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# Algebraic and logical descriptions of generalized trees

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*Abstract*<sup>1</sup> : *Quasi-trees* generalize trees in that the unique "path" between two nodes may be infinite and have any countable order type. They are used to define the rank-width of a countable graph in such a way that it is equal to the least upper-bound of the rank-widths of its finite induced subgraphs. *Join-trees* are the corresponding directed trees. They are useful to define the modular decomposition of a countable graph. We also consider *ordered join-trees*, that generalize rooted trees equipped with a linear order on the set of sons of each node. We define algebras with finitely many operations that generate (via infinite terms) these generalized trees. We prove that the associated regular objects (those defined by regular terms) are exactly the ones that are the unique models of monadic second-order sentences. These results use and generalize a similar result by W. Thomas for countable linear orders.

*Keywords* : Rank-width, quasi-tree, join-tree, ordered tree, algebra, regular term, monadic second-order logic.

## 1 Introduction

We define and study countable generalized trees, called *quasi-trees*, such that the unique "path" between two nodes may be infinite and have any order type, in particular that of rational numbers. Our motivation comes from the notion

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of *rank-width*, a complexity measure of finite graphs investigated first in [Oum] and [OumSey]. Rank-width is based on graph decompositions formalized with finite subcubic trees. In order to extend rank-width to countable graphs in such a way that *the compactness property* holds, i.e., that the rank-width of a countable graph is the least upper-bound of those of its finite induced subgraphs, we base decompositions on *subcubic quasi-trees* [Cou14]. (For a comparison, the natural extension of tree-width to countable graphs has the compactness property [KriTho] without needing quasi-trees.) *Join-trees* can be seen as directed quasi-trees. A join-tree is a partial order  $(N, \leq)$  such that every two elements have a least upper-bound (called their *join*) and each set  $\{y \mid y \geq x\}$  is linearly ordered. The modular decomposition of a countable graph is based on an *ordered join-tree* [CouDel].

Our objective is to obtain finitary descriptions (usable in algorithms) for the following generalized trees : join-trees, ordered join-trees and quasi-trees. For this purpose we will define algebras of such generalized trees that use finitely many operations and such that the finite and infinite terms over these operations define all countable relevant generalized trees. The *regular objects* are those defined by *regular terms*, i.e. that have finitely many different subterms, equivalently, that are the unique solutions of certain finite equation systems. We will prove that a generalized tree is regular if and only if it is *monadic second-order definable*, i.e., is the unique model (up to isomorphism) of a monadic second-order sentence.

A linear order whose elements are labelled by letters from an alphabet is called an *arrangement*. Regular arrangements were defined and studied in [Cou78] and [Hei], and their monadic second-order definability is proved in [Tho86]. We use the latter result for proving its extension to generalized trees.

In Section 2 we review definitions. In Section 3 we first study binary join-trees and then, we extend the definitions and results concerning them to all join-trees. In Section 4, we study ordered join-trees, and in Section 5, we study quasi-trees. An appendix reviews definitions and facts about monadic second-order logic and the application of quasi-trees to the rank-width of countable graphs. An introductory article on these results is [Cou15].

## 2 Definitions, notation and basic facts.

All sets, trees and logical structures are finite or countably infinite. We denote by  $X \uplus Y$  the union of  $X$  and  $Y$  if they are disjoint. Isomorphism of ordered sets, trees and other logical structures is denoted by  $\simeq$ .

For partial orders  $\leq, \preceq, \sqsubseteq, \dots$  we denote respectively by  $<, \prec, \sqsubset, \dots$  the corresponding strict orders and  $X < Y$  means that  $x < y$  for every  $x \in X$  and  $y \in Y$ .

Let  $(V, \leq)$  be a partial order. The least upper bound of  $x$  and  $y$  is denoted by  $x \sqcup y$  if it exists and is called their *join*. The notation  $x \perp y$  means that  $x$  and  $y$

are incomparable. A *line*<sup>2</sup> is a subset  $Y$  of  $V$  that is linearly ordered and satisfies the following *convexity property*: if  $x, z \in Y$ ,  $y \in V$  and  $x \leq y \leq z$ , then  $y \in Y$ . Particular notations for convex sets (not necessarily linearly ordered) are  $[x, y]$  denoting  $\{z \mid x \leq z \leq y\}$ ,  $]x, y]$  denoting  $\{z \mid x < z \leq y\}$ ,  $] - \infty, x]$  denoting  $\{y \mid y \leq x\}$  (even if  $V$  is finite),  $]x, +\infty[$  denoting  $\{y \mid x < y\}$  etc. If  $X \subseteq V$ , then  $\downarrow(X)$  is the union of the sets  $] - \infty, x]$  for  $x$  in  $X$ .

The first infinite ordinal and the linear order  $(\mathbb{N}, \leq)$  are denoted by  $\omega$ .

The restriction of a relation  $R$  or a function  $f$  defined on a set  $V$  to a subset  $W$  is denoted by  $R \upharpoonright W$  or  $f \upharpoonright W$  respectively.

*Monadic second-order logic* (*MS logic* in short) logic is reviewed in the appendix. *MS-definable* and *MS-expressible* mean respectively definable and expressible in monadic second-order logic. A *sentence* is a formula without free variables.

## 2.1 Trees

A *tree* is a possibly empty, finite or countable, undirected graph that is connected and has no cycles. Hence, it has neither loops nor parallel edges. The set of nodes of a tree  $T$  is  $N_T$ .

A *rooted tree* is a nonempty tree equipped with a distinguished node called its *root*. The *level* of a node  $x$  is the number of edges of the path between it and the root and  $Sons(x)$  denotes the set of its sons. We define on  $N_T$  the partial order  $\leq_T$  such that  $x \leq_T y$  if and only if  $y$  is on the unique path between  $x$  and the root. The least upper bound of  $x$  and  $y$ , denoted by  $x \sqcup_T y$  is their least common ancestor. We will specify a rooted tree  $T$  by  $(N_T, \leq_T)$  and we will omit the index  $T$  when the considered tree is clear. For a node  $x$  of  $T$ , the *subtree issued from  $x$* , denoted by  $T/x$ , is defined as  $(N_{T/x}, \leq_T \upharpoonright N_{T/x})$  where  $N_{T/x} := ] - \infty, x]$ .

A partial order  $(N, \leq)$  is  $(N_T, \leq_T)$  for some rooted tree  $T$  if and only if it has a largest element *max* and for each  $x \in N$ , the set  $[x, \text{max}]$  is finite and linearly ordered. These conditions imply that any two nodes have a join.

An *ordered tree* is a rooted tree such that each set  $Sons(x)$  is linearly ordered by an order  $\sqsubseteq_x$ .

## 2.2 Finite and infinite terms

Let  $F$  be a finite set of operations  $f$ , each given with an arity  $\rho(f)$ . We call such a set a *signature*. The maximal arity of a symbol is denoted by  $\rho(F)$ . A *term over  $F$*  is finite or infinite. We denote by  $T^\infty(F)$  the set of all terms over  $F$  and by  $T(F)$  the set of finite ones. A typical example of an infinite term, easily describable linearly, is, with  $f$  binary and  $a$  and  $b$  nullary, the term  $t_\infty := f(a, f(b, f(a, f(b, f(\dots\dots\dots))))))$  that is the unique solution in  $T^\infty(F)$  of the equation  $t = f(a, f(b, t))$ .

<sup>2</sup>In [Cou14] we call *line* a linearly ordered subset, without imposing the convexity property.

Positions in terms are designated by Dewey words. The set  $Pos(t)$  of positions of a term  $t$  is ordered by  $\leq_t$ , the reversal of the prefix order. A term  $t$  can be seen as a labelled, ordered and rooted tree whose set of nodes is  $Pos(t)$ . We have  $Pos(t_\infty) = 2^* \uplus 2^*1$ , where  $2^*$  is the set of occurrences of  $f$ ,  $(22)^*1$  is the set of occurrences of  $a$  and  $(22)^*21$  is that of  $b$ .

There is a canonical structure of  $F$ -algebra on  $T^\infty(F)$ , of which  $T(F)$  is a subalgebra. If  $\mathbb{M} = \langle M, (f_{\mathbb{M}})_{f \in F} \rangle$  is an  $F$ -algebra, a *value mapping* is a homomorphism  $h : T^\infty(F) \rightarrow \mathbb{M}$ . Its restriction to finite terms is uniquely defined.

In some cases, we will use algebras with two sorts. The corresponding modifications of the definitions are straightforward.

*The partial order on terms.*

Let  $F$  contain a special nullary symbol  $\Omega$  intended to be the least term. We define on  $T(F)$  a partial order  $\leq$  as follows:

$$\begin{aligned} \Omega &\leq t \quad \text{for any } t \in T(F), \\ f(t_1, \dots, t_k) &\leq g(t'_1, \dots, t'_k) \quad \text{if and only if } k = k', f = g \text{ and } t_i \leq t'_i \\ &\text{for } i = 1, \dots, k. \end{aligned}$$

For terms in  $T^\infty(F)$ , the definition (subsuming the previous one) is :

$$t \leq t' \text{ if and only if } Pos(t) \subseteq Pos(t') \text{ and every occurrence in } t \text{ of a symbol in } F - \{\Omega\} \text{ is an occurrence in } t' \text{ of the same symbol (and an occurrence in } t \text{ of } \Omega \text{ is an occurrence in } t' \text{ of any symbol).}$$

Every increasing sequence of terms has a least upper bound. More details on terms can be found in [Cou83].

*Regular terms*

A term  $t \in T^\infty(F)$  is *regular* if there is a mapping  $h$  from  $Pos(t)$  into a finite set  $Q$  and a mapping  $\tau : Q \rightarrow F \times Seq(Q)$  (where  $Seq(Q)$  denotes the set of finite sequences of elements of  $Q$ ) such that:

$$\text{if } u \text{ is an occurrence of a symbol } f \text{ of arity } k, \text{ then } \tau(h(u)) = (f, (h(u_1), \dots, h(u_k))) \text{ where } (u_1, \dots, u_k) \text{ is the sequence of sons of } u.$$

Intuitively,  $\tau$  is the transition function of a top-down deterministic automaton with set of states  $Q$ ;  $h(\varepsilon)$  is its initial (root) state and  $h$  defines its unique run. This is equivalent to requiring that  $t$  has finitely many different subterms, or is a component of a finite system of equations that has a unique solution in  $T^\infty(F)$ . (The set  $Q$  can be taken as the set of unknowns of such a system, see [Cou83].)

The above term  $t_\infty$  is regular with  $Q := \{1, 2, 3, 4\}$ ,  $\tau(1) = (f, (2, 3))$ ,  $\tau(2) = (a, ())$ ,  $\tau(3) = (f, (4, 1))$ ,  $\tau(4) = (b, ())$ .

With a term  $t$  we associate the relational structure  $[t] := (Pos(t), \leq_t, (br_i)_{1 \leq i \leq \rho(F)}, (lab_f)_{f \in F})$  where  $br_i(u)$  is true if and only if  $u$  is the  $i$ -th son of his father and  $lab_f(u)$  is true if and only if  $f$  occurs at position  $u$ . A term  $t$  can

be reconstructed in a unique way from any relational structure isomorphic to  $[t]$ .

A term  $t$  is regular if and only if  $[t]$  is MS-definable, i.e., is, up to isomorphism, the unique model of a monadic second-order sentence (see Thomas, [Tho90]).

### 2.3 Arrangements and labelled sets

We review a notion introduced in [Cou78] and further studied in [Hei, Tho86]. Let  $X$  be a set. A linear order  $(V, \leq)$  equipped with a labelling mapping  $lab : V \rightarrow X$  is called an *arrangement over  $X$* . It is *simple* if  $lab$  is injective. We denote by  $\mathcal{A}(X)$  the set of arrangements over  $X$ .

An arrangement over a finite set  $X$  can be considered as a generalized word. A linear order  $(V, \leq)$  is identified with the simple arrangement  $(V, \leq, Id_V)$  such that  $Id_V(v) := v$  for each  $v \in V$ . In the sequel,  $Id$  will denote the identity function on any set.

An *isomorphism of arrangements*  $i : (V, \leq, lab) \rightarrow (V', \leq', lab')$  is an order preserving bijection  $i : V \rightarrow V'$  such that  $lab' \circ i = lab$ . Isomorphism is denoted by  $\simeq$ .

If  $w = (V, \leq, lab) \in \mathcal{A}(X)$  and  $r : X \rightarrow Y$ , then,  $\bar{r}(w) := (V, \leq, r \circ lab)$  is an arrangement over  $Y$ . If  $r$  maps  $V$  into  $Y$ , then  $\bar{r}((V, \leq))$  is the arrangement  $(V, \leq, r)$  over  $Y$  since we identify  $(V, \leq)$  to the simple arrangement  $(V, \leq, Id)$ .

The concatenation of linear orders yield a concatenation of arrangements denoted by  $\bullet$ . We denote by  $\Omega$  the empty arrangement and by  $a$  the one reduced to a single occurrence of  $a \in X$ . Clearly,  $w \bullet \Omega = \Omega \bullet w = w$  for every  $w \in \mathcal{A}(X)$ . The infinite word  $w = a^\omega$  is the arrangement over  $\{a\}$  with underlying order  $\omega$ ; it is described by the equation  $w = a \bullet w$ . Similarly, the arrangement  $w = a^\eta$  over  $\{a\}$  with underlying linear order  $(\mathbb{Q}, \leq)$  (that of rational numbers) is described by the equation  $w = w \bullet (a \bullet w)$ . We will generalize arrangements to tree structures.

Let  $X$  be a set of first-order variables (they are nullary symbols) and  $t \in T^\infty(\{\bullet, \Omega\} \cup X)$ . Hence,  $Pos(t) \subseteq \{1, 2\}^*$ . The *value* of  $t$  is the arrangement  $val(t) := (Occ(t, X), \leq_{lex}, lab)$  where  $Occ(t, X)$  is the set of positions of the elements of  $X$  and  $lab(u)$  is the symbol of  $X$  occurring at position  $u$ . We say that  $t$  *denotes*  $w$  if  $w$  is isomorphic to  $val(t)$ .

For an example,  $t_\bullet := \bullet(a, \bullet(b, \bullet(a, \bullet(b, \bullet(\dots\dots\dots))))))$  denotes the infinite word  $abab\dots$ . Its value is defined from  $Occ(t_\bullet, \{a, b\}) = 2^*1$ , lexicographically ordered (i.e.,  $1 < 21 < 221 < \dots$ ) by taking  $lab(2^i 1) := a$  if  $i$  is even and  $lab(2^i 1) := b$  if  $i$  is odd. The arrangements  $a^\omega$  and  $a^\eta$  are denoted respectively by  $t_1$  and  $t_2$  that are the unique solutions in  $T^\infty(\{\bullet, \Omega, a\})$  of the equations  $t_1 = a \bullet t_1$  and  $t_2 = t_2 \bullet (a \bullet t_2)$ .

An arrangement is *regular* if it is denoted by a regular term. The term  $t_\bullet$  is regular. The arrangements  $a^\omega$  and  $a^\eta$  are regular.

An arrangement is regular if and only if it is a component of the *initial solution of a regular system of equations* over  $F$  [Cou78] or the value of a *regular expression* in the sense of [Hei]. We will use the result of [Tho86] that an arrangement over a finite alphabet is regular if and only if it is MS-definable. For this result, we represent an arrangement  $w = (V, \leq, lab)$  over  $X$  by the relational structure  $\llbracket w \rrbracket := (V, \leq, (lab_a)_{a \in X})$  where  $lab_a(u)$  is true if and only if  $lab(u) = a$ .

An  $X$ -labelled set is a pair  $m = (V, lab)$  where  $lab : V \rightarrow X$ , equivalently, a relational structure  $(V, (lab_a)_{a \in X})$  where each element of  $V$  belongs to a unique set  $lab_a$ . We denote by  $set(w)$  the  $X$ -labelled set obtained by forgetting the linear order of an arrangement  $w$  over  $X$ . Up to isomorphism, an  $X$ -labelled set  $m$  is defined by the cardinalities in  $\mathbb{N} \cup \{\omega\}$  of the sets  $lab_a$ , hence is a finite or countable *multiset of elements of  $X$* : a number in  $\mathbb{N} \cup \{\omega\}$  associated with each  $a \in X$  and represents its number of occurrences in  $m$ .

If  $X$  is finite, each  $X$ -labelled set is  $MS_{fin}$ -definable, *i.e.*, is the unique, finite or countably infinite model up to isomorphism of a sentence of *monadic second-order logic* extended with a set predicate  $Fin(U)$  expressing that the set  $U$  is finite. (See the appendix for details). It is also *regular*, hence, is  $set(val(t))$  for some regular term in  $T^\infty(\{\bullet, \Omega\} \cup X)$ . The notion of regularity is thus trivial for  $X$ -labelled sets when  $X$  is finite.

### 3 Join-trees

Join-trees have been used in [CouDel] for defining the modular decomposition of countable graphs.

#### 3.1 Join-trees, join-forests and their structurings

(3.1) **Definition** : *Join-tree*.

(a) A *join-tree* is a pair  $J = (N, \leq)$  such that:

- 1)  $N$  is a possibly empty, finite or countable set called the set of *nodes*,
- 2)  $\leq$  is a partial order on  $N$  such that, for every node  $x$ , the set  $[x, +\infty[$  (the set of nodes  $y \geq x$ ) is linearly ordered,
- 3) every two nodes  $x$  and  $y$  have a join  $x \sqcup y$ .

A minimal node is a *leaf*. If  $N$  has a largest element, we call it the *root* of  $J$ . The set of strict upper-bounds of a nonempty set  $X \subseteq N$  is a line  $L$ . If  $L$  has a smallest element, we denote it by  $\hat{X}$  and we say that  $\hat{X}$  is *the top* of  $X$ . Note that  $\hat{X} \notin X$ .

(b) A *join-forest* is a pair  $J = (N, \leq)$  that satisfies conditions 1), 2) and the following weakening of 3):

3') if two nodes have an upper bound, they have a join.

The relation that two nodes have a join is an equivalence. Let  $N_s$  for  $s \in S$  be its equivalence classes and  $J_s := (N_s, \leq \upharpoonright N_s)$ , more simply denoted by  $(N_s, \leq)$  by leaving implicit the restriction to  $N_s$ . Then each  $J_s$  is a join-tree, and  $J$  is the union of these pairwise disjoint join-trees, called its *components*.

(c) A join-forest  $J = (N, \leq)$  is *included in* a join-forest  $J' = (N', \leq')$ , denoted by  $J \subseteq J'$ , if  $N \subseteq N'$ ,  $\leq$  is  $\leq' \upharpoonright N$  and  $\sqcup$  is  $\sqcup' \upharpoonright N$ .

(3.2) **Definitions** : *Direction and degree*.

Let  $J = (N, \leq)$  be a join-forest, and  $x$  be one of its nodes. Let  $\sim$  be the equivalence relation on  $] - \infty, x[$  such that  $z \sim y$  if and only if  $z \sqcup y < x$ . Each equivalence class  $C$  is called a *direction of  $J$  relative to  $x$*  and  $\widehat{C} = x$ . The set of directions relative to  $x$  is denoted by  $Dir(x)$  and the *degree* of  $x$  is the number of its directions. The leaves are the nodes of degree 0.

A join-tree is *binary* if its nodes have degree at most 2. We call it a *BJ-tree* for short.

(3.3) **Definition** : *Structured join-trees and join-forests*.

(a) Let  $J = (N, \leq)$  be a join-tree. A *structuring* of  $J$  is a set  $\mathcal{U}$  of nonempty lines forming a partition of  $N$  that satisfies some conditions, stated with the following notation : if  $x \in N$ , then  $U(x)$  denotes the line of  $\mathcal{U}$  containing  $x$ ,  $U_-(x) := U(x) \cap ] - \infty, x[$  and  $U_+(x) := U(x) \cap [x, +\infty[$ . (The set  $[x, +\infty[$  has no top but it can have a greatest element). The conditions are:

- 1) exactly one line  $A$  of  $\mathcal{U}$  is *upwards closed* (i.e.,  $[x, +\infty[ \subseteq A$  if  $x \in A$ ), hence, has no strict upper-bound and no top; we call it the *axis*; each other line  $U$  has a top  $\widehat{U}$ ,
- 2) for each  $x$  in  $N$ , the sequence  $y_0, y_1, y_2, \dots$  such that  $y_0 = x$ ,  $y_{i+1} = \widehat{U(y_i)}$  is finite; its last element is  $y_k \in A$  ( $y_{k+1}$  is undefined) and we call  $k$  the *depth* of  $x$ .

The nodes on the axis are those at depth 0. The lines  $[y_i, y_{i+1}[$  for  $i = 0, \dots, k-1$  and  $[y_k, +\infty[$  are convex subsets of pairwise distinct lines of  $\mathcal{U}$ . We have  $[x, +\infty[ = [y_0, y_1[ \uplus [y_1, y_2[ \uplus \dots \uplus [y_k, +\infty[$ , where  $[y_i, y_{i+1}[ = U_+(y_i)$  for each  $i < k$ ,  $[y_k, +\infty[ = U_+(y_k) \subseteq A$  and the depth of  $y_i$  is  $k - i$ .

We call such a triple  $(N, \leq, \mathcal{U})$  a *structured join-tree*, an *SJ-tree* for short.

Every linear order is an SJ-tree whose elements are all of depth 0.

*Remark* : If  $x < A$  for some  $x$ , then  $A$  has a smallest element, which is the node  $y_k$  of Condition 2) (because if  $z \in A$  is smaller than  $y_k$ , then  $x < z$ , which contradicts the observation that  $[y_{k-1}, y_k[ \subseteq U(y_{k-1})$  and  $U(y_{k-1}) \cap A = \emptyset$ ).



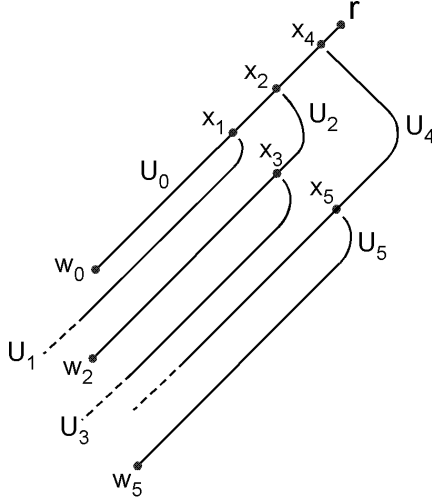


Figure 1: A structured binary join-tree.

(b) Let  $J = (N, \leq)$  be a join-forest whose components are  $J_s$ ,  $s \in S$ . A *structuring* of  $J$  is a set  $\mathcal{U}$  of nonempty lines forming a partition of  $N$  such that, if  $\mathcal{U}_s$  is the set of lines of  $\mathcal{U}$  included in  $N_s$  (every line of  $\mathcal{U}$  is included in some  $N_s$ ), then each triple  $(N_s, \leq, \mathcal{U}_s)$  is a structuring of  $J_s$ .

(3.4) *Example* : Figure 1 shows a structuring  $\{U_0, \dots, U_5\}$  of a binary join-tree. The axis is  $U_0$ . The directions relative to  $x_2$  are  $U_-(x_2) \cup U_1$  and  $U_2 \cup U_3$ . The maximal depth of a node is 2.

(3.5) **Proposition** : Every join-tree and, more generally, every join-forest has a structuring.

**Proof** : Let  $J = (N, \leq)$  be a join-tree. Let us choose an enumeration of  $N$  and a maximal line  $B_0$  ; it is upwards closed. For each  $i > 0$ , we choose a maximal line  $B_i$  containing the first node not in  $B_{i-1} \cup \dots \cup B_0$ . We define  $U_0 := B_0$  and, for  $i > 0$ ,  $U_i := B_i - (U_{i-1} \uplus \dots \uplus U_0) = B_i - (B_{i-1} \cup \dots \cup B_0)$ . We define  $\mathcal{U}$  as the set of lines  $U_i$ . It is a structuring of  $J$ . The axis is  $U_0$ .

If  $J$  is a join-forest, it has a structuring that is the union of structurings of its components.  $\square$

*Remark*: Since each line  $B_i$  is maximal, if  $U_i$  has smallest element, this element is a node of degree 0 in  $J$ .

In view of our use of monadic second-order logic, we give a description of SJ-trees by relational structures.

(3.6) **Definition** : *SJ-trees as relational structures.*

(a) If  $J = (N, \leq, \mathcal{U})$  is an SJ-tree, we define  $S(J)$  as the relational structure  $(N, \leq, N_0, N_1)$  such that  $N_0$  is the set of nodes at even depth and  $N_1 := N - N_0$  is the set of those at odd depth. ( $N_0$  and  $N_1$  are sets but we consider them also as unary relations).

(b) Let  $J = (N, \leq)$  be a join-tree and  $X \subseteq N$ . We say that  $X$  is *laminar* if, for all  $x, y, z \in X$ , if  $[x, z] \cup [y, z] \subseteq X$  (where  $x < z$  and  $y < z$ ), then  $[x, z] \subseteq [y, z]$  or  $[y, z] \subseteq [x, z]$  (the intervals  $[x, z]$  and  $[y, z]$  are relative to  $J$ ). This condition implies that the lines of  $J$  that are included in  $X$  and are maximal with this condition form a partition of  $X$  whose parts will be called its *components*.

(3.7) **Proposition** : For  $J$  and  $S(J)$  as above, the following properties hold:

- 1) the sets  $N_0$  and  $N_1$  are laminar,  $\mathcal{U}$  is the set of their components and the axis  $A$  is a component of  $N_0$ ,
- 2) there is an MS formula  $\varphi(N_0, N_1)$  expressing that a relational structure  $(N, \leq, N_0, N_1)$  is  $S(J)$  for some SJ-tree  $J = (N, \leq, \mathcal{U})$ ,
- 3) there exist MS formulas  $\theta_{Ax}(X, N_0, N_1)$  and  $\theta(u, U, N_0, N_1)$  expressing, respectively, in a structure  $(N, \leq, N_0, N_1) = S(N, \leq, \mathcal{U})$ , that  $X$  is the axis and that  $U \in \mathcal{U} \wedge u = \widehat{U}$ .

**Proof** : 1) is clear from the definitions.

2) That a partial order  $(N, \leq)$  is a join-tree is first-order expressible. The formula  $\varphi(N_0, N_1)$  will include this condition. Let  $J = (N, \leq)$  be a join-tree; let  $N$  be the union of two disjoint laminar sets  $N_0$  and  $N_1$  and  $\mathcal{U}$  be the set of their components. Then,  $J = (N, \leq, \mathcal{U})$  is an SJ-tree and  $S(J) = (N, \leq, N_0, N_1)$  if and only if:

- i) every component of  $N_1$  has a top in  $N_0$  and every component of  $N_0$  except one has a top in  $N_1$ ,
- ii) for each  $U$  in  $\mathcal{U}$ , the sequence  $U_0, U_1, \dots$  of lines of  $\mathcal{U}$  such that  $U_0 = U$ ,  $\widehat{U}_0 \in U_1, \dots, \widehat{U}_i \in U_{i+1}$  terminates at some  $U_k$  that has no top, hence is included in  $N_0$ .

These conditions are necessary. As they rephrase Definition (3.3), they are also sufficient. The integer  $k$  in Condition ii) is the common depth of all nodes in  $U$ .

That a set  $X$  is laminar is MS-expressible, and one can build an MS-formula  $\psi(U, X)$  expressing that  $U$  is a component of  $X$  assumed to be laminar. This formula can be used to express that  $N$  is the union of two disjoint laminar sets  $N_0$  and  $N_1$  that satisfy Conditions i) and ii). For expressing Condition ii), we define for each  $U$  in  $\mathcal{U}$  a set of nodes  $W$  as follows : it is the least set such that  $\widehat{U} \in W$ , and, for each  $w \in W$ , the top of  $U(w)$  belongs to  $W$  if it is defined (where  $U(w)$  is the unique set in  $\mathcal{U}$  that contains  $w$ ). The set  $W$  is linearly ordered (it consists of  $\widehat{U}_0 < \dots < \widehat{U}_i \dots$ ) and Condition ii) says that it must be

finite. To write the formula, we use the fact that the finiteness of a linearly ordered set is MS-expressible (see the Appendix).

3) The construction of  $\varphi$  actually uses the MS formulas  $\theta_{Ax}(X, N_0, N_1)$  and  $\theta(u, U, N_0, N_1)$ .  $\square$

### 3.2 Description schemes of structured binary join-trees

In order to introduce technicalities step by step, we first consider binary join-trees. They are actually sufficient for defining the rank-width of a countable graph. See Section 5 and the appendix.

(3.8) **Definition** : *Structured binary join-trees.*

Let  $J = (N, \leq)$  be a binary join-tree. A *structuring* of  $J$  is a set  $\mathcal{U}$  of lines satisfying the conditions of Definition (3.3) and, furthermore:

- i) if the axis  $A$  has a smallest element, then its degree is 0 or 1,
- ii) each  $x \in N$  is the top of at most one set  $U \in \mathcal{U}$ , denoted by  $U^x$ , and  $U^x := \emptyset$  if  $x$  is the top of no  $U \in \mathcal{U}$ .

We call  $(N, \leq, \mathcal{U})$  a *structured binary join-tree*, an *SBJ-tree* in short.

(3.9) **Proposition**: 1) Every BJ-tree  $J$  has a structuring.

2) The class of structures  $S(J)$  for SBJ-trees  $J$  is monadic second-order definable.

**Proof:** 1) We use the construction of Proposition (3.5) for  $J = (N, \leq)$ . The remark following it implies that, if the axis  $A = U_0$  has a smallest element, this element has degree 0. It implies also that, if  $\widehat{U}_i = x$ , then  $x$  cannot have degree 0 in the BJ-tree  $J_{i-1}$  induced by  $U_{i-1} \uplus \dots \uplus U_0$  because each line  $B_i$  is chosen maximal, and it cannot have degree 2 or more in  $J_{i-1}$  because  $J$  is binary. Hence it has degree 1 in  $J_{i-1}$ . It follows that  $x$  is the top of no line  $U_j$  for  $j < i$ . Hence ii) holds and the construction yields an SBJ-tree  $(N, \leq, \mathcal{U})$ .

2) The formula  $\varphi$  of Proposition (3.7) can easily be modified so as to express that  $(N, \leq, N_0, N_1)$  is  $S(J)$  for some SBJ-tree  $J$ .  $\square$

(3.10) **Definition** : *Description schemes for SBJ-trees.*

(a) A *description scheme for an SBJ-tree*, in short an *SBJ-scheme*, is a triple  $\Delta = (Q, w_{Ax}, (w_q)_{q \in Q})$  such that  $Q$  is a set,  $w_{Ax} \in \mathcal{A}(Q)$  (is an arrangement over  $Q$ ) and  $w_q \in \mathcal{A}(Q)$  for each  $q$ .

It is *regular* if  $Q$  is finite and the arrangements  $w_{Ax}$  and  $w_q$  are regular.

(b) We recall that a linear order  $(V, \leq)$  is identified with the arrangement  $(V, \leq, Id)$ . If  $W \subseteq V$  and  $r : V \rightarrow Q$ , then  $\bar{r}((W, \leq))$  is the arrangement

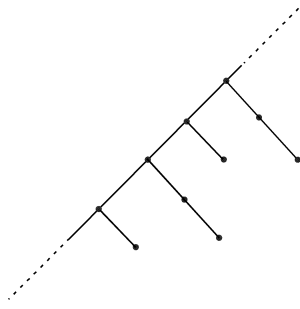


Figure 2: A binary join-tree.

$(W, \leq \upharpoonright W, r \upharpoonright W) \in \mathcal{A}(Q)$  that we will denote more simply by  $(W, \leq, r)$  leaving implicit the restrictions of  $\leq$  and  $r$  to  $W$ .

An SBJ-scheme  $\Delta$  describes an SBJ-tree  $J = (N, \leq, \mathcal{U})$  whose axis is  $A$  if there exists a mapping  $r : N \rightarrow Q$  such that:

$$\bar{r}((A, \leq)) \simeq w_{Ax} \text{ and } \bar{r}((U^x, \leq)) \simeq w_{r(x)} \text{ for every } x \in N.$$

We will also say that  $\Delta$  describes the BJ-tree  $fgs(J) := (N, \leq)$ .

The mapping  $r$  need not be surjective, this means that some elements of  $Q$  and the corresponding arrangements may be useless, and thus can be removed from  $\Delta$ .

(c) Each SBJ-tree  $J = (N, \leq, \mathcal{U})$  has a *standard* description scheme  $\Delta(J) := (N, (A, \leq), ((U^x, \leq))_{x \in N})$ . The identity mapping  $r : N \rightarrow N$  shows that  $\Delta(J)$  describes  $J$ .

Intuitively, one obtains from  $\Delta(J)$  an SBJ-scheme  $\Delta := (Q, w_{Ax}, (w_q)_{q \in Q})$  that describes  $J$  by a kind of quotient construction :

- (i) one chooses a partition  $Q$  of  $N$  satisfying conditions (ii) and (iii) below, where for each  $x \in N$ ,  $r(x)$  denotes the part  $q$  of the partition  $Q$  that contains  $x$ ,
- (ii) for all nodes  $x$  in a part  $q$  of  $Q$ , the arrangements  $(U^x, \leq, r)$  are isomorphic to  $w_q$ ,
- (iii)  $(A, \leq, r) \simeq w_{Ax}$ .

(d) *An example:* Let  $\Delta = (Q, w_{Ax}, (w_q)_{q \in Q})$  be the SBJ-scheme such that  $Q = \{a, b, c\}$ ,  $w_{Ax}$  is  $(\mathbb{Z}, \leq, \ell)$  where  $\ell(i) = a$  if  $i$  is even and  $\ell(i) = b$  if  $i$  is odd,  $w_a = \{c\}$ ,  $w_b = cc$  (two nodes labelled by  $c$ ) and  $w_c = \Omega$ . It describes the BJ-tree of Figure 2.  $\square$

(3.11) **Proposition:** Every SBJ-scheme  $\Delta$  describes a unique SBJ-tree where unicity is up to isomorphism.

We denote this SBJ-tree by  $Unf(\Delta)$ , called the *unfolding of  $\Delta$*  (see the remark following the proof about terminology).

**Proof:** Let  $\Delta = (Q, w_{Ax}, (w_q)_{q \in Q})$  be an SBJ-scheme, defined with arrangements  $w_{Ax} = (V_{Ax}, \preceq, lab_{Ax})$  and  $w_q = (V_q, \preceq, lab_q)$  such that, without loss of generality, the sets  $V_{Ax}$  and  $V_q$  are pairwise disjoint and the same symbol  $\preceq$  denotes their orders.

We construct  $(N, \leq, \mathcal{U}) = Unf(\Delta)$  as follows.

a)  $N$  is the set of finite nonempty sequences  $(v_0, v_1, \dots, v_k)$  such that:

$v_0 \in V_{Ax}, v_i \in V_{q_i}$  for  $1 \leq i \leq k$ , where

$q_1 = lab_{Ax}(v_0), q_2 = lab_{q_1}(v_1), \dots, q_k = lab_{q_{k-1}}(v_{k-1})$ .

b)  $(v_0, v_1, \dots, v_k) \leq (v'_0, v'_1, \dots, v'_j)$  if and only if  $k \geq j$ ,  $(v_0, v_1, \dots, v_{j-1}) = (v'_0, v'_1, \dots, v'_{j-1})$  and  $v_j \preceq v'_j$ .

c) The axis  $A$  is the set of one-element sequences  $(v)$  for  $v \in V_{Ax}$ ; for  $x = (v_0, v_1, \dots, v_k)$ , we define  $U(x)$  as the set of sequences  $(v_0, v_1, \dots, v_{k-1}, v)$  such that  $v \in V_{q_k}$ , hence, we have  $\widehat{U(x)} = (v_0, v_1, \dots, v_{k-1})$ .

Note that  $(v_0, \dots, v_k) < (v_0, \dots, v_j)$  if  $j < k$  and that  $(v_0, \dots, v_{k-1}, v_k) \leq (v_0, \dots, v_{k-1}, v)$  if and only if  $v_k \preceq v$ .

*Claim :*  $\Delta$  describes  $(N, \leq, \mathcal{U})$ .

We define  $r : N \rightarrow Q$  as follows:

if  $x \in A$ , then  $x = (v)$  for some  $v \in V_{Ax}$  and  $r(x) := lab_{Ax}(v)$ ;

if  $x \in N$  has depth  $k \geq 1$ , then  $x = (v_0, v_1, \dots, v_k)$  for some  $v_0, v_1, \dots, v_k$  as in a) and  $r(x) := lab_{q_k}(v_k)$ .

It follows that  $\bar{r}((A, \leq)) \simeq w_{Ax}$  and that, for  $x = (v_0, v_1, \dots, v_k)$  (of depth  $k$ ),  $\bar{r}((U^x, \leq)) \simeq w_{q_k} = w_{r(x)}$ , which proves the claim.  $\square$

We now prove unicity. Assume that  $\Delta$  describes  $J = (N, \leq, \mathcal{U})$  with axis  $A$  and also  $J' = (N', \leq', \mathcal{U}')$  with axis  $A'$ , by means of mappings  $r : N \rightarrow Q$  and  $r' : N' \rightarrow Q$ . We construct an isomorphism  $h : J \rightarrow J'$  as the common extension of bijections  $h_k : N_k \rightarrow N'_k$ , where  $N_k$  (resp.  $N'_k$ ) is the set of nodes of  $J$  (resp. of  $J'$ ) of depth at most  $k$ , and such that :

they map  $\leq$  to  $\leq'$ ,

they map the lines of  $\mathcal{U}$  to those of  $\mathcal{U}'$  of same depth,

$r' \circ h_k = r \upharpoonright N_k$ .

Case  $k = 0$ . We have:

$\bar{r}((A, \leq)) = (A, \leq, r) \simeq w_{Ax} \simeq \bar{r}'((A', \leq')) = (A', \leq', r')$

which gives the order preserving bijection  $h_0 : N_0 = A \rightarrow N'_0 = A'$  such that  $r' \circ h_0 = r \upharpoonright N_0$ .

Case  $k > 0$ . We assume inductively that  $h_{k-1}$  has been constructed.

Let  $U \in \mathcal{U}$  be such that  $x = \widehat{U}$  has depth  $k - 1$ ; hence,  $U \cap N_{k-1} = \emptyset$ . Then  $(U, \leq, r) \simeq w_{r(x)}$ . Let  $x' = h_{k-1}(x)$ ; we have  $r'(x') = r(x)$ . Hence there is  $U' \in \mathcal{U}'$  such that  $x' = \widehat{U'}$ ,  $U' \cap N'_{k-1} = \emptyset$  and  $(U', \leq', r') \simeq w_{r'(x')} = w_{r(x)}$ . Hence, there is an order preserving bijection  $h_U : U \rightarrow U'$  such that  $r' \circ h_U = r \upharpoonright U$ .

We define  $h_k$  as the extension of the injective mappings  $h_{k-1}$  and  $h_U$  for all  $U \in \mathcal{U}$  such that the depth of  $\widehat{U}$  is  $k - 1$ . These mappings have pairwise disjoint domains whose union is  $N_k$ .

The extension to  $N$  of all these mappings  $h_k$  is the desired isomorphism  $h$ .

□

*Remark:* We call *unfolding* the transformation of  $\Delta$  into  $Unf(\Delta)$  because it generalizes the unfolding of a directed graph  $G$  into a finite or countable rooted tree. The unfolding is done from a particular vertex  $s$  of  $G$ , and the nodes of the tree are the sequences of the form  $(x_0, \dots, x_k)$  such that  $s = x_0$  and there is a directed edge in  $G$  from  $x_i$  to  $x_{i+1}$ , for each  $i < k$ . If  $\Delta$  is such that the arrangements  $w_{Ax}$  and  $w_q$  are reduced to a single element, the corresponding directed graph has all its vertices of outdegree one and the tree resulting from the unfolding consists of one infinite path: the SBJ-tree  $Unf(\Delta)$  is the order type  $\omega^-$  of negative integers and the sets in  $\mathcal{U}$  are singletons.

(3.12) **Proposition:** A BJ-tree is monadic second-order definable if it is described by a regular BJ-scheme.

**Proof:** That  $J = (N, \leq)$  is a BJ-tree is first-order expressible. Assume that  $J = fgs(J')$  where  $J' = (N, \leq, \mathcal{U}) \simeq Unf(\Delta)$  for some regular SBJ-scheme  $\Delta = (Q, w_{Ax}, (w_q)_{q \in Q})$  such that  $Q = \{1, \dots, m\}$ . Let  $r$  be the corresponding mapping:  $N \rightarrow Q$  (cf. Definition (3.10.b)). For each  $q \in Q$ , let  $\psi_q$  be an MS sentence that characterizes  $w_q$ , up to isomorphism, by the main result of [Tho86]. Similarly,  $\psi_{Ax}$  characterizes  $w_{Ax}$ . We claim that a relational structure  $(X, \leq)$  is isomorphic to  $J$  if and only if there exist subsets  $N_0, N_1, M_1, \dots, M_m$  of  $X$  such that:

- (i)  $(X, \leq)$  is a BJ-tree and  $(X, \leq, N_0, N_1) = S(J'')$  for some SBJ-tree  $J'' = (X, \leq, \mathcal{U}')$ ,
- (ii)  $(M_1, \dots, M_m)$  is a partition of  $X$ ; we let  $r'$  maps each  $x \in X$  to the unique  $q \in Q$  such that  $x \in M_q$ ,
- (iii) for every  $q$  and node  $x$  in  $M_q$ , the arrangement  $\overline{r'}((U^x, \leq))$  over  $Q$  is isomorphic to  $w_q$  (where  $U^x \in \mathcal{U}'$ ),
- (iv) the arrangement  $\overline{r'}((A', \leq))$  over  $Q$  where  $A'$  is the axis of  $J''$  is isomorphic to  $w_{Ax}$ .

Conditions (ii)-(iv) express that  $\Delta$  describes  $J''$ , hence that  $J''$  is isomorphic to  $J'$ , and so, that  $(X, \leq) \simeq fgs(J') = J$ .

By Proposition (3.9), Condition (i) is expressed by an MS-formula  $\varphi(N_0, N_1)$ , and the property  $U \in \mathcal{U} \wedge x = \widehat{U}$  is expressed in terms of  $N_0, N_1$  by an MS-formula  $\theta(x, U, N_0, N_1)$ . Conditions (iii) and (iv) are expressed by means of the MS-sentences  $\psi_{Ax}$  and  $\psi_q$  suitably adapted to take  $N_0, N_1, M_1, \dots, M_m$  as arguments. Hence,  $J$  is (up to isomorphism) the unique model of an MS sentence of the form:

$$\exists N_0, N_1. [\varphi(N_0, N_1) \wedge \exists M_1, \dots, M_m. \varphi'(N_0, N_1, M_1, \dots, M_m)]$$

where  $\varphi'$  expresses conditions (ii)-(iv).  $\square$

Theorem (3.21) will establish a converse.

### 3.3 The algebra of binary join-trees

We define three operations on *structured binary join-trees* (*SBJ-trees*). The (finite and infinite) terms over these operations define all SBJ-trees.

(3.13) **Definition** : *Operations on SBJ-trees.*

*Concatenation along axes.*

Let  $J = (N, \leq, \mathcal{U})$  and  $J' = (N', \leq', \mathcal{U}')$  be disjoint SBJ-trees, with respective axes  $A$  and  $A'$ . We define:

$$\begin{aligned} J \bullet J' &:= (N \uplus N', \leq'', \mathcal{U}'') \text{ where :} \\ x \leq'' y &:= x \leq y \vee x \leq' y \vee (x \in N \wedge y \in A'), \\ \mathcal{U}'' &:= \{A \uplus A'\} \uplus (\mathcal{U} - \{A\}) \uplus (\mathcal{U}' - \{A'\}). \end{aligned}$$

$J \bullet J'$  is an SBJ-tree with axis  $A \uplus A'$ ; the depth of a node in  $J \bullet J'$  is the same as in  $J$  or  $J'$ .

This operation generalizes the concatenation of linear orders: if  $(N, \leq)$  and  $(N', \leq')$  are disjoint linear orders, then the SBJ-tree  $(N, \leq, \{N\}) \bullet (N', \leq', \{N'\})$  corresponds to the concatenation of  $(N, \leq)$  and  $(N', \leq')$  usually denoted by  $(N, \leq) + (N', \leq')$ .

If  $K = (M, \leq, \mathcal{V})$  is an SBJ-tree with axis  $B$ , and  $B = A \uplus A'$  such that  $A < A'$ , then  $K = J \bullet J'$  where:

$$\begin{aligned} N &:= \downarrow(A), N' := M - N, \\ \mathcal{U} &\text{ is the set of lines of } \mathcal{V} \text{ included in } N - A \text{ together with } A, \\ \mathcal{U}' &\text{ is the set of lines of } \mathcal{V} \text{ included in } N' - A' \text{ together with } A' \text{ and} \\ &\text{the orders of } J \text{ and } J' \text{ are the restrictions of } \leq \text{ to } N \text{ and } N'. \end{aligned}$$

*The empty SBJ-tree:*

The nullary symbol  $\Omega$  denotes the empty SBJ-tree.

*Extension:*

Let  $J = (N, \leq, \mathcal{U})$  be an SBJ-tree, and  $u \notin N$ . Then:

$ext_u(J) := (N \uplus \{u\}, \leq', \{\{u\}\} \uplus \mathcal{U})$  where :

$x \leq' y :\iff x \leq y \vee y = u,$

the axis is  $\{u\}$ .

Clearly,  $ext_u(J)$  is an SBJ-tree. The depth of  $v \in N$  is its depth in  $J$  plus 1. The axis of  $J$  is turned into an "ordinary line" of the structuring of  $ext_u(J)$  with top equal to  $u$ .

When handling SBJ-trees up to isomorphism, we use the notation  $ext(J)$  instead of  $ext_u(J)$ .

*Forgetting structuring:*

If  $J$  is an SBJ-tree as above,  $fgs(J) := (N, \leq)$  is the underlying BJ-tree (binary join-tree).

Anticipating the sequel, we observe that a linear order  $a_1 < \dots < a_n$ , identified with the SBJ-tree  $(\{a_1, \dots, a_n\}, \leq, \{\{a_1, \dots, a_n\}\})$  is defined by the term  $t := ext_{a_1}(\Omega) \bullet ext_{a_2}(\Omega) \bullet \dots \bullet ext_{a_n}(\Omega)$ . The binary (it is even "unary") join-tree  $(\{a_1, \dots, a_n\}, \leq)$  is defined by the term  $fgs(t)$  and also, in a different way, by the term  $fgs(ext_{a_n}(ext_{a_{n-1}}(\dots(ext_{a_1}(\Omega))\dots)))$ .

(3.14) **Definition** : *The algebra SBJT*

We let  $F$  be the signature  $\{\bullet, ext, \Omega\}$ . We obtain an algebra SBJT whose domain is the set of isomorphism classes of SBJ-trees. Concatenation is associative with neutral element  $\Omega$ .

(3.15) **Definition** : *The value of a term.*

(a) In order to define the value of a term  $t$  in  $T^\infty(F)$ , we compare its positions as follows:

$u \approx v$  if and only if every position  $w$  such that  $u <_t w \leq_t u \sqcup_t v$  or

$v <_t w \leq_t u \sqcup_t v$  is an occurrence of  $\bullet$ .

This relation is an equivalence. We will also use the lexicographic order  $\leq_{lex}$  (positions are Dewey words). If  $w$  is an occurrence of a binary symbol, then  $s_1(w)$  is its first (left) son and  $s_2(w)$  its second (right) one.

(b) We define the *value*  $val(t) := (N, \leq, \mathcal{U})$  of  $t$  in  $T^\infty(F)$  as follows:

$N := \text{Occ}(t, ext)$ , the set of occurrences in  $t$  of  $ext$ ,

$u \leq v :\iff u \leq_t w \leq_{lex} v$  for some  $w \in N$  such that  $w \approx v$ ,

$\mathcal{U}$  is the set of equivalence classes of  $\approx$ .



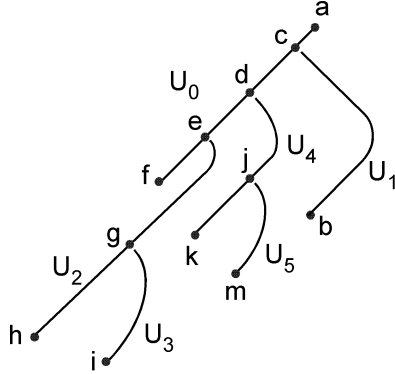


Figure 3: A finite SBJ-tree  $J$ .

Equivalently, we have :

$$u \leq v : \iff u \leq_t v \text{ or } u \leq_t s_1(u \sqcup_t v), v \leq_t s_2(u \sqcup_t v) \text{ and } v \approx u \sqcup_t v$$

(the position  $u \sqcup_t v$  is an occurrence of  $\bullet$ ),

and so (we recall that  $\perp$  denotes incomparability) :

$$u \perp v : \iff u \leq_t s_1(u \sqcup_t v), v \leq_t s_2(u \sqcup_t v) \text{ and there is an occurrence of } ext \text{ between } v \text{ and } u \sqcup_t v \text{ or vice-versa by exchanging } u \text{ and } v.$$

(c) We now consider terms  $t$  written with the operations  $ext_a$  (such that  $a$  is the node created by applying this operation). For each  $a$ , the operation  $ext_a$  must have at most one occurrence in  $t$ . Assuming this condition satisfied, then  $val(t) := (N, \leq, \mathcal{U})$  where :

$N$  is the set of nodes  $a$  such that  $ext_a$  has an occurrence in  $t$  that we will denote by  $u_a$ ,

$$a \approx b : \iff u_a \approx u_b, \text{ with } \approx \text{ as in (a),}$$

$$a \leq b : \iff u_a \leq u_b, \text{ with } \leq \text{ as in (b),}$$

$\mathcal{U}$  is the set of equivalence classes of  $\approx$ .

(3.15.1) *Claim:* The mapping  $val$  in (b) is a value mapping  $T^\infty(F) \rightarrow \text{SBJT}$ .

We say that  $t$  denotes an SBJ-tree  $J$  if  $J$  is isomorphic to  $val(t)$ , and, in this case, we also say that  $fgs(t)$  denotes the BJ-tree  $fgs(J)$ .

(3.16) *Examples and remarks.*

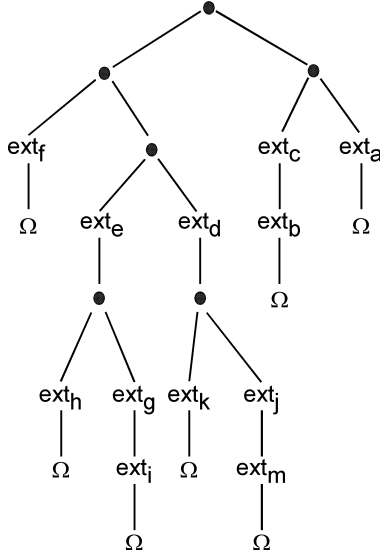


Figure 4: A term  $t$  denoting  $J$ .

(1) The term  $t_0$  that is the unique solution in  $T^\infty(F)$  of the equation  $t_0 = t_0 \bullet t_0$  denotes the empty SBJ-tree  $\Omega$ .

(2) Figure 3 shows a finite SBJ-tree  $J$  whose structuring consists of  $U_0, \dots, U_5$ , and  $U_0$  is the axis. The linear order on  $U_0$  can be described by the word  $fedca$  (with  $f < e < d < \dots$ ). Similarly,  $U_1 = b, U_2 = hg, U_3 = i, U_4 = kj$  and  $U_5 = m$ .

Let us examine the term  $t$  of Figure 4 that denotes  $J$ . A function symbol  $ext_u$  specifies the node  $u$  of  $J$ , and we also denote by  $u$  its occurrence (a position of  $t$ ). The occurrences of  $\bullet$  and  $\Omega$  are denoted by Dewey words. For example, the occurrences of  $\bullet$  above the symbols  $ext$  are the words  $\varepsilon, 1, 2, 12$ . The set  $\{\varepsilon, 1, 2, 12, f, e, d, c, a\}$  is an equivalence class of  $\approx$ . Another one is  $\{1221, k, j\}$ . Each line  $U_i$  is the set of positions of the  $ext$  symbols in some equivalence class of  $\approx$ . Let us now examine how each line is ordered.

The case where  $u < v$  holds because  $u <_t v$  is illustrated, to take a few cases, by  $i < g, g < e, m < j$  and  $j < d$ .

The case where  $u < v$  holds because  $u \perp_t v$ ,  $u \leq_t s_1(u \sqcup_t v)$ ,  $v \leq_t s_2(u \sqcup_t v)$  and  $v \approx u \sqcup_t v$  is illustrated by  $f < e, e < d, d < c$  and  $i < d$ . We have  $i < d$  because  $i \sqcup_t d = 12$ ,  $i <_t s_1(12)$ ,  $d \leq_t s_2(12)$  and  $d \approx 12$ . We do not have  $i < j$  because  $j$  is not  $\approx$ -equivalent to  $12$ , whereas  $i \sqcup_t j = 12$ ,  $i <_t s_1(12)$  and  $j \leq_t s_2(12)$ . This case illustrates the characterization of  $\perp$  in Definition 3.15(c).

(3) Let  $t_1$  be the solution in  $T^\infty(F)$  of the equation  $t_1 = ext(ext(\Omega)) \bullet t_1$ . We write it by naming  $a, b, c, d, e, f, \dots$  the nodes created by the operations  $ext$ , hence,  $t_1 = ext_a(ext_b(\Omega)) \bullet (ext_c(ext_d(\Omega)) \bullet (ext_e(ext_f(\Omega)) \bullet \dots))$ .

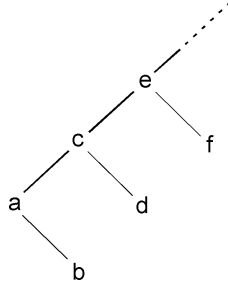


Figure 5: The SBJ-tree  $val(t_1)$ .

Its value is shown in Figure 5. The bold edges link nodes in the axis. The nodes  $b$  and  $f$  are incomparable because the corresponding occurrences of  $ext$ , that are 111 and 2211, have least common ancestor  $\varepsilon$  and 221 is an occurrence of  $ext$  between 2211 and  $\varepsilon$ .

(3.17) **Definition** : *The description scheme associated with a term.*

(a) Let  $t \in T^\infty(F)$  and  $u \in Pos(t)$ . We denote by  $\text{Max}(t, ext, u)$  the set of maximal occurrences of  $ext$  in  $t$  that are below  $u$  or equal to it. Positions are denoted by Dewey words, hence, these sets are linearly ordered by  $\leq_{lex}$ . We denote by  $W(t, u)$  the simple arrangement  $(\text{Max}(t, ext, u), \leq_{lex})$ .

Let  $J = (N, \leq, \mathcal{U})$  be the value of  $t$  (cf. Definition (3.15)) and  $x$  be an occurrence of  $ext$  with son  $u$ . We have  $(U^x, \leq) = (\text{Max}(t, ext, u), \leq_{lex})$ .

For the term  $t$  in Example 3.16(2) we have  $W(t, \varepsilon) = fedca$ ,  $W(t, 1) = fed$ ,  $W(t, e) = hg$ . For  $t_1$  in Example 3.16(3), we have  $W(t_1, \varepsilon) = ace\dots$ ,  $W(t_1, 1) = a$ ,  $W(t_1, 11) = b$  and  $W(t_1, 111) = \Omega$ .

(b) We define  $\Delta(t)$  as the SBJ-scheme

$$(\text{Occ}(t, ext), W(t, \varepsilon), (W(t, s(x)))_{x \in \text{Occ}(t, ext)})$$

where  $s(x)$  is the unique son of an occurrence  $x$  of  $ext$ .

For the term  $t_1$  of Example 3.16(3), we obtain  $\Delta(t_1) = (2^* \uplus 2^*1, ace\dots, (w_x)_{x \in \text{Occ}(t_1, ext)})$  with  $w_a = b$ ,  $w_c = d, \dots, w_b = \Omega$ ,  $w_d = \Omega, \dots$

(3.18) **Lemma** : If  $t \in T^\infty(F)$ , then  $val(t)$  is described by  $\Delta(t)$ .

**Proof:** Let  $val(t) = (N, \leq, \mathcal{U})$ . The conditions of Definition (3.10.b) hold with the identity on  $\text{Occ}(t, ext)$  as mapping  $r$  because

$$(U^x, \leq) = (\text{Max}(t, ext, s(x)), \leq_{lex})$$

as observed in Definition (3.17.a).  $\square$

(3.19) **Proposition** : Every SBJ-tree is the value of a term.

**Proof** : Let  $J = (N, \leq, \mathcal{U})$  be an SBJ-tree. For each  $k$ , we let  $J_k$  be the SBJ-tree  $(N_k, \leq, \mathcal{U}_k)$  where  $N_k$  is the set of nodes of depth at most  $k$  and  $\mathcal{U}_k$  is the set lines  $U \in \mathcal{U}$  of depth at most  $k$ . By induction on  $k$ , we define for each  $k$  a term  $t_k$  that defines  $J_k$  such that  $t_k \leq t_{k'}$  if  $k < k'$ , and then, the least upper bound of the terms  $t_k$  is the desired term  $t$  whose value is  $J$ .

We define terms using the symbols  $ext_a$  where  $a$  names the node created by the corresponding occurrence of the extension operation.

If  $k = 0$ , then  $J_0 = (A, \leq, \{A\})$ . There exists a term  $t \in T^\infty(\{\bullet\}, Ext_A)$  whose value is  $J_0$ , where  $Ext_A$  is the set of terms  $ext_a(\Omega)$  for  $a \in A$  (we use  $Ext_A$  as a set of nullary symbols).

Let  $k \geq 1$ , where  $t_{k-1}$  defines  $J_{k-1}$ . Then  $J_k$  is obtained from  $J_{k-1}$  by adding below some nodes  $x$  at depth  $k - 1$  the line  $U^x$  (if  $U^x = \emptyset$ , there is nothing to add below  $x$ ). Let  $t_x \in T^\infty(\{\bullet\}, Ext_{U^x})$  whose value is  $(U^x, \leq)$ . We obtain  $t_k$  by replacing in  $t_{k-1}$  each subterm  $ext_x(\Omega)$  by  $ext_x(t_x)$ , for  $x$  at depth  $k - 1$  such that  $U^x \neq \emptyset$ .

It is clear that  $t_{k-1} \leq t_k$  and that the least upperbound of the terms  $t_k$  defines  $J$ .  $\square$

For an example, we apply this construction to the SBJ-tree  $J$  of Figure 3. For defining  $J_0$ , we can take:

$$t_0 = ((ext_f(\Omega) \bullet ext_e(\Omega)) \bullet ext_d(\Omega)) \bullet (ext_c(\Omega) \bullet ext_a(\Omega)).$$

To obtain  $t_1$ , we replace  $ext_e(\Omega)$  by  $ext_e(ext_h(\Omega) \bullet ext_g(\Omega))$ ,  $ext_d(\Omega)$  by  $ext_d(ext_k(\Omega) \bullet ext_j(\Omega))$  and  $ext_c(\Omega)$  by  $ext_c(ext_b(\Omega))$ , which gives:

$$t_1 = ((ext_f(\Omega) \bullet ext_e(ext_h(\Omega) \bullet ext_g(\Omega))) \bullet ext_d(ext_k(\Omega) \bullet ext_j(\Omega)) \bullet (ext_c(\Omega) \bullet ext_a(\Omega))).$$

Then, we obtain  $t_2$  that defines  $J$  by replacing  $ext_g(\Omega)$  by  $ext_g(ext_i(\Omega))$  and  $ext_j(\Omega)$  by  $ext_j(ext_m(\Omega))$ .

### 3.4 Regular binary join-trees

(3.20) **Definition** : *Regular BJ- and SBJ-trees.*

A BJ-tree (resp. an SBJ-tree)  $T$  is *regular* if it is denoted by  $fgs(t)$  (resp. by  $t$ ) where  $t$  is a regular term in  $T^\infty(F)$ .

(3.21) **Theorem**: The following properties of a BJ-tree  $J$  are equivalent:

- (1)  $J$  is regular,
- (2)  $J$  is described by a regular scheme,
- (3)  $J$  is MS-definable.

**Proof** : (1) $\implies$ (2) Let  $J = fgs(J')$  with  $J'$  denoted by a regular term  $t$  in  $T^\infty(F)$ . Let  $h : Pos(t) \rightarrow Q$  and  $\tau$  be as in the definition of a regular term in Section 2.2. Without loss of generality, we can assume that  $h(Pos(t)) = Q$ . If this is not the case, we replace  $Q$  by  $h(Pos(t))$  and  $\tau$  by its restriction to this set.

*Claim*: (1) For each  $u \in Pos(t)$ , the arrangement  $\bar{h}(W(t, u)) = (\text{Max}(t, ext, u), \leq_{lex}, h)$  over  $Q$  is regular.

(2) If  $u'$  is another position in  $t$  and  $h(u') = h(u)$ , then  $t/u' = t/u$  and furthermore<sup>3</sup>  $\bar{h}(W(t, u')) \simeq \bar{h}(W(t, u))$ .

Leaving its routine proof, we define  $\Delta := (Q, w_{Ax}, (w_q)_{q \in Q})$  as follows:

(i)  $w_{Ax} := \bar{h}(W(t, \varepsilon))$ ,

(ii) if  $q \in Q$ , then  $w_q := \bar{h}(W(t, s(u)))$  where  $s(u)$  is the unique son of an occurrence  $u$  of  $ext$  such that  $h(u) = q$ ; if  $v$  is another occurrence of  $ext$  such that  $h(v) = q$ , then  $h(s(v)) = h(s(u))$  and so by the claim,  $\bar{h}(W(t, s(v))) \simeq \bar{h}(W(t, s(u)))$ . Hence,  $w_q$  is well-defined up to isomorphism.

Informally,  $\Delta$  is obtained from  $\Delta(t)$  by replacing the labelling mapping  $Id$  of the arrangements  $W(t, u)$  by  $h$ , so that these arrangements are turned into arrangements  $\bar{h}(W(t, u))$  over  $Q$ . Clearly,  $\Delta$  is a regular scheme. As mapping  $r$  showing that it describes  $J'$  (cf. Definition (3.10)), hence also  $J$ , we take the restriction of  $h$  to  $\text{Occ}(t, ext)$  that is the set of nodes of  $J' = val(t)$ .

(2) $\implies$ (3) is proved in Proposition (3.12).

(3) $\implies$ (1) By Definition (3.2), the mapping  $\alpha$  that transforms the relational structure  $[t]$  for  $t$  in  $T^\infty(F)$  into the BJ-tree  $J = (N, \leq) = fgs(val(t))$  is an MS-transduction<sup>4</sup> because an MS formula can identify the nodes of  $J$  among the positions of  $t$  and another one can define  $\leq$ .

Let  $J = (N, \leq)$  be an MS-definable BJ-tree. It is, up to isomorphism, the unique model of an MS sentence  $\beta$ . It follows by a standard argument<sup>5</sup> that the set of terms  $t$  in  $T^\infty(F)$  such that  $\alpha([t]) \models \beta$  is MS-definable and thus, contains a regular term (a result by Rabin, see [Tho90]). This term denotes  $J$ , hence  $J$  is regular.  $\square$

(3.22) **Corollary** : The isomorphism problem for regular BJ-trees is decidable.

**Proof** : A regular BJ-tree can be given, either by a regular term, a regular scheme or an MS sentence. The proof of Theorem (3.21) is effective: algorithms can convert any of these specifications into another one. Hence, two regular BJ-trees can be given, one by an MS sentence  $\beta$ , the other by a regular term

<sup>3</sup>Unless  $u = u'$ , the sets  $\text{Max}(t, ext, u)$  and  $\text{Max}(t, ext, u')$  are not equal, so that the arrangements  $\bar{h}(W(t, u))$  and  $\bar{h}(W(t, u'))$  are isomorphic but not equal.

<sup>4</sup>MS logic, MS-transductions and related definitions and results are reviewed in the appendix.

<sup>5</sup>If  $\alpha$  is an MS-transduction and  $\beta$  is an MS-sentence, then the set of structures  $S$  such that  $\alpha(S) \models \beta$  is MS-definable. See Theorem (A.1) in the appendix.

$t$ . They are isomorphic if and only if  $\alpha(\lfloor t \rfloor) \models \beta$  (cf. the proof of (3) $\implies$ (1) of Theorem (3.21)) if and only if  $t \in L(\beta)$ , which is decidable (see [Tho90]).  $\square$

### 3.5 Logical and algebraic descriptions of join-trees

We now extend to join-trees the definitions and results of the previous sections. Structured join-trees are defined in Section 3.1 (Definition (3.3)). We extend to them the definitions and results of Sections 3.2-3.4. A first novelty will be that the argument of the extension operation  $ext$  will be an SJ-forest, equivalently a set of SJ-trees, instead of a single SBJ-tree. We will need an algebra with two sorts, the sort of SJ-trees and that of SJ-forests. A second one will be the use in monadic second-order formulas of a finiteness predicate (cf. Section 2.3 and the appendix).

(3.23) *Definition : Description schemes for SJ-trees.*

(a) A *description scheme for an SJ-tree*, in short an *SJ-scheme*, is a 5-tuple  $\Delta = (Q, D, w_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$  such that  $Q, D$  are sets,  $w_{Ax} \in \mathcal{A}(Q)$ ,  $w_d \in \mathcal{A}(Q)$  for each  $d \in D$  and  $m_q = (M_q, lab_q)$  is a  $D$ -labelled set (cf. Section 2.3) for each  $q \in Q$ . Without loss of generality, we will assume that the domains  $V_{Ax}$  and  $V_d$  of the arrangements  $w_{Ax}, w_d$  and the sets  $M_q$  are pairwise disjoint, because these arrangements and labelled sets will be used up to isomorphism. Informally,  $M_q$  encodes the different lines  $U$  such that  $\widehat{U} = x$  where  $x$  is labelled by  $q$ , and each of these lines is defined, up to isomorphism, by the arrangement  $w_d$  where  $d$  is its label in  $D$ , defined by  $lab_q$ .

We say that  $\Delta$  is *regular* if  $Q \cup D$  is finite and the arrangements  $w_{Ax}$  and  $w_d$  are regular. The finiteness of  $D$  implies that each  $D$ -labelled set  $m_q$  is regular.

(b) Let  $J = (N, \leq, \mathcal{U})$  be an SJ-tree with axis  $A$ ; for each  $x \in N$ , we denote by  $\mathcal{U}^x$  the set of lines  $U \in \mathcal{U}$  such that  $\widehat{U} = x$ . An SBJ-scheme  $\Delta$  as in a) *describes*  $J$  if there exist mappings  $r : N \rightarrow Q$  and  $\tilde{r} : \mathcal{U} - \{A\} \rightarrow D$  such that:

- (b.1) the arrangement  $(A, \leq, r)$  over  $Q$  is isomorphic to  $w_{Ax}$ ,
- (b.2) for each  $x \in N$ , the  $D$ -labelled set  $(\mathcal{U}^x, \tilde{r})$  is isomorphic to  $m_{r(x)}$ ,
- (b.3) for each  $U \in \mathcal{U} - \{A\}$ , the arrangement  $(U, \leq, r)$  over  $Q$  is isomorphic to  $w_{\tilde{r}(U)}$ .

We will also say that  $\Delta$  *describes* the join-tree  $fgs(J) := (N, \leq)$ .

(c) Each SBJ-tree  $J = (N, \leq, \mathcal{U})$  has a *standard* description scheme  $\Delta(J) := (N, \mathcal{U} - \{A\}, (A, \leq), ((\mathcal{U}^x, Id))_{x \in N}, ((U, \leq))_{U \in \mathcal{U} - \{A\}})$ . The identity mappings  $: N \rightarrow N$  and  $: \mathcal{U} - \{A\} \rightarrow \mathcal{U} - \{A\}$  show that  $\Delta(J)$  describes  $J$ .

(3.24) **Proposition:** Every SJ-scheme  $\Delta$  describes a unique SJ-tree  $Unf(\Delta)$  where unicity is up to isomorphism.

**Proof:** Let  $\Delta = (Q, D, w_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$  be an SJ-scheme, defined with arrangements  $w_{Ax} = (V_{Ax}, \preceq, lab_{Ax})$  and  $w_d = (V_d, \preceq, lab_d)$ , and labelled sets  $m_q = (M_q, lab_q)$  such that the sets  $V_{Ax}$ ,  $V_d$  and  $M_q$  are pairwise disjoint and the same symbol  $\preceq$  denotes the orders of the arrangements  $w_{Ax}$  and  $w_d$ .

We construct  $Unf(\Delta) := (N, \leq, \mathcal{U})$  as follows.

a)  $N$  is the set of finite nonempty sequences  $(v_0, s_1, v_1, s_2, \dots, s_k, v_k)$  such that:

$$\begin{aligned} &v_0 \in V_{Ax}, v_i \in V_{d_i} \text{ and } s_i \in M_{q_{i-1}} \text{ for } 1 \leq i \leq k, \text{ where} \\ &q_0 = lab_{Ax}(v_0), d_1 = lab_{q_0}(s_1), q_1 = lab_{d_1}(v_1), d_2 = lab_{q_1}(s_2), \dots, \\ &q_i = lab_{d_i}(v_i), d_{i+1} = lab_{q_i}(s_{i+1}) \text{ for } 1 \leq i \leq k-1. \end{aligned}$$

b)  $(v_0, s_1, v_1, \dots, s_k, v_k) \leq (v'_0, s'_1, v'_1, \dots, s'_j, v'_j)$  if and only if :

$$k \geq j, (v_0, s_1, v_1, \dots, s_j) = (v'_0, s'_1, v'_1, \dots, s'_j) \text{ and } v_j \preceq v'_j \text{ (} v_j, v'_j \in V_{d_j}\text{)}.$$

c) the axis  $A$  is the set of one-element sequences  $(v)$  for  $v \in V_{Ax}$  and, for  $x = (v_0, s_1, v_1, \dots, s_k, v_k)$ ,  $U(x)$  is the set of sequences in  $N$  of the form  $(v_0, s_1, v_1, s_2, \dots, s_k, v)$  for  $v \in V_{d_k}$ , so that  $\widehat{U(x)} = (v_0, s_1, v_1, \dots, s_{k-1}, v_{k-1})$ .

Note that  $(v_0, s_1, v_1, \dots, v_k) < (v_0, s_1, v_1, \dots, v_j)$  if  $j < k$  and that  $(v_0, s_1, v_1, \dots, s_k, v_k) \leq (v_0, s_1, v_1, \dots, s_k, v)$  if and only if  $v_k \preceq v$ .

In order to prove that  $\Delta$  describes  $J$ , we define  $r : N \rightarrow Q$  and  $\tilde{r} : \mathcal{U} - \{A\} \rightarrow D$  as follows:

$$\begin{aligned} &\text{if } x \in A, \text{ then } x = (v) \text{ for some } v \in V_{Ax} \text{ and } r(x) := lab_{Ax}(v); \\ &\text{if } x \in N \text{ has depth } k \geq 1, \text{ then } x = (v_0, s_1, v_1, \dots, s_k, v_k) \text{ for some} \\ &\quad v_0, s_1, \dots, s_k, v_k \text{ and } r(x) := lab_{d_k}(v_k); \\ &\text{if } U \in \mathcal{U} - \{A\}, \text{ then } U = U(x) \text{ for some } x = (v_0, s_1, v_1, \dots, s_k, v_k), \\ &\quad k \geq 1, \text{ and } \tilde{r}(U) := d_k. \end{aligned}$$

We check the three conditions of Definition (3.23.b). We have  $(A, \leq, r) \simeq w_{Ax}$ , hence (b.1) holds. For checking (b.2), we consider  $x = (v_0, s_1, v_1, \dots, s_k, v_k) \in N, k \geq 1$ . The sets  $U$  in  $\mathcal{U}^x$  are those of the form  $\{(v_0, s_1, v_1, \dots, s_k, v_k, s, v) \mid v \in V_{d_{k+1}}\}$  for all  $s \in M_{q_k}$  where  $q_k = lab_{d_k}(v_k) = r(x)$ , hence (b.2) holds. For checking (b.3), we let  $U = U(x)$  for some  $x = (v_0, s_1, v_1, \dots, s_k, v_k), k \geq 1$ ; it is the set of sequences  $(v_0, s_1, v_1, s_2, \dots, s_k, v)$  for  $v \in V_{d_k}$  ordered by  $\preceq$  on the last components. Hence,  $(U, \leq, lab_{d_k})$  is isomorphic to  $w_{d_k}$ , which proves the property since  $\tilde{r}(U) := d_k$ .

Unicity is proved as in Proposition (3.11).  $\square$

The following proposition extends Proposition (3.12).

(3.25) **Proposition:** A join-tree is  $\text{MS}_{\text{fin}}$ -definable if it is described by a regular SJ-scheme.

**Proof:** Let  $(N, \leq)$  be a join-tree  $J$  (this property is first-order expressible). Assume that  $J = \text{fgs}(J')$  where  $J' = (N, \leq, \mathcal{U}) \simeq \text{Unf}(\Delta)$  for some regular SJ-scheme  $\Delta = (Q, D, w_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$  such that  $Q = \{1, \dots, m\}$  and  $D = \{1, \dots, p\}$ . Let  $r, \tilde{r}$  be the corresponding mappings (cf. Definition 3.23.b). For each  $d \in D$ , let  $\psi_d$  be an MS sentence that characterizes  $w_d$  up to isomorphism, by the main result of [Tho86]. Similarly,  $\psi_{Ax}$  characterizes  $w_{Ax}$ .

A  $D$ -labelled set  $m$  is described up to isomorphism by a  $p$ -tuple  $(m^1, \dots, m^p)$  where  $m^i$  is the number (possibly  $\omega$ ) of elements having label  $i$ .

By Proposition (3.7), there is a bipartition  $(N_0, N_1)$  of  $N$  that describes the structuring  $\mathcal{U}$ ; from this bipartition, we can MS-define the axis  $A$ , the lines forming  $\mathcal{U}$  and the node  $\widehat{U}$  for each  $U \in \mathcal{U} - \{A\}$ .

There is a partition  $(Y_1, \dots, Y_m)$  of  $N$  that describes  $r$  by  $Y_i := r^{-1}(i)$ . There is a partition  $(Z_1, \dots, Z_p)$  where  $Z_j$  is the union of the lines  $U \in \mathcal{U} - \{A\}$  such that  $\tilde{r}(U) = j$ .

Consider a relational structure  $(X, \leq, N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p)$ . By MS formulas, one can express the following properties:

- (i)  $(X, \leq, N_0, N_1)$  is  $S(J'')$  for some SJ-tree  $J'' = (X, \leq, \mathcal{U}')$ ; its axis is denoted by  $A'$ ,
- (ii)  $(Y_1, \dots, Y_m)$  is a partition of  $X$ ; we let  $r(x) := i$  if and only if  $x \in Y_i$ ,
- (iii)  $(Z_1, \dots, Z_p)$  is a partition of  $X$  such that each  $Z_j$  is a union of sets  $U \in \mathcal{U}' - \{A'\}$  such that  $(U, \leq, r) \simeq w_j$ ,
- (iv)  $(A', \leq, r) \simeq w_{Ax}$ ,
- (v) for each  $i \in Q$  and  $x \in Y_i$ , the number of lines  $U \in \mathcal{U}'^x$  that are contained in  $Z_j$  is  $m_i^j$ .

These formulas are constructed as follows:  $\varphi(N_0, N_1)$  for (i) is from Proposition (3.7). The formula for (ii) is standard. All other formulas are constructed so as to express the desired properties when (i) and (ii) do hold. For (iii), we use a suitable adaptation of  $\psi_i$  and the fact from Proposition (3.7) that, if (i) holds, we can MS-define from  $(N_0, N_1)$  the axis  $A'$ , the lines forming  $\mathcal{U}'$  and the node  $\widehat{U}$  for each  $U \in \mathcal{U}'$ . The mapping  $r$  is given by  $(Y_1, \dots, Y_m)$ . For (iv), we do as for (iii) with  $\psi_{Ax}$ .

For (v), we do as follows. We write an MS formula  $\gamma(x, N_0, N_1, Z, W)$  expressing that  $W$  consists of one node of each set  $U \in \mathcal{U}' - \{A'\}$  that is contained in  $Z$  and is such that  $\widehat{U} = x$ . For any  $x$  and  $Z$ , all sets  $W$  satisfying  $\gamma(x, N_0, N_1, Z, W)$  have same cardinality. Then, Property (v) holds if and only if, for all  $i = 1, \dots, m$ ,  $x \in Y_i$  and  $j = 1, \dots, p$ , if  $\gamma(x, N_0, N_1, Z_j, W)$  holds, then  $W$  has cardinality  $m_i^j$ . If some number  $m_i^j$  is  $\omega$ , we need the finiteness predicate  $\text{Fin}(W)$  to express this condition<sup>6</sup>.

<sup>6</sup>If the nodes of  $J$  have degree at most  $a \in \mathbb{N}$ , then  $m_i^j \leq a$  for all  $i, j$  and the finiteness predicate is not needed, hence,  $J$  is MS-definable.



Let  $\beta(N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p)$  express conditions (ii)-(v) in  $(X, \leq)$ . If a join-tree  $(X, \leq)$  satisfies  $\varphi(N_0, N_1) \wedge \beta(N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p)$ , it has a structuring  $\mathcal{U}'$  described by  $N_0, N_1$ : we let  $J'' := (X, \leq, \mathcal{U}')$ . The sets  $Y_1, \dots, Y_m, Z_1, \dots, Z_p$  yield a scheme  $\Delta$  that describes  $J''$  (by Conditions (iii)-(v)), hence  $J''$  is isomorphic to  $J'$  by the unicity property of Proposition (3.24), and so, we have  $(X, \leq) \simeq fgs(J') = J$ .

Hence,  $J$  is (up to isomorphism) the unique model of the  $MS_{fin}$  sentence :

$$\exists N_0, N_1 (\varphi(N_0, N_1) \wedge \exists Y_1, \dots, Y_m, Z_1, \dots, Z_p. \beta(N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p)).$$

□

Theorem (3.30) will establish a converse.

(3.26) **Definition** : *Operations on SJ-trees and SJ-forests.*

We recall from Definition (3.1) that a join-forest is the union of disjoint join-trees. A structured join-forest (an SJ-forest, cf. Definition (3.4)) is the union of disjoint SJ-trees. It has no axis (each of its components has an axis, but we do not single out any of them). We will use objects of three types : join-trees, SJ-trees and SJ-forests, but a 2-sorted algebra will suffice (similarly as above for SBJT, we have not introduced a separate sort for BJ-trees). The two sorts are  $\mathbf{t}$  for SJ-trees and  $\mathbf{f}$  for SJ-forests.

*Concatenation of SJ-trees along axes.*

The *concatenation*  $J \bullet J'$  disjoint SJ-trees  $J$  and  $J'$  is defined exactly as in Definition (3.13) for SBJ-trees.

*The empty SJ-tree* is denoted by the nullary symbol  $\Omega_{\mathbf{t}}$ .

*Extension of an SJ-forest into an SJ-tree.*

Let  $J = (N, \leq, \mathcal{U})$  be an SJ-forest and  $u \notin N$ . Then  $ext_u(J)$  is an SJ-tree defined as in Definition (3.13). When handling SJ-trees up to isomorphism, we will use the notation  $ext(J)$  instead of  $ext_u(J)$ .

*The empty SJ-forest* is denoted by the nullary symbol  $\Omega_{\mathbf{f}}$ .

*Making an SJ-tree into an SJ-forest.*

This is done by the unary operation  $mkf$  that is actually the identity on the triples that define SJ-trees and SJ-forests.

*The union of two disjoint SJ-forests* is denoted by  $\uplus$ .

The types of these operations are thus:

$$\begin{aligned}
\bullet &: \mathbf{t} \times \mathbf{t} \rightarrow \mathbf{t}, \\
\Omega_{\mathbf{t}} &: \mathbf{t}, \\
\uplus &: \mathbf{f} \times \mathbf{f} \rightarrow \mathbf{f}, \\
\Omega_{\mathbf{f}} &: \mathbf{f}, \\
ext &: \mathbf{f} \rightarrow \mathbf{t}, \\
mkf &: \mathbf{t} \rightarrow \mathbf{f}.
\end{aligned}$$

In addition, we have, as in Definition (3.13):

*Forgetting structuring:* If  $J$  is an SJ-tree,  $fgs(J)$  is the underlying join-tree.

(3.27) **Definition** : *The algebra  $\mathbb{SJT}$*

We let  $F'$  be the 2-sorted signature  $\{\bullet, \uplus, ext, mkf, \Omega_{\mathbf{t}}, \Omega_{\mathbf{f}}\}$  where the types of these six operations are as above. We obtain an  $F'$ -algebra  $\mathbb{SJT}$  whose domains are the sets of isomorphism classes of SJ-trees and of SJ-forests. Concatenation is associative with neutral element  $\Omega_{\mathbf{t}}$  and disjoint union is associative and commutative with neutral element  $\Omega_{\mathbf{f}}$ .

(3.28) **Definition** : *The value of a term.*

The definition is actually identical to that for SBJ-trees (Definition (3.15)). We recall it for the reader's convenience. The equivalence relation  $\approx$  is as in this definition. The *value*  $val(t) = (N, \leq, \mathcal{U})$  of  $t \in T^\infty(F')$  is defined as follows:

$$\begin{aligned}
N &:= \text{Occ}(t, ext), \text{ the set of occurrences in } t \text{ of } ext, \\
u \leq v &: \iff u \leq_t w \leq_{lex} v \text{ for some } w \in N \text{ such that } w \approx v, \\
\mathcal{U} &\text{ is the set of equivalence classes of } \approx.
\end{aligned}$$

If  $t$  has sort  $\mathbf{t}$  (resp.  $\mathbf{f}$ ) then  $val(t)$  is an SJ-tree (resp. an SJ-forest). It is clear that we have a value mapping :  $T^\infty(F') \rightarrow \mathbb{SJT}$ .

For terms written with the operations  $ext_a$ , then  $val(t) := (N, \leq, \mathcal{U})$  where :

$$\begin{aligned}
N &\text{ is the set of nodes } a \text{ such that } ext_a \text{ has an occurrence in } t, \text{ actually} \\
&\text{ a unique one, that we will denote by } u_a, \\
a \leq b &: \iff u_a \leq u_b, \\
a \approx b &: \iff u_a \approx u_b, \text{ and} \\
\mathcal{U} &\text{ is the set of equivalence classes of } \approx.
\end{aligned}$$

(3.29) **Definition** : *Regular join-trees.*

A join-tree (resp. an SJ-tree)  $T$  is *regular* if it is denoted by  $fgs(t)$  (resp. by  $t$ ) where  $t$  is a regular term in  $T^\infty(F')$  of sort  $\mathbf{t}$ .

(3.30) **Theorem:** The following properties of a join-tree  $J$  are equivalent:

- (1)  $J$  is regular,
- (2)  $J$  is described by a regular scheme,
- (3)  $J$  is  $\text{MS}_{fin}$ -definable.

**Proof:** (1) $\implies$ (2). Similar to that of Theorem (3.21).

(2) $\implies$ (3) By Proposition (3.25).

(3) $\implies$ (1) As in the proof of Theorem (3.21), the mapping  $\alpha$  that transforms the relational structure  $[t]$  for  $t$  in  $T^\infty(F')_{\mathbf{t}}$  (the set of terms in  $T^\infty(F')$  of sort  $\mathbf{t}$ ) into the join-tree  $J = (N, \leq) = fgs(val(t))$  is an MS-transduction. Let  $J = (N, \leq)$  be an  $\text{MS}_{fin}$ -definable join-tree. It is, up to isomorphism, the unique model of an  $\text{MS}_{fin}$  sentence  $\beta$ . The set  $L$  of terms  $t$  in  $T^\infty(F')_{\mathbf{t}}$  such that  $\alpha([t]) \models \beta$  is thus  $\text{MS}_{fin}$ -definable. However, since the relational structures  $[t]$  have MS-definable linear orderings,  $L$  is also MS-definable (see the Appendix), hence, it contains a regular term. This term denotes  $J$ , hence  $J$  is regular.  $\square$

The same proof as for Corollary (3.22) yields:

(3.31) **Corollary :** The isomorphism problem for regular join-trees is decidable.

The rooted trees of unbounded degree, without order on the sets of sons of their nodes are the join-trees defined by the terms in  $T^\infty(F' - \{\bullet\})_{\mathbf{t}}$ . Theorem (3.30) and Corollary (3.31) hold for them.

## 4 Ordered join-trees

(4.1) **Definition :** *Ordered join-trees and join-hedges.*

The set of directions of a node  $x$  of a join-forest  $(N, \leq)$  is denoted by  $Dir(x)$ , cf. Definition (3.2). We recall that  $x \perp y$  means that  $x$  and  $y$  are incomparable with respect to  $\leq$ , so that  $x < x \sqcup y$  and  $y < x \sqcup y$  if  $x \sqcup y$  is defined.

(a) We say that a join-tree  $J = (N, \leq)$  is *ordered* (is an *OJ-tree*) if each set  $Dir(x)$  is equipped with a linear order  $\sqsubseteq_x$ . (In this way, we generalize the notion of an ordered tree, cf. Section 2.1.) From these orders, we define a single linear order  $\sqsubseteq$  on  $N$  as follows:

$x \sqsubseteq y$  if and only if  $x \leq y$  or,  $x \perp y$  and  $\delta \sqsubseteq_{x \sqcup y} \delta'$  where  $\delta, \delta' \in Dir(x \sqcup y)$ ,  $x \in \delta$  and  $y \in \delta'$ .

(b) The linear order  $\sqsubseteq$  satisfies the following properties, for all  $x, y, x', y'$ :

- (i)  $x \leq y$  implies  $x \sqsubseteq y$ ,
- (ii) if  $x \leq y$ ,  $x' \leq y'$  and  $y \perp y'$ , then  $x \sqsubseteq x'$  if and only if  $y \sqsubseteq y'$ .

(4.1.1) *Claim* : If  $J = (N, \leq)$  is a join-tree and  $\sqsubseteq$  is a linear order on  $N$  satisfying conditions (i) and (ii), then  $J$  is ordered by the family of orders  $(\sqsubseteq_x)_{x \in N}$  such that, for all  $\delta, \delta' \in \text{Dir}(x)$ , we have  $\delta \sqsubseteq_x \delta'$  if and only if  $\delta = \delta'$  or  $y \sqsubset y'$  for some  $y \in \delta$  and  $y' \in \delta'$  (if and only if  $\delta = \delta'$  or  $y \sqsubset y'$  for all  $y \in \delta$  and  $y' \in \delta'$ ).

*Proof sketch*: Consider different directions  $\delta, \delta' \in \text{Dir}(x)$  such that  $y \sqsubset y'$  for some  $y \in \delta$  and  $y' \in \delta'$ . We have also  $y_1 \sqsubset y'_1$  for any  $y_1 \in \delta$  and  $y'_1 \in \delta'$  because  $(y \sqcup y_1) < x, (y' \sqcup y'_1) < x$  and  $(y \sqcup y_1) \perp (y' \sqcup y'_1)$ , hence, Condition (ii) implies that  $y \sqcup y_1 \sqsubset y' \sqcup y'_1$  and  $y_1 \sqsubset y'_1$ .

Hence, each relation  $\sqsubseteq_x$  is a linear order on  $\text{Dir}(x)$ . It is clear that  $\sqsubseteq$  is derived from the relations  $\sqsubseteq_x$  by (a).  $\square$

It follows that an ordered join-tree can be equivalently defined as a triple  $(N, \leq, \sqsubseteq)$  such that  $(N, \leq)$  is a join-tree and  $\sqsubseteq$  is a linear order that satisfies Conditions (i) and (ii). These conditions are first-order expressible.

(c) We define a *join-hedge* as a triple  $H = (N, \leq, \sqsubseteq)$  such that  $(N, \leq)$  is a join-forest and  $\sqsubseteq$  is a linear order that satisfies Conditions (i) and (ii). Let  $J_s$ , for  $s \in S$ , be the join-trees composing  $(N, \leq)$ . Each of them is ordered by  $\sqsubseteq$  according to Claim (4.1.1), and the index set  $S$  is linearly ordered by  $\sqsubseteq_S$  such that  $s \sqsubset_S s'$  if and only if  $s \neq s'$  and  $x \sqsubset y$  for all nodes  $x$  of  $J_s$  and  $y$  of  $J_{s'}$ . Hence  $H$  is also a simple arrangement of pairwise disjoint join-trees.

(4.2) **Definition** : *Structured join-hedges and structured ordered join-trees.*

(a) A *structured join-hedge*, an *SJ-hedge* in short, is a 4-tuple  $J = (N, \leq, \sqsubseteq, \mathcal{U})$  such that  $(N, \leq, \sqsubseteq)$  is a join-hedge and  $\mathcal{U}$  is a structuring of the join-forest  $(N, \leq)$ . A structured ordered join-tree could be defined in the same way, as an OJ-tree  $(N, \leq, \sqsubseteq)$  equipped with a structuring  $\mathcal{U}$ . However, we need a refinement in order to define the operations that construct ordered join-trees and hedges (cf. Definition (4.8)).

(b) Let  $J$  be an OJ-tree  $(N, \leq, \sqsubseteq)$  and  $\mathcal{U}$  be a structuring of  $(N, \leq)$ . For each node  $x$ , the set  $\text{Dir}(x)$  of its directions consists of the following sets:

- the sets  $\downarrow(U)$  for each line  $U \in \mathcal{U}^x$  (we recall that  $\downarrow(U) := \{y \mid y \leq z \in U \text{ for some } z\}$ ),
- the set  $\downarrow(U_-(x))$  (cf. Section 2) if  $U_-(x)$  is not empty; in this case we call it the *central direction* of  $x$ .

If  $x$  is the the smallest element of  $U(x)$ , it has no central direction but  $\mathcal{U}^x$  may be nonempty.

It is clear that  $\downarrow(U) \cap \downarrow(U') = \emptyset$  if  $U$  and  $U'$  are distinct lines in  $\mathcal{U}^x$ . We get a linear order on  $\mathcal{U}^x$  based on that on directions, that we also denote by  $\sqsubseteq_x$ : we have  $U \sqsubset_x U'$  if and only if  $y \sqsubset y'$  for all  $y \in U$  and  $y' \in U'$ .

(c) A *structured ordered join-trees* (an *SOJ-tree*) is a tuple  $(N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$  such that  $(N, \leq, \sqsubseteq)$  is an OJ-tree and  $\mathcal{U} := \{A\} \uplus \mathcal{U}^- \uplus \mathcal{U}^+$  is a structuring of  $(N, \leq)$  with axis  $A$  such that, for each node  $x$  :

if  $U \in \mathcal{U}^x \cap \mathcal{U}^-$  and  $U' \in \mathcal{U}^x \cap \mathcal{U}^+$ , then  $U \sqsubset_x U'$  and furthermore, if  $x$  has a central direction  $\delta$ , then  $U \sqsubset_x \delta \sqsubset_x U'$ .

We define then  $Dir^-(x)$  as the set of directions  $\downarrow(U)$  for  $U \in \mathcal{U}^x \cap \mathcal{U}^-$  and, similarly,  $Dir^+(x)$  with  $U \in \mathcal{U}^x \cap \mathcal{U}^+$ .

Let  $x$  be such that  $[x, +\infty[ \cap U \neq \emptyset$  for some  $U$  in  $\mathcal{U}$  not containing  $x$ . By Condition 2) of Definition (3.3.a), there is a node  $y_i$  in  $U$ , with the notation of that definition. We say that  $x$  is *to the left* (resp. *to the right*) of  $U$  if, for some direction  $\delta$  relative to  $y_i$ , we have  $x \in \delta \in Dir^-(y_i)$  (resp.  $x \in \delta \in Dir^+(y_i)$ ).

As in Propositions (3.5) and (3.9), we have :

(4.3) **Proposition** : Every join-hedge and every ordered join-tree has a structuring.

**Proof:** For a join-hedge  $(N, \leq, \sqsubseteq)$ , we take any structuring  $\mathcal{U}$  of the join-forest  $(N, \leq)$ .

Let  $(N, \leq, \sqsubseteq)$  be an OJ-tree and  $\mathcal{U}$  be any structuring of the join-tree  $(N, \leq)$ . Let  $A$  be its axis. We need only partition each set  $\mathcal{U}^x$  into two sets  $\mathcal{U}^x \cap \mathcal{U}^-$  and  $\mathcal{U}^x \cap \mathcal{U}^+$ .

If  $x$  has a central direction  $\delta$ , we let  $\mathcal{U}^x \cap \mathcal{U}^-$  consist of the lines  $U$  in  $\mathcal{U}^x$  such that  $\downarrow(U) \sqsubset_x \delta$ , and  $\mathcal{U}^x \cap \mathcal{U}^+$  consist of those such that  $\delta \sqsubset_x \downarrow(U)$ . Otherwise, we let  $\mathcal{U}^+$  contain<sup>7</sup>  $\mathcal{U}^x$  so that  $\mathcal{U}^x \cap \mathcal{U}^- = \emptyset$ .  $\square$

We now establish the MS-definability of these structurings. If  $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$  is an SOJ-tree, we define  $S(J)$  as the structure  $(N, \leq, \sqsubseteq, A, N_0^-, N_0^+, N_1^-, N_1^+)$  such that  $A$  is the axis,  $N_0^-$  (resp.  $N_0^+$ ) is the union of the lines  $U \in \mathcal{U}^-$  (resp.  $U \in \mathcal{U}^+$ ) of even depth and  $N_1^-$  (resp.  $N_1^+$ ) is the union of the lines  $U \in \mathcal{U}^-$  (resp.  $U \in \mathcal{U}^+$ ) of odd depth.

(4.4) **Proposition** : Let  $J = (N, \leq, \sqsubseteq)$  be an OJ-tree.

1) There is an MS formula  $\varphi(A, N_0^-, N_0^+, N_1^-, N_1^+)$  expressing that a structure  $(N, \leq, \sqsubseteq, A, N_0^-, N_0^+, N_1^-, N_1^+)$  is  $S(J)$  for some SOJ-tree  $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ .

2) There exists an MS formula  $\theta^-(u, U, N_0^-, N_0^+, N_1^-, N_1^+)$  expressing in a structure  $(N, \leq, \sqsubseteq, A, N_0^-, N_0^+, N_1^-, N_1^+) = S(N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$  that  $U \in \mathcal{U}^- \wedge u = \widehat{U}$ ; similarly, there exists an MS formula  $\theta^+(u, U, N_0^-, N_0^+, N_1^-, N_1^+)$  expressing that  $U \in \mathcal{U}^+ \wedge u = \widehat{U}$ .

**Proof:** Easy modification of the proof of Proposition (3.7).  $\square$

(4.5) **Definition** : *Description schemes for SOJ-trees.*

<sup>7</sup>We might also partition  $\mathcal{U}^x$  into any two sets  $\mathcal{U}^x \cap \mathcal{U}^-$  and  $\mathcal{U}^x \cap \mathcal{U}^+$  such that  $\mathcal{U}^x \cap \mathcal{U}^- \sqsubset_x \mathcal{U}^x \cap \mathcal{U}^+$ .

(a) A *description scheme* for an SOJ-tree, in short an *SOJ-scheme*, is a 6-tuple  $\Delta = (Q, D, w_{Ax}, (w_q^-)_{q \in Q}, (w_q^+)_{q \in Q}, (w_d)_{d \in D})$  such that  $Q, D$  are sets,  $w_{Ax} \in \mathcal{A}(Q)$ ,  $(w_d)_{d \in D}$  is a family of arrangements over  $Q$  and  $(w_q^-)_{q \in Q}$  and  $(w_q^+)_{q \in Q}$  are families of arrangements over  $D$ . Without loss of generality, we will assume that the domains of these arrangements are pairwise disjoint, and the same symbol  $\preceq$  denotes their orders. Informally,  $(w_q^-)_{q \in Q}$  and  $(w_q^+)_{q \in Q}$  encodes the sets of lines, ordered by  $\sqsubseteq_x$  of the two sets  $Dir^-(x)$  and  $Dir^+(x)$  where  $x$  is labelled by  $q$ .

We say that  $\Delta$  is *regular* if  $Q \cup D$  is finite and the arrangements  $w_{Ax}$ ,  $w_d$ ,  $w_q^-$  and  $w_q^+$  are regular.

(b) Let  $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$  be an SOJ-tree. An SOJ-scheme  $\Delta$  as in (a) *describes*  $J$  if there exist mappings  $r : N \rightarrow Q$  and  $\tilde{r} : \mathcal{U}^- \cup \mathcal{U}^+ \rightarrow D$  such that:

$$(b.1) \quad (A, \leq, r) \simeq w_{Ax},$$

(b.2) for each  $x \in N$ , the arrangement  $(\mathcal{U}^x \cap \mathcal{U}^-, \sqsubseteq_x, \tilde{r})$  over  $D$  is isomorphic to  $w_{r(x)}^-$ ,

(b.3) for each  $x \in N$ , the arrangement  $(\mathcal{U}^x \cap \mathcal{U}^+, \sqsubseteq_x, \tilde{r})$  over  $D$  is isomorphic to  $w_{r(x)}^+$ ,

(b.4) for each  $U \in \mathcal{U}^- \cup \mathcal{U}^+$ , the arrangement  $(U, \leq, r)$  over  $Q$  is isomorphic to  $w_{\tilde{r}(U)}$ .

We will also say that  $\Delta$  *describes* the join-tree  $fgs(J) := (N, \leq)$ .

(4.6) **Proposition** : Every SOJ-scheme describes an SOJ-tree that is unique up to isomorphism.

**Proof:** Let  $\Delta = (Q, D, w_{Ax}, (w_q^-)_{q \in Q}, (w_q^+)_{q \in Q}, (w_d)_{d \in D})$  be an SOJ-scheme, defined with arrangements  $w_{Ax} = (V_{Ax}, \preceq, lab_{Ax})$ ,  $w_d = (V_d, \preceq, lab_d)$ ,  $w_q^- = (W_q^-, \preceq, lab_q)$  and  $w_q^+ = (W_q^+, \preceq, lab_q)$  such that the sets  $V_{Ax}, V_d, W_q^-$  and  $W_q^+$  are pairwise disjoint. Furthermore, we extend  $\prec$  by letting  $s \prec s'$  for all  $s \in W_q^-, s' \in W_q^+$  and  $q \in Q$ . We construct  $J = Unf(\Delta) = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$  as follows. Clauses a) to d) are essentially as in Proposition (3.24).

a)  $N$  is the set of finite nonempty sequences  $(v_0, s_1, v_1, s_2, \dots, s_k, v_k)$  such that:

$$\begin{aligned} v_0 \in V_{Ax}, v_i \in V_{d_i} \text{ and } s_i \in W_{q_{i-1}}^- \cup W_{q_{i-1}}^+ \text{ for } 1 \leq i \leq k, \text{ where} \\ q_0 = lab_{Ax}(v_0), d_1 = lab_{q_0}(s_1), q_1 = lab_{d_1}(v_1), d_2 = lab_{q_1}(s_2), \dots, \\ q_i = lab_{d_i}(v_i), d_{i+1} = lab_{q_i}(s_{i+1}) \text{ for } 1 \leq i \leq k-1. \end{aligned}$$

b)  $(v_0, s_1, v_1, \dots, s_k, v_k) \leq (v'_0, s'_1, v'_1, \dots, s'_j, v'_j)$  if and only if :  
 $k \geq j$ ,  $(v_0, s_1, v_1, \dots, s_j) = (v'_0, s'_1, v'_1, \dots, s'_j)$  and  $v_j \preceq v'_j$  ( $v_j, v'_j \in V_{d_j}$ ).

c) The axis  $A$  is the set of one-element sequences  $(v)$  for  $v \in V_{Ax}$ .

d) If  $x = (v_0, s_1, v_1, \dots, s_k, v_k)$ , the line  $U(x)$  is the set of sequences  $(v_0, s_1, v_1, s_2, \dots, s_k, v)$  for  $v \in V_{d_k}$ ; it belongs to  $\mathcal{U}^-$  if  $s_k \in W_{q_{k-1}}^-$  and to  $\mathcal{U}^+$  if  $s_k \in W_{q_{k-1}}^+$ ; in both cases,  $\widehat{U(x)} = (v_0, s_1, v_1, \dots, s_{k-1}, v_{k-1})$ .

e)  $x = (v_0, s_1, v_1, \dots, s_k, v_k) \sqsubseteq y = (v'_0, s'_1, v'_1, \dots, s'_j, v'_j)$  if and only if ,

either  $x \leq y$  or, for some  $\ell < \{j, k\}$ , we have

e.1)  $(v_0, s_1, v_1, \dots, v_\ell) = (v'_0, s'_1, v'_1, \dots, v'_\ell)$  and  $s_{\ell+1} \prec s'_{\ell+1}$ ,

e.2) or  $(v_0, s_1, v_1, \dots, s_\ell) = (v'_0, s'_1, v'_1, \dots, s'_\ell)$ ,  $s_{\ell+1} \in W_{q_\ell}^-$  and  $v'_\ell \prec v_\ell$ ,

e.3) or  $(v_0, s_1, v_1, \dots, s_\ell) = (v'_0, s'_1, v'_1, \dots, s'_\ell)$ ,  $s'_{\ell+1} \in W_{q_\ell}^+$  and  $v_\ell \prec v'_\ell$ .

In Case e.1),  $x$  and  $y$  are in different directions of  $z := (v_0, s_1, v_1, \dots, v_\ell)$  that are not its central direction; in Case e.2),  $x$  is to the left of the central direction  $\delta$  of  $z$  and  $y \leq u$  where  $u := (v_0, s_1, v_1, \dots, v'_\ell)$  is here below  $z$  on  $\delta$ ; in Case e.3),  $y$  is to the right of the central direction  $\delta'$  of  $u$  and  $x \leq z$  where  $z$  is below  $u$  on  $\delta'$ .

In order to prove that  $\Delta$  describes  $J$ , we define  $r : N \rightarrow Q$  and  $\tilde{r} : \mathcal{U}^- \cup \mathcal{U}^+ \rightarrow D$  as follows:

if  $x \in A$ , then  $x = (v)$  for some  $v \in V_{Ax}$  and  $r(x) := lab_{Ax}(v)$ ;

if  $x \in N$  has depth  $k \geq 1$ , then  $x = (v_0, s_1, v_1, \dots, s_k, v_k)$  for some  $v_0, s_1, \dots, s_k, v_k$  and  $r(x) := lab_{d_k}(v_k)$ ;

if  $U \in \mathcal{U}^- \cup \mathcal{U}^+$ , then  $U = U(x)$  for some  $x = (v_0, s_1, v_1, \dots, s_k, v_k)$ , and  $\tilde{r}(U) := d_k$ .

In the last case, as  $d_k = lab_{q_{k-1}}(s_k)$ , it depends only on  $s_k$  and  $v_{k-1}$  (via  $q_{k-1}$ ). It follows that  $\tilde{r}(U)$  is the same if we consider  $U$  as  $U(y)$  with  $y = (v_0, s_1, v_1, \dots, s_k, v)$  hence, is well-defined.

We check the four conditions of Definition (4.5.b). We have  $(A, \leq, r) \simeq w_{Ax}$ , hence (b.1) holds. For (b.2) and (b.3), we consider  $x = (v_0, s_1, v_1, \dots, s_k, v_k) \in N$ . The sets  $U$  in  $\mathcal{U}^x$  are those of the form  $\{(v_0, s_1, v_1, \dots, s_k, v_k, s, v) \mid v \in V_{d_{k+1}}\}$  for all  $s \in W_{q_k}^- \cup W_{q_k}^+$  where  $q_k = lab_{d_k}(v_k) = r(x)$ , hence (b.2) and (b.3) hold.

For checking (b.4), we let  $U = U(x)$  for some  $x = (v_0, s_1, v_1, \dots, s_k, v_k)$ ,  $k > 0$ ; then  $U$  is the set of sequences  $(v_0, s_1, v_1, s_2, \dots, s_k, v)$  such that  $v \in V_{d_k}$  ordered by  $\preceq$  on the last components. Hence,  $(U, \leq, lab_{d_k})$  is isomorphic to  $w_{d_k}$ , which proves the property since  $\tilde{r}(U) := d_k$ .

Unicity is proved as in Proposition (3.11).  $\square$

(4.7) **Proposition:** An SOJ-tree is MS-definable if it is described by a regular SOJ-scheme.

**Proof:** Similar to the proofs of Propositions (3.12) and (3.25).  $\square$

Note that, we need not the finiteness predicate as in Proposition (3.25) because we deal with arrangements that are linearly ordered structures, and not with labelled sets.

Next we define an algebra  $\text{SOJT}$  with two sorts:  $\mathbf{t}$  for SOJ-trees and  $\mathbf{h}$  for SJ-hedges.

(4.8) **Definition** : *Operations on SOJ-trees and SJ-hedges.*

*Concatenation of SOJ-trees along axes.*

Let  $J_1 = (N_1, \leq_1, \sqsubseteq_1, A_1, \mathcal{U}_1^-, \mathcal{U}_1^+)$  and  $J_2 = (N_2, \leq_2, \sqsubseteq_2, A_2, \mathcal{U}_2^-, \mathcal{U}_2^+)$  be disjoint SOJ-trees. We define their concatenation as follows:

$$\begin{aligned} J_1 \bullet J_2 &:= (N_1 \uplus N_2, \leq, \sqsubseteq, A_1 \uplus A_2, \mathcal{U}_1^- \uplus \mathcal{U}_2^-, \mathcal{U}_1^+ \uplus \mathcal{U}_2^+) \text{ where :} \\ x \leq y &: \iff x \leq_1 y \vee x \leq_2 y \vee (x \in N_1 \wedge y \in A_2), \\ x \sqsubseteq y &: \iff x \leq y \vee x \sqsubseteq_1 y \vee x \sqsubseteq_2 y, \\ &\quad \vee (x \perp y \wedge x \in N_1 \wedge y \in N_2 \wedge y \in U \in \mathcal{U}_2^+ \cap \mathcal{U}_2^{\perp y}) \\ &\quad \vee (x \perp y \wedge x \in N_2 \wedge y \in N_1 \wedge x \in U \in \mathcal{U}_2^- \cap \mathcal{U}_2^{\perp y}), \text{ for some } U. \end{aligned}$$

The relations  $x \perp y$  and  $x \sqcup y$  are relative to  $\leq$ .

It is clear that  $J_1 \bullet J_2$  is an SOJ-tree. Its axis is  $A_1 \uplus A_2$ ,  $\mathcal{U}^+ = \mathcal{U}_1^+ \uplus \mathcal{U}_2^+$  and  $\mathcal{U}^- = \mathcal{U}_1^- \uplus \mathcal{U}_2^-$ .

*The empty SOJ-tree* is denoted by the nullary symbol  $\Omega_{\mathbf{t}}$ .

*Extension of two SJ-hedges into a single SOJ-tree:*

Let  $H_1 = (N_1, \leq_1, \sqsubseteq_1, \mathcal{U}_1)$  and  $H_2 = (N_2, \leq_2, \sqsubseteq_2, \mathcal{U}_2)$  be disjoint SJ-hedges and  $u \notin N_1 \uplus N_2$ . Then:

$$\begin{aligned} ext_u(H_1, H_2) &:= (N_1 \uplus N_2 \uplus \{u\}, \leq, \sqsubseteq, \{u\}, \mathcal{U}_1, \mathcal{U}_2) \text{ where :} \\ x \leq y &: \iff x \leq_1 y \vee x \leq_2 y \vee y = u, \\ x \sqsubseteq y &: \iff x \leq y \vee x \sqsubseteq_1 y \vee x \sqsubseteq_2 y \vee (x \in N_1 \wedge y \in N_2). \end{aligned}$$

Clearly,  $ext_u(J)$  is an SOJ-tree. When handling SOJ-trees and SJ-hedges up to isomorphism, we replace the notation  $ext_u(H_1, H_2)$  by  $ext(H_1, H_2)$ .

*The empty SJ-hedge* is denoted by the nullary symbol  $\Omega_{\mathbf{h}}$ .

*Making an SOJ-tree into an SJ-hedge.*

This is done by the unary operation  $mkh$  such that, if  $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$  is an SOJ-tree, then

$$mkh(J) := (N, \leq, \sqsubseteq, \{A\} \uplus \mathcal{U}^- \uplus \mathcal{U}^+).$$



Note that in  $mkh(J)$ , we distinguish neither  $\mathcal{U}^-$  from  $\mathcal{U}^+$  nor the axis  $A$  from the other lines.

*The concatenation of two disjoint SJ-hedges.*

Let  $H_1 = (N_1, \leq_1, \sqsubseteq_1, \mathcal{U}_1)$  and  $H_2 = (N_2, \leq_2, \sqsubseteq_2, \mathcal{U}_2)$  be disjoint SJ-hedges. Their "horizontal" concatenation is:

$$H_1 \otimes H_2 := (N_1 \uplus N_2, \leq_1 \uplus \leq_2, \sqsubseteq, \mathcal{U}_1 \uplus \mathcal{U}_2) \text{ where :}$$

$$x \sqsubseteq y : \iff x \sqsubseteq_1 y \vee x \sqsubseteq_2 y \vee (x \in N_1 \wedge y \in N_2).$$

We let  $F''$  be the 2-sorted signature  $\{\bullet, \otimes, ext, mkh, \Omega_t, \Omega_h\}$  whose operation types are :

$$\begin{aligned} \bullet &: t \times t \rightarrow t, \\ \Omega_t &: t, \\ \otimes &: h \times h \rightarrow h, \\ \Omega_h &: h, \\ ext &: h \times h \rightarrow t, \\ mkh &: t \rightarrow h. \end{aligned}$$

In addition, we have, as in Definitions (3.13) and (3.26) :

*Forgetting the structuring:*

If  $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$  is an SOJ-tree, then  $fgs(J) := (N, \leq, \sqsubseteq)$  is the underlying OJ-tree.

(4.9) **Definition** : *The value of a term.*

If  $u$  is an occurrence of a binary symbol in a term  $t$ , we denote by  $s_1(u)$  its first son and by  $s_2(u)$  the second one (cf. Definition (3.15)).

The value  $val(t) := (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$  of a term  $t \in T^\infty(F'')_t$  is an SOJ-tree defined in a similar way as for  $t \in T^\infty(F')_t$ , cf. Definitions (3.15) and (3.28):

$$\begin{aligned} N &:= \text{Occ}(t, ext), \\ x \leq y &: \iff x \leq_t w \leq_{lex} y \text{ for some } w \in N \text{ such that } w \approx y, \\ A &:= \text{Max}(t, ext, \varepsilon), \end{aligned}$$

with  $\approx$  the equivalence relation on  $N$  defined as in Definition (3.15.a):

$\mathcal{U}^-$  is the set of equivalence classes of  $\approx$  of nodes in  $\text{Max}(t, ext, s_1(u))$  for some occurrence  $u$  of  $ext$ ,

$\mathcal{U}^+$  is the set of equivalence classes of  $\approx$  of nodes in  $\text{Max}(t, ext, s_2(u))$  for some occurrence  $u$  of  $ext$ .

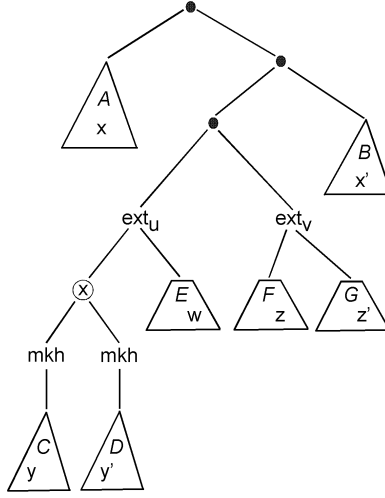


Figure 6: Term  $T$  of Example (4.10).

Hence,  $U(x) \in \mathcal{U}^-$  if  $x \leq_t s_1(\widehat{U(x)})$  and  $U(x) \in \mathcal{U}^+$  if  $x \leq_t s_2(\widehat{U(x)})$ .

Next we define  $\sqsubseteq$ .

$x \sqsubseteq y : \iff x \leq y$  or  $x \perp y$  ( $\perp$  is relative to  $\leq$ , not to  $\leq_t$ ) and we have one of the following cases:

- (i)  $x \sqsubseteq_t y$  is an occurrence of  $\otimes$  or  $ext$ ,  $x \leq_t s_1(x \sqsubseteq_t y)$  and  $y \leq_t s_2(x \sqsubseteq_t y)$ ,
- (ii)  $x \sqsubseteq_t y$  is an occurrence of  $\bullet$ ,  $x \leq_t s_1(x \sqsubseteq_t y)$  and  $y \leq_t s_2(z)$  where  $z$  is the unique maximal occurrence of  $ext$  such that  $y <_t z \leq_t s_2(x \sqsubseteq_t y)$ ,
- (iii)  $x \sqsubseteq_t y$  is an occurrence of  $\bullet$ ,  $y \leq_t s_1(x \sqsubseteq_t y)$  and  $x \leq_t s_1(z)$  where  $z$  is the unique maximal occurrence of  $ext$  such that  $x <_t z \leq_t s_2(x \sqsubseteq_t y)$ .

If  $t \in T^\infty(F'')_{\mathbf{h}}$  its value  $val(t)$  is  $(N, \leq, \sqsubseteq, \mathcal{U})$  with  $(N, \leq, \sqsubseteq)$  defined as above and  $\mathcal{U}$  as in Definition (3.28).

- (4.9.1) *Claim* : (1) The mapping  $val$  is a value mapping  $T^\infty(F'') : \rightarrow \text{SOJT}$ .  
(2) The transformation  $\alpha$  of  $[t]$  into  $(N, \leq, \sqsubseteq)$  is an MS-transduction.

*Proof* : (1) is clear from the definitions.

(2) holds because the conditions of Definition (4.9) are expressible in  $[t]$  by MS formulas.  $\square$

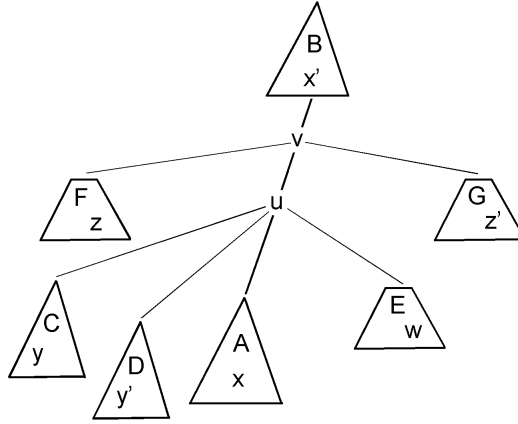


Figure 7: The OJ-tree  $val(T)$  of Example (4.10).

(4.10) *Example:* We now illustrate this definition. Figure 6 shows a term  $T$  where  $A, B, C$  and  $D$  are subterms of type  $t$  and  $E, F$  and  $G$  are subterms of type  $h$ . They contain occurrences of  $ext$  that define nodes  $x, x', y, y', w, z$  and  $z'$  of  $val(T)$ .

The OJ-tree  $val(T)$  is shown on Figure 7, where we designate by  $A, B, \dots, G$  the trees and hedges defined by the terms  $A, B, \dots, G$ .

We have the following comparisons for  $<$  :

$$\{z, z', u\} < v, \text{ because } \{z, z'\} <_T v, u <_{lex} v \text{ and } u \approx v,$$

$$\{y, y', w\} < u, \text{ because } \{y, y', w\} <_T u,$$

$$x \leq \{u, v\} \text{ because } x \leq_T a <_{lex} \{u, v\} \text{ and } a \approx u \approx v \text{ where } a \text{ is the}$$

$$v < x' \text{ if and only if } x' \text{ is on } X, \text{ the axis of } B, \text{ because in this case,}$$

$$v \approx x' \text{ and otherwise } v \text{ and } x' \text{ are incomparable with respect to } \leq;$$

$$\text{in all cases we have } v <_{lex} x'.$$

For  $\sqsubset$  we have:

$$z \sqsubset y \sqsubset y' \sqsubset x \sqsubset w \sqsubset u \sqsubset z' \sqsubset v$$

$$\text{and } x' \sqsubset z \text{ if } x' \text{ is to the left of } X; \text{ otherwise } v \sqsubset x'.$$

All inequalities for  $<$  yield the corresponding inequalities for  $\sqsubset$ . We now compare  $z, y, y', x, w, z'$  that pairwise incomparable for  $<$ .

By Case (i) of Definition (4.9), we get  $\{y, y'\} \sqsubset w, y \sqsubset y'$  and  $z \sqsubset z'$ .

By Case (ii), we get  $x \sqsubset w, \{x, w\} \prec z'$  and  $\{y, y'\} \prec w$ .

By Case(iii) we get  $\{z, y, y'\} \prec x$  and  $z \prec \{y, y'\}$ .

Finally, if  $x'$  is to the left of  $X$ , then Case (iii) gives  $x' \sqsubset z$ , and if it to its right, then Case (ii) gives  $z \sqsubset x'$ .  $\square$

- (4.11) **Theorem:** The following properties of an OJ-tree  $J$  are equivalent:
- (1)  $J$  is regular,
  - (2)  $J$  is described by a regular SOJ-scheme,
  - (3)  $J$  is MS-definable.

**Proof:** (1) $\implies$ (2): Similar to that of Theorem (3.21).

(2) $\implies$ (3): By Proposition (4.7).

(3) $\implies$ (1) As observed in Claim (4.9.2), the mapping  $\alpha$  that transforms the relational structure  $[t]$  for  $t$  in  $T^\infty(F'')_{\mathbf{t}}$  into the OJ-tree  $(N, \leq, \sqsubseteq) = fgs(val(t))$  is an MS-transduction. Let  $J = (N, \leq, \sqsubseteq)$  be an MS-definable OJ-tree. It is, up to isomorphism, the unique model of an MS-sentence  $\beta$ . The set of terms  $t$  in  $T^\infty(F'')_{\mathbf{t}}$  such that  $\alpha([t]) \models \beta$  is thus MS-definable, hence, it contains a regular term. This term denotes  $J$ , hence  $J$  is regular.  $\square$

As in Corollaries (3.22) and (3.31) we deduce that the isomorphism problem for regular OJ-trees is decidable.

## 5 Quasi-trees

Quasi-trees can be viewed intuitively as "undirected join-trees". As in [Cou14], we define them in terms of a ternary betweenness relation.

(5.1) **Definition** : *Betweenness*.

(a) Let  $L = (X, \leq)$  be a linear order. Its *betweenness relation* is the ternary relation on  $X$  such that  $B_L(x, y, z)$  holds if and only if  $x < y < z$  or  $z < y < x$ . It is empty if  $X$  has less than 3 elements.

(b) If  $T$  is a tree, its *betweenness relation* is the ternary relation on  $N_T$ , such that  $B_T(x, y, z)$  holds if and only if  $x, y, z$  are pairwise distinct and  $y$  is on the unique path between  $x$  and  $z$ . If  $R$  is a rooted tree and  $T = Und(R)$  is the tree obtained from  $T$  by forgetting its root and edge directions, then :

$$B_T(x, y, z) \iff x, y, z \text{ are pairwise distinct and } x <_R y \leq_R x \sqcup_R z \\ \text{or } z <_R y \leq_R x \sqcup_R z.$$

(c) If  $B$  is a ternary relation on a set  $X$ , and  $x, y \in X$ , then  $[x, y]_B := \{x, y\} \cup \{z \in X \mid B(x, z, y)\}$ .

(5.2) **Proposition** [Cou14] : (a) The betweenness relation  $B$  of a linear order  $(X, \leq)$  satisfies the following properties for all  $x, y, z, u \in X$ .

$$\begin{aligned}
\text{A1} &: B(x, y, z) \Rightarrow x \neq y \neq z \neq x. \\
\text{A2} &: B(x, y, z) \Rightarrow B(z, y, x). \\
\text{A3} &: B(x, y, z) \Rightarrow \neg B(x, z, y). \\
\text{A4} &: B(x, y, z) \wedge B(y, z, u) \Rightarrow B(x, y, u) \wedge B(x, z, u). \\
\text{A5} &: B(x, y, z) \wedge B(x, u, y) \Rightarrow B(x, u, z) \wedge B(u, y, z). \\
\text{A6} &: B(x, y, z) \wedge B(x, u, z) \Rightarrow \\
& y = u \vee [B(x, u, y) \wedge B(u, y, z)] \vee [B(x, y, u) \wedge B(y, u, z)]. \\
\text{A7}' &: x \neq y \neq z \neq x \Rightarrow B(x, y, z) \vee B(x, z, y) \vee B(y, x, z).
\end{aligned}$$

(b) The betweenness relation  $B$  of a tree  $T$  satisfies the properties A1-A6 for all  $x, y, z, u$  in  $N_T$  together with the following weakening of A7':

$$\begin{aligned}
\text{A7}' &: x \neq y \neq z \neq x \Rightarrow B(x, y, z) \vee B(x, z, y) \vee B(y, x, z) \vee \\
& \exists w. (B(x, w, y) \wedge B(y, w, z) \wedge B(x, w, z)).
\end{aligned}$$

(5.3) **Proposition** : Let  $B$  be a ternary relation on a set  $X$  that satisfies properties A1-A7' for all  $x, y, z, u$  in  $X$ . Let  $a$  and  $b$  be distinct elements of  $X$ . There is a unique linear order  $L = (X, \leq)$  such that  $a < b$  and  $B_L = B$ . It is first-order definable in the logical structure  $(X, B, a, b)$ .

**Proof:** Let  $X, B$  be as in the statement. Let  $Y$  consist of  $n$  elements of  $X$ ,  $n > 2$ . It can be enumerated as  $x_1, \dots, x_n$  in such a way that  $B(x_i, x_j, x_k)$  holds for all  $1 \leq i < j < k \leq n$ ; furthermore, the only other possible enumeration satisfying this property is its reversal,  $x_n, \dots, x_1$ .

This is clear if  $n = 3$ . Otherwise, we use induction. Consider  $x, y, z \in Y$  such that  $B(x, y, z)$  holds. Let  $Y_1 := \{u \in Y \mid B(u, y, z)\}$  and  $Y_2 := \{u \in Y \mid B(x, y, u)\}$ . These sets have less than  $n$  elements as  $z \notin Y_1$  and  $x \notin Y_2$ . Furthermore,  $Y_1 \cap Y_2 = \{y\}$ . The induction hypothesis gives enumerations  $x_1, \dots, y$  of  $Y_1$  and  $y, \dots, x_n$  of  $Y_2$  with the desired properties. The desired enumeration of  $Y$  is  $x_1, \dots, y, \dots, x_n$ . We omit a detailed proof.

If  $a$  and  $b$  are distinct elements of  $X$ , there is a unique linear order  $L = (X, \leq)$  such that  $a < b$  and  $B_L = B$ . This is clear from the above observation if  $X$  is finite. Otherwise, we let  $X$  be the union of an increasing sequence of finite subsets that contain all  $a$  and  $b$ . On each of them, say  $Y$ , we have a unique linear order  $<$  such  $a < b$ , whose betweenness relation is  $B \cap (Y \times Y \times Y)$ . Their union gives the desired linear order, that we denote by  $\leq_{a,b}$ .

We now define it by a first-order formula.

*Case 1* : There are no  $u, v$  such that  $B(u, b, v)$  holds. Then, we have<sup>8</sup> :

$$x \leq_{a,b} y : \Leftrightarrow y \in [x, b]_B \text{ (i.e., } x = y \vee y = b \vee B(x, y, b)).$$

Otherwise and similarly, if there are no  $u, v$  such that  $B(u, a, v)$  holds, we have

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<sup>8</sup>We recall the notation  $[x, y]_B := \{x, y\} \cup \{z \mid B(x, z, y)\}$ .

$$x \leq_{a,b} y : \iff x \in [a, y]_B.$$

Case 2 : Otherwise, we define the following binary relation :

$$\begin{aligned} Z(x, y) : \iff & x \neq y \wedge \\ & [(B(x, a, b) \wedge \neg B(y, x, a)) \vee (x = a \wedge \neg B(y, a, b)) \vee \\ & (B(a, x, b) \wedge \neg B(y, x, b)) \vee (x = b \wedge B(a, b, y)) \vee \\ & (B(a, b, x) \wedge B(b, x, y))]. \end{aligned}$$

It is easy to see that  $x <_{a,b} y$  implies  $Z(x, y)$ . (In particular, that  $Z(a, b)$  holds follows from the clause  $x = a \wedge \neg B(y, a, b)$  with  $y = b$ ).

For the converse, assume that  $Z(x, y)$  holds and  $x <_{a,b} y$  does not. Then, we have  $y <_{a,b} x$ . By looking at the different relative positions of  $x, y, a$  and  $b$ , we get a contradiction. Hence  $x \leq_{a,b} y$  if and only if  $x = y \vee Z(x, y)$ , which is expressed by a first-order formula  $\xi(a, b, x, y)$ .  $\square$

(5.4) **Definition** [Cou14] : *Quasi-trees*.

(a) A *quasi-tree* is a structure  $S = (N, B)$  such that  $B$  is a ternary relation on  $N$ , the set of *nodes*, that satisfies conditions A1-A7. To avoid uninteresting special cases, we also require that  $N$  has at least 3 nodes. In a quasi-tree, the four cases of the conclusion of A7 are exclusive and in the fourth case, there is a unique node  $w$  satisfying  $B(x, w, y) \wedge B(y, w, z) \wedge B(x, w, z)$  (by Lemma 11 of [Cou14]), that is denoted by  $M_S(x, y, z)$ .

A *leaf* (of  $S$ ) is a node  $z$  such that  $B(x, z, y)$  holds for no  $x, y$ . A *line* is set of nodes  $L$  such that  $[x, y]_B \subseteq L$  if  $x, y \in L$  and an *end* of  $L$  is a node  $z$  not in  $[x, y]_B$  for any  $x, y \in L$ .

We say that  $S$  is *discrete* if each set  $[x, y]_B$  is finite.

We say that a quasi-tree  $S = (N, B)$  is a *subquasi-tree* of a quasi-tree  $S' = (N', B')$ , which we denote by  $S \subseteq S'$ , if  $N \subseteq N'$  and  $B = B' \upharpoonright N$ . This condition implies that  $M_S = M_{S'} \upharpoonright N$ .

(b) From a join-tree  $J = (N, \leq)$ , we define a ternary relation  $B_J$  on  $N$  by:

$$B_J(x, y, z) : \iff x \neq y \neq z \neq x \text{ and } (x < y \leq x \sqcup z) \vee (z < y \leq x \sqcup z).$$

(5.5) **Proposition** : (1) The structure  $qt(J) := (N, B_J)$  associated with a join-tree  $J = (N, \leq)$  with at least 3 nodes is a quasi-tree. Every line of  $J$  is a line of  $qt(J)$ . If  $J$  is a subjoin-tree of  $J'$ , then  $qt(J)$  is a subquasi-tree of  $qt(J')$ .

(2) Every quasi-tree  $S$  is  $qt(J)$  for some join-tree  $J$ .

(3) A quasi-tree is discrete if and only if it is  $qt(J)$  for some tree  $J$ .

(4) Let  $S = (N, B)$  be a quasi-tree,  $L$  a line of  $S$  and  $a, b$  distinct nodes of  $L$ . In each of the following two cases:

- (i)  $b$  is the largest node of  $L$  with respect to its linear order<sup>9</sup>  
 $\leq_{a,b}$ ,  
(ii)  $L$  has no largest node and there is no  $w$  in  $N - L$   
such that  $B(u, v, w)$  holds for all  $u, v \in L$  such that  $u <_{a,b}$   
 $v$ ,

there is a unique join-tree  $J = (N, \leq)$  such that  $S = qt(J)$ ,  $a < b$  and  $L$  is an upwards closed<sup>10</sup> line of  $J$ . Its order  $\leq$  is first-order definable in the structure  $(N, B, L, a, b)$ .

**Proof:** (1) Let  $J = (N, \leq)$  be a join-tree with at least 3 nodes.

If it is finite, then it is  $(N_T, \leq_T)$  for a finite tree  $T$ , and thus  $qt(J)$  is a quasi-tree by Proposition (5.2.b).

Otherwise consider distinct elements  $x, y, z, u$  of  $N$ . We want to prove that they satisfy A1-A7. There is a set  $N' \subseteq N$  of cardinality at most 7 that contains  $x, y, z, u$  and is closed under  $\sqcup$ . Then  $J' = (N', \leq \upharpoonright N')$  is a finite join-tree,  $J' \subseteq J$  and  $qt(J') = (N', B_J \upharpoonright N')$  is a quasi-tree by the initial observation, so that  $x, y, z, u$  satisfy A1-A7 for  $B = B_{J'}$  hence for  $B_J$ . (The node  $w$  that may be necessary to satisfy A7 may have to be chosen in the set  $\{x \sqcup y, x \sqcup z, x \sqcup u, \dots\}$ ). As  $x, y, z, u$  are arbitrary, A1-A7 hold for  $B_J$  and all  $x, y, z, u \in N$ . Hence,  $(N, B_J)$  is a quasi-tree.

That every line of  $J$  is a line of  $qt(J)$  follows from the definitions. (The converse does not hold. However, see Assertion (4).) The assertion about subjoin-trees is also easy to prove.

(2) Let  $S = (N, B)$  be a quasi-tree and  $b$  be any element of  $N$ . We define (cf. Proposition (5.3)) :

$$x \leq_b y : \iff y \in [x, r]_B \quad (\iff x = y \vee y = b \vee B(x, y, b)).$$

Then  $(N, \leq_b)$  is a join-tree  $J$  with root  $b$  and  $S = qt(J)$  by Lemma 14 of [Cou14].

(3) is proved in Proposition 17 of [Cou14].

(4) Let us first motivate the technical condition on  $L$ . Let  $J = (N, \leq)$  be a join-tree and  $S = qt(J)$ . If  $L$  is an upwards closed line in  $J$ , then there are two cases: either  $L$  has a maximal element  $m$ , and  $m$  is the root of  $J$  or not; in the latter case, for every  $x \in N$ , there is  $w \in L$  such that  $x < w$ : to prove this, take  $w$  such that  $x \sqcup u < w$  where  $u$  is any element of  $L$ ; if  $x \in N - L$ , we cannot have  $B_J(u, v, x)$  for all  $u, v \in L$  such that  $u < v$ , because, otherwise, letting  $w, w' \in L$  with  $x < w < w'$ , we would have  $B_J(w, w', x)$  and  $B_J(x, w, w')$  contradicting  $A2 \wedge A3$ . The technical conditions on a line  $L$  of  $S$  in the statement are thus

<sup>9</sup>Proposition (5.3) is applicable to  $(L, B \upharpoonright L)$  that satisfies Conditions A1-A7'.

<sup>10</sup>Upwards closed means that  $y \in L$  if  $x < y$  and  $x \in L$ .

necessary for making it into an upwards closed line of a join-tree  $J$  as desired. We now start the proof.

Let  $S = (N, B)$  be a quasi-tree with line  $L$  and distinct elements  $a, b$  of  $L$ . Let  $\leq_{a,b}$  be the linear order on  $L$  from Proposition (5.3).

*Case (i)* :  $b$  is the largest element of  $(L, \leq_{a,b})$ . Then  $(N, \leq_b)$  (cf. (2) for  $\leq_b$ ) is a quasi-tree satisfying the required conditions. If  $J = (N, \leq)$  is another one, then  $[x, +\infty[_J = [x, b]_J$  for all  $x$  and thus,  $x \leq y$  if and only if  $x = y \vee y = b \vee B(x, y, b)$ . Hence,  $\leq = \leq_b$ .

*Case (ii)* :  $L$  has no largest element and there is no  $w$  in  $N - L$  such that  $B(u, v, w)$  holds for all  $u, v \in L$  such that  $u <_{a,b} v$ . It follows that for every  $u \in L$ , there is  $v \in L$  such that  $u <_{a,b} v$ .

We now define, for  $x, y \in N$ :

$$R(x, y) := \iff x \neq y \wedge \exists u, v \in L [u <_{a,b} v \wedge B(x, y, u) \wedge B(y, u, v)],$$

and we will prove that this relation is the desired strict order on  $N$ .

For all  $u, v \in L$  such that  $u <_{a,b} v$ , we define  $N_{u,v} := \{x \in N \mid x = u \vee B(x, u, v)\}$ . Then  $S_{u,v} := (N_{u,v}, B \upharpoonright N_{u,v})$  is a subquasi-tree of  $S$  and  $u$  is the largest element of  $L \cap N_{u,v}$  with respect to  $\leq_{a,b}$ . We define  $\leq_u$  on  $N_{u,v}$  by:

$$x \leq_u y := \iff y \in [x, u]_B.$$

We obtain by Case 1 a join-tree  $J_{u,v} := (N_{u,v}, \leq_u)$  with root  $u$  such that  $S_{u,v} = qt(J_{u,v})$ . Since  $u <_{a,b} v$ , if  $v <_{a,b} u' <_{a,b} v'$ , then  $N_{u,v} \subseteq N_{u',v'}$  and  $\leq_u$  and  $\leq_{u'}$  coincide on  $N_{u,v}$ ; the associated join operations coincide also on  $N_{u,v}$ . Hence  $J_{u,v}$  is a subjoin-tree of  $J_{u',v'}$ .

By the assumption on  $L$ , every  $x \in N$  belongs to some set  $N_{u,v}$ : assume the opposite, then  $x \notin L$ ; assume now that  $x \notin N_{u,v}$ , hence by A7, we have  $B(x, u, v)$  or  $B(u, x, v)$  or  $B(u, v, x)$  or, for some  $w$ ,  $B(u, w, v) \wedge B(x, w, u) \wedge B(x, w, v)$ .

The first case implies  $x \in N_{u,v}$ , the second one  $x \in L$ , and the last one  $w \in L$ ,  $u <_{a,b} w <_{a,b} v$  and  $x \in N_{w,v}$ . All these cases contradict our assumptions on  $x$ . It follows that  $B(u, v, x)$  holds for all  $u, v$  such that  $u <_{a,b} v$ . But this contradicts the assumption on  $L$ .

Hence,  $N$  is the union of the sets  $N_{u,v}$  and  $S$  is the union of the quasi-trees  $S_{u,v}$ . We define  $J = (N, \leq)$  as the union of the join-trees  $J_{u,v}$ . Its order is defined by  $x = y \vee R(x, y)$ . It is a join-tree and  $qt(J) = S$  because  $qt(J_{u,v}) = S_{u,v}$  for all  $u, v$ . It is the unique one satisfying the desired conditions because these conditions determine its restrictions to the sets  $N_{u,v}$  in a unique way.

The property  $x = y \vee R(x, y)$  is defined by a first-order formula  $\theta(L, a, b, x, y)$ .

□

(5.6) **Proposition**: A quasi-tree is  $MS_{fin}$ -definable if it is described by a regular SJ-scheme.

**Proof**: We first explain the idea. From a given quasi-tree  $S$ , we specify by MS-formulas all SJ-trees  $J$  such that  $S = qt(fgs(J))$  and we select one, if



there exists any, that is described by the considered regular SJ-scheme. The construction of the necessary formulas will use Proposition (5.5) and the proof of Proposition (3.25).

Let  $S = (N, B)$  be a quasi-tree (this property is first-order expressible). Assume that  $S = qt(fgs(J))$  where  $J = (N, \leq, \mathcal{U}) \simeq Unf(\Delta)$  for some regular SJ-scheme  $\Delta = (Q, D, w_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$  such that  $Q = \{1, \dots, m\}$  and  $D = \{1, \dots, p\}$  as in the proof of Proposition (3.25). Let  $r, \tilde{r}$  be the corresponding mappings. For each  $d \in D$ , let  $\psi_d$  be an MS sentence that characterizes  $w_d$  up to isomorphism, and similarly  $\psi_{Ax}$  for  $w_{Ax}$ .

The axis  $A$  of  $J$  is upwards closed. If it has a maximal element  $b$  (w.r.t.  $\leq$ ), then  $\leq = \leq_b$  by Proposition (5.5.2). Otherwise,  $\leq = \leq_{a,b}$  where  $a, b$  are any two elements of  $A$  such that  $a < b$ . In this case, there is no  $w$  in  $N - A$  such that  $B_J(u, v, w)$  holds for all  $u, v \in A$  such that  $u \leq_{a,b} v$ . In both cases, by Proposition (5.5) the partial order  $\leq$  on  $N$  is defined by a first-order formula  $\nu(A, a, b, x, y)$  written in such a way that  $b$  is the maximal element of  $A$  if  $a = b$ . We will denote  $\leq$  by  $\leq_{A,a,b}$  to stress that it is defined from  $A, a, b$ .

By Proposition (3.7), there is a bipartition  $(N_0, N_1)$  of  $N$  that describes the structuring  $\mathcal{U}$ , and such that  $A \subseteq N_0$ . From this bipartition, we can MS-define the lines forming  $\mathcal{U}$  and the node  $\hat{U}$  for each  $U \in \mathcal{U} - \{A\}$ .

There is a partition  $(Y_1, \dots, Y_m)$  of  $N$  that describes  $r$  by  $Y_i := r^{-1}(i)$ . There is a partition  $(Z_1, \dots, Z_p)$  of  $N - A$  such that  $Z_j$  is the union of the lines  $U \in \mathcal{U} - \{A\}$  such that  $\tilde{r}(U) = j$ .

Consider relational structures  $R = (X, B, A, N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p, a, b)$  with domain  $X$  such that  $B$  is a ternary relation, the other relations are unary (are subsets of  $X$ ) and  $a, b \in X$ . By MS formulas, one can express in any such  $R$  the following properties:

- (i)  $(X, B)$  is a quasi-tree  $S'$ ,  $A$  is a nonempty line,  $a, b \in A$  and  $a = b$  if and only if  $a$  is an end of  $A$ ,
- (ii) if  $\leq$  is the partial order on  $N$  defined by  $\nu(A, a, b, x, y)$ , then  $(X, \leq, N_0, N_1)$  is  $S(J')$  for some SJ-tree  $J' = (X, \leq, \mathcal{U}')$  with axis  $A$ .
- (iii)  $(Y_1, \dots, Y_m)$  is a partition of  $X$ ; we let  $r(x) := i$  if and only if  $x \in Y_i$ .
- (iv)  $(Z_1, \dots, Z_p)$  is a partition of  $X - A$  such that each  $Z_j$  is a union of sets  $U \in \mathcal{U}' - \{A\}$  such that  $(U, \leq, r) \simeq w_j$ .
- (v)  $(A, \leq, r) \simeq w_{Ax}$ ,
- (vi) for each  $i \in Q$  and  $x \in Y_i$ , the number of lines  $U \in \mathcal{U}'^x$  that are contained in  $Z_j$  is  $m_i^j$ . (A  $D$ -labelled set  $m$  is described by a  $p$ -tuple  $(m^1, \dots, m^p)$  where  $m^j$  is the number of elements having label  $j$ .)

These formulas are constructed as in the proof of Proposition (3.25). The main difference is that  $\leq$  is not in the given structure  $S$  but is defined in it by  $\nu(A, a, b, x, y)$ ; this formula uses auxiliary arguments  $A, a, b$ . As in Proposition (3.25), we need the finiteness predicate if some number  $m_i^j$  is  $\omega$ .

Let  $\varphi'(A, N_0, N_1, a, b)$  express Conditions (i) and (ii) in  $(X, B)$ . Let  $\beta(A, N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p, a, b)$  express conditions (iii)-(vi). If a quasi-tree  $(X, B)$  satisfies :

$$\varphi'(A, N_0, N_1, a, b) \wedge \beta(A, N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p, a, b),$$

the corresponding join-tree  $(X, \leq_{A,a,b})$  has a structuring  $\mathcal{U}'$  with axis  $A$  described by  $N_0, N_1$ . We let  $J' := (X, \leq, \mathcal{U}')$ . The sets  $Y_1, \dots, Y_m, Z_1, \dots, Z_p$  yield a scheme  $\Delta$  that describes  $J'$  (by Conditions (iii)-(vi)), hence  $J'$  is isomorphic to  $J$  by the unicity property of Proposition (3.24), and so, we have  $(X, \leq) = fgs(J') \simeq fgs(J)$  and  $qt(fgs(J')) \simeq qt(fgs(J)) = S$ .

Hence,  $S$  is (up to isomorphism) the unique model  $(X, B)$  of the MS sentence:

$$\begin{aligned} \exists A, N_0, N_1, a, b. [\varphi'(A, N_0, N_1, a, b) \wedge \\ \exists Y_1, \dots, Y_m, Z_1, \dots, Z_p. \beta(A, N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p, a, b)]. \end{aligned}$$

□

The next theorem establishes a converse. As algebra for quasi-trees, we take the algebra  $\mathbb{SJT}$  of join-trees together with the (external) *forgetting operation*  $qt$  (similar to  $fgs$ ). We say that a quasi-tree  $S$  is *described* by an SJ-scheme if this scheme describes a join-tree  $J$  such that  $qt(J) = S$ . It is *regular* if it is  $qt(J)$  for some regular join-tree  $J$ .

(5.7) **Theorem:** The following properties of a quasi-tree  $S$  are equivalent:

- (1)  $S$  is regular,
- (2)  $S$  is described by a regular SJ-scheme,
- (3)  $S$  is  $\text{MS}_{fin}$ -definable.

The isomorphism of regular quasi-trees is decidable.

**Proof:** (1) $\implies$ (2): Similar to that of Theorem (3.21).

(2) $\implies$ (3): By Proposition (5.6).

(3) $\implies$ (1): The mapping  $\alpha$  that transforms the relational structure  $[t]$  for  $t$  in  $T^\infty(F')_{\mathbf{t}}$  into the quasi-tree  $S = qt(fgs(val(t)))$  is an MS-transduction by Claim (4.9.2). The proof continues as in Theorem (3.21).

The decidability of the isomorphism problem is as in Corollary (3.22). □

We make these results more precise for subcubic quasi-trees, that are useful for defining the rank-width of countable graphs, see the appendix.

(5.8) **Definition :** *Directions* ([Cou14])

Let  $S = (N, B)$  be a quasi-tree and  $x$  a node of  $S$ .

(a) We say that  $y, z \in N - \{x\}$  are the *same direction relative to  $x$*  (or of  $x$ ) if, either  $y = z$  or  $B(y, z, x)$  or  $B(z, y, x)$  or  $B(y, u, x) \wedge B(z, u, x)$  for some node  $u$ . Equivalently,  $y \sqcup_x z <_x x$  ( $<_x$  is as in Proposition (5.5)). Hence, if  $B(y, x, z)$  holds,  $y$  and  $z$  are in different directions relative to  $x$ . This relation is an equivalence, denoted by  $y \sim_x z$ , and its classes are the *directions of  $x$* .

(b) The *degree* of  $x$  is the number of classes of  $\sim_x$ . A node has degree 1 if and only if it is a leaf. We say that  $S$  is *subcubic* if its nodes have degree at most 3. If  $S = Q(T)$  for a tree  $T$ , then a direction of  $x$  is associated with each neighbour  $y$  of  $x$  and is the set of nodes of the connected component of  $T - \{x\}$  that contains  $y$ .

(c) If  $S = qt(J)$  for a join tree  $J = (N, \leq)$ , then, the directions of  $x$  in  $S$  are those of  $x$  in  $J$  together with  $]x, +\infty[$  if this set is not empty. It follows that  $S$  is subcubic if  $J$  is a BJ-tree.

(5.9) **Lemma** : Every subcubic quasi-tree is  $qt(fgs(J))$  for some SBJ-tree  $J$ .

**Proof:** We choose a maximal line  $A$  of the given subcubic quasi-tree  $S$  and distinct element  $a, b$  of  $A$ . By proposition (5.5.4), the partial order  $\leq_{A, a, b}$  gives a binary join-tree  $K$ . By using the method of Proposition (3.5) with  $U_0 := A$ , we obtain a structuring  $J$  of  $K$ , making it into an SBJ-tree as defined in Definition (3.8).  $\square$

(5.10) **Theorem:** The following properties of a subcubic quasi-tree  $S$  are equivalent:

- (1)  $S$  is regular,
- (2)  $S$  is described by a regular SBJ-scheme,
- (3)  $S$  is MS-definable.

**Proof:** By Lemma (5.9) and Proposition (3.19), every subcubic quasi-tree  $S$  is  $qt(fgs(val(t)))$  for some term  $t \in T^\infty(F)$ .

Property (1) means that  $S = qt(fgs(val(t)))$  for some regular term in  $T^\infty(F)_t$ . Let (1') mean that  $S = qt(fgs(val(t)))$  for some regular term in  $T^\infty(F)$ . Then (1')  $\implies$  (2) by the similar implication in Theorem (3.21).

(2)  $\implies$  (3) by the similar implication in Theorem (3.21) and the observation that, in a quasi-tree  $S$ , the SBJ-trees  $J$  such that  $S = qt(fgs(J))$  can be specified by MS formulas in terms of a 5-tuple  $(A, N_0, N_1, a, b)$  satisfying the formula  $\varphi'(A, N_0, N_1, a, b)$  of the proof of Proposition (5.6).

(3)  $\implies$  (1') by the observation that the mapping  $\alpha$  that transforms the relational structure  $[t]$  for  $t$  in  $T^\infty(F)$  into the subcubic quasi-tree  $qt(fgs(val(t)))$  is an MS-transduction. The proof goes then as in Theorem (3.21).

The implication (1')  $\implies$  (1) is trivial and (1) implies that  $S$  is  $MS_{fin}$  definable by Theorem (5.6). But a term  $t \in T^\infty(F)$  that defines  $S$  is MS-definable, and the relational structure representing a term has an MS-definable linear order. It follows that  $S$  has an MS-definable linear order, hence that  $S$  is MS-definable (cf. the appendix).  $\square$

## 6 Conclusion

We have defined quasi-trees and join-trees of different kinds from regular terms. These terms have finitary descriptions. There are other infinite terms that have finitary descriptions: the algebraic ones [Cou83] and more generally, those of Caucal's hierarchy [Blu+]. Such terms also yield effective (algorithmically usable) notions of join-trees and quasi-trees. It is unclear whether the corresponding isomorphism problems are decidable.

## 7 References

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## 8 Appendix

### A.1: Monadic second-order logic and related notions.

*Monadic second-order logic* extends first-order logic by the use of *set variables*  $X, Y, Z \dots$  denoting subsets of the domain of the considered logical structure, and the atomic formulas  $x \in X$  expressing membership of  $x$  in  $X$ . We call *first-order* a formula where set variables are not quantified. For example, a first-order formula can express that  $X \subseteq Y$ . A *sentence* is a formula without free variables.

Logical structures, graphs, trees, partial orders are finite or countably infinite.

Let  $\mathcal{R}$  be a finite set of relation symbols, each symbol  $R$  being given with an arity  $\rho(R)$ . We call it a *relational signature*. For every set of variables  $\mathcal{W}$ , we denote by  $MS(\mathcal{R}, \mathcal{W})$  the set of MS formulas written with  $\mathcal{R}$  and free variables in  $\mathcal{W}$ . An  $\mathcal{R}$ -*structure* is a tuple  $S = (D_S, (R_S)_{R \in \mathcal{R}})$  where  $D_S$  is a finite or countable set, called its *domain*, and  $R_S$  is a relation on  $D_S$  of arity  $\rho(R)$ . A property  $P$  of  $\mathcal{R}$ -structures is *monadic second-order definable* (*MS-definable*) if it is equivalent to the validity, in every  $\mathcal{R}$ -structure  $S$ , of a monadic second-order sentence  $\varphi$ , which we denote by  $S \models \varphi$ .

For example, a graph  $G$  is identified with the  $\{edg\}$ -structure  $(V_G, edg_G)$  where  $V_G$  is its vertex set and  $edg_G(x, y)$  means that there is an edge from  $x$  to  $y$ , or between  $x$  and  $y$  if  $G$  is undirected. To take an example, 3-colorability is expressed by the MS-sentence :

$$\exists X, Y [X \cap Y = \emptyset \wedge \neg \exists u, v (edg(u, v) \wedge [(u \in X \wedge v \in X) \vee (u \in Y \wedge v \in Y) \wedge (u \notin X \cup Y \wedge v \notin X \cup Y)])].$$

Many properties of partial orders can also be expressed by MS sentences. We take examples that are useful in our proofs.

(a) The formula  $Lin(X)$  defined as  $\forall x, y. [(x \in X \wedge y \in X) \implies (x \leq y \vee y \leq x)]$  expresses that a subset  $X$  of  $N$ , partially ordered by  $\leq$ , is linearly ordered.

(b) The formula  $Lin(X) \wedge \exists a, b. [Min(X, a) \wedge Max(X, b) \wedge \theta(X, a, b)]$  expresses that  $X$  is linearly ordered and finite, where  $Min(X, a)$  and  $Max(X, b)$  are first-order formulas expressing respectively that  $X$  has a least element  $a$  and a largest one  $b$ , and  $\theta(X, a, b)$  is an MS formula expressing that :

- (i) each element  $x$  of  $X$  except  $b$  has a successor  $c$  in  $X$  (i.e.,  $c$  is the least element of  $\{y \in X \mid y > x\}$ ), and
- (ii)  $(a, b) \in Suc^*$ , where  $Suc$  is the above defined successor relation (depending on  $X$ ) and  $Suc^*$  is its reflexive and transitive closure.

Assertion (ii) is expressed by the MS formula:

$$\forall U [U \subseteq X \wedge a \in U \wedge \forall x, y ((x \in U \wedge (x, y) \in Suc) \implies y \in U) \implies b \in U].$$

The reader will easily build first-order formulas expressing  $U \subseteq X$ ,  $(x, y) \in Suc$  and Property (i). Without a linear order, the finiteness of a set  $X$  is not MS-expressible. It is thus useful, in some cases, to enrich MS logic with a *finiteness predicate*  $Fin(X)$  expressing that  $X$  is finite. We denote by  $MS_{fin}$  the corresponding extension of MS logic.

If  $S$  is a relational structure  $(N, \leq_t, (br_i)_{1 \leq i \leq \rho(F)}, (lab_f)_{f \in F})$  isomorphic to  $[t]$  that represents a term  $t \in T^\infty(F)$ , then a linear order  $\sqsubseteq$  on  $N$  is MS-definable as follows:

$$x \sqsubseteq y : \iff x \leq_t y \vee (x \perp_t y \text{ and } x \text{ is below the } i\text{-th son of } x \sqcup_t y \\ \text{and } y \text{ is below the } j\text{-th son of } x \sqcup_t y \text{ where } i < j).$$

The definability of linear orders by MS-formulas is studied in [CouBlu].

*Monadic second-order transductions* (MS transductions) are transformations of logical structures specified by MS or  $MS_{fin}$  formulas. We use them in the proofs of Theorems (3.21), (3.30), (4.11), (5.6) and (5.10). For these proofs, we only need very simple MS transductions, said to be *noncopying and parameterless* in [CouEng]. We simply call them *MS transductions*.

Let  $\mathcal{R}$  and  $\mathcal{R}'$  be two relational signatures. A *definition scheme* of type  $\mathcal{R} \rightarrow \mathcal{R}'$  is a tuple of formulas of the form  $\mathcal{D} = \langle \chi, \delta, (\theta_R)_{R \in \mathcal{R}'} \rangle$  such that  $\chi \in MS(\mathcal{R})$ ,  $\delta \in MS(\mathcal{R}, \{x\})$  and  $\theta_R \in MS(\mathcal{R}, \{x_1, \dots, x_{\rho(R)}\})$  for each  $R$  in  $\mathcal{R}'$ . We define  $\widehat{\mathcal{D}}(S) := S' = (D_{S'}, (R_{S'})_{R \in \mathcal{R}'})$  as follows:

$$\begin{aligned} S' \text{ is defined if and only if } S \models \chi, \\ D_{S'} \text{ is the set of elements } d \text{ of } D_S \text{ such that } S \models \delta(d), \\ R_{S'} \text{ is the set of tuples } (d_1, \dots, d_{\rho(R)}) \text{ of elements of } D_S \\ \text{such that } S \models \theta_R(d_1, \dots, d_{\rho(R)}). \end{aligned}$$

The mapping that associates the join-tree  $(N, \leq)$  with  $[t]$  for  $t \in T^\infty(F)$  (cf. Definition (3.15)) is an MS-transduction defined by  $\mathcal{D} = \langle \chi, \delta, \theta_{\leq} \rangle$  such that  $\chi$  expresses that the considered input structure  $S$  is isomorphic to  $[t]$  for some  $t \in T^\infty(F)$ ,  $\delta(x)$  is  $lab_{ext}(x)$  (expressing that  $x$  is in  $N$ ) and  $\theta_{\leq}(x, y)$  expresses that  $x \leq y$ , cf. Definition (3.15.b).

Our main tool is the following (well-known) result:

(A. 1) **Theorem** : Let  $\mathcal{D}$  be a definition scheme as above and  $\varphi \in MS_{fin}(\mathcal{R}', \mathcal{X})$ . There exists a formula  $\varphi^{\mathcal{D}} \in MS_{fin}(\mathcal{R}, \mathcal{X})$  such that, for every  $\mathcal{R}$ -structure  $S$ , for every  $\mathcal{X}$ -assignment  $\nu$  in  $D_S$ , we have  $(S, \nu) \models \varphi^{\mathcal{D}}$  if and only if:

- (i)  $S \models \chi$  (so that  $\widehat{\mathcal{D}}(S) = S'$  is well-defined),
- (ii)  $\nu$  is an  $\mathcal{X}$ -assignment in  $D_{S'}$  (that is  $\nu(x) \in D_{S'}$  and  $\nu(X) \subseteq D_{S'}$  for  $x, X \in \mathcal{X}$ ) and
- (iii)  $(S', \nu) \models \varphi$ .

**Proof** : The proof is given in [CouEng] (*Backwards Translation Theorem*, Theorem 7.10) for finite structures, so that the finiteness predicate  $Fin(X)$  is of no use. However, it works for infinite structures as well and the predicate  $Fin(X)$ , that translates back to itself (under the assumption that  $\nu(X) \subseteq D_{S'}$ ).

The formula  $\varphi^{\mathcal{D}}$  is the conjunction of  $\chi$ , a formula expressing (ii) and a formula  $\varphi'$  obtained from  $\varphi$  by replacing each atomic formula  $R(x_1, \dots, x_r)$  by  $\theta_R(x_1, \dots, x_{\rho(R)})$ , i.e., by its definition given by  $\mathcal{D}$ .  $\square$

It follows that, if the monadic theory of a class of structures  $\mathcal{S}$  is decidable and  $\mathcal{S}' = \widehat{\mathcal{D}}(\mathcal{S})$  for some definition scheme  $\mathcal{D}$ , then the monadic theory of  $\mathcal{S}'$  is decidable, because  $S' \models \varphi$  holds for all  $S'$  in  $\mathcal{S}'$  if and only if  $S \models \varphi^{\mathcal{D}}$  holds for all  $S$  in  $\mathcal{S}$ .

## A.2 : Rank-width of countable graphs.

Rank-width and modular decomposition (cf. [Cou14, CouDel]) motivate the study of quasi-trees and join-trees respectively. We now review rank-width for countable graphs. *Rank-width* is a width measure on finite graphs investigated first in [Oum] and [OumSey]. Here is its generalization to countable graphs.

We consider finite or countable, loop-free, undirected graphs without parallel edges. Let  $G$  be such a graph. Its *adjacency matrix* is  $M_G : V_G \times V_G \rightarrow \{0, 1\}$  with  $M_G[x, y] = 1$  if and only if  $x$  and  $y$  are adjacent. If  $U$  and  $W$  are disjoint sets of vertices, we denote by  $M_G[U, W]$  the matrix that is the restriction of  $M_G$  to  $U \times W$ . Ranks are over  $GF(2)$ . The *rank* of  $M_G[U, W]$  defined as the maximum cardinality of an independent set of rows (equivalently, of columns) is denoted by  $rk(M_G[U, W])$ ; it belongs to  $\mathbb{N} \cup \{\omega\}$ . It is convenient to take  $rk(M_G[\emptyset, W]) = rk(M_G[U, \emptyset]) := 0$ .

*Fact*: If  $X \uplus Y$  is infinite, then  $rk(M_G[X, Y]) = \sup\{rk(M_G[U, W]) \mid U \subseteq X, W \subseteq Y \text{ and, } U \text{ and } W \text{ are finite}\}$ .

In [Cou14] we define a *layout* of  $G$  as a subcubic quasi-tree  $T$  whose set of leaves is  $V_G$ . Its *rank* is the least upper-bound of the ranks  $rk(M_G[X \cap V_G, X^c \cap V_G])$  such that  $X$  and  $X^c := N_T - X$  are two convex subsets of  $N_T$ . ( $X$  is *convex* if  $x, z \in X \wedge B(x, y, z) \implies y \in X$ .) The *rank-width* of  $G$ , denoted by  $rw(G)$ , is the smallest rank of a layout. Its *discrete rank-width*, denoted by  $rw^{dis}(G)$ , is similar except that layouts are subcubic trees. Hence,  $rw(G) \leq rw^{dis}(G)$ . For finite graphs, we get the rank-width of [Oum]. The notation  $G \subseteq_i H$  means that  $G$  is an induced subgraph of  $H$ .

(A.2) **Theorem** [Cou14]: For every graph  $G$ :

- (1) if  $H \subseteq_i G$ , then  $rw(H) \leq rw(G)$  and  $rw^{dis}(H) \leq rw^{dis}(G)$ ,
- (2) *Compactness* :  $rw(G) = \text{Sup}\{rw(H) \mid H \subseteq_i G \text{ and } H \text{ is finite}\}$ ,

(3) *Compactness with gap* :  $rd^{dis}(G) \leq 2 \cdot \text{Sup}\{rd(H) \mid H \subseteq_i G \text{ and } H \text{ is finite}\}$ .

The *gap function* in (3) is  $n \mapsto 2n$ , showing a weak form of compactness. A related gap concerns the clique-width of countable graphs [Cou04].

**Proof sketch:** (1) is clear from the definitions.

(2) is proved by Koenig's Lemma.

(3) is based on the representation of a countable linear order as the set of leaves of an ordered binary tree; this construction is adapted from [CouDel]. $\square$