ}(m_1, \dots, m_r)$ , where  $\sum_{i=1}^r m_i = 4g - 4$  and each  $m_i$  is either positive or equal to  $-1$ .

Let  $S$  be a finite subset of  $X$  of cardinal  $n$ , so that  $(X, S)$  gives a point in the marked Teichmüller space  $\mathcal{T}_{g,n}$  (genus  $g$  with  $n$  marked points). The cotangent space of  $\mathcal{T}_{g,n}$  at  $X$  is exactly the space  $\mathcal{Q}_S(X)$  of holomorphic quadratic differentials on  $X \setminus S$  with poles of order one at some points of  $S$ . There is a norm on  $\mathcal{Q}_S(X)$  given by  $\|q\| = \int_X |q|$ , as well as a dual norm on  $\mathcal{Q}_S(X)^*$ . The corresponding distance on  $\mathcal{T}_{g,n}$  is the Teichmüller metric.

Let us fix  $(X, S)$  as well as an element  $q$  in  $\mathcal{Q}_S(X)$ . There is a complex linear form  $\mu_q$  on  $\mathcal{Q}_S(X)$  given by scalar product with the  $L^\infty$  Beltrami differential  $\frac{|q|}{q}$ :

$$\mu_q(\tilde{q}) = \int_X \tilde{q} \frac{|q|}{q}.$$

Note that  $\mu_q(q) = \int_X |q| > 0$  so that  $\mu_q$  is nonzero. Besides, we have  $\|\mu_q\| = 1$ . The map  $q \rightarrow \mu_q$  gives a non-linear isomorphism between the unit spheres of  $\mathcal{Q}_S(X)$  and  $\mathcal{Q}_S(X)^*$ , hence between the unitary cotangent space  $U^*\mathcal{T}_{g,n}$  and the unitary tangent space  $U\mathcal{T}_{g,n}$ .

If  $(X, q)$  is given and  $S$  is the set of poles of  $q$ , the Teichmüller flow of  $(X, q)$  introduced formerly is the geodesic flow (for the Teichmüller metric) on  $\mathcal{T}_{g,n}$  starting from  $X$  in the direction  $\mu_q$ .

## 2.2. The period mapping.

For any compact Riemann surface  $X$ ,  $H^1(X, \mathbb{C})$  is the orthogonal sum (for the intersection form) of  $\Omega(X)$  and  $\overline{\Omega(X)}$ . Besides, the composition

$$\psi : H^1(X, \mathbb{R}) \rightarrow H^1(X, \mathbb{C}) \xrightarrow{\text{pr}_1} \Omega(X)$$

is an isomorphism. The Hodge norm  $\|\cdot\|_{\text{Hodge}}$  is the unique norm on  $H^1(X, \mathbb{R})$  making  $\psi$  an isometry.

Let us now consider a local holomorphic family of curves, that is a proper holomorphic submersion  $\pi : \mathcal{X} \rightarrow \mathcal{B}$  whose fibers  $(X_b)_{b \in \mathcal{B}}$  are compact Riemann surfaces of some genus  $g$ , where  $\mathcal{B}$  is a small ball in  $\mathbb{C}^n$ . The Hodge bundle  $\mathcal{H}$  is a holomorphic vector bundle on  $\mathcal{B}$  of rank  $g$  whose fiber at each point  $b$  is the vector space  $\mathcal{H}_b := \Omega(X_b)$ . The local system  $R^1\pi_* \mathbb{R}_{\mathcal{X}}$  is trivial, which means

that we can canonically identify all the vector spaces  $H^1(X_b, \mathbb{R})$  to some fixed real vector space  $\mathbb{V}$  of dimension  $2g$ . The local period map

$$\xi: \mathcal{B} \rightarrow \text{Gr}(g, \mathbb{V}^{\mathbb{C}})$$

associates to any  $b$  the subspace  $\mathcal{H}_b$  in the Grassmannian of  $g$ -dimensional complex subspaces of  $\mathbb{V}^{\mathbb{C}}$ . The derivative of  $\xi$  at a point  $b$  in  $\mathcal{B}$  is a linear map from  $T_b^{1,0} \mathcal{B}$  to  $\text{End}(\mathcal{H}_b, \mathbb{V}^{\mathbb{C}}/\mathcal{H}_b)$ , which is isomorphic to  $\text{End}(\mathcal{H}_b, \overline{\mathcal{H}}_b)$ .

The differential of  $\xi$  can be explicitly computed:  $\xi$  induces a classifying map  $\xi_{\text{Teich}}: \mathcal{B} \rightarrow \mathcal{T}_g$ . Then we have the following formula due to Ahlfors: for any vector  $v$  in  $T_b \mathcal{B}$  and any elements  $\alpha$  and  $\beta$  in  $\Omega(X_b)$ ,

$$(\beta, \overline{\xi'_v(\alpha)}) = \int_X \alpha \otimes \beta \cdot \xi'_{\text{Teich}}(v).$$

In this formula,  $\xi'_{\text{Teich}}(v)$  is a tangent vector to  $\mathcal{T}_g$ , hence represented by a Beltrami differential which is a tensor field on  $X$  of type  $(-1, 1)$ . Thus, the integrand in the above formula is of type  $(2, 0) + (-1, 1) = (1, 1)$ . We can also think of  $\xi'_{\text{Teich}}(v)$  as a linear form on  $Q(X)$ ; in this case the above formula reads

$$(\beta, \overline{\xi'_v(\alpha)}) = \xi'_{\text{Teich}}(v) \{ \alpha \otimes \beta \}.$$

It is also possible to give another interpretation on  $\xi'$ . For this we consider the exact sequence of holomorphic vector bundles

$$0 \rightarrow \mathcal{H} \rightarrow \mathbb{V} \otimes \mathcal{O}_{\mathcal{B}} \rightarrow \overline{\mathcal{H}} \rightarrow 0.$$

The bundle  $\mathbb{V} \otimes \mathcal{O}_{\mathcal{B}}$  carries a natural flat connection (the Gauß-Manin connection), but  $\mathcal{H}$  is not in general a flat sub-bundle of  $\mathbb{V} \otimes \mathcal{O}_{\mathcal{B}}$ . A precise way to measure this (see formula (2) below) is the second fundamental form  $\sigma$  associated with this exact sequence and the Gauß-Manin connection; it is a  $(1, 0)$ -form with values in  $\text{Hom}(\mathcal{H}, \overline{\mathcal{H}})$ . A simple calculation shows that

$$(1) \quad \sigma = \xi'.$$

The Hodge bundle  $\mathcal{H}$  carries a natural metric given by the intersection form, its curvature form is given by the formula

$$(2) \quad \Theta_{\mathcal{H}} = \sigma^* \wedge \sigma.$$

By " $\wedge$ " we mean composition on the fiber and wedge-product on the base. In particular,  $i \text{Tr} \Theta_{\mathcal{H}}$  is a positive  $(1, 1)$ -form on  $\mathcal{B}$ .

For any compact half-translation surface  $(X, q)$ , Forni's B-form is a bilinear form on  $\Omega(X)$  defined by

$$B_q(\alpha, \beta) = \int_X \alpha \otimes \beta \frac{|q|}{q}.$$

If  $\xi'_{\text{Teich}}(v)$  has unit norm, we can write it as  $\mu_q$  for some holomorphic quadratic differential on  $X$ . Then we have  $(\beta, \overline{\xi'_v(\alpha)}) = B_q(\alpha, \beta)$ . In the case of a Teichmüller orbit  $(X_t, q_t)$ , if we differentiate along the vector field  $\frac{\partial}{\partial t}$ , we get the formula

$$(3) \quad (\beta, \overline{\xi'_{\partial_t}(\alpha)}) = B_{q_t}(\alpha, \beta).$$

Applying Cauchy-Schwarz inequality,  $|B_q(\alpha, \beta)| \leq \|\alpha\| \times \|\beta\|$  with equality if and only if there exists a holomorphic one-form  $\omega$  and two complex constants  $c$  and  $c'$  such that  $q = \omega^2$ ,  $\alpha = c\omega$  and  $\beta = c'\omega$ . In particular, if  $q$  is meromorphic with simple poles,  $\|B_q\| < 1$  where  $\|\cdot\|$  denotes the operator norm.

We recall now Forni's inequality: let  $(X, q)$  be a half-translation surface,  $(X_t, q_t)$  be its orbit under the Teichmüller flow,  $v$  be in  $H^1(X, \mathbb{R})$  and  $t \rightarrow v_t$  be its parallel transport under the Teichmüller flow for the Gauß-Manin connection. We write  $v_t = \chi_t + \overline{\chi_t}$  where  $\chi_t$  is in  $\Omega(X_t)$ . Then a simple calculation gives

$$\partial_t \|v_t\|_{\text{Hodge}} = B_{q_t}(\chi_t, \chi_t).$$

Combined with the inequality  $\|B\| \leq 1$ , this gives Forni's inequality

$$(4) \quad \left| \partial_t \{\log \|v_t\|_{\text{Hodge}}\} \right| \leq 1.$$

### 2.3. Lyapunov exponents of the KZ cocycle.

The parallel transport for the Gauß-Manin connection of vectors of  $H^1(X, \mathbb{R})$  under the Teichmüller flow is called the Kontsevich-Zorich cocycle. Recall that the Teichmüller flow is ergodic on every connected component  $\mathfrak{D}_1$  of the normalized stratum  $\mathcal{Q}_1(m_1, \dots, m_r)$ . By Osceledets' theorem, it is possible to associate  $2g$  Lyapunov exponents to this cocycle.

Forni's inequality (4) implies that the KZ cocycle is log-integrable, so that the Lyapunov exponents are well-defined. Since the cocycle is symplectic, the Lyapunov spectrum is of the form  $\{-\lambda_1, -\lambda_2, \dots, -\lambda_g, \lambda_g, \lambda_{g-1}, \dots, \lambda_1\}$  where  $\lambda_1 \geq \dots \geq \lambda_g$ .

Note that the exponents  $\lambda_i$  are called  $\lambda_i^+$  in numerous papers (e.g. in [EKZ11b], [FMZ11]). The exponents  $\lambda_i^-$  will never be considered in the article.

By (4), all  $\lambda_i$ 's are at most one. If the component  $\mathfrak{D}_1$  is orientable, which means that every quadratic differential occurring in the stratum is the square of an abelian differential, then the top Lyapunov exponent  $\lambda_1$  equals one. If not, the norm of Forni's B form is strictly smaller than one so that  $\lambda_1 < 1$ .

For any  $(X, q)$  in a stratum  $\mathcal{PQ}(m_1, \dots, m_r)$ , the Poincaré metric on  $\mathbb{H}$  induces a metric on the Teichmüller disc passing through  $(X, q)$ . The corresponding volume element defines a relative  $(1, 1)$  form  $dV_{\text{Teich}}$ , where by "relative" we mean relative with respect to the foliation by Teichmüller discs. If  $\Theta$  is the curvature of the Hodge bundle on  $\mathcal{PQ}(m_1, \dots, m_r)$ , its trace is also a relative  $(1, 1)$  form on the projective stratum. Let  $\Lambda: \mathcal{PQ}(m_1, \dots, m_r) \rightarrow \mathbb{R}$  be defined by the formula

$$\Lambda = \frac{\text{Tr } \Theta}{dV_{\text{Teich}}}.$$

Then Kontsevich-Forni's main formula for the Lyapunov exponents is

$$\lambda_1 + \dots + \lambda_g = \int_{\mathfrak{D}} \Lambda(X, q) dV$$

where  $\mathfrak{D}$  is the projection of  $\mathfrak{D}_1$  in the projective stratum and  $dV$  is the normalized volume element on  $\mathfrak{D}$  of total mass one. For any  $(X, q)$  in  $\mathfrak{D}$ , let  $\theta_1, \dots, \theta_g$  be the eigenvalues of Forni's B-form in the direction of the Teichmüller flow when diagonalized in an orthonormal basis for the intersection form. Using formulæ (2), (1) and (3), we see that

$$(5) \quad \lambda_1 + \dots + \lambda_g = \int_{\mathfrak{D}} \left\{ |\theta_1(X, q)|^2 + \dots + |\theta_g(X, q)|^2 \right\} dV$$

Forni's inequality implies that  $|\theta_i(X, q)| \leq 1$  for all  $i$  so that  $\lambda_1 + \dots + \lambda_g \leq g$ .

Thanks to the main result of [EM13], any closed  $\mathrm{SL}(2; \mathbb{R})$ -invariant locus in the projective stratum  $\mathrm{PQ}(m_1, \dots, m_r)$  is affine in period coordinates, hence carries a natural  $\mathrm{SL}(2; \mathbb{R})$ -invariant probability measure. It is also possible to define Lyapunov exponents for this measure, and formula (5) holds.

If  $(X, q)$  is any half-translation surface, the closure of its  $\mathrm{SL}(2; \mathbb{R})$ -orbit in the normalized stratum is affine in period coordinates. It follows from [CE13] that almost every direction  $\theta$ , the real Teichmüller flow of  $(X, e^{i\theta}q)$  is Osceledets-generic for the corresponding natural probability measure. Therefore it makes sense to consider Lyapunov exponents of  $(X, q)$ , and formula (5) is still valid if we integrate on the closure of the  $\mathrm{PGL}(2; \mathbb{R})$ -orbit.

### 3. THE FORNI LOCUS

**3.1. General properties.** Let  $\mathfrak{D}$  be a connected component of the projective stratum  $\mathrm{PQ}(m_1, \dots, m_r)$ .

**Definition 1.** *The Forni locus of  $\mathfrak{D}$  is the set of elements  $(X, q)$  in  $\mathfrak{D}$  such that for all holomorphic 1-forms  $\alpha$  and  $\beta$  on  $X$ ,  $B_q(\alpha, \beta) = 0$ .*

**Remark 1.** *In the terminology used in [For06], the Forni locus is called the rank 0 locus.*

Let us now recall Noether's theorem (see [FK92, p. 104 & 159]):

**Proposition 1.** *Let  $X$  be a compact Riemann surface of genus  $g$  and*

$$\tau: \mathrm{Sym}^2 \Omega^1(X) \rightarrow Q(X)$$

*be the multiplication map.*

- (i) *If  $X$  is not hyperelliptic or if  $g \leq 2$ ,  $\tau$  is surjective.*
- (ii) *If  $X$  is hyperelliptic,  $\mathrm{Im}(\tau)$  has codimension  $g - 2$  in  $Q(X)$  and consists of the quadratic differentials invariant by the hyperelliptic involution.*

Since  $\tau$  is the transpose of the derivative of the period map, Noether's result has the following geometric interpretation:



**Proposition 2** (Infinitesimal Torelli's theorem).

Let  $\xi: \mathcal{T}_g \rightarrow \mathbb{H}_g$  be the period map. Then  $\xi$  is an immersion outside the hyperelliptic locus or everywhere if  $g \leq 2$ , and the restriction of  $\xi$  to the hyperelliptic locus is also an immersion.

Remark that Forni's  $B$ -form factors through  $\text{Im } \tau$ , and can be extended naturally to  $\mathcal{Q}(X)$  by the formula  $B_q(\tilde{q}) = \int_X \tilde{q} \frac{|q|}{q}$ .

The key proposition of this section is:

**Proposition 3.** *Let  $(X, q)$  be a half-translation surface,  $n$  the number of poles of  $q$ , and  $\mathbb{D}$  be its Teichmüller disc. Then the following are equivalent:*

- (i)  $\mathbb{D}$  lies in the Forni locus.
- (ii) The forgetful map  $\mathcal{T}_{g,n} \rightarrow \mathcal{T}_g$  maps  $\mathbb{D}$  to a point.
- (iii) For any  $(X_t, q_t)$  in  $\mathbb{D}$ , the extension of  $B_{q_t}$  to  $\mathcal{Q}(X_t)$  vanishes.
- (iv) All Lyapunov exponents of  $(X, q)$  are zero.

*Proof.*

(i)  $\Rightarrow$  (ii) Using (3), the composite map  $\mathbb{D} \hookrightarrow \mathcal{T}_{g,n} \rightarrow \mathcal{T}_g \xrightarrow{\xi} \mathbb{H}_g$  has zero derivative. Assume that  $\mathbb{D}$  is not contained in the hyperelliptic locus. Thanks to the infinitesimal Torelli theorem,  $\mathbb{D}$  is mapped to a point via the forgetful map  $\mathcal{T}_{g,n} \rightarrow \mathcal{T}_g$ . Assume now that  $\mathbb{D}$  is contained in the hyperelliptic locus. Then the restriction of  $\xi$  to this locus is again an immersion, and we can apply the same argument.

(ii)  $\Rightarrow$  (iii) If  $(X_t, q_t)$  is a point in  $\mathbb{D}$ , the derivative of projection of the Teichmüller flow of  $(X_t, q_t)$  on  $\mathcal{T}_g$  is the linear form  $\tilde{q} \rightarrow B_{q_t}(\tilde{q})$  on  $\mathcal{Q}(X_t)$ .

(iii)  $\Rightarrow$  (i) Obvious.

(i)  $\Leftrightarrow$  (iv) Let  $\mathcal{V}$  be the closure of the  $\text{PSL}(2; \mathbb{R})$ -orbit of  $X$  and  $\nu$  the corresponding  $\text{PSL}(2; \mathbb{R})$ -invariant probability measure. If  $\lambda_1, \dots, \lambda_g$  are the Lyapunov exponents of  $(X, q)$ , then

$$\lambda_1 + \dots + \lambda_g = \int_{\mathcal{V}} \{\theta_1(X, q) + \dots + \theta_g(X, q)\} d\nu.$$

Since all  $\theta_i$ 's are nonnegative and continuous functions,  $\lambda_1 = \dots = \lambda_g = 0$  if and only if all  $\theta_i$ 's vanish on  $\mathbb{D}$ . □

**Corollary 1.** *If  $q$  is a holomorphic quadratic differential on  $X$ , the Teichmüller disc of  $(X, q)$  is not included in the Forni locus. In particular some Lyapunov exponents are non-zero.*

*Proof.* If  $q$  is holomorphic,  $B_q(q) > 0$  and we apply Proposition 3. □

By Proposition 3 the vanishing along a Teichmüller disc of the  $B$ -form and of its extension to the quadratic differentials are equivalent. This is no longer the case if we consider these properties for a single half-translation surface  $(X, q)$ , as shown by the example below.

Let  $X$  be a hyperelliptic surface of genus at least 3, let  $j$  be the hyperelliptic involution, and let  $q$  be an anti-invariant holomorphic quadratic differential (if  $X$  is the Riemann surface of a polynomial  $w^2 - P(z)$ , we can take  $q = w^{-1}dz^{\otimes 2}$ ). Since any holomorphic 1-form on  $X$  is anti-invariant under  $j^*$ ,  $B_q = 0$ . Hence we see that the following properties are satisfied:

- $q$  is holomorphic.
- $(X, q)$  is in the Forni locus.
- The extension of  $B_q$  to  $Q(X)$  is nonzero since  $B_q(q) > 0$ .
- The Teichmüller disc of  $(X, q)$  is not contained in the Forni locus (see Corollary 1).

This also gives a concrete example showing that the Forni locus is not  $SL(2; \mathbb{R})$ -invariant.

**3.2. Galois coverings and pillow-tiled surfaces.** Let  $(X, q)$  be a half-translation surface and  $(Y, \pi)$  be an arbitrary finite covering of  $X$  with branching locus  $S$ . Assume that for any point  $y$  in  $Y$  above  $S$ , the ramification index of  $\pi$  at  $y$  is at least 2. Then  $\pi^*q$  is holomorphic, so that  $B_{\pi^*q}$  is non zero on  $Q(Y)$ . Thanks to Corollary 1, the Teichmüller disc of  $(Y, \pi^*q)$  doesn't lie in the Forni locus. Using this observation, we have the following result:

**Proposition 4.** *Let  $(X, q)$  be a half-translation surface and  $(Y, \pi)$  be a finite Galois covering of  $X$  with branch locus  $S$ . If the Teichmüller disc of  $(Y, \pi^*q)$  lies in the Forni locus, then at least one pole of  $q$  does not belong to  $S$ .*

As a particular by-product, we get:

**Corollary 2.** *Let  $(X, q, \pi)$  be a pillow-tiled surface such that  $\pi$  is Galois. Then the Teichmüller disc of  $(X, q)$  lies in the Forni locus if and only if the branching locus of  $\pi$  contains at most three points.*

*Proof.* Let  $q_{\text{st}}$  be the standard meromorphic differential on  $\mathbb{P}^1$  with four simple poles such that  $q = \pi^*q_{\text{st}}$ . By definition the branching locus of  $\pi$  lies in the set of poles of  $q_{\text{st}}$ . If  $X$  is in the Forni locus, according to Proposition 4, one of the poles of  $q_{\text{st}}$  is not a branching point of  $\pi$ .

Conversely, assume that the branching locus of  $\pi$  has less than four points. If  $\{z_1, z_2, z_3, z_4\}$  are the four poles of  $q_{\text{st}}$ , let us assume that  $z_4$  is not a branch point of  $\pi$ . The complex Teichmüller flow of  $(\mathbb{P}^1, q_{\text{st}})$  is of the form  $(\mathbb{P}^1, q_t)$  where  $q_t$  has poles at  $z_1, z_2, z_3$  and another point  $z_4(t)$  such that  $[z_1, z_2, z_3, z_4(t)] = t$ . Let  $\mathring{X}$  be the open Riemann surface obtained by removing  $\pi^{-1}\{z_1, z_2, z_3\}$ . Then  $\mathring{X}$  is an unramified covering of  $\mathbb{P}^1 \setminus \{z_1, z_2, z_3\}$ . It follows that  $(\mathring{X} \setminus \pi^{-1}(z_4(t)), \pi^*q_t)$  parametrizes the Teichmüller disc of  $(X, q)$  in  $T^*\mathcal{T}_{g,n}$  (where  $n$  is the number of poles of  $q$ ). This disc maps to  $\{X\}$  via the forgetful map  $\mathcal{T}_{g,n} \rightarrow \mathcal{T}_g$ . Thanks to Proposition 3, the Teichmüller disc of  $(X, q)$  lies in the Forni locus.  $\square$

Let us now consider pillow-tiled surfaces arising as *cyclic* coverings of the projective line. They are given by a combinatorial datum  $(N, a_1, a_2, a_3, a_4)$  where

$0 < a_i \leq N$ ,  $\gcd(a_1, a_2, a_3, a_4, N) = 1$  and  $\sum_{i=1}^4 a_i \equiv 0 \pmod{N}$ : the associated cyclic covering is the Riemann surface of the polynomial

$$w^N - (z - z_1)^{a_1}(z - z_2)^{a_2}(z - z_3)^{a_3}(z - z_4)^{a_4}.$$

In topological terms, if  $(\gamma_i)_{1 \leq i \leq 4}$  are small loops around the  $z_i$ 's for  $1 \leq i \leq 4$ , then the kernel of the group morphism

$$\pi_1(\mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}) \rightarrow \mathbb{Z}/N\mathbb{Z}$$

given by  $\gamma_i \rightarrow a_i$  defines a true cyclic covering of  $\mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}$  of degree  $N$ , which extends to a branched cyclic covering of the projective line.

In [FMZ11, Thm. 35], the authors prove that all Lyapunov exponents of the Teichmüller curve corresponding to a cyclic covering are 0 if one of the integers  $a_i$  equals  $N$ .

**Proposition 5.** *If  $(X, q)$  is a pillow-tiled surface obtained by a cyclic covering of  $\mathbb{P}^1$  with combinatorial datum  $(N, a_1, a_2, a_3, a_4)$ , then the Teichmüller disc of  $(X, q)$  lies in the Forni locus if and only if one of the  $a_i$ 's equals  $N$ .*

*Proof.* Thanks to Corollary 2, it suffices to prove that the projection  $\pi$  of the covering is branched at three points or less if and only if one of the  $a_i$ 's equals  $N$ . If  $\{z_1, z_2, z_3, z_4\}$  are the four points defining the cyclic cover, the ramification index of  $\pi$  at any point of  $\pi^{-1}(z_i)$  is  $\frac{N}{\gcd(N, a_i)}$ .  $\square$

### 3.3. Construction of Forni subspaces.

In this section, we provide the precise statement underlying Theorem 1 as well as its proof.

Let  $m_1, \dots, m_r$  and  $k$  be positive integers such that  $(\sum_{i=1}^r m_i) - k = -4$ , and let  $\mathcal{S}$  be the set of couples  $(q, x_1, \dots, x_{k-3})$  such that  $q$  is a meromorphic differential on  $\mathbb{P}^1$  with simple poles at  $0, 1, \infty$  and the  $x_i$ 's, and  $q$  has  $r$  zeroes of order  $m_1, \dots, m_r$ . It is a smooth  $\mathrm{GL}(2; \mathbb{R})$ -invariant submanifold of  $T^*\mathcal{M}_{0,[k]}$  (where the bracket means that the points are ordered).

Let us fix a covering  $(Y, \pi)$  of  $\mathbb{P}^1$  ramified over  $0, 1$  and  $\infty$ , and let  $g$  be the genus of  $Y$ . Put

$$n = \#\{y \in \pi^{-1}\{0, 1, \infty\} \text{ such that } \pi \text{ is unramified at } y\} + \deg(\pi) \times (k - 3)$$

We have a natural map

$$\chi: \mathcal{S} \rightarrow T_{\mathrm{orb}}^* \mathcal{M}_{g,n}$$

given by  $\chi(q) = (Y, \pi^* q)$ , where  $T_{\mathrm{orb}}^*$  denotes the *orbifold* cotangent bundle.

**Theorem 2.** *Let  $\mathcal{W}$  be the image of  $\chi$ .*

- (1) *The map  $\chi: \mathcal{S} \rightarrow \mathcal{W}$  is a holomorphic orbifold map, which is a local immersion. Besides,  $\mathcal{W}$  is a suborbifold<sup>2</sup> of the orbifold cotangent bundle of  $\mathcal{M}_{g,n}$  of dimension  $r + k - 2$ .*

<sup>2</sup>By suborbifold, we mean as usually done in this theory "locally finite union of suborbifolds".

- (2)  $\mathcal{W}$  is  $\mathrm{GL}(2; \mathbb{R})$ -invariant and lies in the Forni locus, and the projection of  $\mathcal{W}$  by the map  $T_{\mathrm{orb}}^* \mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,n} \rightarrow \mathcal{M}_g$  is  $\{Y\}$ .
- (3) The Lyapunov spectrum of  $\mathcal{W}$  is fully degenerate.

*Proof.* Let  $q$  be a point in  $\mathcal{S}$ , and  $U$  be a small neighborhood of  $q$  in  $\mathcal{S}$ . It is possible to lift locally  $\chi$  to a smooth map  $\widehat{\chi}$  from  $U$  to  $T^* \mathcal{T}_{g,n}$ , so that  $\chi$  is a smooth orbifold map.

If  $q_1, q_2$  are two elements in  $U$  such that  $\chi(q_1) = \chi(q_2)$ , then there exists  $\varphi$  in  $\mathrm{Aut}(Y)$  such that  $\varphi^*(Y, \pi^* q_1) = (Y, \pi^* q_2)$ . Thus the fibers of  $\widehat{\chi}|_U$  are finite. But  $\widehat{\chi}$  is affine in period coordinates, so that it is an immersion on  $U$ .

Point (2) is proved using the same argument as in Corollary 2, which corresponds to the particular case  $r = 0$ .

Lastly, the fact that the Lyapunov spectrum of  $\mathcal{W}$  is totally degenerate results from the implication (ii)  $\Rightarrow$  (iv) in Proposition 3.  $\square$

**Remark 2.** In view of the existing terminology in the abelian case, it is natural to call  $\mathcal{W}$  a Forni subspace of the stratum.

#### 4. NUMERICAL CONSTRAINTS

In this section, given a half-translation surface  $(X, q)$  whose Teichmüller disc lies in the Forni locus, we give some numerical constraints relating the genus  $g$  of  $X$ , the number of  $n$  of simple poles of  $q$  and the degree  $d$  when  $(X, q)$  is a pillow-tiled surface.

**Proposition 6.** Let  $(X, q)$  be a half-translation surface of genus  $g \geq 1$  whose Teichmüller disc lies in the Forni locus, and let  $n$  be the number of poles of  $q$ . Then  $n \geq \max(2g - 2, 2)$ .

*Proof.* The fact that the number  $n$  of poles of  $q$  must be at least two follows from [Kra81, Thm 4']. To get the lower bound  $2g - 2$  in the proposition, we use [EKZ11b, Thm 2] for the closure of the  $\mathrm{SL}(2; \mathbb{R})$ -orbit  $\mathcal{O}$  of  $(X, q)$ , which is contained in a stratum  $\mathrm{PQ}((-1)^n, m_1, \dots, m_r)$ : we get

$$\lambda_1 + \dots + \lambda_g = \frac{1}{24} \sum_{j=1}^r \frac{m_j(m_j + 4)}{m_j + 2} - \frac{n}{8} + \frac{\pi^2}{3} C_{\mathrm{area}}(\overline{\mathcal{O}})$$

where  $C_{\mathrm{area}}(\overline{\mathcal{O}})$  is a Siegel-Veech constant of the locus  $\overline{\mathcal{O}}$  which is nonnegative. Thus, if  $\lambda_1 + \dots + \lambda_g = 0$ ,

$$\sum_{j=1}^r \frac{m_j(m_j + 4)}{m_j + 2} \leq 3n.$$

Since  $(\sum m_j) - n = 4g - 4$ ,

$$2g - 2 \leq \sum_{j=1}^r \frac{m_j}{m_j + 2} + 2g - 2 \leq n.$$

$\square$

**Remark 3.** *We will see that this bound is asymptotically sharp at the end of this section.*

We now focus on the case of pillow-tiled surfaces. Let us start with a technical result:

**Proposition 7.** *Let  $X$  be a Riemann surface of genus  $g$ ,  $B(t_0, \varepsilon)$  a small ball in  $\mathbb{C} \setminus \{0, 1, \infty\}$ , and  $\varphi: B(t_0, \varepsilon) \times X \rightarrow \mathbb{P}^1$  be a holomorphic map satisfying the following conditions:*

- (1) *For any  $t$  in  $B(t_0, \varepsilon)$ ,  $\varphi_t$  is non-constant and the branching locus of  $\varphi_t$  is  $\{0, 1, \infty, t\}$ .*
- (2) *The configuration of the ramification points of  $\varphi_t$  remains constant with  $t$ .*

*If  $d$  is the degree of the branched covers  $\varphi_t$ , then  $d \geq 3(g - 1)$ .*

*Proof.* For any  $x$  in  $X$ , let  $s(x) = \frac{\partial}{\partial t}|_{t=t_0} \varphi_t(x) \in T_{\varphi_{t_0}(x)} \mathbb{P}^1$ . Then  $s$  is a holomorphic section of the holomorphic line bundle  $\varphi_{t_0}^* \mathbb{TP}^1$ . Let  $x_0$  be a ramification point of  $\varphi_{t_0}$  such that  $\varphi_{t_0}(x_0) = 0$ . Let us assume that  $s(x_0) \neq 0$ . By the implicit function theorem, the equation  $\varphi_t(x) = 0$  has a unique solution  $(t(x), x)$  depending holomorphically on  $x$  for  $(t, x)$  near  $(t_0, x_0)$ . Since  $\varphi_{t(x)}(x) = 0$ , we get

$$\partial_t \varphi(t(x), x) t'(x) + (\varphi_{t(x)})'(x) = 0.$$

By hypothesis,  $x$  is a ramification point of  $\varphi_{t(x)}$ , i.e.  $(\varphi_{t(x)})'(x) = 0$ . Besides, since  $\partial_t \varphi(t(x), x) \rightarrow s(x_0)$  as  $x \rightarrow x_0$ ,  $t'$  vanishes. Hence  $\varphi_{t_0}(x)$  vanishes for  $x$  near  $x_0$ , so that  $\varphi_{t_0}$  is constant and we get a contradiction. It follows that  $s$  vanishes at  $x_0$ . The same result also holds over any ramification point of  $\varphi_{t_0}$  lying over 1 and  $\infty$ . Lastly, if  $\psi_t(x) = \varphi_t(x) - t$ , the argument we used proves that for any ramification point  $x$  of  $\psi_{t_0}$  lying over 0,  $\frac{\partial}{\partial t}|_{t=t_0} \psi_t(x) = 0$ , which means that  $s(x) = 1$ . In particular  $s$  is nonzero.

We can now decompose the ramification divisor  $\mathcal{R}$  of the branched covering  $\varphi_{t_0}$  (which is the divisor of the section  $d\varphi_{t_0}$  of  $\varphi_{t_0}^* \mathbb{TP}^1$ ) as the sum  $\mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_\infty + \mathcal{R}_t$ . Besides, we can assume that  $\deg \mathcal{R}_t$  is smaller than  $\deg \mathcal{R}_0$ ,  $\deg \mathcal{R}_1$  and  $\deg \mathcal{R}_\infty$ , otherwise we move the points 0, 1,  $\infty$  and  $t$  by a suitable homographic transformation. Besides, thanks to the Riemann-Hurwitz formula, we have

$$\deg \mathcal{R} = 2(g + d - 1)$$

Now  $s$  is a nonzero section of the line bundle  $\mathcal{L} = \varphi_{t_0}^* \mathbb{TP}^1(-\mathcal{R}_0 - \mathcal{R}_1 - \mathcal{R}_\infty)$ ; hence

$$0 \leq \deg \mathcal{L} = 2d - \deg \mathcal{R} + \deg \mathcal{R}_t \leq 2d - \frac{3}{4} \deg \mathcal{R} = \frac{d}{2} - \frac{3g}{2} + \frac{3}{2}.$$

The result follows.  $\square$

**Corollary 3.** *Let  $(X, q, \pi)$  be a pillow-tiled surface of genus  $g$  whose Teichmüller disc lies in the Forni locus, and let  $d$  be the degree of  $\pi$ . If the ramification locus of  $\pi$  consists of 4 points, then  $d \geq 3(g - 1)$ .*

We end this section by proving that the estimate  $n \geq 2g - 2$  of Proposition 6 is asymptotically sharp and that the condition on the ramification locus of  $\pi$  in Corollary 3 is necessary.

Let  $p$  be a prime number, and let  $Y$  be a cyclic covering of degree  $p$  of the projective line fully ramified at three distinct points  $z_1, z_2$  and  $z_3$  obtained by taking the Riemann surface of the polynomial

$$w^p - (z - z_1)^{a_1}(z - z_2)^{a_2}(z - z_3)^{a_3}$$

where  $1 \leq a_1, a_2, a_3 \leq p - 1$  and  $a_1 + a_2 + a_3 = p$ . Then the genus of  $Y$  is  $\frac{p-1}{2}$ . If  $q$  is the pull-back of a meromorphic differential on the sphere with four simple poles at  $z_1, z_2, z_3$  and another point  $z_4$ , then  $\pi^*q$  has exactly  $p$  poles, and three zeros of order  $p - 2$ . Therefore, we have

$$\underbrace{n}_p = \underbrace{2g - 2}_{p-3} + \underbrace{\sum_j \frac{m_j}{m_j + 2}}_{3 - \frac{6}{p}} + \underbrace{\frac{\pi^2}{3} C_{\text{area}}}_{\frac{6}{p}}$$

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I2M, UNIVERSITÉ D’AIX-MARSEILLE, 39 RUE F. JOLIOT-CURIE, 13453 MARSEILLE CEDEX 20  
*E-mail address:* jgrivaux@math.cnrs.fr

I2M, UNIVERSITÉ D’AIX-MARSEILLE, 39 RUE F. JOLIOT-CURIE, 13453 MARSEILLE CEDEX 20  
*E-mail address:* pascal.hubert@univ-amu.fr