

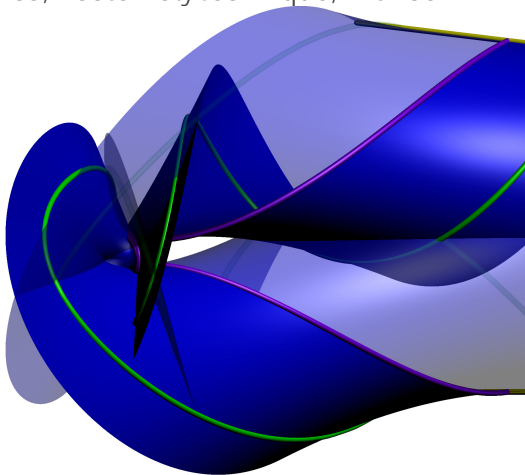
# NON-SMOOTH MODAL ANALYSIS OF PIECEWISE-LINEAR IMPACT OSCILLATORS

Seminar, Lab. of Solid Mechanics, École Polytechnique, France

March 17, 2016

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# INTRODUCTION

# PHASE SPACE

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Equivalence:

Mechanical state



Point in the phase space

# LINEAR MODAL ANALYSIS

# LINEAR MODE 1

# LINEAR MODE 2

# LINEAR MODES: SUMMARY

## Linear modes:

- ▶ rely on **superposition principle**
- ▶ structure of vector space (base: linear modes)
- ▶ linear algebra ( $\sim$  end of 19th)
- ▶ eigenvalues: resonance



# WHAT ABOUT NONLINEAR SYSTEMS?

## Nonlinear modes:

- ▶ superposition principle no longer holds
- ▶ impossible to calculate all the trajectories

# WHAT ABOUT NONLINEAR SYSTEMS?

## Nonlinear modes:

- ▶ superposition principle no longer holds
- ▶ impossible to calculate all the trajectories
- ▶ *but* maths results (Lyapunov theorem (1907) + central manifold theorem (~ 1970))
- ▶ existence of fix points, periodic orbits, attractors, continuums of periodic orbits in the vicinity of fixed points
- ▶ applied to dynamical systems (Rosenberg, Shaw, Pierre, Vakakis, etc.)
- ▶ practical interest: local behaviour + prediction of resonance

# NONLINEAR MODES

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# NONLINEAR MODES



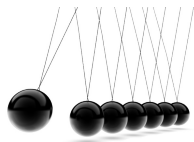
# LIMITATION OF NONLINEAR MODAL ANALYSIS

## Limitation

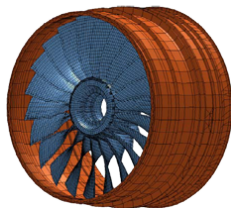
Computation of nonlinear modes relies on *smoothness*.

→ What about non-smooth systems?

- ▶ Non-smooth system: system undergoing impact and/or dry friction.



Newton cradle [Inria]



Turbomachine [McGill]

- ▶ **Long-term objective:** extend non-linear modal analysis to non-smooth systems
- ▶ Literature in (ultra) short: nonlinear smooth modes **or** time-evolution of non-smooth systems

# OUTLINE

## 1. Introduction

## 2. Formulation

## 3. Resolution

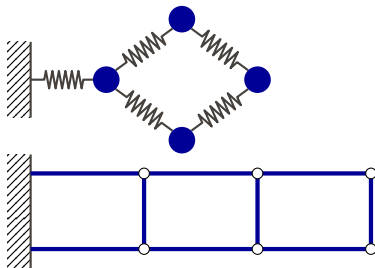
## 4. Illustrations

## 5. Additional properties

## 6. Conclusions

# FORMULATION

# INVESTIGATED SYSTEMS: ASSUMPTIONS

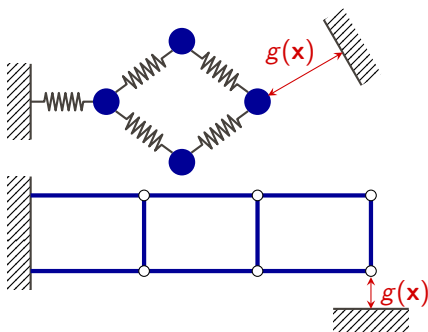


$$\mathbf{M} = \begin{bmatrix} \bullet & & 0 \\ & \ddots & \\ 0 & & \bullet \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \bullet & \dots & \bullet \\ \vdots & & \vdots \\ \bullet & \dots & \bullet \end{bmatrix}$$

- ▶ linear system (material + geometry)
- ▶ autonomous (= unforced)
- ▶ conservative (= undamped)

# INVESTIGATED SYSTEMS: ASSUMPTIONS

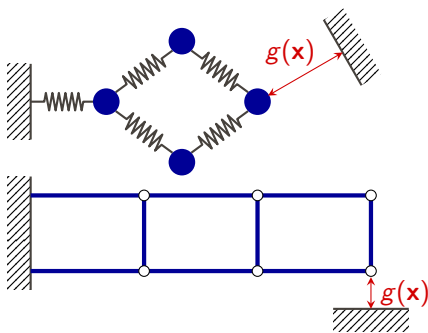


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- ▶ autonomous (= unforced)
- ▶ conservative (= undamped)
- ▶ unique contact condition

# INVESTIGATED SYSTEMS: ASSUMPTIONS

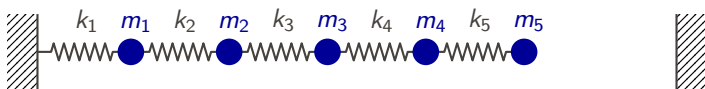


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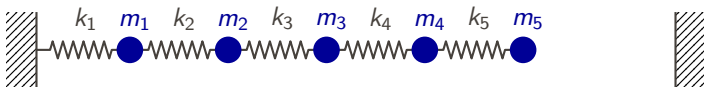
- ▶ linear system (material + geometry) *piecewise*
- ▶ autonomous (= unforced)
- ▶ conservative (= undamped)
- ▶ unique contact condition
- ▶ perfectly elastic impact law

## FREE FLIGHT



- ▶ Dynamics  $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$
- ▶ First-order  $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$  with  $\mathbf{X} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}$  and  $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix}$
- ▶ Solutions  $\mathbf{X}(t) = e^{t\mathbf{A}}\mathbf{X}(0)$

## FREE FLIGHT



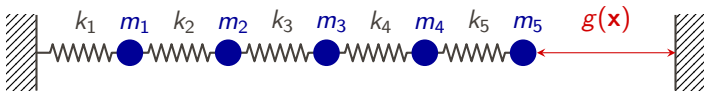
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- ▶ Solutions  $\mathbf{X}(t) = e^{t\mathbf{A}}\mathbf{X}(0)$
- ▶ After a change of basis:  $\bar{\mathbf{X}}(t) = \mathbf{S}(t)\bar{\mathbf{X}}(0)$  where  $\mathbf{S}(t)$  is block-diagonal:

$$\begin{bmatrix} \cos(\omega_i t) & -\sin(\omega_i t) \\ \sin(\omega_i t) & \cos(\omega_i t) \end{bmatrix}$$

$\implies \mathbf{S}(t)$  acts like  $n$  rotations in  $n$  orthogonal planes

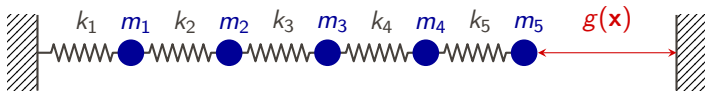


## IMPACT



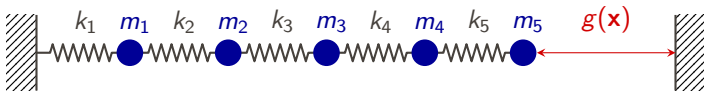
- ▶ gap:  $g(\mathbf{x}) = g_0 - \mathbf{x}$  with  $g_0$  unconstrained gap
- ▶ Newton impact law:  $\dot{\mathbf{x}}_n(t_{\text{imp}}^+) = -e \dot{\mathbf{x}}_n(t_{\text{imp}}^-)$  with  $e = 1$ 
  - ▷ conservative when  $\mathbf{M}$  is diagonal:  $\dot{\mathbf{x}}^{+\top} \mathbf{M} \dot{\mathbf{x}}^+ = \sum_i m_{i,i} \dot{x}_i^{+2} = \dot{\mathbf{x}}^{-\top} \mathbf{M} \dot{\mathbf{x}}^-$

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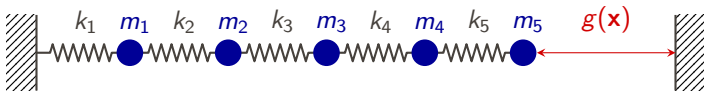
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  - ▶ non-conservative in general:  $\dot{\mathbf{x}}^{+\top} \mathbf{M} \dot{\mathbf{x}}^+ \neq \dot{\mathbf{x}}^{-\top} \mathbf{M} \dot{\mathbf{x}}^-$

# IMPACT



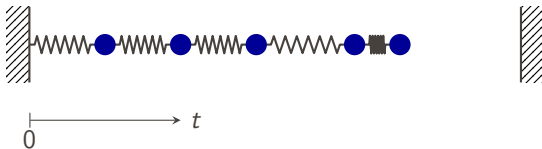
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- ▶ more generally:  $\dot{\mathbf{x}}^+ = \mathbf{N} \dot{\mathbf{x}}^-$  with  $\mathbf{N} = \mathbf{M}^{-1/2} (\mathbf{I} - 2\mathbf{r}\mathbf{r}^\top) \mathbf{M}^{1/2}$

## IMPACT



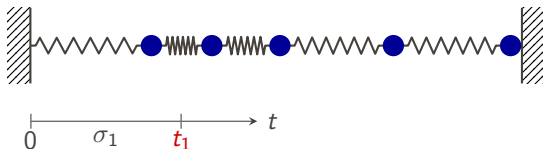
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  - $\implies \mathbf{N}$  acts as a reflection w.r.t to a hyperplane

# LINEAR MAPPING AFTER $K$ IMPACTS



► State at time  $t$  :

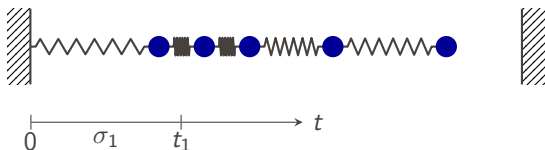
$$\mathbf{X}(t) = \mathbf{S}(t)\mathbf{X}_0$$

LINEAR MAPPING AFTER  $K$  IMPACTS

- State at time  $t$  :

$$\mathbf{X}(t) = \mathbf{NS}(\sigma_1)\mathbf{X}_0$$

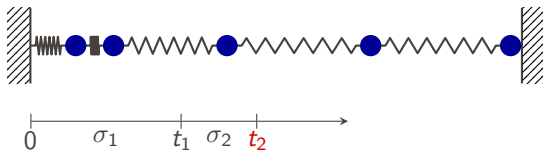
# LINEAR MAPPING AFTER $K$ IMPACTS



- State at time  $t$  :

$$\mathbf{X}(t) = \mathbf{S}(t - t_1)\mathbf{N}\mathbf{S}(\sigma_1)\mathbf{X}_0$$

# LINEAR MAPPING AFTER $K$ IMPACTS

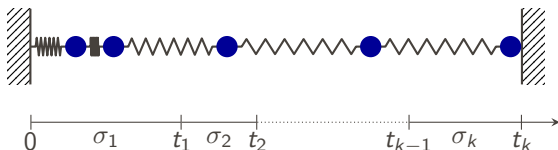


► State at time  $t$  :

$$\mathbf{X}(t) = \mathbf{NS}(\sigma_2)\mathbf{NS}(\sigma_1)\mathbf{X}_0$$



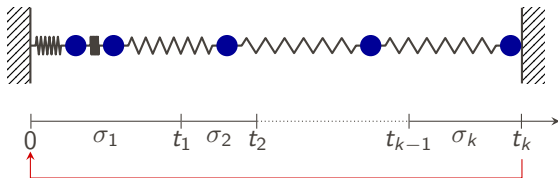
# LINEAR MAPPING AFTER $K$ IMPACTS



- State at time  $t$  :

$$\mathbf{X}(t) = \mathbf{NS}(\sigma_k) \dots \mathbf{NS}(\sigma_2) \mathbf{NS}(\sigma_1) \mathbf{X}_0$$

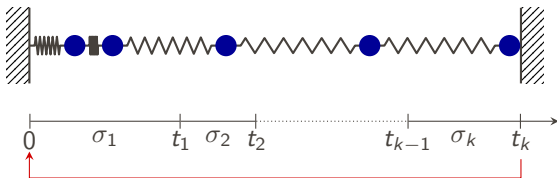
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► State at time  $t$  :

$$\mathbf{X}(t) = \mathbf{NS}(\sigma_k) \dots \mathbf{NS}(\sigma_2) \mathbf{NS}(\sigma_1) \mathbf{X}_0 = \mathbf{X}_0$$

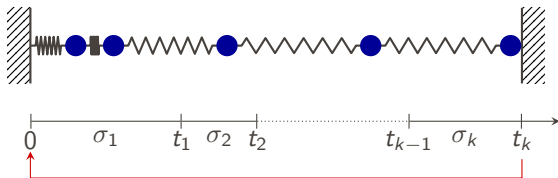
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- Formulation: For a given  $k$ , find  $\mathbf{X}_0$  and  $t_1, \dots, t_k$  such that

$$\begin{cases} u(\mathbf{X}_0) = \mathbf{X}_0 & \text{NC1 (dynamics + periodicity)} \\ \mathbf{X} \text{ determined by } \mathbf{X}_0 \text{ such that:} \\ \forall i \in \llbracket 1, k \rrbracket, g(\mathbf{X}(t_i)) = 0 & \text{NC2 (gap closures)} \\ \forall t \in [0, T], g(\mathbf{X}(t)) \geq 0 & \text{NC3 (non-interpenetration)} \end{cases}$$

RESOLUTION

## MAIN RESULTS

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- Example: for  $n = 2$ ,  $k = 2$ , NC1:

$$\mathbf{NS}(\sigma_2)\mathbf{NS}(\sigma_1)\mathbf{X}_0 = \mathbf{X}_0$$

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- Example: for  $n = 2$ ,  $k = 2$ , NC1:

$$\mathbf{N} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \quad \mathbf{S}(t) = \mathbf{P} \begin{bmatrix} \cos(\omega_1 t) & -\sin(\omega_1 t) & & \\ \sin(\omega_1 t) & \cos(\omega_1 t) & & \\ & & \cos(\omega_2 t) & -\sin(\omega_2 t) \\ & & \sin(\omega_2 t) & \cos(\omega_2 t) \end{bmatrix} \mathbf{P}^{-1}$$

Very computationally expensive!



# MAIN RESULTS

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## Theorem (NC1)

$\mathbf{X}_0$  is invariant by  $u$  iff  $\exists \lambda \in \mathbb{R}^k$  such that

$$\left\{ \begin{array}{l} \mathbf{\Pi}(t_1, \dots, t_k)\lambda = \mathbf{0} \\ \mathbf{X}_0 = \varphi(\lambda) \end{array} \right.$$

The applications  $\mathbf{\Pi}$  and  $\varphi$  are **known** !

$\mathbf{\Pi}$  is a skew-symmetric  $k \times k$  matrix.

## MAIN RESULTS

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## Theorem (NC2)

$\mathbf{X}_0$  verifies NC2 iff  $\exists \lambda \in \mathbb{R}^k$  such that

$$\left\{ \begin{array}{l} \Sigma(t_1, \dots, t_k) \lambda = g_0 \mathbf{j} \\ \mathbf{X}_0 = \varphi(\lambda) \end{array} \right.$$

with  $\mathbf{j} = [1, \dots, 1]^\top$ .

The applications  $\Sigma$  and  $\varphi$  are **known**, and  $\varphi$  is the same as in Theorem 1.  
 $\Sigma$  is a symmetric  $k \times k$  matrix.

## MAIN RESULTS

- For a given  $k$ , find  $\mathbf{X}_0 \in \mathbb{R}^{2n}$  and  $t_1, \dots, t_k$  such that

$$\begin{cases} u(\mathbf{X}_0) = \mathbf{X}_0 & \text{NC1} \\ \mathbf{X} \text{ determined by } \mathbf{X}_0 \text{ such that:} & \\ \forall i \in \llbracket 1, k \rrbracket, g(\mathbf{X}(t_i)) = 0 & \text{NC2} \\ \forall t \in [0, T], g(\mathbf{X}(t)) \geq 0 & \text{NC3} \end{cases}$$

becomes:

- For a given  $k$ , find  $\boldsymbol{\lambda} \in \mathbb{R}^k$  and  $t_1, \dots, t_k$  such that

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# ILLUSTRATIONS

CASE  $k = 1$ 

- For a given  $k$ , find  $\lambda \in \mathbb{R}^k$  and  $t_1, \dots, t_k$  such that

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- Example  $k = 1$ :

▷  $\mathbf{\Pi}(t_1) = [0]$ , so NC1 is always satisfied.

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- ▷  $\mathbf{\Pi}(t_1) = [0]$ , so NC1 is always satisfied.
- ▷  $\mathbf{\Sigma}(t_1) = [f(t_1)]$  so NC2 reads:

$$f(t_1)\lambda = g_0.$$

If  $g_0 \neq 0$ ,  $\lambda = g_0/f(t_1)$



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$$f(t_1)\lambda = g_0.$$

If  $g_0 \neq 0$ ,  $\lambda = g_0/f(t_1)$  so  $\mathbf{X}_0 = \varphi(g_0/f(t_1))$ .

⇒ vector of initial conditions  $\mathbf{X}_0$  is parametrised by  $t_1 = T$ !

CASE  $k = 1$ 

- For a given  $k$ , find  $\lambda \in \mathbb{R}^k$  and  $t_1, \dots, t_k$  such that

$$\begin{cases} \mathbf{\Pi}(t_1, \dots, t_k)\lambda = \mathbf{0} & \text{NC1} \\ \mathbf{\Sigma}(t_1, \dots, t_k)\lambda = g_0\mathbf{j} & \text{NC2} \\ \mathbf{X} \text{ determined by } \mathbf{X}_0 = \varphi(\lambda) \text{ such that:} \\ \forall t \in [0, T], g(\mathbf{X}(t)) \geq 0 & \text{NC3} \end{cases}$$

- Example  $k = 1$ :

- ▷  $\mathbf{\Pi}(t_1) = [0]$ , so NC1 is always satisfied.
- ▷  $\mathbf{\Sigma}(t_1) = [f(t_1)]$  so NC2 reads:

$$f(t_1)\lambda = g_0.$$

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⇒ vector of initial conditions  $\mathbf{X}_0$  is parametrised by  $t_1 = T!$

- ▷ NC3 has to be checked numerically.

# CASE $k = 1$

- ▶ Parametrisation  $\mathbf{X}_0 = \varphi(g_0/f(\mathbf{t}_1))$ .

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# TERMINOLOGY

## Definition (Potential solution)

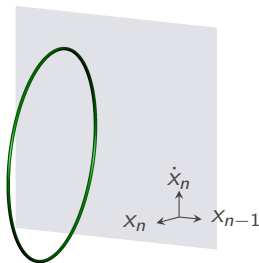
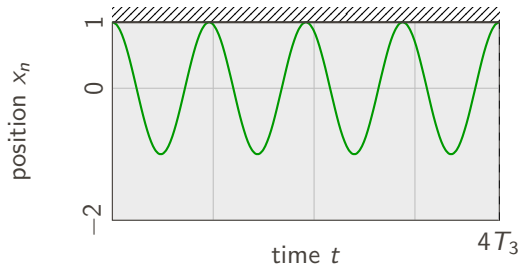
Initial conditions satisfying  $NC1 + NC2$

## Definition (Admissible solution)

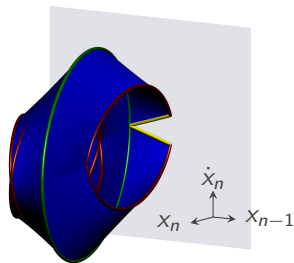
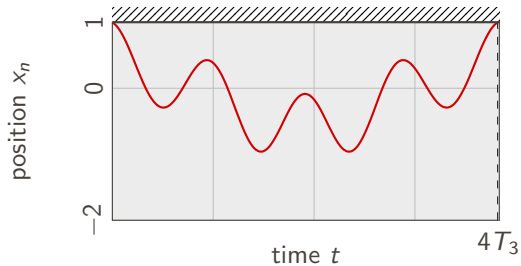
Initial conditions satisfying  $NC1 + NC2 + NC3$

In practice, seek potential solutions and check  $NC3$  numerically.

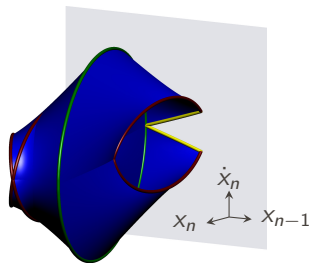
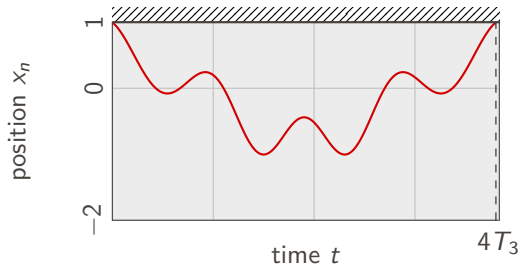
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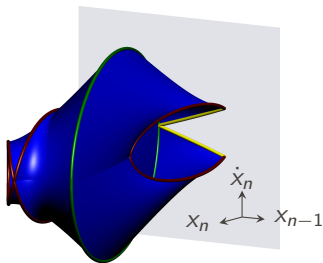
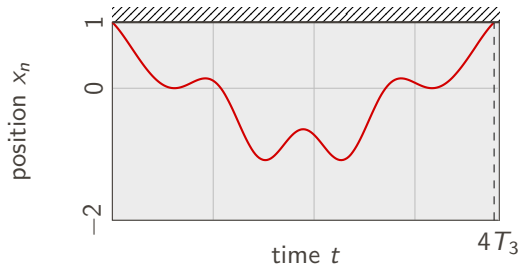
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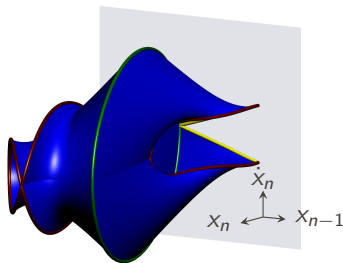
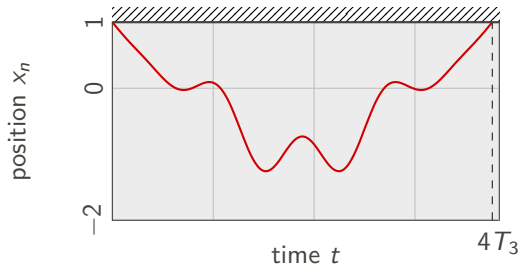
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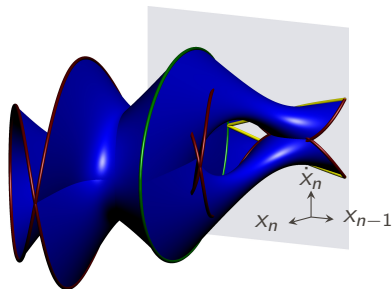
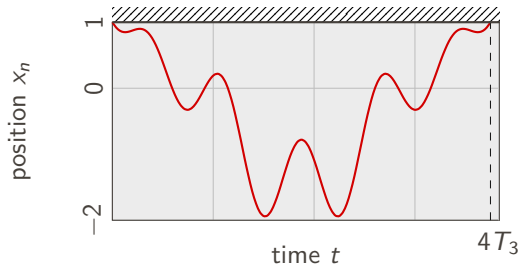


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CASE 2 IMPACTS PER PERIOD ( $k = 2$ )

► For a given  $k$ , find  $\lambda \in \mathbb{R}^k$  and  $t_1, \dots, t_k$  such that

$$\left\{ \begin{array}{ll} \mathbf{\Pi}(t_1, \dots, t_k)\lambda = \mathbf{0} & \text{NC1} \\ \mathbf{\Sigma}(t_1, \dots, t_k)\lambda = g_0\mathbf{j} & \text{NC2} \\ \mathbf{X} \text{ determined by } \mathbf{X}_0 = \varphi(\lambda) \text{ such that:} & \\ \forall t \in [0, T], g(\mathbf{X}(t)) \geq 0 & \text{NC3} \end{array} \right.$$

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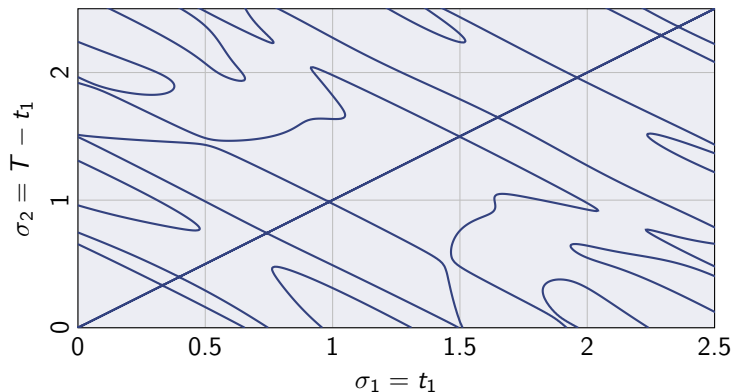
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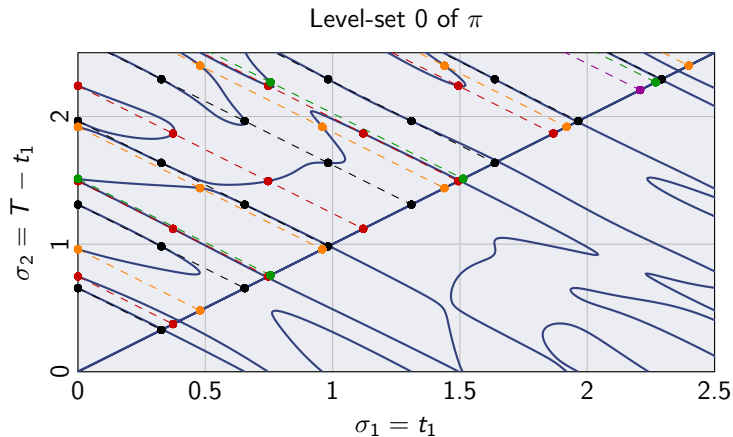
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Level-set 0 of  $\pi$ 

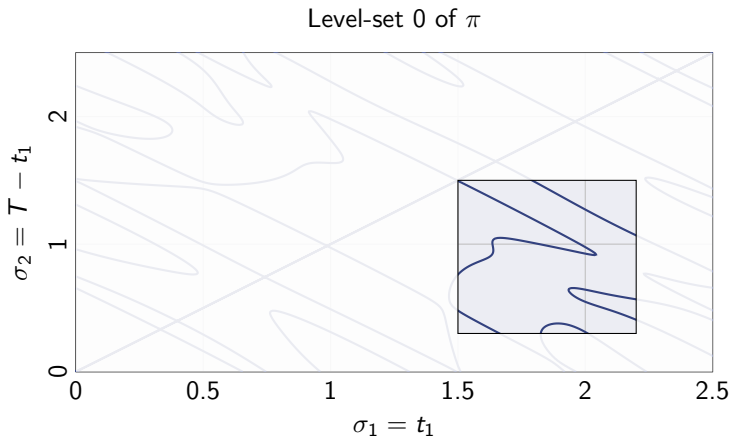
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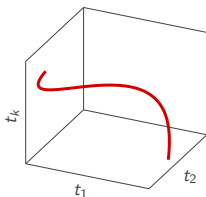
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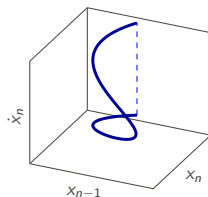
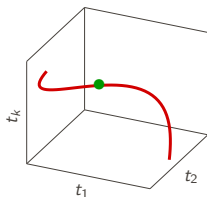


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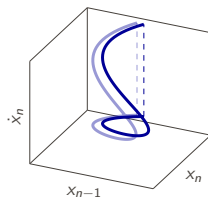
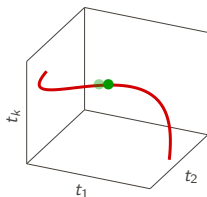


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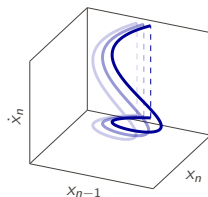
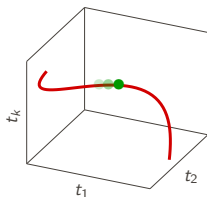


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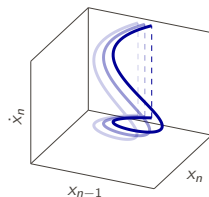
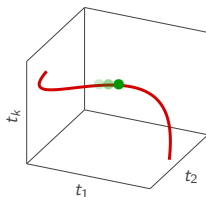


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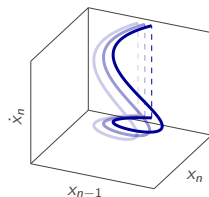
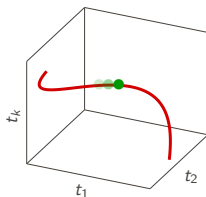
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# METHODOLOGY

1. Find a potential solution (solve  $\mathbf{\Pi}(\mathbf{s})\mathbf{\Sigma}^{-1}(\mathbf{s})\mathbf{j} = \mathbf{0}$  for  $\mathbf{s} \in \mathbb{R}_+^k$ ).
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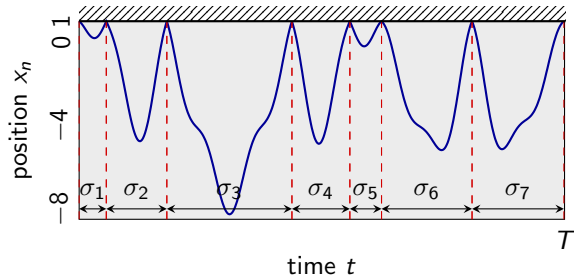
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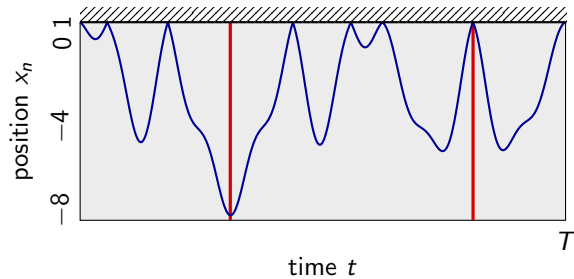
# ADDITIONAL PROPERTIES



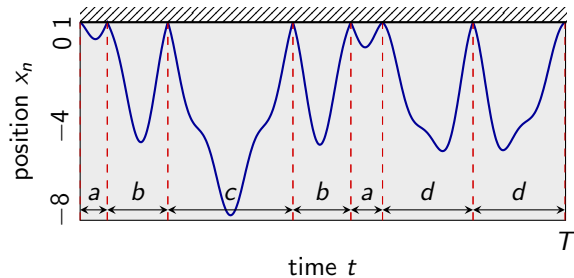
## SYMMETRIES



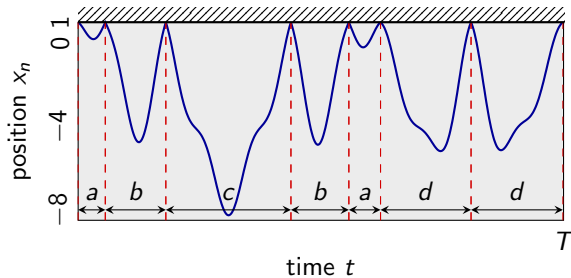
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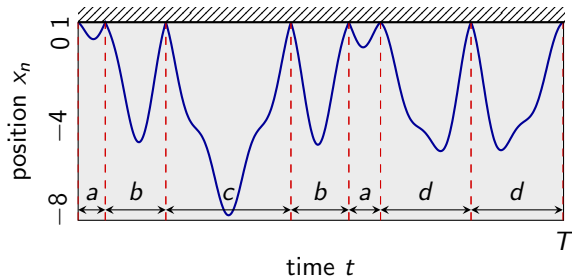
- symmetry of the periodic sequence  $\sigma_1, \dots, \sigma_k$ :

$$\text{for some } p \in \mathbb{Z}, \forall i \in \mathbb{Z}, \sigma_i = \sigma_{p-i}$$

## Theorem (Symmetries if $\Sigma$ invertible)

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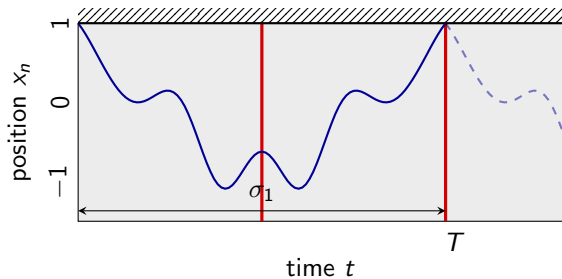
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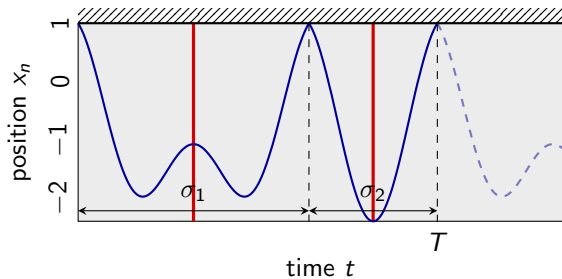
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# SINGULARITY OF $\Sigma(\mathbf{s})$

- ▶ If  $\Sigma(\mathbf{s})$  is invertible: 1-parameter continuum of periodic orbits...
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If so, then  $\Sigma(\mathbf{s})\lambda = g_0\mathbf{j}$  has:

- ▶ either no solutions;
- ▶ either a line, plane, etc. of solutions.

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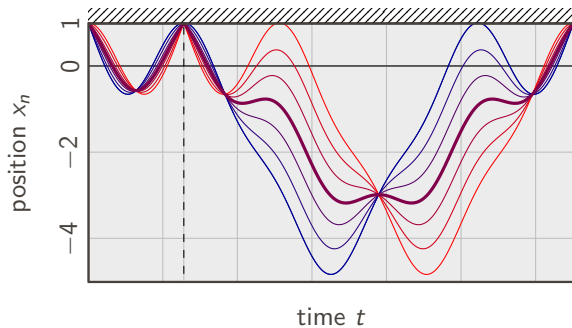
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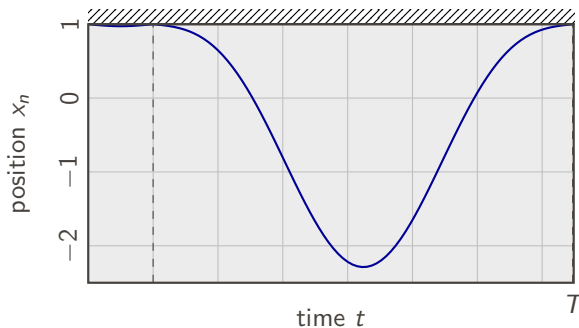
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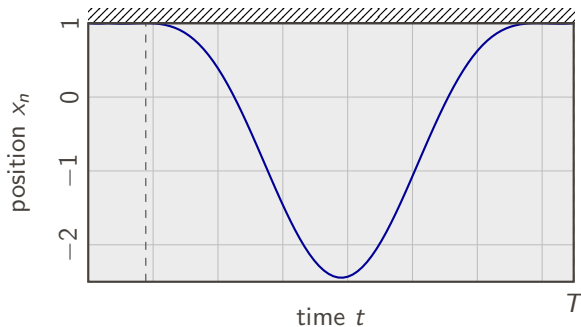
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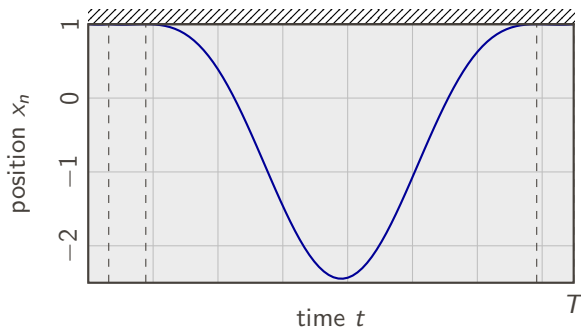
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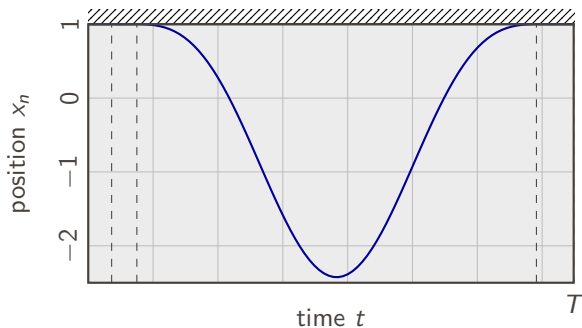
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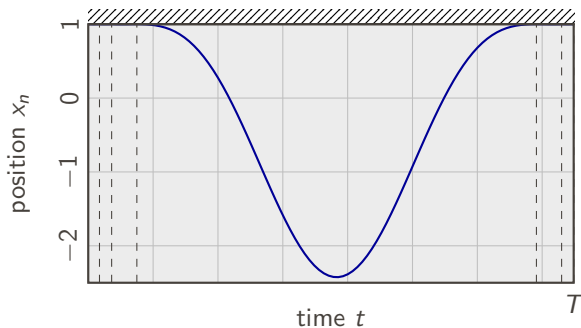
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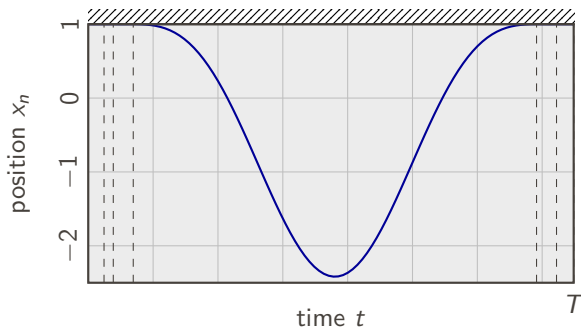
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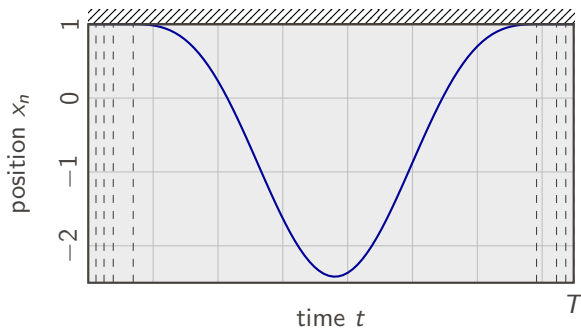
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Other lessons:

- ▶ large variety of behaviours induced by contact
- ▶ importance of mathematics and multidisciplinary collaborations

Thank you for your attention

