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Non-smooth modal analysis of piecewise-linear impact oscillators

Anders THORIN¹, Pierre DELEZOIDE², Mathias LEGRAND¹

¹ Department of Mechanical Engineering, McGill University, Canada

² Independent researcher, France

ABSTRACT Periodic solutions of autonomous and conservative second-order dynamical systems of finite dimension n undergoing *one* unilateral contact condition are investigated in continuous time. The unilateral constraint is complemented with a purely elastic impact law which preserves total energy. The dynamics is linear when there is no contact. The number k of impacts per period arises as a natural parameter of the proposed formulation. Interestingly, the existence of the targeted periodic solutions is essentially governed by a system of only $k - 1$ nonlinear equations with k unknowns, regardless of the number of degrees-of-freedom. This serves to prove that the phase-space is populated by one-dimensional continua of periodic solutions generating invariant manifolds which can be understood as *non-smooth modes of vibration* in the context of vibration analysis. Additionally, these equations provide an efficient and systematic way of calculating non-smooth modes of vibration. They also demonstrate the existence of interesting properties: symmetries of trajectories and emergence of unique features such as manifolds supporting constant-frequency orbits. All results are illustrated through a simple in-line spring–mass system whose last mass undergoes the unilateral impact law.

KEYWORDS Impact dynamics; nonlinear oscillators; nonlinear modal analysis

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1. Background In this work, $2n$ -dimensional autonomous dynamical systems are considered in the framework of classical structural mechanics. Through Newton’s second law of motion, the dynamics is known to be of second order. It is furthermore assumed that the resulting governing Ordinary Differential Equations have constant coefficients. At a given time, such systems are fully described by their generalised positions and velocities, corresponding to a point in the phase space of dimension $2n$. Their time-evolution can be represented as a curve parametrised by time, in the phase space. Such a curve is referred to as a *trajectory* or equivalently, an *orbit*.

In such a framework, solutions to linear dynamics exhibit a vector space structure of dimension n , a basis of which is provided by linear modes of vibration, also called *eigenmodes*. In the phase space, a linear mode can be seen as a planar trajectory. In particular, in the conservative case, autonomous solutions are linear combinations of planar elliptic trajectories. Hence, every linear mode defines a plane,

which is *invariant* in the sense that for any initial condition on the plane, the solution remains on it as time unfolds. This fully takes advantage of basic linear algebra, developed in the late nineteenth century [9], to describe the dynamics globally.

For *nonlinear* dynamics governed by Ordinary Differential Equations, the above linear space structure no longer holds and methods able to systematically characterize the global dynamics are not available. However, it is known that fixed points, also called equilibrium points, limit cycles, and chaos may exist. The centre manifold theorem [10, 13] together with Lyapunov’s center theorem [4, p. 5] prove that for second-order dynamics away from resonance conditions, two-dimensional manifolds of the phase space exist and are tangent to the linear modes of the system linearised in the vicinity of its fixed points. Such invariant two-dimensional manifolds in the phase space were later defined as *nonlinear modes of vibration* [20, 22] in the vibration community.

Nonlinear modes have been primarily applied to smooth systems, *i.e.* systems described by ODEs. Though not as informative as their linear counterparts, nonlinear modes still reveal significant features such as potential vibratory resonances, stability properties, and step up the ground for modal reduction [22]. However, many mechanical systems, involving unilateral contact conditions or dry friction, are known to exhibit a certain degree of non-smoothness: velocities or accelerations are not differentiable with respect to time. Such so-called *non-smooth systems* cannot be described with ODEs but require weaker formalisms, such as Measure Differential Inclusions, Linear Complementarity Problems or Variational Inequalities [1, 8].

The present paper deals with autonomous and conservative n -degree-of-freedom oscillators subject to a single unilateral contact condition complemented with an elastic impact law. Existence results for such systems are provided in [17] for grazing bifurcations as well as in [19] for oscillators with bilateral rigid barriers where each contact is activated at most once per period. A two-degree-of-freedom system with an elastic stop is treated with one impact per period in [18]. However, no general analytic minimal expression of periodic solutions for such systems with $k \in \mathbb{N}^*$ impacts per period is available. Recently, a spring–mass system subject to a purely elastic impact law with one impact per period has been thoroughly investigated [15]. The present work extends the results of the latter—stability analysis set aside—to more complex geometries, non-diagonal mass matrices (such as in Finite–Element models) and any number of impacts per period.

Because the dynamics is deterministic, the problem is formulated as finding the initial conditions $\bar{\mathbf{x}}_0 \in \mathbb{R}^{2n}$ from which emanates T -periodic solutions which satisfies a succession of “free flights”, governed by linear dynamics, separated by k impacts. Enforcing periodicity comes down to finding an initial condition invariant by a linear mapping which is explicitly exhibited. Although challenging, it nicely reduces to solving two independent systems of equations, nonlinear in the k times of impacts and linear in k scalar unknowns, irrespective of the number of degrees-of-freedom n . These k scalar quantities then determine potential initial conditions $\bar{\mathbf{x}}_0$ via an explicit isomorphism; $\bar{\mathbf{x}}_0$ is a solution of the problem when it satisfies a simple (but restrictive) inequality. The proofs rely on a particular basis of the phase space called “effective basis” in this work. In this basis, the free dynamics is seen as n uncoupled planar orthogonal rotations, while the impact law simply reads as a Householder matrix.

Remark 1.1. Nonlinear modes are, by definition, continua of periodic orbits of a nonlinear system. By extension, non-smooth modes are defined as nonlinear modes of non-smooth systems. In the same spirit, linear modes can be conceived as continua of periodic elliptical orbits of a linear system.

2. Assumptions and formulation

2.1. General framework In this work, the following assumptions are considered: the mechanical system of interest lies in a one, two or three-dimensional space, with a finite number n of degrees-of-freedom (dof). It embeds neither damping nor external forces but a single unilateral constraint corresponding to a frictionless impact with a rigid still obstacle.

With $\mathbf{x} \in \mathbb{R}^n$ the vector of generalised positions, let $g(\mathbf{x})$ denote the gap, *i.e.* the signed distance function separating the contacting dof and the rigid obstacle. By convention, it is chosen positive when there is no contact. Its time-derivative, the contact velocity, is denoted γ . The unilateral constraint preventing interpenetration writes

$$g(\mathbf{x}) \geq 0 \tag{2.1}$$

which does not explicitly depend on time¹. At a given time, there are exactly two distinct possible configurations:

1. Contact is not activated, that is $g(\mathbf{x}) > 0$ and the dynamics obeys

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad (2.2)$$

where $\mathbf{M} \in \mathbb{R}^{n,n}$ is the mass matrix and $\mathbf{K} \in \mathbb{R}^{n,n}$, the stiffness matrix; \mathbf{M} and \mathbf{K} are symmetric positive-definite. This situation is referred to as “free flight”. By convention, the unconstrained rest positions are chosen to be zero.

2. Contact is activated, that is $g(\mathbf{x}) = 0$ and the dynamics is governed by [21]:

$$\mathbf{M}(\dot{\mathbf{x}}^+ - \dot{\mathbf{x}}^-) = \mathbf{w}\Lambda \quad (2.3)$$

where \mathbf{w}^\top is the gradient² of g , Λ is the contact impulse along the outward-pointing normal direction of the contact surface, $\dot{\mathbf{x}}^-$ and $\dot{\mathbf{x}}^+$ are the pre- and post-impact velocities: if t is a time of impact, $\dot{\mathbf{x}}^+(t) = \lim_{\varepsilon \rightarrow 0^+} \dot{\mathbf{x}}(t + \varepsilon)$ and $\dot{\mathbf{x}}^-(t) = \lim_{\varepsilon \rightarrow 0^-} \dot{\mathbf{x}}(t + \varepsilon)$. Since this work involves a single unilateral constraint, $g(\mathbf{x}) \in \mathbb{R}$ and \mathbf{w} is a column vector. Sticking is not considered, so necessarily $\Lambda \geq 0$. The impulse Λ is the integral of a generalised contact force λ .

Inequality (2.1) together with Eqs. (2.2) and (2.3) shall then be reformulated as

$$\begin{cases} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{w}\lambda & (2.4a) \\ g(\mathbf{x}) \geq 0 & (2.4b) \\ \lambda \geq 0 & (2.4c) \\ g(\mathbf{x})\lambda = 0 & (2.4d) \end{cases}$$

where Eqs. (2.4b)–(2.4d) are the Signorini complementarity conditions preventing any interpenetration with the obstacle [12].

Because the contact condition can induce discontinuities of $\dot{\mathbf{x}}$, Problem (2.4) is to be understood in the weak sense where $\ddot{\mathbf{x}}$ and λ might include Dirac measures [1]. Prescribing the initial positions and velocities $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ in Problem (2.4) is not sufficient to ensure the uniqueness of a solution: in contrast to the space-continuous framework for which existence and uniqueness hold without any impact law [7, 14], discrete mechanical systems do not incorporate the constitutive description of the impact process and the formulation should be supplemented with an *impact law* [3, 6]. It maps $(g(\mathbf{x}), \gamma^-)$ to γ^+ [3] where γ^- and γ^+ are the pre- and post-impact normal contact velocities, respectively. Since unforced periodic solutions are targeted, the only possible choice is a conservative impact law of the form

$$g(\mathbf{x}) = 0 \implies \gamma^+ = -\gamma^-. \quad (2.5)$$

The impact law can also be expressed in terms of the generalised velocities $\dot{\mathbf{x}}^-$ and $\dot{\mathbf{x}}^+$ through the following linear relationship which introduces the matrix $\mathbf{N} \in \mathbb{R}^{n,n}$:

$$g(\mathbf{x}) = 0 \implies \dot{\mathbf{x}}^+ = \mathbf{N}\dot{\mathbf{x}}^-. \quad (2.6)$$

In subsection 2.3, it is shown that \mathbf{N} has a very specific expression but the condensed form (2.6) is kept for the formulation. The gap g does not depend on time explicitly, so the geometric relation between γ and $\dot{\mathbf{x}}$ is linear:

$$\gamma(t) = \frac{d}{dt}g(\mathbf{x}(t)) = \mathbf{w}^\top \dot{\mathbf{x}}(t) \quad (2.7)$$

corresponding in terms of positions to

$$g(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + g_0. \quad (2.8)$$

The real number g_0 stands for the signed distance between the unconstrained rest position of the system and the position of the obstacle, see Figure 1.

¹ Such a constraint is sometimes referred to as a *scleronomic constraint*.

² In general, \mathbf{w}^\top is the Jacobian matrix of the gaps; here, there is a single gap so \mathbf{w}^\top reduces to a gradient.

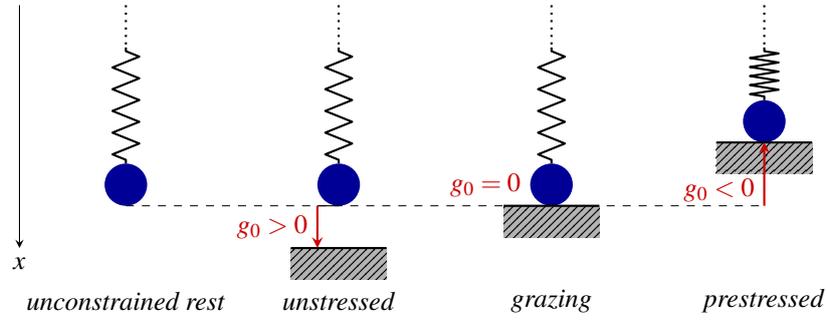


Figure 1: Positions of the obstacle illustrated in a unidimensional case.

Following [1, 5], the formulation becomes: Find displacements \mathbf{x} and generalised force λ satisfying

$$\begin{cases} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{w}\lambda, & (2.9a) \\ g(\mathbf{x}) \geq 0, & (2.9b) \\ \lambda \geq 0, & (2.9c) \\ g(\mathbf{x})\lambda = 0, & (2.9d) \\ g(\mathbf{x}) = 0 \implies \dot{\mathbf{x}}^+ = \mathbf{N}\dot{\mathbf{x}}^-. & (2.9e) \end{cases}$$

2.2. Formulation for periodic solutions Non-smooth modes of vibration are defined as families of periodic solutions to Problem (2.9). Here we focus on solutions corresponding to $t \mapsto g(\mathbf{x}(t))$ of finite support, excluding phenomena such as chattering or sticking. This naturally introduces two quantities of interest: the unknown period T and the (finite) number of impacts per period (ipp) k . Solutions with $k = 0$ are smooth solutions of the linear problem and are of course not of interest here, hence $k \in \mathbb{N}^*$. As Problem (2.9) is autonomous and the unilateral constraint is scleronomic, we decide, without loss of generality, that contact is activated at $t = 0$, *i.e.* $g(\mathbf{x}(0)) = 0$. With T the assumed period of a solution \mathbf{x} , the times of impact in $[0, T]$ are denoted by t_1, \dots, t_k with $t_k = T$ and, by convention, $t_0 = 0$. The problem is now expressed as: Given T and k , find \mathbf{x} and $t_1, \dots, t_k \in [0, T]$ satisfying

$$\begin{cases} \forall i \in \llbracket 1, k \rrbracket, \forall t \in (t_{i-1}, t_i), \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0} & \text{free flight,} & (2.10a) \\ \forall i \in \llbracket 1, k \rrbracket, g(\mathbf{x}(t_i)) = 0 & \text{gap closure,} & (2.10b) \\ \forall t \in [0, T], g(\mathbf{x}(t)) \geq 0 & \text{non-penetration,} & (2.10c) \\ \forall i \in \llbracket 1, k \rrbracket, \dot{\mathbf{x}}^+(t_i) = \mathbf{N}\dot{\mathbf{x}}^-(t_i) & \text{impact law,} & (2.10d) \\ \mathbf{x}(0) = \mathbf{x}(T) & \text{periodicity.} & (2.10e) \end{cases}$$

The generalised force λ no longer appears explicitly but can be deduced from the solution: it is determined at the times of impact t_1, \dots, t_k through Eq. (2.3) and vanishes everywhere else, see Eq. (2.10a). We immediately check that a solution of Problem (2.10) is a solution of Problem (2.9). The orthogonality condition $g(\mathbf{x})\lambda = 0$ in Eq. (2.9d) follows from Eq. (2.10b). The positiveness of λ expressed in Eq. (2.9c) follows from the impact law Eq. (2.10d) and Ineq. (2.10c). Lastly, Eq. (2.9e) is satisfied at t_1, \dots, t_k and if $g(\mathbf{x}(t)) = 0$ for some t during a free flight phase (t_{i-1}, t_i) , then $\gamma = 0$ because of Ineq. (2.10c) and because γ is continuous. Hence, there is no discontinuity in velocity and, though uninformative, the condition $\dot{\mathbf{x}}^+ = \mathbf{N}\dot{\mathbf{x}}^-$ still holds.

2.2.1. Endomorphism μ : mapping from the initial state to the state after k impacts A solution \mathbf{x} of Problem (2.10) is fully determined by the initial positions $\mathbf{x}(0)$ and the initial velocities $\dot{\mathbf{x}}^+(0)$ of the system; the motion consists in a succession of k free flights described by linear dynamics, separated by k impacts described wby the impact law. Introducing $\bar{\mathbf{x}} = [\mathbf{x}^\top, \dot{\mathbf{x}}^\top]^\top$ is advantageously used to reformulate Problem (2.10) in terms of the unknowns $\bar{\mathbf{x}}_0 = \bar{\mathbf{x}}(0)$, the periodicity condition reducing to the fact that $\bar{\mathbf{x}}_0$ is invariant under a linear transformation to be defined in the following.

First, Eq. (2.10a) can be expressed in the first-order form

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} \quad \text{with} \quad \bar{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} \quad (2.11)$$

which also reads $\bar{\mathbf{x}}(t) = \bar{\mathbf{S}}(t - \tilde{t})\bar{\mathbf{x}}(t_0)$ for an arbitrary time \tilde{t} with

$$\bar{\mathbf{S}}(t) = \exp(t\bar{\mathbf{A}}). \quad (2.12)$$

Therefore, every free flight from t_{i-1} to t_i is a linear mapping from $\bar{\mathbf{x}}^+(t_{i-1})$ to $\bar{\mathbf{x}}^-(t_i)$, that is

$$\forall i \in \llbracket 1, k \rrbracket, \quad \bar{\mathbf{x}}^-(t_i) = \bar{\mathbf{S}}(t_i - t_{i-1})\bar{\mathbf{x}}^+(t_{i-1}). \quad (2.13)$$

In the same spirit, the impact law reads

$$\forall i \in \llbracket 1, k \rrbracket, \quad \bar{\mathbf{x}}^+(t_i) = \bar{\mathbf{N}}\bar{\mathbf{x}}^-(t_i) \quad \text{with} \quad \bar{\mathbf{N}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix} \quad (2.14)$$

which is also a linear mapping between two states. Accordingly, the overall dynamics results in the following composite linear mapping acting on $\bar{\mathbf{x}}_0 := \bar{\mathbf{x}}(t_0)$:

$$\left\{ \begin{array}{l} \forall i \in \llbracket 2, k \rrbracket, \forall t \in [t_{i-1}, t_i], \bar{\mathbf{x}}(t) = \bar{\mathbf{S}}(t - t_{i-1})\bar{\mathbf{N}} \dots \bar{\mathbf{N}}\bar{\mathbf{S}}(t_1 - t_0)\bar{\mathbf{x}}_0 \\ \bar{\mathbf{x}}(t_k) = \bar{\mathbf{N}}\bar{\mathbf{S}}(t_k - t_{k-1})\bar{\mathbf{N}} \dots \bar{\mathbf{N}}\bar{\mathbf{S}}(t_1 - t_0)\bar{\mathbf{x}}_0 \end{array} \right. \quad (2.15a)$$

$$\bar{\mathbf{x}}(t_k) = \bar{\mathbf{N}}\bar{\mathbf{S}}(t_k - t_{k-1})\bar{\mathbf{N}} \dots \bar{\mathbf{N}}\bar{\mathbf{S}}(t_1 - t_0)\bar{\mathbf{x}}_0 \quad (2.15b)$$

Definition 2.1 [Endomorphism u] u is the endomorphism of matrix $\bar{\mathbf{N}}\bar{\mathbf{S}}(t_k - t_{k-1})\bar{\mathbf{N}} \dots \bar{\mathbf{N}}\bar{\mathbf{S}}(t_1 - t_0)$ in the canonical basis $\bar{\mathcal{C}}$ of \mathbb{R}^{2n} .

2.2.2. Formulation in terms of initial conditions The question of finding periodic solutions to Problem (2.10) with impact times t_1, \dots, t_k is formulated in three necessary conditions (NC) to be satisfied by $\bar{\mathbf{x}}_0$: Given $T \in \mathbb{R}_+^*$ and $k \in \mathbb{N}^*$, find $\bar{\mathbf{x}}_0 \in \mathbb{R}^{2n}$ and t_1, \dots, t_k such that:

$$\left\{ \begin{array}{ll} u(\bar{\mathbf{x}}_0) = \bar{\mathbf{x}}_0 & \text{NC1 (dynamics + periodicity)} \quad (2.16a) \\ \bar{\mathbf{x}} \text{ determined from } \bar{\mathbf{x}}_0 \text{ through (2.15) satisfies:} & \\ \left\{ \begin{array}{ll} \forall i \in \llbracket 1, k \rrbracket, \bar{g}(\bar{\mathbf{x}}(t_i)) = 0 & \text{NC2 (gap closure)} \quad (2.16b) \\ \forall t \in [0, T], \bar{g}(\bar{\mathbf{x}}(t)) \geq 0 & \text{NC3 (non-penetration)} \quad (2.16c) \end{array} \right. & \end{array} \right.$$

with $\bar{g}(\bar{\mathbf{x}}) := g(\mathbf{x})$. Eq. (2.16a) enforces periodicity through a sequence of free flights over (t_{i-1}, t_i) punctuated by perfectly elastic impacts at $t_i, i \in \llbracket 1, k \rrbracket$. Eq. (2.16b) guarantees that the impact law applies only when the separating gap is closed. Ineq. (2.16c) is necessary to ensure that the unilateral contact condition is not violated during the free flights.

The fact that NC3 is an inequality while NC1 and NC2 are equalities introduces the distinction between “potential solutions” and “admissible solutions” defined below.

Definition 2.2 [Potential solution] A potential solution is an initial condition $\bar{\mathbf{x}}_0$ which satisfies NC1 and NC2 for some t_1, \dots, t_k . By extension, a time-function $\bar{\mathbf{x}}$ determined by a potential solution $\bar{\mathbf{x}}_0$ using relations (2.15) also qualifies as such.

Definition 2.3 [Admissible solution] An admissible solution is a potential solution which satisfies NC3.

2.3. Reformulation of the impact law with a Householder matrix It is now demonstrated that the impact law (2.5) corresponds to a very particular linear mapping \mathbf{N} between the pre-impact and post-impact generalised velocities $\dot{\mathbf{x}}^-$ and $\dot{\mathbf{x}}^+$.

Proposition 2.1 The impact law (2.5) can be reformulated as

$$\dot{\mathbf{x}}^+ = \mathbf{N}\dot{\mathbf{x}}^- \quad \text{with} \quad \mathbf{N} = \mathbf{M}^{-1/2}(\mathbf{I} - 2\mathbf{r}\mathbf{r}^\top)\mathbf{M}^{1/2} \quad (2.17)$$

where \mathbf{r} is the vector defined by $\mathbf{r} = \mathbf{M}^{-1/2}\mathbf{w}(\mathbf{w}^\top\mathbf{M}^{-1}\mathbf{w})^{-1/2}$.

Eq. (2.17) is very convenient: it readily provides the purely elastic impact law satisfying Eq. (2.5) given a mass matrix \mathbf{M} together with a unilateral constraint $g(\mathbf{x}) = 0$.

Proof. Taking the derivative of $t \mapsto g(\mathbf{x}(t))$ defines the normal contact velocity $\gamma = J_g(\mathbf{x}) \dot{\mathbf{x}} = \mathbf{w}^\top \dot{\mathbf{x}}$. Through Eq. (2.3), the difference between the pre- and post-impact normal contact velocities γ^+ and γ^- is

$$\gamma^+ - \gamma^- = \mathbf{w}^\top (\dot{\mathbf{x}}^+ - \dot{\mathbf{x}}^-) = \mathbf{w}^\top \mathbf{M}^{-1} \mathbf{w} \Lambda = G \Lambda. \quad (2.18)$$

The quantity $G = \mathbf{w}^\top \mathbf{M}^{-1} \mathbf{w}$ is often referred to as the Delassus matrix [16]; it reduces here to a scalar because \mathbf{w} is a vector. The matrix \mathbf{M} being symmetric positive-definite, it has a symmetric positive-definite inverse \mathbf{M}^{-1} and since $\mathbf{w} \neq \mathbf{0}$, then $G > 0$ so G has an inverse ($G \neq 0$). Inserting (2.5) into (2.18) yields $-2\gamma^- = G \Lambda$ hence $\Lambda = -2G^{-1}\gamma^-$ and thus, using Eq. (2.3), $\dot{\mathbf{x}}^+ = (\mathbf{I} - 2\mathbf{M}^{-1}\mathbf{w}G^{-1}\mathbf{w}^\top)\dot{\mathbf{x}}^-$. This provides an expression for \mathbf{N} introduced in Eq. (2.6) through $\dot{\mathbf{x}}^+ = \mathbf{N}\dot{\mathbf{x}}^-$. the quantity $\mathbf{P} := \mathbf{M}^{-1}\mathbf{w}G^{-1}\mathbf{w}^\top$, satisfying $\mathbf{N} = \mathbf{I} - 2\mathbf{P}$, is a projector since $\mathbf{P}\mathbf{P} = \mathbf{M}^{-1}\mathbf{w}G^{-1}\mathbf{w}^\top\mathbf{M}^{-1}\mathbf{w}G^{-1}\mathbf{w}^\top = \mathbf{M}^{-1}\mathbf{w}G^{-1}GG^{-1}\mathbf{w}^\top = \mathbf{P}$. Accordingly, G is strictly positive and has an invertible square root $G^{1/2}$ and $\mathbf{N} = \mathbf{I} - 2\mathbf{M}^{-1}(\mathbf{w}G^{-1/2})(\mathbf{w}G^{-1/2})^\top$. \mathbf{M} is symmetric positive-definite and thus has a unique positive-definite square root $\mathbf{M}^{1/2}$ such that

$$\mathbf{M}^{1/2}\mathbf{N}\mathbf{M}^{-1/2} = \mathbf{I} - 2\mathbf{M}^{-1/2}(\mathbf{w}G^{-1/2})(\mathbf{w}G^{-1/2})^\top\mathbf{M}^{-1/2} \quad (2.19a)$$

$$= \mathbf{I} - 2(\mathbf{M}^{-1/2}\mathbf{w}G^{-1/2})(\mathbf{M}^{-1/2}\mathbf{w}G^{-1/2})^\top \quad (2.19b)$$

and $\mathbf{r} = \mathbf{M}^{-1/2}\mathbf{w}G^{-1/2}$ yields $\mathbf{N} = \mathbf{M}^{-1/2}(\mathbf{I} - 2\mathbf{r}\mathbf{r}^\top)\mathbf{M}^{1/2}$. \blacksquare

The following proposition is a consequence of the energy conservation in the impact law.

Proposition 2.2 *The vector $\mathbf{r} = \mathbf{M}^{-1/2}\mathbf{w}G^{-1/2}$ is a unit vector.*

Proof. $\|\mathbf{r}\|^2 = \mathbf{r}^\top \mathbf{r} = G^{-1/2}\mathbf{w}^\top\mathbf{M}^{-1/2}\mathbf{M}^{-1/2}\mathbf{w}G^{-1/2} = G^{-1/2}\mathbf{w}^\top\mathbf{M}^{-1}\mathbf{w}G^{-1/2} = G^{-1/2}GG^{-1/2} = 1$. \blacksquare

This proves that \mathbf{N} is similar to $\mathbf{I} - 2\mathbf{r}\mathbf{r}^\top$ which is the matrix of a reflection with respect to a hyperplane of normal \mathbf{r} . \mathbf{N} is symmetric and involutory. Form (2.17) of the impact law will be used in the exposition of the main results summarized in section 3. It is also essential to the provided proofs.

Remark 2.1. From $\mathbf{M}\mathbf{P} = \mathbf{P}^\top\mathbf{M} = \mathbf{P}^\top\mathbf{M}\mathbf{P}$, the impact law is indeed conservative since

$$\mathbf{N}^\top\mathbf{M}\mathbf{N} = (\mathbf{I} - 2\mathbf{P}^\top)\mathbf{M}(\mathbf{I} - 2\mathbf{P}) = \mathbf{M} - 2\mathbf{M}\mathbf{P} - 2\mathbf{P}^\top\mathbf{M} + 4\mathbf{P}^\top\mathbf{M}\mathbf{P} = \mathbf{M}. \quad (2.20)$$

As a consequence, \mathbf{N} is an isometry for the quadratic form of the kinetic energy $T : \dot{\mathbf{x}} \mapsto \frac{1}{2}\dot{\mathbf{x}}^\top\mathbf{M}\dot{\mathbf{x}}$, that is $T(\mathbf{x}) = T(\mathbf{N}\mathbf{x})$.

Remark 2.2. Matrices of the form $\mathbf{I} - 2\mathbf{r}\mathbf{r}^\top$ are known as Householder matrices [11].

Remark 2.3. Impact law (2.17) is not restricted to a mass impacting a rigid obstacle. It also handles two impacting masses, for instance.

3. Main results

3.1. Preliminary notations Efficiently exposing the main results necessitates a few quantities which require various definitions. The resulting heavy notations must not mask the fact that their computations are very inexpensive and they are all determined solely by the mass matrix \mathbf{M} , the stiffness matrix \mathbf{K} , the gap g and the times of impact t_1, \dots, t_k . Let us define:

- $\mathbf{K}^{1/2}$, the positive-definite square-root of the symmetric positive-definite matrix \mathbf{K} . Recall that $\mathbf{M}^{1/2}$ is the square-root of \mathbf{M} .
- $\mathbf{B} = -\mathbf{M}^{-1/2}\mathbf{K}^{1/2}$; \mathbf{Q} an orthogonal matrix satisfying $\mathbf{B}\mathbf{B}^\top = \mathbf{Q}\mathbf{\Delta}^2\mathbf{Q}^{-1}$ with the diagonal matrix $\mathbf{\Delta} = \text{diag}(\omega_1, \dots, \omega_n)$.
- The period $T \in \mathbb{R}_+^*$, assumed not to be a multiple of a $2\pi/\omega_i$, that is $\forall i \in \llbracket 1, n \rrbracket, T \not\equiv 0 \pmod{2\pi/\omega_i}$.
- $k \in \mathbb{N}^*$ the number of impacts per period and $(t_1, \dots, t_k) \in \mathbb{R}_+^{*k}$ the times of impact with $t_{i-1} < t_i$. The duration between two successive impacts is denoted by $\sigma_i = t_i - t_{i-1}$ with $t_0 = 0$ and $T = t_k$.
- The unit vector $\mathbf{r} \in \mathbb{R}^{2n}$ of odd components 0 and even components those of $\mathbf{Q}^{-1}\mathbf{r}$ where $\mathbf{r} \in \mathbb{R}^n$ is defined in Prop. 2.1 by $\mathbf{r} = \mathbf{M}^{-1/2}\mathbf{w}(\mathbf{w}^\top\mathbf{M}^{-1}\mathbf{w})^{-1/2}$; \mathbf{w} is the transpose of the gradient of g .
- The vector $\underline{\mathbf{w}}$ of even components 0 and whose odd components are the ones (in the same order) of the vector $-\mathbf{\Delta}^{-1}\mathbf{Q}^\top\mathbf{M}^{-1/2}\mathbf{w}$.

- The block-diagonal matrix $\underline{\mathbf{S}} \in \mathbb{R}^{2n,2n}$:

$$\underline{\mathbf{S}}(t) = \mathbf{diag} \left(\left[\begin{array}{cc} \cos(\omega_i t) & -\sin(\omega_i t) \\ \sin(\omega_i t) & \cos(\omega_i t) \end{array} \right]_{i=1,\dots,n} \right). \quad (3.1)$$

- The function

$$\Upsilon_T(\sigma) = \sum_{i=1}^n \frac{\sin(\omega_i \sigma / 2)}{\sin(\omega_i T / 2)} r_{2i}^2 \quad (3.2)$$

where r_{2i} is the $2i$ -th entry of $\underline{\mathbf{r}}$.

- The skew-symmetric matrix $\underline{\mathbf{\Pi}}$ populated by the elements

$$\Pi_{i,j}(t_1, \dots, t_k) = \Upsilon_T(2(t_j - t_i) - T), \quad i < j \leq k. \quad (3.3)$$

- The symmetric matrix $\underline{\mathbf{\Sigma}}$ populated by the elements

$$\Sigma_{i,j}(t_1, \dots, t_k) = 2\underline{\mathbf{w}}^\top \left(\underline{\mathbf{S}}\left(\frac{T}{2}\right) - \underline{\mathbf{S}}\left(-\frac{T}{2}\right) \right)^{-1} \underline{\mathbf{S}}\left(t_i - t_j + \frac{T}{2}\right) \underline{\mathbf{r}}, \quad i \leq j \leq k. \quad (3.4)$$

- The n vectors $\underline{\mathbf{w}}_p \in \mathbb{R}^{2n}$, for $p \in \llbracket 1, n \rrbracket$:

$$\underline{\mathbf{w}}_p := - \begin{bmatrix} \sum_{i=1}^n (\mathbf{M}^{-1/2})_{p,i} Q_{i,1} / \omega_1 \\ 0 \\ \vdots \\ \sum_{i=1}^n (\mathbf{M}^{-1/2})_{p,i} Q_{i,n} / \omega_n \\ 0 \end{bmatrix}. \quad (3.5)$$

- The functions $\phi_{p,T}$ and $\psi_{p,T}$ for $p \in \llbracket 1, n \rrbracket$:

$$\phi_{p,T}(t) = 2\underline{\mathbf{w}}_p^\top \left(\underline{\mathbf{S}}\left(\frac{T}{2}\right) - \underline{\mathbf{S}}\left(-\frac{T}{2}\right) \right)^{-1} \underline{\mathbf{S}}\left(\frac{t}{2}\right) \underline{\mathbf{r}} \quad \text{and} \quad \psi_{p,T}(t) = 2 \frac{d\phi_{p,T}}{dt}(t). \quad (3.6)$$

- The linear application

$$\varphi : \mathbb{R}^k \rightarrow \mathbb{R}^{2n} \\ \lambda \mapsto 2 \begin{bmatrix} -\mathbf{M}^{-1/2} \mathbf{Q} \mathbf{\Delta}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1/2} \mathbf{Q} \end{bmatrix} \mathbf{P}_B^\beta (\underline{\mathbf{S}}(T) - \mathbf{I})^{-1} \sum_{j=1}^k \lambda_j \underline{\mathbf{S}}(T - t_j) \underline{\mathbf{r}}. \quad (3.7)$$

where \mathbf{P}_B^β is the transpose of the permutation matrix associated to permutation

$$\begin{pmatrix} 1 & 2 & 3 & \dots & 2n-1 & 2n \\ 1 & n+1 & 2 & \dots & n & 2n \end{pmatrix}. \quad (3.8)$$

- The constant vector $\underline{\mathbf{j}} = [1, \dots, 1]^\top \in \mathbb{R}^k$.

3.2. Important theorems NC1, defined in Eq. (2.16a), consists in finding invariant elements of the endomorphism u introduced in Definition 2.1 as the product of $2k$ matrices, k of which are matrix exponentials. This is a challenging problem even for k and n as small as $k = 2$ and $n = 3$. Theorem indicates a very powerful simplification of this question.

Theorem 3.1 $\bar{\mathbf{x}}_0$ is invariant by u iff there exists a vector $\underline{\lambda} \in \mathbb{R}^k$ such that:

$$\begin{cases} \underline{\mathbf{\Pi}}(t_1, \dots, t_k) \underline{\lambda} = \mathbf{0} \\ \bar{\mathbf{x}}_0 = \varphi(\underline{\lambda}). \end{cases} \quad (3.9a)$$

$$(3.9b)$$

Theorem 3.1 relies on the existence of an explicit isomorphism between the maximal invariant subspace of u and the kernel of the matrix $\underline{\mathbf{\Pi}}$. One direct consequence is that calculating the dimension of the maximal invariant subspace reduces to calculating the kernel of the $k \times k$ matrix $\underline{\mathbf{\Pi}}$. The proof is provided in section 5.

Condition NC2 of gap closure (Eq. (2.9b)) can also be expressed as another condition on the same vector $\underline{\lambda} \in \mathbb{R}^k$.

Theorem 3.2 $\bar{\mathbf{x}}_0$ is such that $\forall i \in \llbracket 1, k \rrbracket$, $\bar{g}(\bar{\mathbf{x}}(t_i)) = 0$ iff there exists $\boldsymbol{\lambda} \in \mathbb{R}^k$ such that:

$$\begin{cases} \boldsymbol{\Sigma}(t_1, \dots, t_k)\boldsymbol{\lambda} = -g_0 \mathbf{j} & (3.10a) \\ \bar{\mathbf{x}}_0 = \varphi(\boldsymbol{\lambda}). & (3.10b) \end{cases}$$

Theorems 3.1 and 3.2 give a practical way of finding potential solutions: it is sufficient to find $\boldsymbol{\lambda} \in \mathbb{R}^k$ verifying Eqs. (3.9a) and (3.10a) for some (t_1, \dots, t_k) ; $\bar{\mathbf{x}}_0$ can then be explicitly exhibited using (3.9b). The proof of Theorem 3.2 is provided in section 6.

The following theorem, which is a direct consequence of Theorems 3.1 and 3.2, addresses the existence of potential solutions and thus of non-smooth modes of vibration. It implies that, if $g_0 \neq 0$, then non-smooth modes are two-dimensional surfaces of the phase space in the neighbourhood of a generic point, as in the smooth case.

Denote by \mathcal{S} the set of impact times for which a potential solution exists:

$$\mathcal{S} := \{(t_1, \dots, t_k) \mid \exists \boldsymbol{\lambda} \in \mathbb{R}^k \text{ solution of } \boldsymbol{\Pi}(t_1, \dots, t_k)\boldsymbol{\lambda} = \mathbf{0} \text{ and } \boldsymbol{\Sigma}(t_1, \dots, t_k)\boldsymbol{\lambda} = -g_0 \mathbf{j}\} \quad (3.11)$$

Then, there exists a one-parameter continuum of potential solutions, for some specific t_1, \dots, t_k .

Theorem 3.3 [Dimension of the space of potential solutions] *A generic point of \mathcal{S} is:*

- a point of a curve of \mathcal{S} if $g_0 \neq 0$;
- an isolated point of \mathcal{S} if $g_0 = 0$.

The proof of this theorem is given in section 8.

Corollary 3.1 [Dimension of non-smooth modes of vibration] *If $\bar{\mathbf{x}}$ is an admissible solution for some generic impact times $s = (t_1, \dots, t_k)$, then:*

- s is a point of a curve in the space of the impact times, and the neighbourhood of s on the curve corresponds to admissible solutions, if $g_0 \neq 0$;
- s is an isolated point if $g_0 = 0$.

Corollary 3.1 shows that when $g_0 \neq 0$, an admissible solution is necessarily part of a one-parameter continuum of periodic trajectories, which is by definition a non-smooth mode. This one-parameter continuum of trajectories is a two-dimensional surface in the phase space, since every trajectory is a curve. In the end, it shows that periodic solutions of the initial non-smooth problem form, when they exist, two-dimensional manifolds, as in the linear and smooth nonlinear counterparts. Moreover, Theorem 3.4 gives the explicit expressions of the n positions and n velocities as functions of time. Its proof is provided in section 7.

Theorem 3.4 [Expression of the positions and velocities] *If $\bar{\mathbf{x}}_0$ is a potential solution for impact times t_1, \dots, t_k , then the time-evolution of the components of \mathbf{x} are given $\forall \tau \in [0, \sigma_{i+1}]$ and $\forall p \in \llbracket 1, n \rrbracket$ by*

$$x_p(\tau + t_i) = \sum_{j=1}^i \lambda_j \phi_{p,T}(2(\tau + t_i - t_j) - T) + \sum_{j=i+1}^k \lambda_j \phi_{p,T}(2(\tau + t_i - t_j) + T) \quad (3.12)$$

where λ_j is the j -th component of $\boldsymbol{\lambda} := \varphi^{-1}(\bar{\mathbf{x}}_0)$. Similarly, the components of $\dot{\mathbf{x}}$ are given $\forall \tau \in [0, \sigma_{i+1}]$ and $\forall p \in \llbracket 1, n \rrbracket$ by

$$y_p(\tau + t_i) = \sum_{j=1}^i \lambda_j \psi_{p,T}(2(\tau + t_i - t_j) - T) + \sum_{j=i+1}^k \lambda_j \psi_{p,T}(2(\tau + t_i - t_j) + T). \quad (3.13)$$

An insight of these theorems is shown in Figure 2 for $g_0 \neq 0$. For a generic point $s = (t_1, \dots, t_k)$ of the space of the impact times, $\boldsymbol{\Sigma}(s)$ is invertible and Theorems 3.1 and 3.2 indicate that s must satisfy $\boldsymbol{\Pi}(s)\boldsymbol{\Sigma}^{-1}(s)\mathbf{j} = \mathbf{0}$ for a potential solution to exist. Such an s determines a vector $\boldsymbol{\lambda} = -g_0 \boldsymbol{\Sigma}^{-1}(s)\mathbf{j}$ and the time-evolutions of positions and velocities are given by Theorem 3.4 as functions of $\boldsymbol{\lambda}$. They correspond to trajectories in the phase space. If the unilateral constraints are satisfied by the potential solution, the latter qualifies as admissible solution. Corollary 3.1 shows that such a point s is generically a point of a curve in the space of impact times; the neighbourhood of s on the curve also corresponds to

Theo. 1&2: $s \mid \mathbf{\Pi}(s)\mathbf{\Sigma}^{-1}(s)\mathbf{j} = \mathbf{0}$
 Theo. 3: \mathcal{S} is a curve

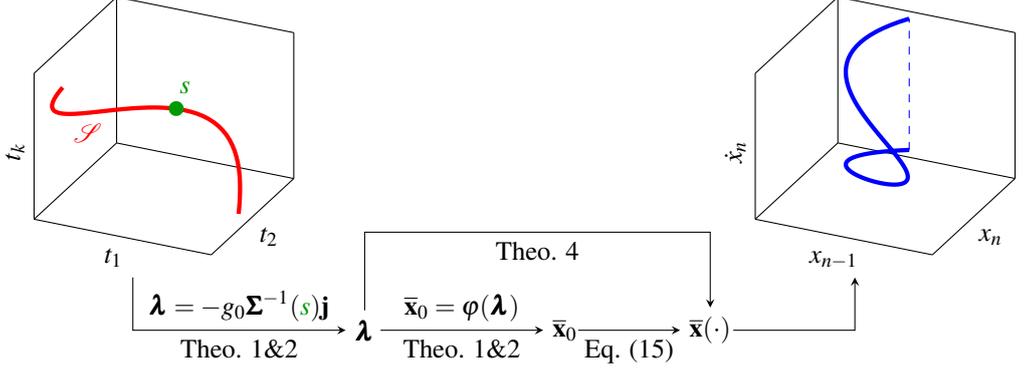


Figure 2: Summary of the main results for a generic point $s \in \mathcal{S}$ with $g_0 \neq 0$. When $s = (t_1, \dots, t_k)$ travels along the red curve \mathcal{S} , the periodic solution describes a non-smooth mode with k impacts per period.

admissible solutions. Along the neighbourhood of s in \mathcal{S} , the corresponding admissible solutions describe a k -ipp non-smooth mode in the phase space.

Remark 3.1. As shown in the proof of Theorem 3.3, finding λ and thus \bar{x}_0 requires solving $k - 1$ equations. However, the set of equations depends on the parity of k . Because $\mathbf{\Pi}$ is a skew-symmetric matrix, solving the $k - 1$ equations can be significantly simplified using the Pfaffian operator, especially when k is odd. Finally:

- If k is odd, for almost every s , $\text{rank}(\mathbf{\Pi}(s)) = k - 1$ and it can be shown that its kernel is spanned by the vector \mathbf{k} of the Pfaffians of the diagonal $k - 1 \times k - 1$ submatrices of $\mathbf{\Pi}$ with alternating signs. Hence, \mathbf{k} writes as a function of the k impact times. Nevertheless $\mathbf{\Pi}\lambda = \mathbf{0}$ so there exists a $\mu \neq 0$ such that $\lambda = \mu\mathbf{k}$. Eq. (3.10a) implies that all the rows of $\mathbf{\Sigma}(t_1, \dots, t_k)\mathbf{k}$ have identical value; this provides the $k - 1$ equations. Then $\lambda = \mu\mathbf{k}$ where μ is determined by $\mathbf{\Sigma}(t_1, \dots, t_k)\mathbf{k} = -\frac{g_0}{\mu}\mathbf{j}$ and the sought initial solution is given by $\bar{x}_0 = \varphi(\lambda)$.
- If k is even, for almost every s , $\text{rank}(\mathbf{\Pi}(s)) = k$ and there is no $\lambda \neq \mathbf{0}$ solution of Eq. (3.9a). Hence one must solve $\text{pf}(\mathbf{\Pi}) = 0$ for (t_1, \dots, t_k) . When this equation is satisfied, for almost every s , $\text{rank}(\mathbf{\Pi}(s)) = k - 2$, and $\mathbf{\Pi}(t_1, \dots, t_k)\lambda = \mathbf{0}$ provides $k - 2$ independent equations. Again, a total of $k - 1$ equations are to be solved.

Remark 3.2. Theorem 3.3 illustrates the effectiveness of Theorems 3.1 and 3.2, but the latter are also constructive. For example, let us treat the problem for $k = 1$. Then, $\mathbf{\Pi} = [0]$ because it is skew-symmetric, and the only condition on $\lambda \in \mathbb{R}$ and $t_1 = T$ is $\mathbf{\Sigma}(T)\lambda = -g_0\mathbf{j}$. This leads to the existence of a continuum of potential solutions whenever $\mathbf{\Sigma}(T) \neq 0$, that is almost everywhere in \mathbb{R}_+^* . Interestingly, this result does not depend on n . Theorems 3.1 and 3.2 hence allow to solve the periodic problem with one impact per period in a few lines only. This is further detailed on a simple example in subsection 10.2.

Remark 3.3. There is no general result regarding the last necessary condition NC3. In practice, the first step is to compute potential solutions and then check NC3. However, it follows from Theorem 3.1 that a general admissible solution is an element of a continuum of admissible solutions. As a consequence, the corresponding non-smooth mode can be numerically retrieved from *one* admissible solution using continuation techniques. It is important to note that potential solutions can be straightforwardly filtered out by checking that the sign of the pre-impact velocities corresponds to non-penetration.

4. Introduction of the effective basis $\underline{\mathcal{B}}$ As shown in Eq. (2.15), the dynamics is a combination of free flights, captured by $\bar{\mathbf{S}}$, and impacts, accounted for with $\bar{\mathbf{N}}$. Matrices $\bar{\mathbf{S}}$ and $\bar{\mathbf{N}}$ respectively correspond to two endomorphisms of fundamentally different nature: the free flight can be regarded as n rotations in n orthogonal planes (the linear modes) of the phase space, while, as already known, the impact dynamics is a reflection with respect to a hyperplane, which only affects the coordinates appearing explicitly in the expression of $g(\mathbf{x})$. The first step for the proofs is to define a basis $\underline{\mathcal{B}}$, called the “effective basis”, in which both the free flights and the impacts have simpler expressions.

To this end, is introduced the quadratic form Q of the total energy of the system

$$Q([\mathbf{x}, \mathbf{y}]) = \frac{1}{2} \mathbf{x}^\top \mathbf{K} \mathbf{x} + \frac{1}{2} \mathbf{y}^\top \mathbf{M} \mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{x}^\top & \mathbf{y}^\top \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}. \quad (4.1)$$

From Remark 2.1, \mathbf{N} is an isometry for the kinetic energy $\dot{\mathbf{x}} \mapsto \frac{1}{2} \dot{\mathbf{x}}^\top \mathbf{M} \dot{\mathbf{x}}$, hence $\bar{\mathbf{N}} = \mathbf{diag}(\mathbf{I}, \mathbf{N})$ is an isometry for the total energy Q . As \mathbf{N} is similar to a reflection with respect to a hyperplane (cf. Proposition 2.1), it is an indirect isometry. Hence, $\bar{\mathbf{N}}$ is also an indirect isometry. The matrix $\bar{\mathbf{S}}(t)$, defined in Eq. (2.12), is an isometry as well for the total energy, but a direct isometry, as proved below.

Proposition 4.1 $\forall t \in \mathbb{R}$, $e^{t\bar{\mathbf{A}}}$ is a direct isometry for Q .

Proof. Denote $\mathbf{F} = \mathbf{diag}(\mathbf{K}, \mathbf{M})$. $\bar{\mathbf{F}}\bar{\mathbf{A}}$ is a skew-symmetric matrix and \mathbf{F} is symmetric so

$$-\bar{\mathbf{A}} = -\mathbf{F}^{-1} \bar{\mathbf{F}} \bar{\mathbf{A}} = \mathbf{F}^{-1} (\bar{\mathbf{F}} \bar{\mathbf{A}})^\top = \mathbf{F}^{-1} \bar{\mathbf{A}}^\top \mathbf{F}. \quad (4.2)$$

Taking the exponential,

$$\forall t \in \mathbb{R}, \quad e^{t\mathbf{F}^{-1} \bar{\mathbf{A}}^\top \mathbf{F}} = e^{-t\bar{\mathbf{A}}} = \mathbf{F}^{-1} (e^{t\bar{\mathbf{A}}})^\top \mathbf{F} \quad (4.3)$$

or $\forall \bar{\mathbf{x}} \in \mathbb{R}^{2n}$, $\forall t \in \mathbb{R}$

$$(e^{t\bar{\mathbf{A}}} \bar{\mathbf{x}})^\top \mathbf{F} e^{t\bar{\mathbf{A}}} \bar{\mathbf{x}} = \bar{\mathbf{x}}^\top (e^{t\bar{\mathbf{A}}})^\top \mathbf{F} e^{t\bar{\mathbf{A}}} \bar{\mathbf{x}} = \bar{\mathbf{x}}^\top \mathbf{F} e^{-t\bar{\mathbf{A}}} \mathbf{F}^{-1} \mathbf{F} e^{t\bar{\mathbf{A}}} \bar{\mathbf{x}} = \bar{\mathbf{x}}^\top \mathbf{F} \bar{\mathbf{x}} \quad (4.4)$$

which proves that $e^{t\bar{\mathbf{A}}}$ is an isometry for $\bar{\mathbf{S}}$. The determinant of $e^{t\bar{\mathbf{A}}}$ is therefore either -1 or $+1$. Its value is $+1$ for $t = 0$ and it is conserved by continuity. $e^{t\bar{\mathbf{A}}}$ is thus a direct isometry for Q . ■

We now proceed with the ‘‘effective basis’’.

4.1. Construction Let a denote the endomorphism of matrix $\bar{\mathbf{A}}$ in the canonical basis $\bar{\mathcal{C}}$. Both matrices \mathbf{K} and \mathbf{M} are positive definite and have unique square root matrices, denoted by $\mathbf{K}^{1/2}$ and $\mathbf{M}^{1/2}$ respectively, which also are positive-definite. Consider the basis \mathcal{B}_1 such that the change of basis matrix from the canonical basis to \mathcal{B}_1 is $\mathbf{diag}(\mathbf{K}^{-1/2}, \mathbf{M}^{-1/2})$. The matrix of Q in \mathcal{B}_1 is

$$\begin{bmatrix} \mathbf{K}^{-1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1/2} \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{K}^{-1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1/2} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad (4.5)$$

and so \mathcal{B}_1 is an orthonormal basis for Q . The matrix of a in this basis is

$$\text{Mat}_{\mathcal{B}_1}(a) = \begin{bmatrix} \mathbf{K}^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{1/2} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1} \mathbf{K} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{K}^{-1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1/2} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{B}^\top \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \quad (4.6)$$

with $\mathbf{B} = -\mathbf{M}^{-1/2} \mathbf{K}^{1/2}$. Being a real invertible matrix, \mathbf{B} has a unique (right) polar decomposition $\mathbf{B} = \boldsymbol{\nu} \boldsymbol{\sigma}$ ($\boldsymbol{\nu}$ is orthogonal and $\boldsymbol{\sigma}$ is symmetric positive-definite). The matrix

$$\mathbf{B}^\top \mathbf{B} = \mathbf{M}^{-1/2} \mathbf{K}^{1/2} \mathbf{K}^{1/2} \mathbf{M}^{-1/2} = \mathbf{M}^{1/2} (\mathbf{M}^{-1} \mathbf{K}) \mathbf{M}^{-1/2} \quad (4.7)$$

is symmetric positive-definite, diagonalisable in an orthonormal basis. It is similar to $\mathbf{M}^{-1} \mathbf{K}$, so $\mathbf{M}^{-1} \mathbf{K}$ is diagonalisable and has non-negative eigenvalues.

Since $\mathbf{M}^{1/2}$ is a real symmetric matrix (square-root of a diagonal matrix), a diagonalisation of $\mathbf{M}^{-1} \mathbf{K}$ can be made orthogonal. This yields a decomposition $\mathbf{B} \mathbf{B}^\top = \mathbf{Q} \boldsymbol{\Delta}^2 \mathbf{Q}^{-1}$ where $\boldsymbol{\Delta}$ is a diagonal matrix of non-negative coefficients $\omega_1, \dots, \omega_n$ and \mathbf{Q} is orthogonal. Also, $\mathbf{B} \mathbf{B}^\top = \boldsymbol{\sigma} \boldsymbol{\nu} \boldsymbol{\nu}^\top \boldsymbol{\sigma}^\top = \boldsymbol{\sigma}^2$ since $\boldsymbol{\nu}$ is orthogonal and $\boldsymbol{\sigma}$ symmetric. In the same line, $\mathbf{B} \mathbf{B}^\top = \mathbf{Q} \boldsymbol{\Delta}^2 \mathbf{Q}^{-1} = (\mathbf{Q} \boldsymbol{\Delta} \mathbf{Q}^{-1})^2$ and, by uniqueness of the square root of $\mathbf{B} \mathbf{B}^\top$, $\boldsymbol{\sigma} = \mathbf{Q} \boldsymbol{\Delta} \mathbf{Q}^{-1}$. Then

$$-\mathbf{M}^{-1/2} \mathbf{K}^{1/2} = \mathbf{B} = \boldsymbol{\sigma} \boldsymbol{\nu} = \mathbf{Q} \boldsymbol{\Delta} \mathbf{Q}^{-1} \boldsymbol{\nu} = \mathbf{Q} \boldsymbol{\Delta} \mathbf{Q}_1^{-1} \quad (4.8a)$$

$$\text{and } \mathbf{B}^\top = -\mathbf{K}^{1/2} \mathbf{M}^{-1/2} = \mathbf{Q}_1 \boldsymbol{\Delta} \mathbf{Q}^{-1} \quad (4.8b)$$

with $\mathbf{Q}_1 = \boldsymbol{\nu}^{-1} \mathbf{Q}$. It comes:

$$\text{Mat}_{\mathcal{B}_1}(a) = \begin{bmatrix} \mathbf{0} & -\mathbf{Q}_1 \boldsymbol{\Delta} \mathbf{Q}^{-1} \\ \mathbf{Q} \boldsymbol{\Delta} \mathbf{Q}_1^{-1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\boldsymbol{\Delta} \\ \boldsymbol{\Delta} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix}^{-1}. \quad (4.9)$$

Let $\mathcal{B} = (\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_{n+1}, \dots, \mathbf{e}_{2n})$ be the basis such that the change of basis matrix from \mathcal{B}_1 to \mathcal{B} is $\text{diag}(\mathbf{Q}_1, \mathbf{Q})$. It follows from (4.9) that

$$\text{Mat}_{\mathcal{B}}(a) = \begin{bmatrix} \mathbf{0} & -\mathbf{\Delta} \\ \mathbf{\Delta} & \mathbf{0} \end{bmatrix} \quad (4.10)$$

and thus $a(\mathbf{e}_i) = \omega_i \mathbf{e}_{n+i}$ and $a(\mathbf{e}_{n+i}) = -\omega_i \mathbf{e}_i$, $\forall i \in \llbracket 1, n \rrbracket$. The *effective basis* $\underline{\mathcal{B}}$ is obtained by reordering the basis elements:

$$\underline{\mathcal{B}} = (\mathbf{e}_1, \mathbf{e}_{n+1}, \dots, \mathbf{e}_n, \mathbf{e}_{2n}) = (\underline{\mathbf{e}}_1, \dots, \underline{\mathbf{e}}_{2n}) \quad (4.11)$$

so that $\text{Mat}_{\underline{\mathcal{B}}}(a)$ is block-diagonal, the diagonal blocks being

$$\forall i \in \llbracket 1, n \rrbracket, \quad \mathbf{A}_i = \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}. \quad (4.12)$$

Proposition 4.2 $\underline{\mathcal{B}}$ is an orthonormal basis for Q .

Proof. \mathcal{B}_1 is a orthonormal basis for Q as $(\mathbf{P}_{\bar{\mathcal{C}}}^{\mathcal{B}_1})^\top \bar{\mathbf{S}} \mathbf{P}_{\bar{\mathcal{C}}}^{\mathcal{B}_1} = \mathbf{I}$. $\mathbf{P}_{\mathcal{B}_1}^{\mathcal{B}}$ is an orthogonal matrix because \mathbf{Q} and \mathbf{Q}_1 are orthogonal, so \mathcal{B} is a orthonormal basis for Q . So is $\underline{\mathcal{B}}$ because $\mathbf{P}_{\underline{\mathcal{B}}}^{\mathcal{B}}$ is a permutation matrix. ■

The matrix $\underline{\mathbf{S}}(t)$ of e^{ta} in $\underline{\mathcal{B}}$ is the block-diagonal matrix whose diagonal blocks are

$$\forall i \in \llbracket 1, n \rrbracket, \quad \begin{bmatrix} \cos(\omega_i t) & -\sin(\omega_i t) \\ \sin(\omega_i t) & \cos(\omega_i t) \end{bmatrix} \quad (4.13)$$

and it follows that $\underline{\mathbf{S}}(t)$ acts as n independent rotations in n orthogonal planes. It is indeed a direct isometry. The purpose of basis $\underline{\mathcal{B}}$ is to simplify the matrix expression of the endomorphism u . A priori, the reflection with respect to the hyperplane defining the impact law should affect all the components of $\underline{\mathbf{x}}$. Here, $\underline{\mathcal{B}}$ is constructed in such a way that its odd-index vectors are elements of the hyperplane defining the impact law. This can be understood as rotating the pair of vectors spanning each stable plane such that the first vector is aligned with the intersection between the stable plane and the hyperplane, see Fig. 3. Not only does $\underline{\mathcal{B}}$ eventually take advantage of the linear modes of the unconstrained system (*i.e.* the n orthogonal stable planes), but it is also such that the impact law does not affect the odd components of $\underline{\mathbf{x}}$.

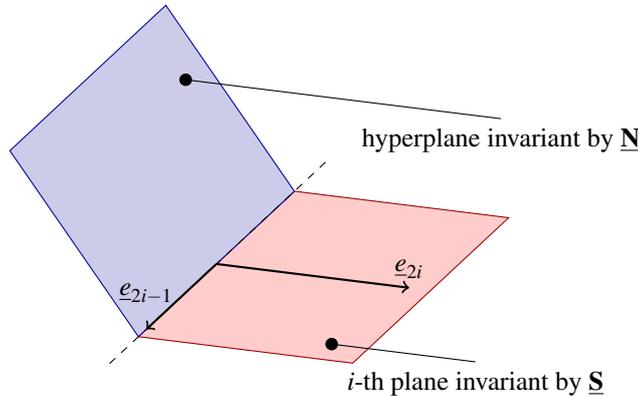


Figure 3: The effective basis $\underline{\mathcal{B}}$ is constructed in such a way that the hyperplane invariant by $\underline{\mathbf{N}}$ intersects the n orthogonal planes invariant by $\underline{\mathbf{S}}$ along a line spanned by the odd-index vectors of $\underline{\mathcal{B}}$.

4.2. From the canonical basis $\bar{\mathcal{C}}$ to $\underline{\mathcal{B}}$ We now exhibit the change of basis matrix from $\bar{\mathcal{C}}$ and $\underline{\mathcal{B}}$. This is useful to express the quantities $\bar{\mathbf{N}}$, $\bar{\mathbf{S}}$, and $\bar{\mathbf{x}}$ in the effective basis.

Proposition 4.3 Let \vec{v} be a column vector of components

$$V_{\bar{\mathcal{C}}}(\vec{v}) = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \quad V_{\underline{\mathcal{B}}}(\vec{v}) = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \quad (4.14)$$

in the canonical basis $\bar{\mathcal{C}}$ and $\underline{\mathcal{B}}$, respectively. Then

$$\begin{cases} \mathbf{x} = -\mathbf{M}^{-1/2} \mathbf{Q} \mathbf{\Delta}^{-1} \mathbf{u} \\ \mathbf{y} = \mathbf{M}^{-1/2} \mathbf{Q} \mathbf{v}. \end{cases} \quad (4.15)$$

Proof.

$$V_{\mathcal{B}}(\vec{v}) = \sum_{i=1}^n u_i \mathbf{e}_i + \sum_{i=1}^n v_i \mathbf{e}_{n+i} = \sum_{i=1}^n u_i \mathbf{e}_{2i-1} + \sum_{i=1}^n v_i \mathbf{e}_{2i} \quad (4.16)$$

$$= u_1 \mathbf{e}_1 + v_1 \mathbf{e}_2 + \cdots + u_n \mathbf{e}_{2n-1} + v_n \mathbf{e}_{2n}. \quad (4.17)$$

Then

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{K}^{-1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1/2} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{K}^{-1/2} \mathbf{Q}_1 \mathbf{u} \\ \mathbf{M}^{-1/2} \mathbf{Q} \mathbf{v} \end{bmatrix} \quad (4.18)$$

so $\mathbf{x} = \mathbf{K}^{-1/2} \mathbf{Q}_1 \mathbf{u}$ and $\mathbf{y} = \mathbf{M}^{-1/2} \mathbf{Q} \mathbf{v}$. Using (4.8), $\mathbf{Q}_1 = -\mathbf{K}^{1/2} \mathbf{M}^{-1/2} \mathbf{Q} \mathbf{\Delta}^{-1}$ so $\mathbf{x} = -\mathbf{M}^{-1/2} \mathbf{Q} \mathbf{\Delta}^{-1} \mathbf{u}$ and $\mathbf{y} = \mathbf{M}^{-1/2} \mathbf{Q} \mathbf{v}$. ■

Proposition 4.3 provides the expression of the change-of-basis matrix from $\bar{\mathcal{C}}$ to \mathcal{B} :

$$\mathbf{P}_{\bar{\mathcal{C}}}^{\mathcal{B}} = \begin{bmatrix} -\mathbf{M}^{-1/2} \mathbf{Q} \mathbf{\Delta}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1/2} \mathbf{Q} \end{bmatrix}. \quad (4.19)$$

The components of \vec{v} in $\underline{\mathcal{B}}$ are easily deduced from those in \mathcal{B} : they simply have to be reordered through the permutation matrix $\mathbf{P}_{\underline{\mathcal{B}}}^{\mathcal{B}}$.

4.3. Expression of the impact law in $\underline{\mathcal{B}}$ The relevance of basis $\underline{\mathcal{B}}$ to efficiently express the impact law is now demonstrated. Indeed, in this basis, not only does the free flight dynamics reduce to rotations in orthogonal planes, but the impact law also highly simplifies.

Proposition 4.4 In $\underline{\mathcal{B}}$, the matrix of the impact law (2.6) writes

$$\underline{\mathbf{N}} = \mathbf{I} - 2\underline{\mathbf{r}}\underline{\mathbf{r}}^{\top} \quad (4.20)$$

where $\underline{\mathbf{r}} \in \mathbb{R}^{2n}$ is the vector of odd components 0 and even components those of $\mathbf{Q}^{-1} \mathbf{r}$. Moreover, $\underline{\mathbf{r}}$ is a unit vector.

Proof. This follows from Eq. (4.14). Using $\dot{\mathbf{x}} = \mathbf{M}^{-1/2} \mathbf{Q} \mathbf{v}$, $\dot{\mathbf{x}}^+ = \underline{\mathbf{N}} \dot{\mathbf{x}}^-$ becomes

$$\begin{aligned} \mathbf{v}^+ &= (\mathbf{M}^{-1/2} \mathbf{Q})^{-1} (\mathbf{M}^{-1/2} (\mathbf{I} - 2\underline{\mathbf{r}}\underline{\mathbf{r}}^{\top}) \mathbf{M}^{1/2}) \mathbf{M}^{-1/2} \mathbf{Q} \mathbf{v}^- \\ &= \mathbf{Q}^{-1} (\mathbf{I} - 2\underline{\mathbf{r}}\underline{\mathbf{r}}^{\top}) \mathbf{Q} \mathbf{v}^- = (\mathbf{I} - 2\mathbf{Q}^{-1} \underline{\mathbf{r}} \underline{\mathbf{r}}^{\top} \mathbf{Q}) \mathbf{v}^- \\ &= (\mathbf{I} - 2(\mathbf{Q}^{-1} \underline{\mathbf{r}})(\mathbf{Q}^{-1} \underline{\mathbf{r}})^{\top}) \mathbf{v}^- \end{aligned} \quad (4.21)$$

because \mathbf{Q} is an orthogonal matrix ($\mathbf{Q}^{-1} = \mathbf{Q}^{\top}$). Since the impact law does not affect the positions, $\mathbf{Q}^{-1} \mathbf{r} \in \mathbb{R}^n$ can be extended to \mathbb{R}^{2n} in $\underline{\mathcal{B}}$ by inserting n zeros on the position indices, *i.e.* the odd ones, yielding $\underline{\mathbf{r}}$. The unit length of $\underline{\mathbf{r}}$ follows from the unit length of \mathbf{r} : $\|\underline{\mathbf{r}}\|^2 = \underline{\mathbf{r}}^{\top} \underline{\mathbf{r}} = (\mathbf{Q}^{-1} \mathbf{r})^{\top} \mathbf{Q}^{-1} \mathbf{r} = \mathbf{r}^{\top} \mathbf{Q} \mathbf{Q}^{-1} \mathbf{r} = \mathbf{r}^{\top} \mathbf{r} = 1$. ■

4.4. Expression of the endomorphism u in $\underline{\mathcal{B}}$ Recall that a k -ipp motion consists in a free flight from $t_0 = 0$ to t_1 , an impact at t_1 , a free flight from t_1 to t_2 , an impact at t_2 , etc. The free-flight durations $\sigma_1, \dots, \sigma_k$, defined as $t_1 - t_0, \dots, t_k - t_{k-1}$, as illustrated in Fig. 4. Now that both the free flight and the impact law are known in $\underline{\mathcal{B}}$, the endomorphism u introduced in Eq. (2.15), mapping a vector of initial conditions to the state of the system after k impacts, can be expressed in $\underline{\mathcal{B}}$:

$$\underline{\mathbf{U}} = \underline{\mathbf{N}} \mathbf{S}(\sigma_k) \underline{\mathbf{N}} \mathbf{S}(\sigma_{k-1}) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \mathbf{S}(\sigma_1). \quad (4.22)$$

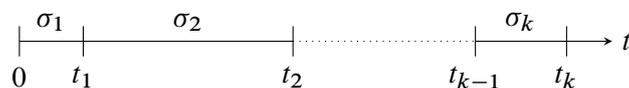


Figure 4: Impact times t_1, \dots, t_k and free flight durations $\sigma_1, \dots, \sigma_k$ with period $T = t_k$.

4.5. Summary of the main notations The main notations and bases used in the remainder are listed in Table 1 where the associated change of basis matrices are

$$\mathbf{P}_{\bar{\mathcal{C}}}^{\mathcal{B}} = \begin{bmatrix} -\mathbf{M}^{-1/2}\mathbf{Q}\mathbf{\Delta}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1/2}\mathbf{Q} \end{bmatrix} \quad (4.23)$$

and $\mathbf{P}_{\underline{\mathcal{B}}}^{\mathcal{B}}$, the matrix associated to permutation

$$\begin{pmatrix} 1 & 2 & 3 & \dots & 2n-1 & 2n \\ 1 & n+1 & 2 & \dots & n & 2n \end{pmatrix}. \quad (4.24)$$

basis	\mathcal{C}	$\bar{\mathcal{C}}$	\mathcal{B}	$\underline{\mathcal{B}}$
cardinality	n	$2n$	$2n$	$2n$
linear combination	\mathbf{x} or $\dot{\mathbf{x}}$	$\bar{\mathbf{x}}$		$\underline{\mathbf{x}}$
gap	$g(\mathbf{x})$	$\bar{g}(\bar{\mathbf{x}})$		
gradient of g	\mathbf{w}	$\bar{\mathbf{w}}$		
normal vector to hyperplane				\mathbf{r}
free flight matrix		$\bar{\mathbf{S}}$		$\underline{\mathbf{S}}$
impact matrix	\mathbf{N}	$\bar{\mathbf{N}}$		$\underline{\mathbf{N}}$
matrix of endomorphism u				$\underline{\mathbf{U}}$

Table 1: Summary of main notations in the different bases; change-of-basis matrices.

5. Explicit parametrization of the maximal invariant subspace of u [proof of Theorem 3.1]

Are now introduced two technical lemmas needed to prove Theorems 3.1 and 3.2.

Lemma 5.1 [Interversion lemma] Let \mathbf{A}, \mathbf{B} be two real matrices of size $p \times q$. The application $\mathbf{x} \mapsto \mathbf{Ax}$ is a linear isomorphism between the kernel of $\mathbf{I}_q - \mathbf{B}^\top \mathbf{A}$ and the kernel of $\mathbf{I}_p - \mathbf{AB}^\top$.

Proof. $\mathbf{A} \mapsto \mathbf{xA}$ is a linear application.

$$(\mathbf{I}_q - \mathbf{B}^\top \mathbf{A})\mathbf{x} = \mathbf{0} \implies (\mathbf{I}_p - \mathbf{AB}^\top)\mathbf{Ax} = \mathbf{Ax} - \mathbf{AB}^\top \mathbf{Ax} = \mathbf{Ax} - \mathbf{Ax} = \mathbf{0} \quad (5.1)$$

so $\mathbf{A} \mapsto \mathbf{xA}$ is an application from $\ker(\mathbf{I}_q - \mathbf{B}^\top \mathbf{A})$ to $\ker(\mathbf{I}_p - \mathbf{AB}^\top)$. It is injective because $(\mathbf{I}_q - \mathbf{B}^\top \mathbf{A})\mathbf{x} = \mathbf{0}$ implies $\mathbf{Ax} = \mathbf{0}$, hence $\mathbf{x} = \mathbf{B}^\top \mathbf{Ax} = \mathbf{0}$. It is also surjective because if $(\mathbf{I}_p - \mathbf{AB}^\top)\mathbf{y} = \mathbf{0}$, then noting $\mathbf{x} = \mathbf{B}^\top \mathbf{y}$ yields $\mathbf{y} = \mathbf{AB}^\top \mathbf{y} = \mathbf{Ax}$ and $(\mathbf{I}_q - \mathbf{B}^\top \mathbf{A})\mathbf{x} = \mathbf{x} - \mathbf{B}^\top \mathbf{AB}^\top \mathbf{y} = \mathbf{x} - \mathbf{B}^\top \mathbf{y} = \mathbf{0}$. ■

Lemma 5.2 If s is an isometry with no non-zero invariant vector,

$$(\text{Id} - s)^{-1} + (\text{Id} - s^*)^{-1} = \text{Id}. \quad (5.2)$$

Proof. s, s^* and Id commute so the fractional notation can be used:

$$(\text{Id} - s)^{-1} = \frac{\text{Id}}{\text{Id} - s} = \frac{\text{Id} - s^*}{(\text{Id} - s)(\text{Id} - s^*)} = \frac{\text{Id} - s^*}{2\text{Id} - (s + s^*)} \quad (5.3)$$

and so

$$(\text{Id} - s)^{-1} + (\text{Id} - s^*)^{-1} = \frac{\text{Id} - s^*}{2\text{Id} - (s + s^*)} + \frac{\text{Id} - s}{2\text{Id} - (s + s^*)} = \text{Id}. \quad (5.4)$$

We can now proceed with the two following propositions, involved in the proof of Theorem 3.1. Two quantities \mathbf{c}_i and \mathbf{l}_i are first introduced:

$$\mathbf{c}_i = \underline{\mathbf{S}}(T - t_i)\underline{\mathbf{r}} \quad \text{and} \quad \mathbf{l}_i = \underline{\mathbf{r}}^\top \underline{\mathbf{S}}(\sigma_i)\underline{\mathbf{N}} \dots \underline{\mathbf{N}}\underline{\mathbf{S}}(\sigma_1). \quad (5.5)$$

Proposition 5.3 The following equality holds:

$$\underline{\mathbf{U}} = \underline{\mathbf{S}}(T) - 2 \sum_{i=1}^k \mathbf{c}_i \mathbf{l}_i \quad (5.6)$$

Proof. Let us introduce

$$\underline{\mathbf{U}}_i = \underline{\mathbf{S}}(\sigma_k + \dots + \sigma_{i+1}) \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_i) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) = \underline{\mathbf{S}}(T - t_i) \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_i) \dots \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1). \quad (5.7)$$

Note that $\underline{\mathbf{U}}_k = \underline{\mathbf{U}}$ as $\underline{\mathbf{S}}(0) = \mathbf{I}$. Replacing the left $\underline{\mathbf{N}}$ with $\mathbf{I} - 2\underline{\mathbf{r}}\underline{\mathbf{r}}^\top$ in Eq. (5.7) yields

$$\underline{\mathbf{U}}_i = \underline{\mathbf{U}}_{i-1} - 2\underline{\mathbf{S}}(T - t_i) \underline{\mathbf{r}}\underline{\mathbf{r}}^\top \underline{\mathbf{S}}(\sigma_i) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) \quad (5.8)$$

with $\underline{\mathbf{U}}_0 = \underline{\mathbf{S}}(T)$, because $\underline{\mathbf{S}}(T - t_i) \underline{\mathbf{S}}(\sigma_i) = \underline{\mathbf{S}}(T - t_i + \sigma_i) = \underline{\mathbf{S}}(T - t_{i-1})$. It follows that

$$2 \sum_{i=1}^k \mathbf{c}_i \mathbf{l}_i = \sum_{i=1}^k (\underline{\mathbf{U}}_{i-1} - \underline{\mathbf{U}}_i) = \underline{\mathbf{U}}_0 - \underline{\mathbf{U}}_k \quad (5.9)$$

hence Eq. (5.6). ■

Proposition 5.4 *The following equality holds:*

$$\mathbf{l}_i + \sum_{j=1}^{i-1} 2\langle \mathbf{c}_i, \mathbf{c}_j \rangle \mathbf{l}_j = \mathbf{c}_i^\top \underline{\mathbf{S}}(T). \quad (5.10)$$

Proof. Set down $\mathbf{h}_{i,j} = \underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_j) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1)$, $\forall j < i$ so that $\mathbf{l}_i = \mathbf{h}_{i,i-1}$. Let us develop the left $\underline{\mathbf{N}}$ using again equality $\underline{\mathbf{N}} = \mathbf{I} - 2\underline{\mathbf{r}}\underline{\mathbf{r}}^\top$:

$$\mathbf{h}_{i,j} = \mathbf{h}_{i,j-1} - 2\underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}}\underline{\mathbf{r}}^\top \underline{\mathbf{S}}(\sigma_j) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) = \mathbf{h}_{i,j-1} - 2(\underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}}) \mathbf{l}_j \quad (5.11)$$

For $j = 1$, this leads to $\mathbf{h}_{i,1} = \underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i - t_1) \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) = \underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i) - 2(\underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i - t_1) \underline{\mathbf{r}}) \mathbf{l}_1$. With $\mathbf{h}_{i,0} = \underline{\mathbf{r}}^\top \underline{\mathbf{S}}(\sigma_i)$ this yields:

$$\sum_{j=1}^{i-1} (\mathbf{h}_{i,j-1} - \mathbf{h}_{i,j}) = \mathbf{h}_{i,0} - \mathbf{h}_{i,i-1} = \sum_{j=1}^{i-1} 2(\underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}}) \mathbf{l}_j \quad (5.12)$$

or

$$\mathbf{l}_i + \sum_{j=1}^{i-1} 2(\underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}}) \mathbf{l}_j = \underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i) = \mathbf{c}_i^\top \underline{\mathbf{S}}(T). \quad (5.13)$$

One conclude by observing that $\underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}} = \underline{\mathbf{r}}^\top \underline{\mathbf{S}}(t_i - T) \underline{\mathbf{S}}(T - t_j) \underline{\mathbf{r}} = \mathbf{c}_i^\top \mathbf{c}_j = \langle \mathbf{c}_i, \mathbf{c}_j \rangle$. ■

We now introduce three matrices which soon will become handy. Let \mathbf{T} denote the lower-triangular matrix k of diagonal elements 1 and whose element (i, j) , $j < i$, is

$$T_{i,j} = 2\langle \mathbf{c}_i, \mathbf{c}_j \rangle. \quad (5.14)$$

Denote by \mathbf{C} the matrix whose k columns are $\mathbf{c}_1, \dots, \mathbf{c}_k$ and \mathbf{L} the matrix whose rows are $\mathbf{l}_1, \dots, \mathbf{l}_k$.

Proposition 5.5 *The following equality holds:*

$$\mathbf{T} + \mathbf{T}^\top = 2\mathbf{C}^\top \mathbf{C}. \quad (5.15)$$

Proof. With \mathbf{C} and \mathbf{L} , equality (5.10) of Prop. 5.4 becomes

$$\mathbf{T}\mathbf{L} = \mathbf{C}^\top \underline{\mathbf{S}}(T). \quad (5.16)$$

With these notations,

$$\sum_{i=1}^k \mathbf{c}_i \mathbf{l}_i = \sum_{i=1}^k \mathbf{C} \mathbf{e}_i^\top \mathbf{e}_i \mathbf{L} = \mathbf{C} \left(\sum_{i=1}^k \mathbf{e}_i^\top \mathbf{e}_i \right) \mathbf{L} = \mathbf{C} \mathbf{I} \mathbf{L} = \mathbf{C} \mathbf{L} \quad (5.17)$$

so, using Proposition 5.3 and because \mathbf{T} is invertible as it is lower-triangular with ones on the diagonal,

$$\underline{\mathbf{U}} = \underline{\mathbf{S}}(T) - 2\mathbf{C}\mathbf{L} = \underline{\mathbf{S}}(T) - 2\mathbf{C}\mathbf{T}^{-1} \mathbf{C}^\top \underline{\mathbf{S}}(T) = (\mathbf{I} - 2\mathbf{C}\mathbf{T}^{-1} \mathbf{C}^\top) \underline{\mathbf{S}}(T). \quad (5.18)$$

Since $\mathbf{c}_i = \underline{\mathbf{S}}(T - t_i) \underline{\mathbf{r}}$ is a unit vector, $2 = 2\langle \mathbf{c}_i, \mathbf{c}_i \rangle$ and so $\mathbf{T} + \mathbf{T}^\top = 2(\langle \mathbf{c}_i, \mathbf{c}_j \rangle)_{i,j} = 2\mathbf{C}^\top \mathbf{C}$. ■

Let's keep rearranging the matrix equations. Let $\mathbf{\Omega} = \mathbf{I} - \underline{\mathbf{S}}(-T)$ and $\mathbf{\Pi} = \mathbf{T} - 2\mathbf{C}^\top \mathbf{C} \mathbf{\Omega}^{-1} \mathbf{C}$. The following proposition proves that $\mathbf{\Pi}$ is the matrix defined in Eq. (3.3).

Proposition 5.6 For $T \notin \{0, 2\pi/\omega_1, \dots, 2\pi/\omega_n\}$, $\Pi_{i,j} = \Upsilon_T(2(t_j - t_i) - T)$, $\forall i < j$ with

$$\Upsilon_T(\sigma) = \sum_{i=1}^n \frac{\sin(\omega_i \sigma/2)}{\sin(\omega_i T/2)} r_{2i}^2. \quad (5.19)$$

Proof. From (5.18),

$$\underline{\mathbf{U}} - \mathbf{I} = (\mathbf{I} - \underline{\mathbf{S}}(-T) - 2\mathbf{C}\mathbf{T}^{-1}\mathbf{C}^\top)\underline{\mathbf{S}}(T). \quad (5.20)$$

Since $T \notin \{0, 2\pi/\omega_1, \dots, 2\pi/\omega_n\}$ then $\mathbf{\Omega} = \mathbf{I} - \underline{\mathbf{S}}(-T) = (\underline{\mathbf{S}}(T) - \mathbf{I})\underline{\mathbf{S}}(-T)$ is invertible. Moreover, factorizing (5.20) by $\mathbf{\Omega}$ reads

$$\underline{\mathbf{U}} - \mathbf{I} = \mathbf{\Omega}(\mathbf{\Omega}^{-1}(\mathbf{I} - \underline{\mathbf{S}}(-T)) - 2\mathbf{\Omega}^{-1}\mathbf{C}\mathbf{T}^{-1}\mathbf{C}^\top)\underline{\mathbf{S}}(T) \quad (5.21)$$

$$= \mathbf{\Omega}(\mathbf{I} - 2\mathbf{\Omega}^{-1}\mathbf{C}\mathbf{T}^{-1}\mathbf{C}^\top)\underline{\mathbf{S}}(T) \quad (5.22)$$

$$= \mathbf{\Omega}(\mathbf{I} - \mathbf{A}\mathbf{B}^\top)\underline{\mathbf{S}}(T) \quad (5.23)$$

with $\mathbf{A} = 2\mathbf{\Omega}^{-1}\mathbf{C}$ a $2n \times k$ matrix and $\mathbf{B}^\top = \mathbf{T}^{-1}\mathbf{C}^\top$ a $k \times 2n$ matrix. Lemma 5.1 proves there is an explicit isomorphism between $\ker(\mathbf{I} - \mathbf{A}\mathbf{B}^\top)$ and $\ker(\mathbf{I}_k - \mathbf{B}^\top \mathbf{A})$. It follows that

$$\mathbf{I}_k - \mathbf{B}^\top \mathbf{A} = \mathbf{I}_k - 2\mathbf{T}^{-1}\mathbf{C}^\top \mathbf{\Omega}^{-1} \mathbf{C} = \mathbf{T}^{-1}(\mathbf{T} - 2\mathbf{C}^\top \mathbf{\Omega}^{-1} \mathbf{C}) = \mathbf{T}^{-1} \mathbf{\Pi} \quad (5.24)$$

but $\mathbf{\Pi}$ is skew-symmetric because

$$\mathbf{\Pi} + \mathbf{\Pi}^\top = \mathbf{T} + \mathbf{T}^\top - 2\mathbf{C}^\top \mathbf{\Omega}^{-1} \mathbf{C} - 2\mathbf{C}^\top \mathbf{\Omega}^{-\top} \mathbf{C} \quad (5.25)$$

$$= 2\mathbf{C}^\top \mathbf{C} - 2\mathbf{C}^\top (\mathbf{\Omega}^{-1} + \mathbf{\Omega}^{-\top}) \mathbf{C} \quad \text{using Eq. (5.15)} \quad (5.26)$$

$$= \mathbf{0} \quad \text{from Lemma 5.2.} \quad (5.27)$$

By definition, \mathbf{T} is lower-triangular so

$$\forall i < j, \quad \Pi_{i,j} = -2\mathbf{c}_i^\top \mathbf{\Omega}^{-1} \mathbf{c}_j = -2\mathbf{r}^\top \underline{\mathbf{S}}(t_i - T)(\mathbf{I} - \underline{\mathbf{S}}(-T))^{-1} \underline{\mathbf{S}}(T - t_j) \mathbf{r} \quad (5.28)$$

and since $t \mapsto \underline{\mathbf{S}}(t)$ is a group homomorphism,

$$\Pi_{i,j} = -2\mathbf{r}^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(T/2 - (t_j - t_i)). \quad (5.29)$$

Now, considering $\sigma \mapsto \Upsilon_T(\sigma) = 2\mathbf{r}^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(\sigma/2) \mathbf{r}$ and since $(\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(\sigma/2)$ is a block-diagonal matrix of diagonal blocks

$$\begin{aligned} & \begin{bmatrix} 0 & -2\sin(\omega_i \sigma/2) \\ 2\sin(\omega_i \sigma/2) & \end{bmatrix}^{-1} \begin{bmatrix} \cos(\omega_i \sigma/2) & -\sin(\omega_i \sigma/2) \\ \sin(\omega_i \sigma/2) & \cos(\omega_i \sigma/2) \end{bmatrix} \\ &= \frac{1}{2\sin(\omega_i T/2)} \begin{bmatrix} \sin(\omega_i \sigma/2) & \cos(\omega_i \sigma/2) \\ -\cos(\omega_i \sigma/2) & \sin(\omega_i \sigma/2) \end{bmatrix} \end{aligned} \quad (5.30)$$

it comes that

$$\Upsilon_T(\sigma) = \sum_{i=1}^n \frac{\sin(\omega_i \sigma/2)}{\sin(\omega_i T/2)} r_{2i}^2 \quad (5.31)$$

because the odd components of \mathbf{r} are zero. Finally,

$$\forall i < j, \quad \Pi_{i,j} = -\Upsilon_T(T - 2(t_j - t_i)) = \Upsilon_T(2(t_j - t_i) - T) \quad (5.32)$$

since $\sigma \mapsto \Upsilon_T(\sigma)$ is an odd function. ■

Remark 5.1. The physical meaning of $T \notin \{0, 2\pi/\omega_1, \dots, 2\pi/\omega_n\}$ is that T cannot be a period of a linear mode. It does not exclude cases where T is in the punctured neighbourhood of $\{0, 2\pi/\omega_1, \dots, 2\pi/\omega_n\}$. As such, this condition is not restrictive and does not prevent the investigation of the resonant behaviour.

Remark 5.2. The quantity $2(t_j - t_i) - T$ for $i < j$ is the signed sum of $\sigma_1, \dots, \sigma_k$ where the $j - i$ variables σ_q for $i < q \leq j$ have the sign $-$ while the $k - j + i$ other variables have the sign $+$.

We now prove Theorem 3.1.

Proof. The kernel of $\mathbf{\Pi} = \mathbf{T} - 2\mathbf{C}^\top \mathbf{\Omega}^{-1} \mathbf{C}$ is the kernel of $\mathbf{T}^{-1} \mathbf{\Pi} = \mathbf{I}_k - \mathbf{B}^\top \mathbf{A}$ where $\mathbf{A} = 2\mathbf{\Omega}^{-1} \mathbf{C}$ and $\mathbf{B}^\top = \mathbf{T}^{-\top} \mathbf{C}$. From Lemma 5.1, $\lambda \mapsto \mathbf{A} \lambda$ is an isomorphism between $\ker \mathbf{\Pi}$ and $\ker(\mathbf{I}_n - \mathbf{A} \mathbf{B}^\top)$. Using Eq. (5.23), the isomorphism from $\ker \mathbf{\Pi}$ to $\ker(\mathbf{U} - \mathbf{I})$ is $\lambda \mapsto \mathbf{S}(T)^{-1} \mathbf{A} \lambda = 2\mathbf{S}(-T) \mathbf{\Omega}^{-1} \mathbf{C} \lambda$ and its inverse reads $\underline{\mathbf{x}} \mapsto \mathbf{B}^\top \mathbf{S}(T) \underline{\mathbf{z}} = \mathbf{T}^{-\top} \mathbf{C} \mathbf{S}(T) \underline{\mathbf{x}} = \mathbf{L} \underline{\mathbf{x}}$ because $\mathbf{T} \mathbf{L} = \mathbf{C}^\top \mathbf{S}(T)$ (cf. Eq. (5.16)) and so $\mathbf{L} = \mathbf{T}^{-\top} \mathbf{C} \mathbf{S}(T)$. Accordingly, $\lambda \in \mathbb{R}^k$ is such that $\mathbf{\Pi} \lambda = \mathbf{0}$ iff there is a $\underline{\mathbf{x}} \in \mathbb{R}^{2n}$ such that $\mathbf{U} \underline{\mathbf{x}} = \underline{\mathbf{x}}$

$$\underline{\mathbf{x}} = 2\mathbf{S}(-T)(\mathbf{I} - \mathbf{S}(-T))^{-1} \mathbf{C} \lambda = 2(\mathbf{S}(T) - \mathbf{I})^{-1} \sum_{j=1}^k \lambda_j \mathbf{S}(T - t_j) \underline{\mathbf{r}} \quad (5.33)$$

and then $\forall j, \lambda_j = \mathbf{1}_j \underline{\mathbf{x}}$. Then, the vector expressed as $\underline{\mathbf{x}}$ in $\underline{\mathcal{B}}$ can be written in the canonical basis $\underline{\mathcal{C}}$ using $\bar{\mathbf{x}} = \mathbf{P}_{\underline{\mathcal{C}}}^{\underline{\mathcal{B}}} \mathbf{P}_{\underline{\mathcal{B}}}^{\underline{\mathcal{C}}} \underline{\mathbf{x}}$ where $\mathbf{P}_{\underline{\mathcal{C}}}^{\underline{\mathcal{B}}}$ is given by Proposition 4.3, and this yields expression (3.7) of φ and proves Theorem 3.1. Proposition 5.6 provides a simple expression for the elements of $\mathbf{\Pi}$. ■

6. Gap closure at impact times [proof of Theorem 3.2] The idea of the proof is to write the conditions NC2 as a linear system and introduce the isomorphism between the kernels of $\mathbf{\Pi}$ and $\mathbf{U} - \mathbf{I}$. Introducing the vector $\bar{\mathbf{w}} = [\mathbf{w}^\top \ 0 \ \dots \ 0]^\top \in \mathbb{R}^{2n}$ and using (2.8), NC2 (2.16b) becomes

$$\forall i \in \llbracket 1, k \rrbracket, \quad \bar{\mathbf{w}}^\top \bar{\mathbf{x}}(t_i) = -\bar{g}(\mathbf{0}). \quad (6.1)$$

Inserting the expression of $\bar{\mathbf{x}}(t_i)$ given by Eq. (2.15) reads

$$\forall i \in \llbracket 1, k \rrbracket, \quad \bar{\mathbf{w}}^\top \bar{\mathbf{N}} \mathbf{S}(\sigma_i) \bar{\mathbf{N}} \mathbf{S}(\sigma_{i-1}) \bar{\mathbf{N}} \dots \bar{\mathbf{N}} \mathbf{S}(\sigma_1) \bar{\mathbf{x}}_0 = -\bar{g}(\mathbf{0}). \quad (6.2)$$

Noting $\underline{\mathbf{x}}_0 = \mathbf{P}_{\underline{\mathcal{B}}}^{\underline{\mathcal{C}}} \bar{\mathbf{x}}_0$ and $\underline{\mathbf{w}} = \mathbf{P}_{\underline{\mathcal{C}}}^{\underline{\mathcal{B}}} \bar{\mathbf{w}}$, Eq. (6.2) can be rewritten as

$$\forall i \in \llbracket 1, k \rrbracket, \quad \underline{\mathbf{w}}^\top \underline{\mathbf{N}} \mathbf{S}(\sigma_i) \underline{\mathbf{N}} \mathbf{S}(\sigma_{i-1}) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \mathbf{S}(\sigma_1) \underline{\mathbf{x}}_0 = -\bar{g}(\mathbf{0}). \quad (6.3)$$

Note that the even components of $\underline{\mathbf{w}}$ are zero because the last n components of $\bar{\mathbf{w}}$ are zero: NC2 remains a geometric condition on the *positions*.

Setting

$$\forall i \in \llbracket 1, k \rrbracket, \quad \mathbf{f}_i = \underline{\mathbf{w}}^\top \underline{\mathbf{N}} \mathbf{S}(\sigma_i) \underline{\mathbf{N}} \mathbf{S}(\sigma_{i-1}) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \mathbf{S}(\sigma_1) \quad \text{and} \quad \mathbf{f}_0 = \underline{\mathbf{w}}^\top \quad (6.4)$$

$$\forall j \leq i, \quad \mathbf{k}_{i,j} = \underline{\mathbf{w}}^\top \mathbf{S}(t_i - t_j) \underline{\mathbf{N}} \mathbf{S}(\sigma_j) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \mathbf{S}(\sigma_1) \quad (6.5)$$

and replacing the left $\underline{\mathbf{N}}$ with $\mathbf{I} - 2\underline{\mathbf{r}} \underline{\mathbf{r}}^\top$ in Eq. (6.5) yields

$$\mathbf{k}_{i,j} = \mathbf{k}_{i,j-1} - 2\underline{\mathbf{w}}^\top \mathbf{S}(t_i - t_j) \underline{\mathbf{r}} \underline{\mathbf{r}}^\top \mathbf{S}(t_i - t_j) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \mathbf{S}(\sigma_1), \quad (6.6)$$

$$= \mathbf{k}_{i,j-1} - 2\underline{\mathbf{w}}^\top \mathbf{S}(t_i - t_j) \underline{\mathbf{r}} \mathbf{l}_j. \quad (6.7)$$

With $j = 1$, this decomposition becomes $\mathbf{k}_{i,1} = \underline{\mathbf{w}}^\top \mathbf{S}(t_i - t_1) \underline{\mathbf{N}} \mathbf{S}(\sigma_1) = \underline{\mathbf{w}}^\top \mathbf{S}(t_i) - 2\underline{\mathbf{w}}^\top \mathbf{S}(t_i - t_1) \underline{\mathbf{r}} \mathbf{l}_1$ such that with $\mathbf{k}_{i,0} = \underline{\mathbf{w}}^\top \mathbf{S}(t_i)$:

$$\sum_{j=1}^i (\mathbf{k}_{i,j-1} - \mathbf{k}_{i,j}) = \mathbf{k}_{i,0} - \mathbf{k}_{i,i} = \sum_{j=1}^i 2\underline{\mathbf{w}}^\top \mathbf{S}(t_i - t_j) \underline{\mathbf{r}} \mathbf{l}_j \quad (6.8)$$

and since $\mathbf{k}_{i,i} = \mathbf{f}_i$,

$$\mathbf{f}_i = \underline{\mathbf{w}}^\top \mathbf{S}(t_i) - 2 \sum_{j=1}^i \underline{\mathbf{w}}^\top \mathbf{S}(t_i - t_j) \underline{\mathbf{r}} \mathbf{l}_j. \quad (6.9)$$

If $\underline{\mathbf{x}}_0$ is invariant by u , $\underline{\mathbf{x}}_0$ can be expressed as a function of $\lambda_1, \dots, \lambda_k$, see Eq. (5.33), and we are now showing that the gap closure at t_1, \dots, t_k can be explicitly expressed as equations on $\lambda_1, \dots, \lambda_k$.

By definition of \mathbf{f}_i , Equation (6.3) can be written as

$$\forall i \in \llbracket 1, k \rrbracket, \quad \mathbf{f}_i \underline{\mathbf{x}}_0 = -g_0 \quad (6.10)$$

or, using (6.9):

$$\forall i \in \llbracket 1, k \rrbracket, \quad \underline{\mathbf{w}}^\top \underline{\mathbf{S}}(t_i) \underline{\mathbf{x}}_0 - 2 \sum_{j=1}^i \underline{\mathbf{w}}^\top \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}} \lambda_j \underline{\mathbf{x}}_0 = -g_0. \quad (6.11)$$

These equalities can be expanded by inserting the expression of $\underline{\mathbf{x}}_0$ in (5.33):

$$2 \sum_{j=1}^k \underline{\mathbf{w}}^\top \underline{\mathbf{S}}(t_i) (\underline{\mathbf{S}}(T) - \mathbf{I})^{-1} \underline{\mathbf{S}}(T - t_j) \underline{\mathbf{r}} \lambda_j - 2 \sum_{j=1}^i \underline{\mathbf{w}}^\top \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}} \lambda_j = -g_0 \quad (6.12)$$

or

$$2 \sum_{j=1}^k \underline{\mathbf{w}}^\top (\mathbf{I} - \underline{\mathbf{S}}(T))^{-1} \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}} \lambda_j - 2 \sum_{j=1}^i \underline{\mathbf{w}}^\top \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}} \lambda_j = -g_0 \quad (6.13)$$

that is

$$2 \sum_{j=i+1}^k \underline{\mathbf{w}}^\top (\mathbf{I} - \underline{\mathbf{S}}(T))^{-1} \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}} \lambda_j + 2 \sum_{j=1}^i \underline{\mathbf{w}}^\top ((\mathbf{I} - \underline{\mathbf{S}}(-T))^{-1} - \mathbf{I}) \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}} \lambda_j = -g_0. \quad (6.14)$$

This can be simplified using Lemma 5.2 which proves that $(\mathbf{I} - \underline{\mathbf{S}}(-T))^{-1} + (\mathbf{I} - \underline{\mathbf{S}}(T))^{-1} = \mathbf{I}$ and so $\forall i \in \llbracket 1, k \rrbracket$:

$$2 \sum_{j=i+1}^k \underline{\mathbf{w}}^\top (\mathbf{I} - \underline{\mathbf{S}}(T))^{-1} \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}} \lambda_j + 2 \sum_{j=1}^i \underline{\mathbf{w}}^\top (\underline{\mathbf{S}}(T) - \mathbf{I})^{-1} \underline{\mathbf{S}}(t_i - t_j) \underline{\mathbf{r}} \lambda_j = -g_0 \quad (6.15)$$

or, because $t \mapsto \underline{\mathbf{S}}(t)$ is a group homomorphism,

$$\begin{aligned} & 2 \sum_{j=1}^i \underline{\mathbf{w}}^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(t_i - t_j - T/2) \underline{\mathbf{r}} \lambda_j + \\ & 2 \sum_{j=i+1}^i \underline{\mathbf{w}}^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(t_i - t_j + T/2) \underline{\mathbf{r}} \lambda_j = -g_0. \end{aligned} \quad (6.16)$$

Linear system (6.16) can be written in a matrix form $\underline{\Sigma} \underline{\lambda} = -g_0 \mathbf{j}$ with $\mathbf{j} = [1 \ \dots \ 1]^\top$ as

$$\Sigma_{i,j} = \begin{cases} \underline{\mathbf{w}}^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(t_i - t_j + T/2) \underline{\mathbf{r}} & \text{if } i < j \\ \underline{\mathbf{w}}^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(t_i - t_j - T/2) \underline{\mathbf{r}} & \text{if } i \geq j. \end{cases} \quad (6.17)$$

Proposition 6.1 *The matrix $\underline{\Sigma}$ defined in Eq. (6.17) is symmetric.*

Proof. From Eq. (5.30),

$$(\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(\sigma) = \mathbf{diag} \left(\frac{1}{2 \sin(\omega_i T/2)} \begin{bmatrix} \sin(\omega_i \sigma) & \cos(\omega_i \sigma) \\ -\cos(\omega_i \sigma) & \sin(\omega_i \sigma) \end{bmatrix} \right) \quad (6.18)$$

so $(\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} (\underline{\mathbf{S}}(\sigma) - \underline{\mathbf{S}}(-\sigma))$ is a diagonal matrix, as cosine is an even function. Yet, the odd components of $\underline{\mathbf{r}}$ are zero, so the product of a diagonal matrix and $\underline{\mathbf{r}}$ has also zero odd components. Recalling that the even components of $\underline{\mathbf{w}}$ are zero yields $\underline{\mathbf{w}}^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} (\underline{\mathbf{S}}(\sigma) - \underline{\mathbf{S}}(-\sigma)) \underline{\mathbf{r}} = 0$. Taking $\sigma = t_i - t_j + T/2$ for $i < j$ proves the equality $\Sigma_{i,j} = \Sigma_{j,i}$. ■

Accordingly, linear system (6.10) now reads as $\underline{\Sigma} \underline{\lambda} = -g_0 \mathbf{j}$ where $\lambda_j = \mathbf{I}_j \underline{\mathbf{x}}_0$ which concludes the proof of Theorem 3.2.

7. Expression of the modal positions and velocities [proof of Theorem 3.4] In this section, the positions and velocities corresponding to a solution $\bar{\mathbf{x}}_0$ in the canonical basis $\bar{\mathcal{C}}$ for impact times t_1, \dots, t_k are expressed as functions of time. The principle is to use the advantageous properties of matrix $\underline{\mathbf{S}}$ and the simple expression of $\underline{\mathbf{N}}$, giving its name to the “effective basis”, and then retrieve the corresponding generalised coordinates $\bar{\mathbf{x}}$ in the canonical basis.

7.1. Explicit relationships between bases If \vec{v} is a vector whose $\bar{\mathcal{C}}$ representation is $[\mathbf{x}^\top \mathbf{y}^\top]^\top$, whose \mathcal{B} -representation is $[\mathbf{u}^\top \mathbf{v}^\top]^\top$ and whose $\underline{\mathcal{B}}$ -representation is $\underline{\mathbf{x}}$, then using Proposition 4.3:

$$\begin{cases} \mathbf{x} = -\mathbf{M}^{-1/2} \mathbf{Q} \mathbf{\Delta}^{-1} \mathbf{u} \\ \mathbf{y} = \mathbf{M}^{-1/2} \mathbf{Q} \mathbf{v} \end{cases}$$

or

$$\forall p \in \llbracket 1, n \rrbracket, \quad \begin{cases} x_p = -\sum_{i,j=1}^n (\mathbf{M}^{-1/2})_{p,i} Q_{i,j} \omega_j^{-1} u_j \\ y_p = \sum_{i,j=1}^n (\mathbf{M}^{-1/2})_{p,i} Q_{i,j} v_j \end{cases} \quad (7.1)$$

where x_p is the p -th component of \mathbf{x} , y_p , the p -th component of \mathbf{y} , and $Q_{i,j}$, entry (i, j) of \mathbf{Q} . Therefore, the relations between x_p , y_p and $\underline{\mathbf{x}}$ can be written $\forall p \in \llbracket 1, n \rrbracket$ as:

$$\begin{cases} x_p = \langle \underline{\mathbf{w}}_p, \underline{\mathbf{x}} \rangle \\ y_p = \langle \underline{\mathbf{r}}_p, \underline{\mathbf{x}} \rangle \end{cases} \quad (7.2)$$

with

$$\underline{\mathbf{w}}_p := - \begin{bmatrix} \sum_{i=1}^n (\mathbf{M}^{-1/2})_{p,i} Q_{i,1} / \omega_1 \\ 0 \\ \vdots \\ \sum_{i=1}^n (\mathbf{M}^{-1/2})_{p,i} Q_{i,n} / \omega_n \\ 0 \end{bmatrix}, \quad \underline{\mathbf{r}}_p := \begin{bmatrix} 0 \\ \sum_{i=1}^n (\mathbf{M}^{-1/2})_{p,i} Q_{i,1} \\ \vdots \\ 0 \\ \sum_{i=1}^n (\mathbf{M}^{-1/2})_{p,i} Q_{i,n} \end{bmatrix}. \quad (7.3)$$

Remark 7.1. It can be checked that $\underline{\mathbf{w}}, \bar{\mathbf{w}}$ and $\underline{\mathbf{w}}_1, \dots, \underline{\mathbf{w}}_p$ are related by the expression

$$\underline{\mathbf{w}} = \mathbf{P}_{\underline{\mathcal{B}}}^{\underline{\mathcal{B}}} [\underline{\mathbf{w}}_1, \dots, \underline{\mathbf{w}}_p, \mathbf{0}, \dots, \mathbf{0}] \bar{\mathbf{w}}. \quad (7.4)$$

7.2. Explicit expression of the positions and velocities We are now going to derive the expressions of the n positions x_p and n velocities y_p as functions of the impact times t_1, \dots, t_k where k is given. By definition of $\underline{\mathbf{S}}, \underline{\mathbf{N}}$ and using Eq. (7.2), $\forall p \in \llbracket 1, n \rrbracket$:

$$\forall \tau \in [0, \sigma_1], \quad x_p(\tau) = \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau) \underline{\mathbf{x}}_0 \quad (7.5a)$$

$$\forall \tau \in [0, \sigma_2], \quad x_p(\tau + t_1) = \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau) \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) \underline{\mathbf{x}}_0 \quad (7.5b)$$

$$\vdots$$

$$\forall \tau \in [0, \sigma_k], \quad x_p(\tau + t_{k-1}) = \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau) \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_{k-1}) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) \underline{\mathbf{x}}_0 \quad (7.5c)$$

Writing $s_{i,j} = \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{N}} \underline{\mathbf{S}}(t_j) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) \underline{\mathbf{x}}_0$ so that $x_p(\tau + t_i) = s_{i,i}$ and replacing the left $\underline{\mathbf{N}}$ with $\mathbf{I} - 2\underline{\mathbf{r}}\underline{\mathbf{r}}^\top$ yields:

$$s_{i,j} = s_{i,j-1} - 2\underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}}\underline{\mathbf{r}}^\top \underline{\mathbf{S}}(\sigma_j) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) \underline{\mathbf{x}}_0 \quad (7.6a)$$

$$= s_{i,j-1} - 2\underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} \mathbf{l}_j \underline{\mathbf{x}}_0. \quad (7.6b)$$

In particular for $j = 1$, it comes that:

$$s_{i,1} = \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i - t_1) \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) \underline{\mathbf{x}}_0 \quad (7.7a)$$

$$= \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i) \underline{\mathbf{x}}_0 - 2\underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i - t_1) \underline{\mathbf{r}}\underline{\mathbf{r}}^\top \underline{\mathbf{S}}(\sigma_1) \underline{\mathbf{x}}_0 \quad (7.7b)$$

$$= \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i) \underline{\mathbf{x}}_0 - 2\underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i - t_1) \underline{\mathbf{r}} \mathbf{l}_1 \underline{\mathbf{x}}_0. \quad (7.7c)$$

With $s_{i,0} = \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i) \underline{\mathbf{x}}_0$:

$$\sum_{j=1}^i (s_{i,j-1} - s_{i,j}) = s_{i,0} - s_{i,i} = 2 \sum_{j=1}^i \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} \mathbf{l}_j \underline{\mathbf{x}}_0 \quad (7.8)$$

so that

$$\forall t \in [0, \sigma_{i+1}], \quad x_p(\tau + t_i) = \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i) \underline{\mathbf{x}}_0 - 2 \sum_{j=1}^i \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} \mathbf{l}_j \underline{\mathbf{x}}_0 \quad (7.9)$$

and replacing the left $\underline{\mathbf{x}}_0$ with its expression (5.33) and using $\mathbf{l}_j \underline{\mathbf{x}}_0 = \lambda_j$:

$$x_p(\tau + t_i) = 2 \sum_{j=1}^k \lambda_j \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i) (\underline{\mathbf{S}}(T) - \mathbf{I})^{-1} \underline{\mathbf{S}}(T - t_j) \underline{\mathbf{r}} - 2 \sum_{j=1}^i \lambda_j \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} \quad (7.10)$$

$$= 2 \sum_{j=1}^k \lambda_j \underline{\mathbf{w}}_p^\top (\mathbf{I} - \underline{\mathbf{S}}(-T))^{-1} \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} - 2 \sum_{j=1}^i \lambda_j \underline{\mathbf{w}}_p^\top \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} \quad (7.11)$$

because $\underline{\mathbf{S}}(\tau + t_i) (\underline{\mathbf{S}}(T) - \mathbf{I})^{-1} \underline{\mathbf{S}}(T - t_j) = (\underline{\mathbf{S}}(0) - \underline{\mathbf{S}}(-T))^{-1} \underline{\mathbf{S}}(\tau + t_i - t_j)$. The first sum can be decomposed as:

$$\begin{aligned} x_p(\tau + t_i) &= 2 \sum_{j=i+1}^k \lambda_j \underline{\mathbf{w}}_p^\top (\mathbf{I} - \underline{\mathbf{S}}(-T))^{-1} \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} \\ &\quad - 2 \sum_{j=1}^i \lambda_j \underline{\mathbf{w}}_p^\top (\mathbf{I} - (\mathbf{I} - \underline{\mathbf{S}}(-T))^{-1}) \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} \end{aligned} \quad (7.12)$$

and Lemma 5.2 induces:

$$\begin{aligned} x_p(\tau + t_i) &= 2 \sum_{j=i+1}^k \lambda_j \underline{\mathbf{w}}_p^\top (\underline{\mathbf{S}}(T) - \mathbf{I})^{-1} \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} \\ &\quad + 2 \sum_{j=1}^i \lambda_j \underline{\mathbf{w}}_p^\top (\mathbf{I} - \underline{\mathbf{S}}(-T))^{-1} \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} \end{aligned} \quad (7.13)$$

so that finally, $\forall \tau \in [0, \sigma_{i+1}]$:

$$\begin{aligned} x_p(\tau + t_i) &= 2 \sum_{j=i+1}^k \lambda_j \underline{\mathbf{w}}_p^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(\tau + t_i - t_j) \underline{\mathbf{r}} \\ &\quad + 2 \sum_{j=1}^i \lambda_j \underline{\mathbf{w}}_p^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(\tau + t_i - t_j + T/2) \underline{\mathbf{r}}. \end{aligned} \quad (7.14)$$

Setting

$$\phi_{p,T}(\sigma) = 2 \underline{\mathbf{w}}_p^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(\sigma/2) \underline{\mathbf{r}} \quad (7.15)$$

Eq. (7.14) can be recast into

$$x_p(\tau + t_i) = \sum_{j=1}^i \lambda_j \phi_{p,T}(2(\tau + t_i - t_j) - T) + \sum_{j=i+1}^k \lambda_j \phi_{p,T}(2(\tau + t_i - t_j) + T) \quad (7.16)$$

Similarly, the velocity of the p -th mass can be expressed in terms of

$$\psi_{p,T}(\sigma) = 2 \frac{d\phi_{p,T}}{dt}(\sigma) \quad (7.17)$$

by differentiating Eq. (7.16) with respect to τ : $\forall \tau \in [0, \sigma_{i+1}]$,

$$y(\tau + t_i) = \sum_{j=1}^i \lambda_j \psi_{p,T}(2(\tau + t_i - t_j) - T) + \sum_{j=i+1}^k \lambda_j \psi_{p,T}(2(\tau + t_i - t_j) + T) \quad (7.18)$$

This concludes the proof of Theorem 3.4.

8. Dimension of the solution space and non-smooth modes [proof of Theorem 3.3] We now prove Theorem 3.3 which justifies the notion of non-smooth modes as two-dimensional surfaces in the phase space. It implies the existence of continua of potential solutions.

Proof. Assume $g_0 \neq 0$. Given a generic point $s \in \mathcal{S}$, $\Sigma(s)$ is invertible and then $\lambda = -g_0 \Sigma^{-1}(s)\mathbf{j}$. So in the neighbourhood of s , \mathcal{S} is the set of (t_1, \dots, t_k) such that $\Pi(t_1, \dots, t_k)\Sigma^{-1}(t_1, \dots, t_k)\mathbf{j} = \mathbf{0}$. This gives *a priori* k equations $\sum_{j=1}^k \Pi_{i,j}\lambda_j = 0$ for $i \in \llbracket 1, k \rrbracket$. But $\lambda \neq \mathbf{0}$ so there is an index i_0 such that $\lambda_{i_0} \neq 0$. And Π is skew-symmetric so

$$0 = \lambda^\top \Pi \lambda = \sum_{i=1}^k \lambda_i \sum_{j=1}^k \Pi_{i,j} \lambda_j = \lambda_{i_0} \sum_{j=1}^k \Pi_{i_0,j} \lambda_j + \sum_{i \neq i_0} \lambda_i \sum_{j=1}^k \Pi_{i,j} \lambda_j. \quad (8.1)$$

Hence, the nullity of the $k-1$ rows $i \neq i_0$ of $\Pi \Sigma^{-1} \mathbf{j}$ implies the nullity of row i_0 , which proves that, in the neighbourhood of s , \mathcal{S} is the set of solutions of $k-1$ equations in the k unknowns t_1, \dots, t_k . These $k-1$ equations can be linearised around s which yields $k-1$ linear equations $E_i = 0$ with $i \neq i_0$ in t_1, \dots, t_k . In the following, we assume w.l.o.g. that $i_0 = k$. The Jacobian matrix of $(t_1, \dots, t_k) \mapsto (E_1, \dots, E_{k-1})$ is generically of rank k , because otherwise the k submatrices of dimension $(k-1) \times (k-1)$ should be singular, corresponding to k additional equations. Hence, generically, there exists a linear form F such that (E_1, \dots, E_{k-1}, F) is of rank k , hence the Jacobian matrix of $(t_1, \dots, t_k) \mapsto (E_1, \dots, E_{k-1}, F)$ is invertible. The inverse function theorem implies that (E_1, \dots, E_{k-1}, F) yields locally a system of curvilinear coordinates and \mathcal{S} is the set of points whose first $k-1$ coordinates are zero, corresponding to curves. This ends the proof for the case $g_0 \neq 0$.

If $g_0 = 0$, $\Sigma \lambda = \mathbf{0}$ with $\lambda \neq \mathbf{0}$ so necessarily Σ is singular. This gives one equation $\det(\Sigma(t_1, \dots, t_k)) = 0$ and when this condition is satisfied, Σ is generically of rank $k-1$. Its adjunct matrix is of rank 1 and the kernel of Σ is spanned by one column \mathbf{c} of the transpose of its adjunct matrix. The other condition to satisfy is $\Pi \mathbf{c} = \mathbf{0}$ which generically yields, as previously, $k-1$ independent equations. A total of k equations to be solved in k unknowns is available. There is generically a countable set of solutions. ■

The subgeneric case where the sequence (σ_i) is invariant by a transformation $n \mapsto n - a$ for some $a \in \mathbb{N}$ yields $k-1$ linearly dependent equations $E_i = 0$, $i = 1, \dots, k-1$. However, from Proposition 9.6, the solution space is still of dimension 1, provided $g_0 \neq 0$.

We now prove Corollary 3.1 concerning admissible solutions, which is a direct consequence of Theorem 3.3. It proves the existence of non-smooth modes *as soon as an admissible solution exists*, in the generic case.

Proof. Admissible solutions are potential solutions satisfying NC3 so if an admissible solutions has impact times $s \in \mathbb{R}^k$, then $s \in \mathcal{S}$. If $g_0 \neq 0$, Theorem 3.3 implies that a generic s lies on a curve of impact times corresponding to some potential solutions. Since NC3 is satisfied for $\bar{\mathbf{x}}$, it is satisfied in the neighbourhood of s by continuity of $t \mapsto g(\mathbf{x}(t))$ hence the curve of potential solutions is also locally a curve of admissible solutions. If $g_0 = 0$, from Theorem 3.3 \mathcal{S} forms, generically, a set of isolated points, hence s is an isolated point of \mathbb{R}^k . ■

9. Symmetries in the modal orbits Theorems 3.1 and 3.2 state that for a given k , potential solutions $\bar{\mathbf{x}}_0 \in \mathbb{R}^{2n}$ can be sought as the images by φ of vectors $\lambda \in \mathbb{R}^k$ solutions of

$$\begin{cases} \Pi(t_1, \dots, t_k)\lambda = \mathbf{0} \\ \Sigma(t_1, \dots, t_k)\lambda = -g_0\mathbf{j}. \end{cases} \quad (9.1a) \quad (9.1b)$$

When $g_0 \neq 0$, non-trivial solutions may exist only if $\Sigma(t_1, \dots, t_k)$ is invertible and then the two vectorial conditions reduce to $\Pi(t_1, \dots, t_k)\Sigma^{-1}(t_1, \dots, t_k)\mathbf{j} = \mathbf{0}$ which yields *a priori* $k-1$ equations in the k unknowns t_1, \dots, t_k , and then a potential solution for t_1, \dots, t_k is given by $\bar{\mathbf{x}}_0 = \varphi(-g_0 \Sigma^{-1} \mathbf{j})$.

However, some potential solutions $\bar{\mathbf{x}}_0$ correspond to free-flight durations $\sigma_i = t_i - t_{i-1}$ which are not all independent. For example, Fig. 5 illustrates the position of a contacting node of a 7-ipp solution for which the sequence of free-flight durations $\dots, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \dots$ is of the form $\dots, c, b, a, b, c, d, d, \dots$. For more detailed about Fig. 5, refer to subsection 10.4 (Fig. 5 is the same as Fig. 16③). The sequence (σ_i) is invariant, modulo 7, by the transformation $n \mapsto 6 - n$, as shown in Tab. 2. The position features two axes of symmetry corresponding to the symmetry in the sequence (σ_i) .

The current section shows in general how the symmetry in the sequence is reflected by symmetries in the motion.

n	1	2	3	4	5	6	7
σ_n	c	b	a	b	c	d	d
$6-n$	5	4	3	2	1	7	6
σ_{6-n}	c	b	a	b	c	d	d

Table 2: Invariance of the sequence $\dots, c, b, a, b, c, d, d, \dots$ by the transformation $n \mapsto 6-n$.

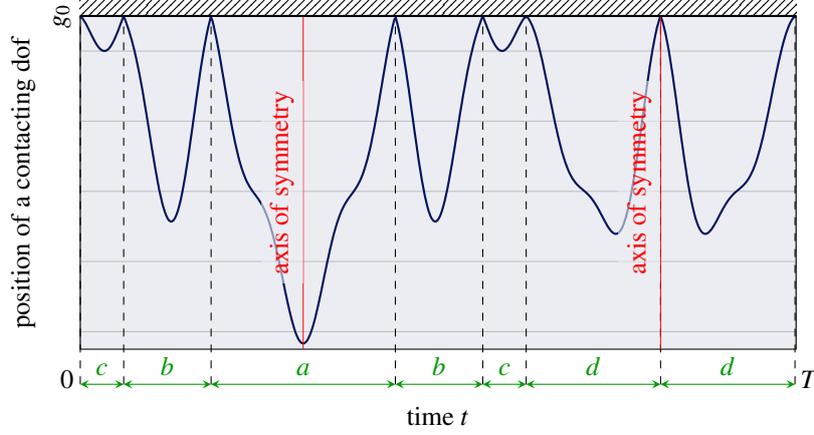


Figure 5: Example of motions with symmetries of the free-flight durations sequence.

9.1. Symmetry of a k -periodic sequence

Let $(\sigma_n)_{n \in \mathbb{Z}}$ denote a sequence of period $k \in \mathbb{N}^*$.

Proposition 9.1 *If (σ_n) is invariant by the transformation $n \mapsto a-n$ for some $a \in \mathbb{Z}$, then a is unique modulo k .*

Proof. The sequence (σ_n) , which is invariant by the transformation $n \mapsto a-n$, is also invariant by $n \mapsto b-n$ iff it is invariant by $n \mapsto b-a+n$, i.e. iff $b-a$ is a multiple of k . Hence if such an a exists, it is unique modulo k . ■

The transformation $n \mapsto a-n$ is involutory so its cycle decomposition only involves transpositions. The following proposition lists the number of invariant elements of $n \mapsto a-n$, which explains several properties proven later.

Proposition 9.2 [Number of invariant elements] *The equation $a-i=i$ of unknown i has:*

- exactly one solution modulo k if k is odd;
- no solution if k is even and a is odd;
- exactly two solutions modulo k if both k and a are even.

Proof. If $k = 2r - 1$, in $\mathbb{Z}/k\mathbb{Z}$, $2r = 1$ implies that, the equivalence class $[2]$ is invertible, of inverse $[r]$. Hence $a = 2i$ iff $ra = 2ri = i$ which gives exactly one solution modulo k . If $k = 2r$, $a = 2i$ has no solution if a is odd. If a is even, $i = a/2$ modulo r so there are exactly two solutions: $a/2$ and $r + a/2$. ■

9.2. Extension of the impact times sequence to \mathbb{Z}

In order to properly explore symmetries, the sequence (t_i) is extended from $i \in \mathbb{N}$ to $i \in \mathbb{Z}$. The sequence of impact times $(t_i)_{i \in \mathbb{N}}$ introduced in subsection 2.2 is such that $t_0 = 0$ and $\forall i \in \mathbb{N}$, $t_{i+1} = t_i + \sigma_{i+1}$. The sequence of free-flight durations $(\sigma_i)_n$ can be defined over \mathbb{Z} by periodicity, so the definition of $(t_i)_n$ can also be extended to \mathbb{Z} with $t_0 = 0$ and $\forall i \in \mathbb{Z}$, $t_{i+1} = t_i + \sigma_{i+1}$.

It follows that the definition of $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ can be extended too. Recall that the skew-symmetric matrix $\mathbf{\Pi}$ is defined for $0 < i < j \leq k$ by $\Pi_{i,j} = \Upsilon_T(2(t_j - t_i) - \sigma)$. Introducing the sign function δ defined on \mathbb{Z} with the convention $\delta(0) = 0$, this can be rewritten as

$$\forall (i, j) \in \llbracket 1, k \rrbracket^2, \quad \Pi_{i,j} = \Upsilon_T(2(t_j - t_i) - \delta(j - i)T). \quad (9.2)$$

Similarly, the symmetric matrix $\mathbf{\Sigma}$ is defined for $i < j \leq k$ by

$$\Sigma_{i,j} = 2\mathbf{w}^\top (\mathbf{S}(\frac{T}{2}) - \mathbf{S}(-\frac{T}{2}))^{-1} \mathbf{S}(t_i - t_j + \frac{T}{2}) \mathbf{r} := \Gamma_T(2(t_j - t_i) - T). \quad (9.3)$$

Introducing the sign function ϵ on $\llbracket -(k+1), k+1 \rrbracket$ with the convention $\epsilon(0) = 1$, it can be rewritten as

$$\forall (i, j) \in \llbracket 1, k \rrbracket^2, \quad \Sigma_{ij} = \Gamma_T(2(t_j - t_i) - \epsilon(j - i)T) \quad (9.4)$$

because $\Gamma_T(\cdot)$ is an even function. Finally, Definitions (9.2) and (9.4) can be extended from $\llbracket 1, k \rrbracket^2$ to \mathbb{Z}^2 by defining δ and ϵ on \mathbb{Z} as:

$$\forall p \in \mathbb{Z}, \quad \delta(p+k) = 2 + \delta(p) \quad \text{and} \quad \epsilon(p+k) = 2 + \epsilon(p) \quad (9.5)$$

Then (Π_{ij}) and (Σ_{ij}) define two sequences which are k -periodic with respect to both indices.

9.3. Symmetries of positions in the time domain Let $(\sigma_i)_{i \in \mathbb{Z}}$ denote a k -periodic sequence invariant by $n \mapsto a - n$ for some $a \in \mathbb{Z}$. From the previous subsection, this sequence uniquely defines the sequence $(t_i)_{i \in \mathbb{Z}}$.

Proposition 9.3 *The sequence $(t_i + t_{a-i-1})_i$ is of constant value t_{a-1} .*

Proof. $\forall i < j$,

$$t_j - t_i = \sum_{p=i+1}^j \sigma_p = \sum_{p=i+1}^j \sigma_{a-p} = \sum_{p=a-j}^{a-i-1} \sigma_p = t_{a-i-1} - t_{a-j-1} \quad (9.6)$$

so that $(t_i + t_{a-i-1})_i$ is constant of value $t_0 + t_{a-1} = t_{a-1}$. ■

We can verify Proposition 9.3 with the example in Fig. 5, corresponding to $a = 6$:

$$t_2 + t_{6-2-1} = t_2 + t_3 = (c + b) + (c + b + a) = c + b + a + b + c = t_5 \quad (9.7)$$

or

$$t_4 + t_{6-4-1} = t_4 + t_1 = (c + b + a + b) + (c) = c + b + a + b + c = t_5. \quad (9.8)$$

Theorem 9.4 *Let \mathbf{x} denote the position defined over \mathbb{R} corresponding to free dynamics of durations $(\sigma_i)_{i \in \mathbb{Z}}$. If $g_0 \neq 0$ and Σ is invertible, then*

$$\forall t \in \mathbb{R}, \quad \mathbf{x}(t) = \mathbf{x}(t_{a-1} - t). \quad (9.9)$$

Proof. From Theorems 3.1 and 3.2, when $g_0 \neq 0$, a solution $\bar{\mathbf{x}}_0$ is determined by the vector $\boldsymbol{\lambda} = -g_0 \Sigma^{-1} \mathbf{j}$ solution of $\Pi \boldsymbol{\lambda} = \mathbf{0}$. Hence, for a given (t_1, \dots, t_k) such that $\boldsymbol{\lambda}$ is solution of $\Pi(t_1, \dots, t_k) \boldsymbol{\lambda} = \mathbf{0}$, the motion is unique. But $t \mapsto \mathbf{x}(t)$ and $t \mapsto \mathbf{x}(t_{a-1} - t)$, which are both solutions of $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$, correspond to the same g_0 and the same free dynamics durations: $\sigma_1 = \sigma_{a-1}$, $\sigma_2 = \sigma_{a-2}$, etc., therefore $\mathbf{x}(t) = \mathbf{x}(t_{a-1} - t)$ for all $t \in \mathbb{R}$. ■

Corollary 9.1 $\forall t \in \mathbb{R}$, let $\mathbf{x}(t)$ denote the positions corresponding to the free dynamics of durations $(\sigma_i)_{i \in \mathbb{Z}}$ with $g_0 \neq 0$. Then \mathbf{x} has the following axes of symmetry:

$$2t = t_{a-1+pk}, \quad p \in \mathbb{Z}. \quad (9.10)$$

These correspond to two axes of symmetry on every period.

Proof. From Theorem 9.4

$$\mathbf{x}\left(t + \frac{t_{a-1+pk}}{2}\right) = \mathbf{x}\left(t + \frac{t_{a-1} + pT}{2}\right) = \mathbf{x}\left(t_{a-1} - \left(t + \frac{t_{a-1} + pT}{2}\right)\right) \quad (9.11a)$$

$$= \mathbf{x}\left(\frac{t_{a-1} - pT}{2} - t\right) = \mathbf{x}\left(\frac{t_{a-1} + pT}{2} - pT - t\right) \quad (9.11b)$$

$$= \mathbf{x}\left(\frac{t_{a-1} + pT}{2} - t\right) \quad (9.11c)$$

where the T -periodicity of \mathbf{x} is used in the last equality. This proves that $t_{a-1+pk}/2$ is an axis of symmetry of \mathbf{x} , for any $p \in \mathbb{Z}$. This corresponds to two axes of symmetry in $[0, T)$ because if $t_{a-1+pk}/2 \in [0, T)$ for some p , then exactly one of $t_{a-1+pk}/2 + T/2$ and $t_{a-1+pk}/2 - T/2$ is in $[0, T)$ too. ■

Corollary 9.2 Under the assumptions of Theorem 9.4, the following lines are axes of symmetry for the position \mathbf{x} , $\forall p \in \mathbb{Z}$:

- if $k = 2r - 1$ and $a = 2b$,

$$t = \frac{t_{b-1} + t_b}{2} + pT \quad \text{and} \quad t_{b+r-1} + pT \quad (9.12)$$

- if $k = 2r - 1$ and $a = 2b + 1$,

$$t = \frac{t_{b+r-1} + t_{b+r}}{2} + pT \quad \text{and} \quad t_b + pT \quad (9.13)$$

- if $k = 2r$ and $a = 2b$,

$$t = \frac{t_{b-1} + t_b}{2} + pT \quad \text{and} \quad \frac{t_{b+r-1} + t_{b+r}}{2} + pT \quad (9.14)$$

- if $k = 2r$ and $a = 2b$,

$$t = t_b + pT \quad \text{and} \quad t_{b+r} + pT \quad (9.15)$$

Proof. If $k = 2r - 1$ and $a = 2b$, Proposition 9.3 with $i = b$ in Corollary 9.1 implies that

$$t = \frac{t_{a-1}}{2} = \frac{t_{2b-b-1} + t_b}{2} = \frac{t_{b-1} + t_b}{2} \quad (9.16)$$

is an axis of symmetry. Another one is given by

$$t = \frac{t_{a-1}}{2} + \frac{T}{2} = \frac{t_{a-1}}{2} + \frac{\sigma_{b-r+1} + \dots + \sigma_{b-1} + \sigma_b + \sigma_{b+1} + \dots + \sigma_{b+r-1}}{2} \quad (9.17a)$$

$$= \frac{t_{a-1}}{2} + \frac{\sigma_b + 2(\sigma_{b+1} + \dots + \sigma_{b+r-1})}{2} = t_b + \sigma_{b+1} + \dots + \sigma_{b+r-1} = t_{b+r-1} \quad (9.17b)$$

- If $k = 2r - 1$ and $a = 2b + 1$, then $a + k$ is even and $(k + a)/2 = r + b$ which yields the axis of symmetry.
- If $k = 2r$ and $a = 2b$, as in Eq. (9.16), $t = (t_{b-1} + t_b)/2$ is an axis of symmetry. The other axis over the period is given by

$$t = \frac{t_{a-1}}{2} + \frac{T}{2} \quad (9.18a)$$

$$= \frac{t_{a-1}}{2} + \frac{(\sigma_{b-r+1} + \dots + \sigma_{b-1}) + \sigma_b + (\sigma_{b+1} + \dots + \sigma_{b+r-1}) + \sigma_{b+r}}{2} \quad (9.18b)$$

$$= \frac{t_b + t_{b-1} + \sigma_b + 2(\sigma_{b+1} + \dots + \sigma_{b+r-1}) + \sigma_{b+r}}{2} \quad (9.18c)$$

$$= t_{b+r-1} + \frac{\sigma_{b+r}}{2} = \frac{t_{b+r-1} - t_{b+r}}{2} \quad (9.18d)$$

- If $k = 2r$ and $a = 2b - 1$,

$$t = \frac{t_{a-1}}{2} = \frac{t_{2b-b} + t_b}{2} = t_b \quad (9.19)$$

which provides an axis of symmetry. And $a' = k - a = 2r - 2b - 1 = 2(r - b) - 1$ is odd too, so $t = t_{r-b} = t_{r-b+k} = t_{b+r}$ is also an axis of symmetry. ■

The conclusions of Corollary 9.2 apply to the introductory example in Fig. 5 where $k = 7 = 2r - 1$ with $r = 4$ and $a = 6 = 2b$ with $b = 3$: there are indeed two axes of symmetry, of equations $t = (t_{b-1} + t_b)/2 = (t_2 + t_3)/2 = c + b + a/2$ and $t = t_{b+r-1} = t_6 = c + b + a + b + c + d$.

9.4. Dimension of the solution space in the symmetric case The consequences of the symmetries of (t_i) on the matrices $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ are now investigated.

Proposition 9.5

$$\Pi_{i,j} = \Pi_{\tau(j),\tau(i)} \quad \text{and} \quad \Sigma_{i,j} = \Sigma_{\tau(j),\tau(i)} \quad (9.20)$$

where $\tau : p \mapsto a - p - 1$ (modulo k).

Proof. Using Proposition 9.3 in Eq. (9.2):

$$\Pi_{i,j} = \Upsilon_T(2(t_j - t_i) - \delta(j - i)T) \quad (9.21a)$$

$$= \Upsilon_T(2(t_{a-i-1} - t_{a-j-1}) - \delta((a - i - 1) - (a - j - 1))T) \quad (9.21b)$$

$$= \Pi_{a-j-1, a-i-1} \quad (9.21c)$$

In the exact same way, $\Sigma_{i,j} = \Sigma_{a-j-1, a-i-1}$. ■

The transformation τ is an involution since $\tau(\tau(i)) = a - (a - i - 1) - 1 = i$ and if \mathbf{P}_τ is the associated permutation matrix, $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ are invariant by the transformation \mathcal{T} defined for a generic matrix \mathbf{A} by $\mathcal{T}(\mathbf{A}) = (A_{\tau(j), \tau(i)})_{i,j} = \mathbf{P}_\tau^{-1} \mathbf{A}^\top \mathbf{P}_\tau$. Let $\boldsymbol{\lambda}$ be such that $\mathbf{\Sigma} \boldsymbol{\lambda} = -g_0 \mathbf{j}$. $\mathbf{\Sigma}$ is symmetric so $\mathbf{\Sigma} = \mathbf{P}_\tau^{-1} \mathbf{\Sigma}^\top \mathbf{P}_\tau = \mathbf{P}_\tau^{-1} \mathbf{\Sigma} \mathbf{P}_\tau$ and $\mathbf{P}_\tau \mathbf{\Sigma} = \mathbf{\Sigma} \mathbf{P}_\tau$. \mathbf{P}_τ is a permutation matrix so $\mathbf{P}_\tau \mathbf{j} = \mathbf{j}$ and

$$-g_0 \mathbf{j} = -g_0 \mathbf{P}_\tau \mathbf{j} = \mathbf{P}_\tau \mathbf{\Sigma} \boldsymbol{\lambda} = \mathbf{\Sigma} \mathbf{P}_\tau \boldsymbol{\lambda} \quad (9.22)$$

hence if $g_0 \neq 0$, $\mathbf{P}_\tau \boldsymbol{\lambda} = -g_0 \mathbf{\Sigma}^{-1} \mathbf{j} = \boldsymbol{\lambda}$. Additionally, $\mathbf{\Pi} \boldsymbol{\lambda} = \mathbf{0} := \mathbf{E}$ and $\mathbf{\Pi} = \mathbf{P}_\tau^{-1} \mathbf{\Pi}^\top \mathbf{P}_\tau = -\mathbf{P}_\tau^{-1} \mathbf{\Pi} \mathbf{P}_\tau$ so that

$$\mathbf{P}_\tau \mathbf{E} = \mathbf{P}_\tau \mathbf{\Pi} \boldsymbol{\lambda} = -\mathbf{\Pi} \mathbf{P}_\tau \boldsymbol{\lambda} = -\mathbf{\Pi} \boldsymbol{\lambda} = -\mathbf{E} \quad (9.23)$$

which gives *a priori* k equations. The equations corresponding to invariant elements of τ are always satisfied (for some i , $\mathbf{E}_i = -\mathbf{E}_i \implies \mathbf{E}_i = 0$). The others are in even number and half of them can be dropped ($\mathbf{E}_i = -\mathbf{E}_j \implies \mathbf{E}_j = -\mathbf{E}_i$). The number of remaining equations is the number of transpositions in the decomposition of the permutation τ .

From Theorem 3.3, $\mathbf{\Pi} \mathbf{\Sigma}^{-1} \mathbf{j} = \mathbf{0}$ yields $k - 1$ equations for the k unknowns t_1, \dots, t_k corresponding, generically, to a solution space of dimension 1. However, when the sequence $(\sigma_i)_i$ is invariant by a transformation $\tau : i \mapsto a - i - 1$, the number of independent unknowns is reduced. It was just shown that this reduction comes together with the reduction of the number of independent equations. The following proposition indicates that, even in the subgeneric case where symmetries in the flight durations sequence reduce the number of independent unknowns, the solution space dimension is 1.

Proposition 9.6 *If the k -periodic sequence $(\sigma_i)_{i \in \mathbb{Z}}$ is invariant by the transformation $i \mapsto a - i - 1$, the solution space of $\mathbf{\Pi}(t_1, \dots, t_k) \mathbf{\Sigma}^{-1}(t_1, \dots, t_k) \mathbf{j} = \mathbf{0}$ is of dimension 1.*

Proof. The number of unknowns is determined by the involution $i \mapsto a - i$; the number of equations is determined by the involution $i \mapsto a - 1 - i$.

- If $k = 2r - 1$, both transformations have one invariant element (see Prop. 9.2) and are composed of $r - 1$ transpositions, hence the number of unknowns is $(r - 1) + 1$ and the number of equations is $r - 1$. The space of solutions is of dimension 1.
- If $k = 2r$ and a is even, $i \mapsto a - i - 1$ has no invariant elements and is made of r transpositions, so the number of equations is r . The involution $i \mapsto a - i$ has two invariant elements and is made of $r - 1$ transpositions, so the number of unknowns is $r - 1 + 2 = r + 1$. The space of solutions is of dimension 1.
- If $k = 2r$ and a is odd, $i \mapsto a - i$ has two invariant elements and is made of $r - 1$ transpositions, which corresponds to $r - 1$ equations. $i \mapsto a + i - 1$ has no invariant element and is made of r transpositions, yielding r unknowns so, again, a space of solutions of dimension 1. ■

10. Illustration on a simple serial oscillator The mathematical results of sections 3 and 9 are now illustrated on a very simple oscillator: a unidimensional spring–mass system with n masses m_1, \dots, m_n and n springs of positive stiffnesses k_1, \dots, k_n as depicted in Figure 6. For this system, the derivations slightly simplify. However, the methodology to calculate non-smooth modes and the computational cost would be similar for more sophisticated systems. The positions of the masses are denoted x_1, \dots, x_n and by choice, $x_1 = \dots = x_n = 0$ at the resting position. The motion of the last mass (in red) is constrained by an obstacle. The gap between the contacting mass and the obstacle is $g(\mathbf{x}) = d - x_n$, corresponding to $g_0 = d$, which can be positive or negative. The perfectly elastic Newton impact law for the description of contact reads

$$d - x_n(t_i) = 0 \implies \dot{x}_n^+(t_i) = -\dot{x}_n^-(t_i). \quad (10.1)$$

All illustrations are provided with $n = 5$, $m_i = 1/n$, $k_i = n$ and $g_0 = d = 1$.

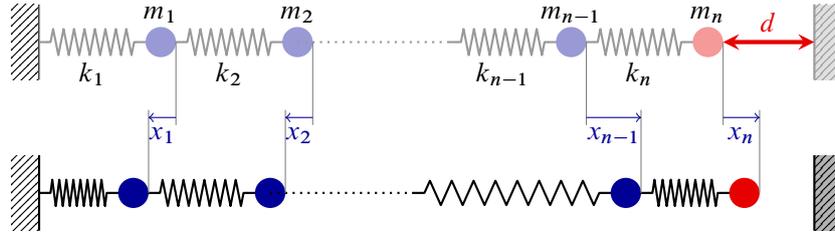


Figure 6: Base model for the study of non-smooth modes. Top: at rest. Bottom: in motion.

10.1. Simplified governing equations

10.1.1. Mass and stiffness matrices The equilibrium equation for each mass writes, with the convention $x_0 = x_{n+1} = 0$ and $k_{n+1} = 0$:

$$\forall i \in \llbracket 1, n \rrbracket, \quad m_i \ddot{x}_i = k_{i+1}(x_{i+1} - x_i) - k_i(x_i - x_{i-1}) \quad (10.2)$$

or in a matrix form

$$\mathbf{M} \ddot{\mathbf{x}} = -\mathbf{K} \mathbf{x} \quad (10.3)$$

with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \ddots & & \\ & & & m_{n-1} & \\ & & & & m_n \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n \end{bmatrix}$$

which provides the \mathbf{M} and \mathbf{K} matrices of Problem (2.16). It is noteworthy that \mathbf{M} is positive-definite as a diagonal matrix of coefficients $m_1, \dots, m_n > 0$ and so is \mathbf{K} : $k_1, \dots, k_n > 0$ and the associated quadratic form $\sum_{i=1}^n k_i (x_i - x_{i-1})^2$ is positive, and zero iff $x_0 = \dots = x_n = 0$.

10.1.2. Change of basis Since \mathbf{M} is diagonal, $\sum_i (\mathbf{M}^{-1/2})_{p,i} Q_{i,j} / \omega_j = Q_{p,j} / (\sqrt{m_p} \omega_j)$ and $\underline{\mathbf{w}}_p$ reduces to

$$\underline{\mathbf{w}}_p = -\frac{1}{\sqrt{m_p}} \begin{bmatrix} Q_{p,1}/\omega_1 \\ 0 \\ \vdots \\ Q_{p,n}/\omega_n \\ 0 \end{bmatrix} \quad \text{and similarly} \quad \underline{\mathbf{r}}_p = \frac{1}{\sqrt{m_p}} \begin{bmatrix} 0 \\ Q_{p,1} \\ \vdots \\ 0 \\ Q_{p,n} \end{bmatrix}. \quad (10.4)$$

10.1.3. Gap and impact law The gap definition $g(\mathbf{x}) = d - x_n$ implies $\mathbf{w} = J_g(\mathbf{x})^\top = [0 \ \dots \ 0 \ -1]^\top$ which induces $\underline{\mathbf{w}} = -\underline{\mathbf{w}}_n$ because

$$\underline{\mathbf{w}} = \mathbf{P}_{\underline{\mathbf{B}}}^\beta \begin{bmatrix} \Delta^{-1} \mathbf{Q}^\top \mathbf{M}^{-1/2} \mathbf{w} \\ \mathbf{0} \end{bmatrix} = \mathbf{P}_{\underline{\mathbf{B}}}^\beta \begin{bmatrix} (-\omega_i^{-1} Q_{n,i} m_n^{-1/2})_{i=1,n} \\ \mathbf{0} \end{bmatrix} = -\underline{\mathbf{w}}_n. \quad (10.5)$$

Through the definition of $\underline{\mathbf{r}}$ and replacing \mathbf{w} by its expression:

$$\underline{\mathbf{r}} = \mathbf{M}^{-1/2} \mathbf{w} (\mathbf{w}^\top \mathbf{M}^{-1} \mathbf{w})^{-1/2} = \begin{bmatrix} 0 & \dots & 0 & m_n^{-1/2} \end{bmatrix}^\top (m_n^{-1})^{-1/2} = -\underline{\mathbf{w}} \quad (10.6)$$

from which the expression of $\underline{\mathbf{r}}$ is deduced:

$$\underline{\mathbf{r}} = \mathbf{P}_{\underline{\mathbf{B}}}^\beta \begin{bmatrix} 0 \\ \mathbf{Q}^\top \underline{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} 0 & Q_{n,1} & \dots & 0 & Q_{n,n} \end{bmatrix}^\top = \sqrt{m_n} \underline{\mathbf{r}}_n. \quad (10.7)$$

10.1.4. Positions and velocities The function ϕ in Eq. (3.6) can be simplified by taking advantage of the structure of $\underline{\mathbf{w}}$ and $\underline{\mathbf{r}}$. Indeed, invoking Eq. (5.30), $(\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1}\underline{\mathbf{S}}(\sigma)$ is a block-diagonal matrix so $\forall p \in \llbracket 1, n \rrbracket$,

$$\begin{aligned} & \underline{\mathbf{w}}_p^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(\sigma) \underline{\mathbf{r}} \\ &= \sum_{i=1}^n \frac{1}{2 \sin(\omega_i T/2)} [w_{p2i-1} \ 0] \begin{bmatrix} \sin(\omega_i \sigma/2) & \cos(\omega_i \sigma/2) \\ -\cos(\omega_i \sigma/2) & \sin(\omega_i \sigma/2) \end{bmatrix} \begin{bmatrix} 0 \\ r_{2i} \end{bmatrix} \\ &= \sum_{i=1}^n w_{p2i-1} r_{2i} \frac{\cos(\omega_i \sigma)}{\sin(\omega_i T/2)} = - \sum_{i=1}^n \frac{Q_{p,i}}{\omega_i \sqrt{m_n}} Q_{n,i} \frac{\cos(\omega_i \sigma)}{\sin(\omega_i T/2)} \end{aligned} \quad (10.8)$$

that is

$$\phi_{p,T}(\sigma) = 2 \underline{\mathbf{w}}_p^\top (\underline{\mathbf{S}}(T/2) - \underline{\mathbf{S}}(-T/2))^{-1} \underline{\mathbf{S}}(\sigma/2) \underline{\mathbf{r}} = - \frac{1}{\sqrt{m_p}} \sum_{j=1}^n \frac{\cos(\omega_j \sigma/2)}{\omega_j \sin(\omega_j T/2)} Q_{p,j} Q_{n,j}. \quad (10.9)$$

Similarly, the velocity of the p -th mass reads

$$\psi_{p,T}(\sigma) = 2 \frac{d\phi_{p,T}}{dt}(t) = \frac{1}{\sqrt{m_p}} \sum_{j=1}^n \frac{\sin(\omega_j \sigma/2)}{\sin(\omega_j \sigma/2)} Q_{p,j} Q_{n,j} \quad (10.10)$$

and following on Theorem 3.4, the positions and velocities are, $\forall \tau \in [0, \sigma_{i+1}]$:

$$x_p(\tau + t_i) = \sum_{j=1}^i \lambda_j \phi_{p,T}(2(\tau + t_i - t_j) - T) + \sum_{j=i+1}^k \lambda_j \phi_{p,T}(2(\tau + t_i - t_j) + T), \quad (10.11a)$$

$$y_p(\tau + t_i) = \sum_{j=1}^i \lambda_j \psi_{p,T}(2(\tau + t_i - t_j) - T) + \sum_{j=i+1}^k \lambda_j \psi_{p,T}(2(\tau + t_i - t_j) + T). \quad (10.11b)$$

10.1.5. Components of the governing matrices Since $\underline{\mathbf{w}} = -\underline{\mathbf{w}}_n$, the matrix $\underline{\Sigma}$ is populated by the elements

$$\Sigma_{i,j} = \Gamma_T(2(t_j - t_i) - T) = -\phi_{n,T}(2(t_i - t_j) + T) \quad (10.12a)$$

$$= \frac{1}{\sqrt{m_n}} \sum_{i=1}^n \frac{\cos(\omega_i(t_i - t_j + T/2))}{\omega_i \sin(\omega_i T/2)} Q_{n,i}^2. \quad (10.12b)$$

For comparison, recall that matrix $\underline{\Pi}$'s entries are $\Upsilon_T(2(t_j - t_i) - T)$ where Υ , defined in Eq. (3.2) reduces to

$$\Pi_{i,j} = \Upsilon_T(2(t_j - t_i) - T) = \sum_{i=1}^n \frac{\sin(\omega_i(t_i - t_j + T/2))}{\omega_i \sin(\omega_i T/2)} Q_{n,i}^2 \quad (10.13)$$

because $r_{2i} = Q_{n,i}$ from Eq. (10.7).

10.1.6. Physical interpretation of the kernel isomorphic to the invariant subspace of u The physical interpretation of the parameters λ_j is straightforward. By definition

$$\mathbf{l}_j \underline{\mathbf{x}} = \underline{\mathbf{r}}^\top \underline{\mathbf{S}}(\sigma_j) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) \underline{\mathbf{x}} \quad (10.14)$$

and $\lambda_j = \mathbf{l}_j \underline{\mathbf{x}}$. The vector $\underline{\mathbf{x}}_j = \underline{\mathbf{S}}(\sigma_j) \underline{\mathbf{N}} \dots \underline{\mathbf{N}} \underline{\mathbf{S}}(\sigma_1) \underline{\mathbf{x}}$ is the state of the system just before the j -th impact. Accordingly, the velocity of the n -th mass just before the j -th impact is given by

$$y_n^-(t_j) = \langle \underline{\mathbf{r}}_n, \underline{\mathbf{x}}_j \rangle = \frac{1}{\sqrt{m_n}} \langle \underline{\mathbf{r}}, \underline{\mathbf{x}}_j \rangle = \frac{1}{\sqrt{m_n}} \underline{\mathbf{r}}^\top \underline{\mathbf{x}}_j = \frac{1}{\sqrt{m_n}} \mathbf{l}_j \underline{\mathbf{x}} = \frac{\lambda_j}{\sqrt{m_n}} \quad (10.15)$$

and $\lambda_j = \sqrt{m_n} y_n^-(t_j)$; in particular, λ_j and $y_n^-(t_j)$ have the same sign. This observation suggests an efficient way of detecting potential solutions which are not admissible solutions, since admissible solutions necessarily have non-negative pre-impact velocities.

10.2. One-impact-per-period solutions In order to facilitate the coming discussions, the following definition of harmonic is used.

Definition 10.1 [Harmonic] A harmonic is a multiple of $T_i := 2\pi/\omega_i$ for one $i \in \llbracket 1, n \rrbracket$.

10.2.1. Invariance by u : Theorem 3.1 (NC1) If $k = 1, t_1 = T$ is the period. The skew-symmetric matrix $\mathbf{\Pi}$ is of dimension $k = 1$, hence $\mathbf{\Pi} = [0]$ and $\mathbf{j} = [1], \boldsymbol{\lambda} = [\lambda]$. Any $\lambda \in \mathbb{R}$ satisfies $\mathbf{\Pi}\boldsymbol{\lambda} = 0$ and Theorem 3.1 stipulates that any maximal invariant subspace of u is always of dimension one and given by $\varphi(\lambda), \lambda \in \mathbb{R}$.

10.2.2. Potential solutions: Theorem 3.2 (NC2) According to Theorem 3.2, NC2 can be expressed as $\boldsymbol{\Sigma}(T)\boldsymbol{\lambda} = -g_0\mathbf{j} = -d\mathbf{j}$, which gives a condition on $\boldsymbol{\lambda} = [\lambda]$. Here, $\boldsymbol{\Sigma} = [-\phi_{n,T}(T)]$ and the condition becomes $\phi_{n,T}(T)\lambda = d$. The existence and uniqueness of potential solutions is determined by the values of $\phi_{n,T}(T)$ and the nullity of d .

- If $d = 0$, there is generically no non-trivial potential solution. If T is solution of $\phi_{n,T}(T) = 0$, which occurs for a countable set of values, there is an infinity of potential solutions with the same period T given by $\varphi(\lambda), \lambda \in \mathbb{R}$.
- If $d \neq 0$, there is no potential solution if $\phi_{n,T}(T) = 0$ and a unique potential solution corresponding to $\lambda = d/\phi_{n,T}(T)$ whenever $\phi_{n,T}(T) \neq 0$. Generically, $\phi_{n,T}(T) \neq 0$ hence, as implied by Theorem 3.3, there is locally a one-parameter family of periodic trajectories. This family can obviously be parametrized by T .

10.2.3. Admissible solutions for a non-vanishing unconstrained gap at rest We focus on the case $d \neq 0$ in the neighbourhood of a T such that $\phi_{n,T}(T) \neq 0$. The unique potential solution for T is given by $\varphi(d/\phi_{n,T}(T))$. It is an admissible solution if the gap remains non-negative (NC3). Following the results from subsection 10.1.6, many time intervals can be easily excluded using the necessary (but not sufficient) criterion $\lambda \geq 0$ or equivalently $\text{sign}(d) \text{sign}(\phi_{n,T}(T)) = 1$. From Eq. (10.9), the following is reminded:

$$\phi_{n,T}(T) = -\frac{1}{\sqrt{m_n}} \sum_{j=1}^n \cot(\omega_j T/2) \frac{Q_{n,j}^2}{\omega_j}. \quad (10.16)$$

Since \cot is a decreasing function on any of its intervals of definition, $T \mapsto \phi_{n,T}(T)$ increases on any open interval bounded by two consecutive harmonics. On such an interval, its left limit is $-\infty$ and its right limit is $+\infty$. It is continuous, hence necessarily has a unique root T_0 on every such interval. The condition $\lambda \geq 0$ comes down to seeking periods T of admissible solutions in intervals delimited:

- on the the left, by a harmonic and on the right, by the next root T_0 of $\phi_{n,T}(T)$, if $d > 0$;
- on the the left, by a root T_0 of $\phi_{n,T}(T)$ and the right, by the next harmonic, if $d < 0$.

The sufficient condition NC3 ($x_n \leq d$ on $(0, T)$ with x_n given by Eq. (10.11)) has to be tested numerically on every such intervals. If it is satisfied for one T , $\varphi(d/\phi_{n,T}(T))$ is the unique admissible solution and there is a one-parameter family of admissible solutions in the neighbourhood of T through Corollary 3.1.

It is worthy to note that there is a complementarity between the open gap ($d > 0$) and prestressed system ($d < 0$) configurations, in the sense that for any given period, the impact velocity is always compatible with the unilateral constraints for either $d > 0$ or $d < 0$, but not simultaneously. This complementarity is illustrated in Fig. 7.

10.2.4. Closed-form expression of the non-smooth modes and illustrations From Eq. (10.11) consequence of Theorem 3.4, the positions and the velocities are known explicitly: $\forall \tau \in [0, T]$,

$$\left\{ \begin{array}{l} x_p(\tau) = -\frac{\lambda}{\sqrt{m_p}} \sum_{j=1}^n \frac{\cos(\omega_j(\tau - T/2))}{\omega_j \sin(\omega_j T/2)} Q_{p,j} Q_{n,j} \end{array} \right. \quad (10.17a)$$

$$\left\{ \begin{array}{l} \dot{x}_p(\tau) = -\frac{\lambda}{\sqrt{m_p}} \sum_{j=1}^n \frac{\sin(\omega_j(\tau - T/2))}{\sin(\omega_j T/2)} Q_{p,j} Q_{n,j}. \end{array} \right. \quad (10.17b)$$

On an interval of periods corresponding to admissible solutions, Eq. (10.17) gives a closed-form expression of 1-ipp non-smooth modes parametrized by the period T and time τ . The motion of the n -th mass is plotted in Fig. 8 for several periods in the interval delimited by ① and ⑥ in Fig. 7. The non-smooth mode corresponding to this continuum of admissible solutions is projected in the $(x_{n-1}, x_n, \dot{x}_n)$ space in Fig. 9.

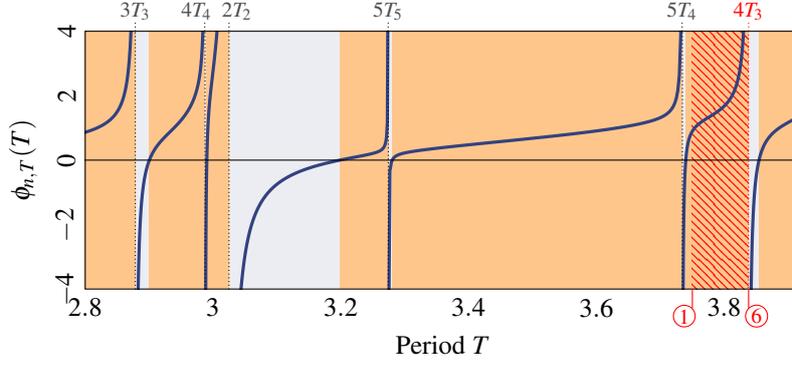


Figure 7: Graph of $T \mapsto \phi_{n,T}(T)$ [—]. Harmonics [---]. Grey area [■] corresponds to $\lambda < 0$, i.e. zones with no admissible solutions if $d > 0$. Orange area [■] corresponds to $\lambda > 0$, i.e. zones with no admissible solutions if $d < 0$. ① and ⑥ delimit an interval of periods corresponding to admissible solutions, see Fig. 8.

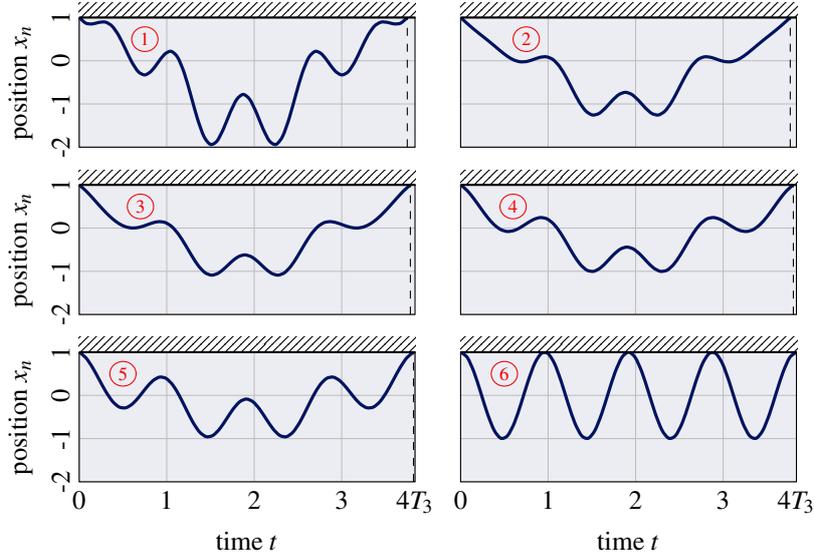


Figure 8: Continuum of solutions with 1 impact per period.

Remark 10.1.

- The sequence of free flights is 1-periodic, hence Corollary 9.1 with $a = 0$ implies that $t = t_0/2 = 0$ and $t = t_1/2 = T/2$ are axes of symmetry for the positions, hence the velocities are symmetric with respect to the points $(0, 0)$ and $(T/2, 0)$. This is clear in Fig. 8 and corresponds to the symmetry with respect to the plane $\dot{x}_n = 0$ in Fig. 9.
- All the velocities \dot{x}_p vanish at $\tau = T/2$, which is a direct consequence of the symmetry.
- For $p \neq n$, $\dot{x}_p^+(0) = \dot{x}_p^-(T)$ because the lines of \mathbf{Q} are orthogonal: $\sum_j Q_{p,j} Q_{n,j} = 0$ which implies from Eq. (10.17) the continuity of positions of the non-constrained masses.
- For $p = n$, $\dot{x}_n(0^+) = \lambda_1/m_n = -\dot{x}_n(T^-)$ which corresponds to the impact law.
- For larger T , the probability of meeting the obstacle between impacts increases together with the number of oscillations during one period. This indicates that the density of non-smooth modes decreases with the period.

10.3. Two-impact-per-period solutions

10.3.1. Invariance by u : Theorem 3.1 (NC1) The impact times are t_1, t_2 , $T = t_2$ being the period. The free flight durations are $\sigma_1 = t_1$ and $\sigma_2 = T - t_1$. The skew-symmetric matrix 2×2 - $\mathbf{\Pi}$ reads:

$$\mathbf{\Pi} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \quad (10.18)$$

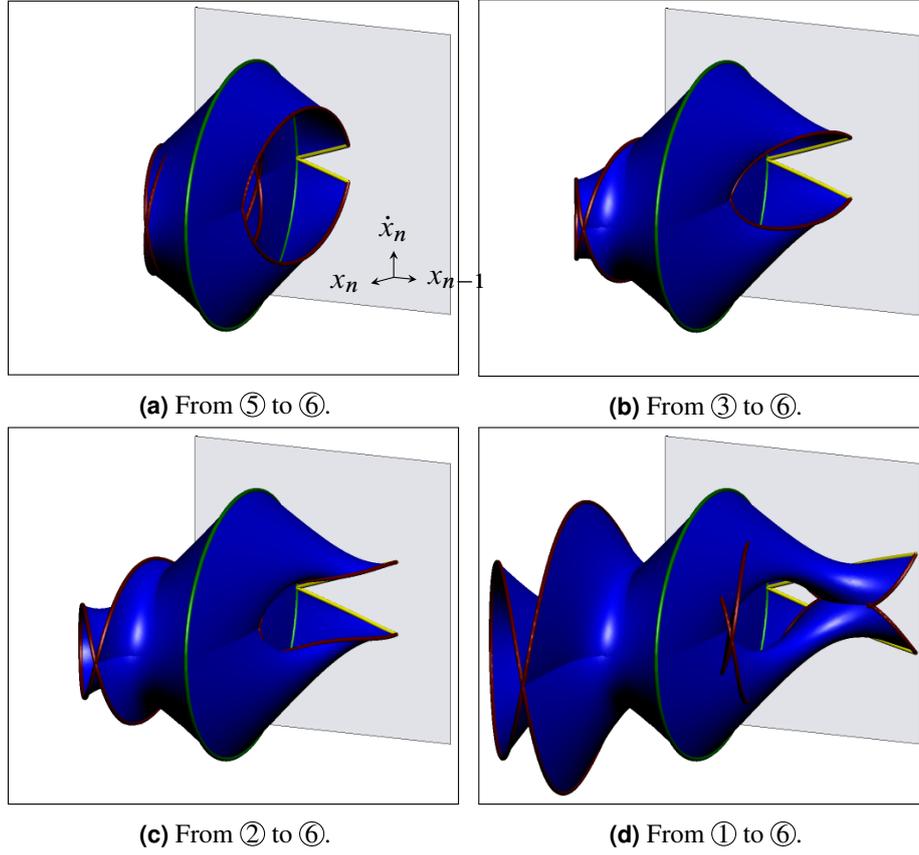


Figure 9: Projection in $(x_{n-1}, x_n, \dot{x}_n)$ of the 1-ipp non-smooth mode in the neighbourhood of $4T_3$. The plots correspond to various intervals of T . The green ellipse shows the motion ⑥ in Fig. 8, *i.e.* the grazing orbit of the third linear mode. The red curves correspond to motions ⑤, ③, ②, and ①. The yellow curve highlights the intersection between the non-smooth mode and the hyperplane $x_n = d$.

with

$$a = Q_{\sigma_1 + \sigma_2}(\sigma_2 - \sigma_1) = \sum_{j=1}^n \frac{\sin(\frac{1}{2}\omega_j(\sigma_2 - \sigma_1))}{\sin(\frac{1}{2}\omega_j(\sigma_1 + \sigma_2))} Q_{n,j}^2 = \sqrt{m_n} \psi_{n, \sigma_1 + \sigma_2}(\sigma_2 - \sigma_1). \quad (10.19)$$

Using Theorem 3.1, NC1 comes down to $\mathbf{\Pi}\boldsymbol{\lambda} = \mathbf{0}$ hence $\boldsymbol{\lambda} = \mathbf{0}$ or $a = 0$. Non-trivial solutions can be found only if $a = 0$, which yields the following nonlinear equation in σ_1, σ_2 :

$$\psi_{n, \sigma_1 + \sigma_2}(\sigma_2 - \sigma_1) = 0. \quad (10.20)$$

The set of roots of a is denoted by \mathcal{S} .

Solution curves of Eq. (10.20) There is generally no closed-form expressions for the solutions to Eq. (10.20). It is yet possible to describe the behaviour of the roots of $\psi_{n, \sigma_1 + \sigma_2}(\sigma_2 - \sigma_1)$ when $\sigma_1 + \sigma_2$ is in the neighbourhood of a harmonic. Indeed, the function $(\sigma_1, \sigma_2) \mapsto \psi_{n, \sigma_1 + \sigma_2}(\sigma_2 - \sigma_1)$ is defined when $\sigma_1 + \sigma_2$ is not a harmonic. It shares the roots of

$$\tilde{a}(\sigma_1, \sigma_2) := a(\sigma_1, \sigma_2) \prod_{j=1}^n \sin(\frac{1}{2}\omega_j(\sigma_1 + \sigma_2)) \quad (10.21)$$

which is a linear combination of $\sin(\omega_j(\sigma_2 - \sigma_1)/2)$ for fixed T , hence the equation $\tilde{a}(\sigma_1, \sigma_2) = 0$ is easier to solve numerically. Roots of \tilde{a} are depicted in Fig. 10.

Obvious roots are given by $\sigma_1 = \sigma_2$, which corresponds to NC1 for 1-ipp motions seen as 2-ipp motions. The set $\sigma_2 = \sigma_1$ emerges clearly in Fig. 10. Some other particular roots can be very easily computed. When $\sigma_1 + \sigma_2$ is in the vicinity of a harmonic pT_i , $a(\sigma_1, \sigma_2)$ is dominated by the i -th element of the sum and its roots are in the neighbourhood of the roots of

$$\sin(\frac{1}{2}\omega_i(\sigma_2 - \sigma_1)) = \sin(\frac{1}{2}\omega_i(T - 2\sigma_1)) = \sin(\frac{1}{2}\omega_i T) \cos(\omega_i \sigma_1) - \cos(\frac{1}{2}\omega_i T) \sin(\omega_i \sigma_1) \quad (10.22)$$

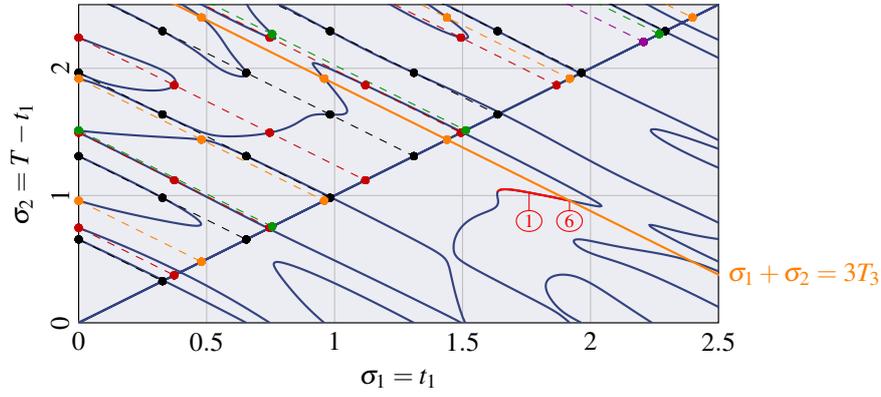


Figure 10: Level set 0 of $Q_{\sigma_1 + \sigma_2}(\sigma_2 - \sigma_1)$, which corresponds to potential solutions [—]. Line $\sigma_1 + \sigma_2 = 3T_3$ [—]. Lower-right triangle: Curve corresponding to an continuum of admissible solutions [—]. ① and ⑥ correspond to the curves reported in Fig. 11. Upper-left triangle: Dashed lines are iso-periods when $\sigma_1 + \sigma_2$ is harmonic. The dots correspond to the intersection between the roots and iso-period lines and they are regularly spaced on each line.

which converges to $-\sin(\omega_i \sigma_1)$ when T approaches pT_i . After all, the solutions of Eq. (10.20) are such that $\sigma_1 = pT_i/2$ with $p \in \mathbb{N}^*$ when $\sigma_1 + \sigma_2 \rightarrow pT_i$. The intersections between the harmonics lines $\sigma_2 = qT_i - \sigma_1$ and the roots of a is the set

$$\left\{ \left(\frac{p}{2}T_i, \left(q - \frac{p}{2} \right)T_i \right), i, p, q \in \mathbb{N}^* \right\}. \quad (10.23)$$

This is illustrated by colored dots in Fig. 10 and produces obvious roots of \tilde{a} . This result will become very handy in view of subsection 10.3.3.

Remark 10.2.

- Because $a(\sigma_1, \sigma_2) = -a(\sigma_2, \sigma_1)$, $\sigma_1 = \sigma_2$ is an axis of symmetry for the roots of a (see Fig. 10).
- Since \tilde{a} is analytic, the implicit function theorem applies almost everywhere and it is therefore possible to calculate locally a parametrization $T(\sigma_1)$.
- Generically, the roots of a are arranged along curves, in agreement with Theorem 3.3.

10.3.2. Potential solutions: Theorem 3.2 (NC2) The set of times of impact (t_1, t_2) for which there exists a potential solution is a subset of the set of the roots of a . Using Theorem 3.2, NC2 writes

$$\Sigma \lambda = -d \mathbf{j} \quad \text{where} \quad \Sigma = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (10.24)$$

with $\alpha = -\phi_{n, \sigma_1 + \sigma_2}(\sigma_1 + \sigma_2)$ and $\beta = -\phi_{n, \sigma_1 + \sigma_2}(\sigma_2 - \sigma_1)$. For σ_1, σ_2 solutions of Eq. (10.20), the existence and uniqueness of potential solutions is determined by the value of α, β and the nullity of d . The following illustrates Theorem 3.3 for $k = 2$.

Non-vanishing algebraic gap $d \neq 0$

- If $\alpha = \beta \neq 0$, any λ such that $\lambda_2 = -\lambda_1 - d/\alpha$ satisfies (10.24), hence a solution space of dimension 1.
- If $\alpha = -\beta$, $\Sigma \lambda = -d \mathbf{j}$ has no solution.
- If $\alpha \neq \beta$, Σ is invertible and there is a unique solution $\lambda = -d \Sigma^{-1} \mathbf{j}$. This is the generic case.

Case $d = 0$

- If $\alpha = \beta = 0$, then λ satisfies $\Sigma \lambda = -d \mathbf{j}$ and the solution space is two-dimensional.
- If $\alpha = \beta \neq 0$, any λ such that $\lambda_1 = -\lambda_2$ satisfies NC2 so the set of solutions is of dimension 1.
- If $\alpha = -\beta \neq 0$, $\lambda_1 = \lambda_2$ is sufficient and the solution space is of dimension 1.
- If $\alpha \neq \beta$, $\lambda = -d \Sigma^{-1} \mathbf{j} = \mathbf{0}$ and there is no non-trivial solution (generic case).

For $d \neq 0$, admissible solutions of the generic case $\alpha \neq \beta$ and the non-generic case $\alpha = \beta \neq 0$ are investigated in depth in subsection 10.3.3.

10.3.3. Admissible solution when $d \neq 0$

Generic case $\alpha \neq \beta$ The general setting $\alpha \neq \beta$ is considered and λ satisfies both $\mathbf{\Pi}\lambda = \mathbf{0}$ and $\mathbf{\Sigma}\lambda = -d\mathbf{j}$. Hence, $\lambda = -d\mathbf{\Sigma}^{-1}\mathbf{j}$ but \mathbf{j} is obviously an eigenvector of $\mathbf{\Sigma}$ corresponding to the eigenvalue $\alpha + \beta$ so $\lambda_1 = \lambda_2 := \lambda$ and $\lambda = 1/\alpha + \beta$. The physical interpretation is that the mass n has the same impact velocity at t_1 and T . The expressions of the positions and velocities result from Eq. (10.11): $\forall \tau \in [0, \sigma_1]$,

$$\begin{cases} x_p(\tau) = \lambda(\phi_{p,\sigma_1+\sigma_2}(2\tau + \sigma_2 - \sigma_1) + \phi_{p,\sigma_1+\sigma_2}(2\tau - \sigma_1 - \sigma_2)) & (10.25a) \\ \dot{x}_p(\tau) = \lambda(\psi_{p,\sigma_1+\sigma_2}(2\tau + \sigma_2 - \sigma_1) + \psi_{p,\sigma_1+\sigma_2}(2\tau - \sigma_1 - \sigma_2)) & (10.25b) \end{cases}$$

and $\forall \tau \in [0, \sigma_2]$,

$$\begin{cases} x_p(\tau + \sigma_1) = \lambda(\phi_{p,\sigma_1+\sigma_2}(2\tau - \sigma_1 - \sigma_2) + \phi_{p,\sigma_1+\sigma_2}(2\tau + \sigma_1 - \sigma_2)) & (10.26a) \\ \dot{x}_p(\tau + \sigma_1) = \lambda(\psi_{p,\sigma_1+\sigma_2}(2\tau - \sigma_1 - \sigma_2) + \psi_{p,\sigma_1+\sigma_2}(2\tau + \sigma_1 - \sigma_2)). & (10.26b) \end{cases}$$

As previously, the admissible solutions necessarily correspond to non-negative pre-impact velocities: $\lambda_1 \geq 0, \lambda_2 \geq 0$ which reduces to $\lambda \geq 0$. Besides, the sign of λ switches when T crosses a harmonic. This sign is the product between the sign of d and the sign of

$$\alpha + \beta = -\phi_{n,T}(T) - \phi_{n,T}(T - 2t_1) = \frac{1}{\sqrt{m_n}} \sum_{j=1}^n \frac{Q_{n,j}^2 \cos(\frac{\omega_j}{2}T) + \cos(\frac{\omega_j}{2}(T - 2t_1))}{\omega_j \sin(\frac{\omega_j}{2}T)}. \quad (10.27)$$

When T is in a punctured neighbourhood of a harmonic pT_i and t_1 is not a multiple of T_i , the term i of the above sum dominates since $\sin(\omega_i T/2) \rightarrow 0$ when $T \rightarrow T_i$. It tends to $+\infty$ when $T \rightarrow T_i^+$ and $-\infty$ when $T \rightarrow T_i^-$. Therefore, when $d > 0$, $\lambda \geq 0$ for T in the left-neighbourhood of a harmonic and $\lambda \leq 0$ in the right-neighbourhood. When $d < 0$, $\lambda \geq 0$ in the right-neighbourhood of a harmonic and $\lambda \leq 0$ in the left-neighbourhood.

Remark 10.3. Recall that the intersection points between the set \mathcal{S} of impact times leading to potential solutions and harmonic lines ($\sigma_1 + \sigma_2 = pT_i$) are given by Eq. (10.23). One strategy to search for admissible solutions is hence to check the admissibility of potential solutions in the left-neighbourhood (resp. right-neighbourhood) of these points if $d > 0$ (resp. $d < 0$).

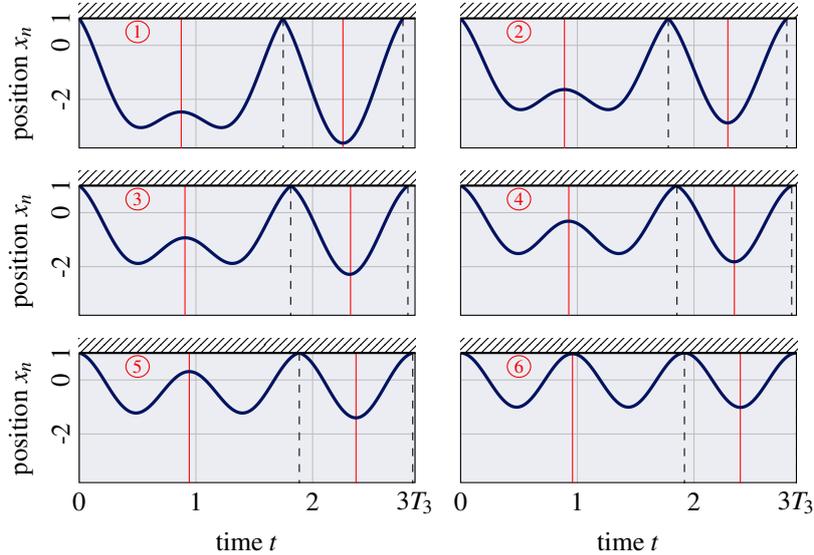


Figure 11: Positions of the n -th mass of on a continuum of admissible solutions with 2 impacts per period. Circled numbers refer to Figs. 10 and 12. Impact times [- -]. Axes of symmetry [—].

The position x_n of a continuum of admissible solutions obtained for T in the left-neighbourhood of $3T_3$ is pictured in Fig. 11. In Fig. 10, the admissible solution ⑥ is indeed at the intersection of the line $\sigma_2 = 3T_3 - \sigma_1$ and \mathcal{S} . This continuum constitutes a non-smooth mode illustrated in Fig. 12.

Remark 10.4. Theorem 9.4 holds here because the sequence of free-flight durations $\dots, \sigma_1, \sigma_2, \dots$ is invariant by the transformation $n \mapsto 2 - n$. The sets $t = t_1/2$ and $t = (t_1 + T)/2$ are indeed axes of symmetry for every x_p , in particular for x_n as shown in Fig. 11.

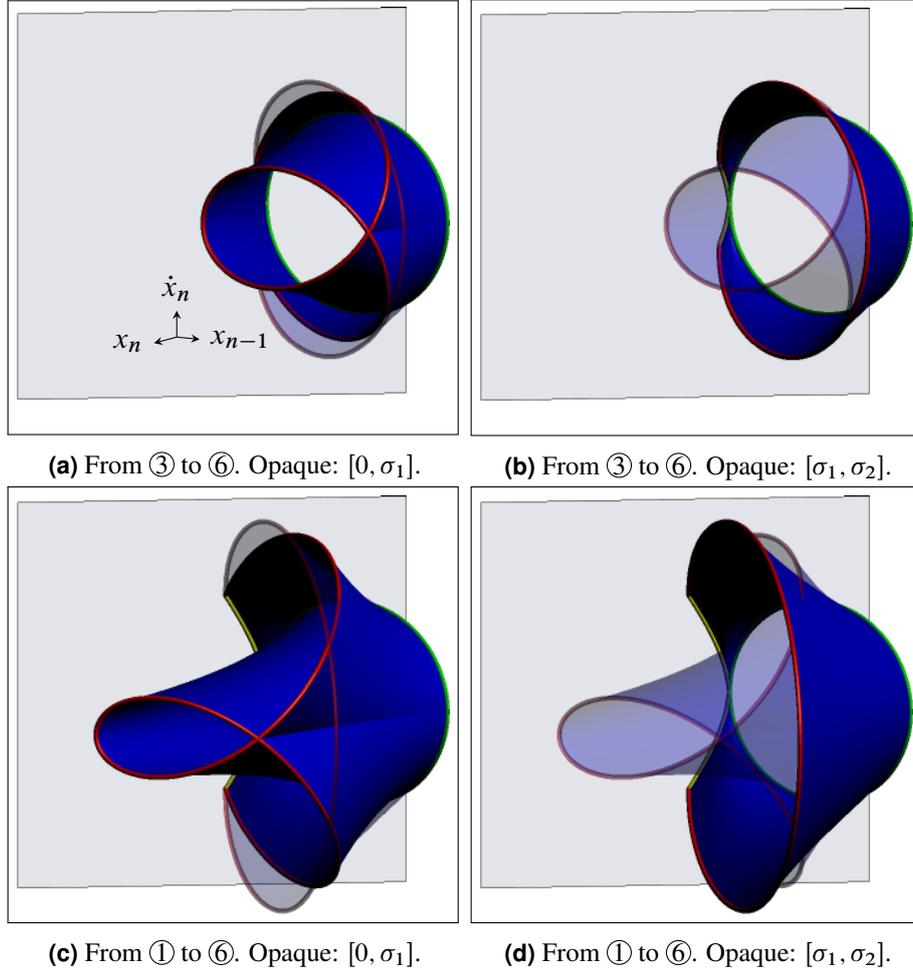


Figure 12: 2-ipp non-smooth mode. Circled numbers refer to Figs. 10 and 11.

Non-generic case $\alpha = \beta \neq 0$ As opposed to the generic case for which continua of solutions emanate from a curve in the space of impact times (t_1, T) (or $(\sigma_1, \sigma_1 + \sigma_2)$), when $\alpha = \beta \neq 0$ continua emanate from a single point in this space. The condition $\Sigma \lambda = -d\mathbf{j}$ reads $\lambda_1 + \lambda_2 = -d/\alpha$ and the expression of the positions stems from Eq. (10.11):

$$x_p(\tau) = \begin{cases} \lambda_1 \phi_{p,T}(2\tau + \sigma_2 - \sigma_1) - \left(\frac{d}{\alpha} + \lambda_1\right) \phi_{p,T}(2\tau - T) & \text{if } \tau \in [0, \sigma_1] \\ \lambda_1 \phi_{p,T}(2\tau - T) - \left(\frac{d}{\alpha} + \lambda_1\right) \phi_{p,T}(2\tau + \sigma_1 - \sigma_2) & \text{if } \tau \in (\sigma_1, T] \end{cases} \quad (10.28)$$

Examples of such solutions are proposed in Fig. 13, which illustrates a continuum of admissible solutions sharing the same impact times t_1 and T . In contrast to the generic case where Σ is invertible, it shows that positions do not display any axis of symmetry, except the blue one ($\lambda_1 = 0$, axes at 0 and $T/2$), red one ($\lambda_2 = 0$, axes at t_1 and $t_1 + T/2$) and the purple one ($\lambda_1 = \lambda_2$, axes at $t_1/2$ and $(t_1 + T)/2$). This highlights the necessity of an invertible Σ in Theorem 9.4.

The geometric properties of the position curves in Fig. 13 are now interpreted.

- The pre-impact velocities λ_1 and $-\lambda_1 - d/\alpha$ are bounded by 0 and $-d/\alpha$. Since \dot{x}_n is a continuous function of λ_1 , the pre-impact velocities continuously vary from $(0, -d/\alpha)$ (blue curve) to $(-d/\alpha, 0)$ (red curve), which are trajectories with two grazing contacts and one (non-grazing) impact. The sum of the two pre-impact velocities is constant.
- Even though every position x_n has generically no symmetry, the set of all admissible positions for (t_1, T) has two axes of symmetry: $t = t_1/2$ and $t = (t_1 + T)/2$. Indeed, if \mathbf{x} is an admissible solution for λ_1 , then, for $\tau \in [0, t_1]$,

$$x_p(\sigma_1 - \tau) = -\left(\frac{d}{\alpha} + \lambda_1\right) \phi_{n,T}(2\tau + \sigma_2 - \sigma_1) + \lambda_1 \phi_{n,T}(2\tau - T), \quad (10.29)$$

hence $\tau \mapsto \mathbf{x}(\sigma_1 - \tau)$ is an admissible solution for $-\lambda_1 - d/\alpha \geq 0$. This proves that $t = \sigma_1/2 = t_1/2$ is an axis of symmetry for all positions; similarly, $t = (t_1 + T)/2$ is another axis of symmetry.

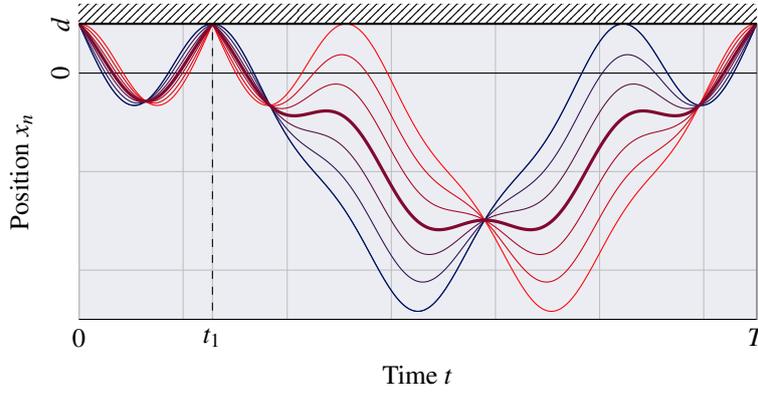


Figure 13: Continuum of 2-ipp admissible solutions with constant impact times. The continuum is delimited by the blue and the red curves, which both correspond to one impact and two grazing contacts. The thick purple curve has two axes of symmetry. All the other curves have no axis of symmetry, because Σ is not invertible (see Theorem 9.4).

If $\lambda_1 = -d/(2\alpha)$, then $\lambda_2 = \lambda_1$ and these two axes are axes of symmetry for the corresponding position.

- It can also be shown that the two limiting configurations $\lambda_1 = 0$ and $\lambda_2 = 0$ are associated to the same solution, but with a time-shift of σ_1 (and by interchangeability of σ_1 and σ_2 , of σ_2 too).
- The continuum of positions has fixed points, *i.e.* points common to all trajectories. For example,

$$x_p(t_1/2) = \lambda_1 \phi_{n,T}(\sigma_2) - \left(\frac{d}{\alpha} + \lambda_1\right) \phi_{p,T}(\sigma_1 - T) = -\frac{d}{\alpha} \phi_{n,T}(\sigma_2) \quad (10.30)$$

which does not depend on λ_1 .

- Lastly, positions x_p and velocities y_p are linear functions of λ_1 , for all $p \in \llbracket 1, n \rrbracket$. This means that the manifold of solutions is a ruled surface of the phase space.

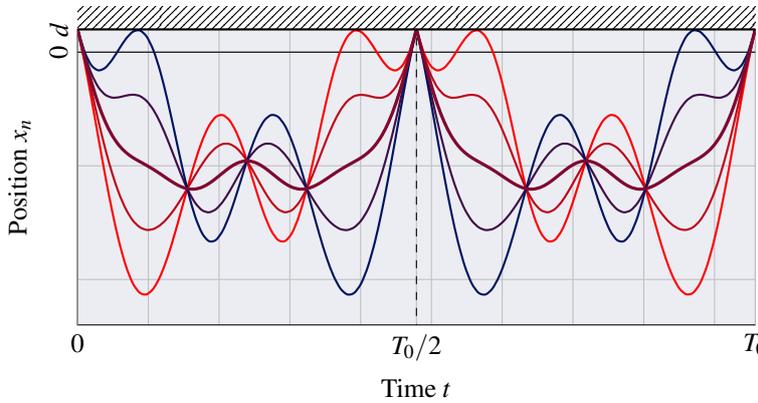


Figure 14: Continuum of 2-ipp admissible solutions with constant impact times and $t_1 = T_0/2$. The continuum includes a 1-ipp admissible solution [—]: it is a *bridge* between 1-ipp and 2-ipp solutions.

Interestingly, some manifolds with constant impact times exhibit the additional property of *joining* a 1-ipp manifold to a 2-ipp manifold. Such a situation is referred to as *bridge* between 1-ipp and 2-ipp and arises when $\alpha = \beta$. The positions then reduce to

$$x_p(\tau) = \begin{cases} \lambda_1 \phi_{p,T}(2\tau) - \left(\frac{d}{\alpha} + \lambda_1\right) \phi_{p,T}(2\tau - T) & \text{if } \tau \in [0, T/2], \\ \lambda_1 \phi_{p,T}(2\tau - T) - \left(\frac{d}{\alpha} + \lambda_1\right) \phi_{p,T}(2\tau) & \text{if } \tau \in (T/2, T]. \end{cases} \quad (10.31)$$

An illustration is provided in Fig. 14. The corresponding manifold is shown in Fig. 15. Such solutions have the following attributes:

- when $\lambda_1 = -d/(2\alpha)$, the solution is certainly a 1-ipp of period $T/2$,
- if this 1-ipp solution is admissible, it is generically part of a 1-ipp non-smooth mode (Theorem 3.3), hence its trajectory both belongs to a 2-ipp NSM and a 1-ipp NSM,
- all positions have an axis of symmetry at $t = T_0/2$.

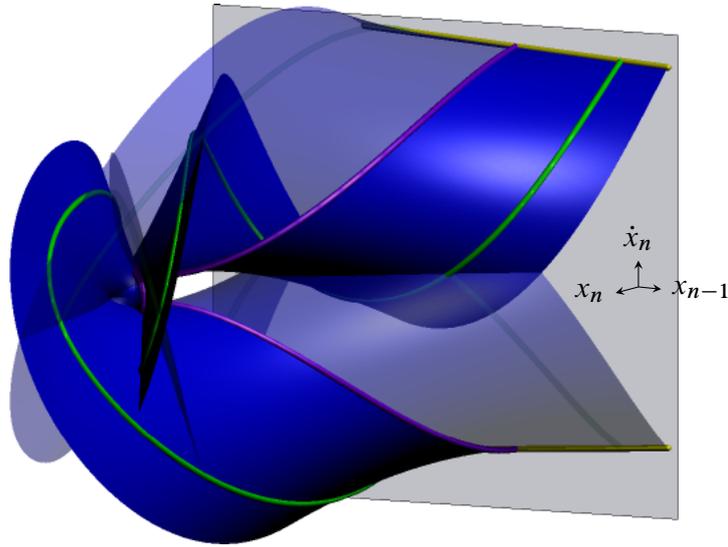


Figure 15: NSM bridge connecting 1-ipp and 2-ipp manifolds. All trajectories of this manifold share the same period. Opaque: part corresponding to $[0, T_0/2]$. Transparent: part corresponding to $[T_0/2, T_0]$. 1-ipp solution [—]. The purple curve surface intersects 1-ipp non-smooth mode which is not represented for clarity purposes.

10.4. Multiple-impact-per-period solutions In the more general case $k > 2$, the principle is identical to $k = 1$ or $k = 2$. Numerical difficulties emerge from the higher number of equations to be solved. From Theorems 3.1 and 3.2, it is known that potential solutions can be sought as impact times $s = (t_1, \dots, t_k)$ such that there exists a $\lambda \in \mathbb{R}^k$ solution of $\mathbf{\Pi}(s)\lambda = \mathbf{0}$ and $\mathbf{\Sigma}(s)\lambda = -d\mathbf{j}$. Following up on Remark 3.1 when k is odd, then generically $\text{rank}(\mathbf{\Pi}(s)) = k - 1$ and the kernel of $\mathbf{\Pi}$ is spanned by the vector $\mathbf{k}(s)$ of the Pfaffians of the diagonal $(k - 1) \times (k - 1)$ submatrices of $\mathbf{\Pi}$ with alternating signs. It therefore suffices to find s such that the k lines of $\mathbf{\Sigma}(s)\mathbf{k}(s)$ are all equal to the same arbitrary value $\mu \neq 0$ ($k - 1$ equations) and for such an s , λ is given by $\lambda = -d\mathbf{k}(s)/\mu$. In practice, one possibility to find admissible solutions is to seek s with a root-finding numerical solver starting from a random initial guess. The vector $\lambda = -d\mathbf{\Sigma}^{-1}(s)\mathbf{j}$ corresponding to the numerical solution s must have only non-negative components. If satisfied, the only remaining condition is that the unilateral constraint should not be violated during the free flights, which has to be checked numerically as well. When such a point is found, Theorem 3.3 ensures that s is a point of a curve \mathcal{S} of impact times corresponding to admissible solutions. This curve can be computed using continuation techniques [2].

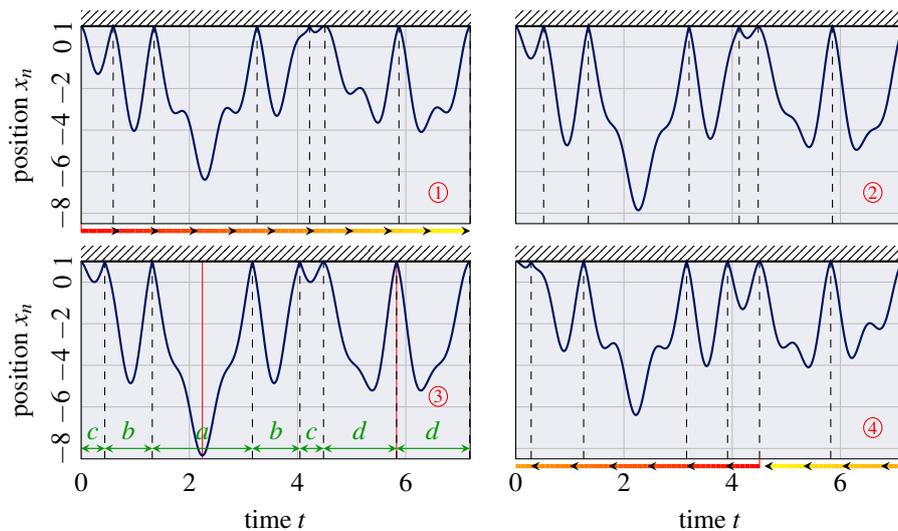


Figure 16: Continuum of 7-ipp admissible solutions. Bottom-left: symmetric motion of free-flight durations c, b, a, b, c, d, d . The bottom horizontal arrows in ① and ④ illustrate the relationship between the two trajectories: they correspond to the same orbit but with different time-orientation.

Fig. 16 depicts four positions of a 7-ipp non-smooth mode, identified through the procedure described above. Diagrams ① and ④ represent limiting cases of the admissible curve: indeed, they both have a

zero pre-impact velocity, which would become negative if s followed the curve of *potential* solutions. In other words, ① and ④ each have a grazing contact (the fifth and the seventh, respectively); on the outer-neighbourhood of the admissible domain, the motion violates the contact condition.

Interestingly, ③ exhibits an axis of symmetry. Indeed, the periodic sequence of free-flight durations $\sigma_1, \sigma_2, \dots, \sigma_7$ is of the form c, b, a, b, c, d, d , which is invariant by the transformation $n \mapsto 6 - n$. Invoking Theorem 9.4 (or its Corollary 9.2), it follows that the positions have two axes of symmetry per period: $t = (t_2 + t_3)/6$ and $t = t_6$, as observed in diagram ③.

Lastly, ① and ④ correspond to the same curve in the phase space, one parametrised in increasing time and the other one, in decreasing time (see arrows in Fig. 16, ① and ④).

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