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# Optimal scaling of the Random Walk Metropolis algorithm under $L^p$ mean differentiability

Alain Durmus\*   Sylvain Le Corff†   Eric Moulines‡   Gareth O. Roberts§

## Abstract

This paper considers the optimal scaling problem for high-dimensional random walk Metropolis algorithms for densities which are differentiable in  $L^p$  mean but which may be irregular at some points (like the Laplace density for example) and / or are supported on an interval. Our main result is the weak convergence of the Markov chain (appropriately rescaled in time and space) to a Langevin diffusion process as the dimension  $d$  goes to infinity. Because the log-density might be non-differentiable, the limiting diffusion could be singular. The scaling limit is established under assumptions which are much weaker than the one used in the original derivation of [6]. This result has important practical implications for the use of random walk Metropolis algorithms in Bayesian frameworks based on sparsity inducing priors.

## 1 Introduction

A wealth of contributions have been devoted to the study of the behaviour of high-dimensional Markov chains. One of the most powerful approaches for that purpose is the scaling analysis, introduced by [6]. Assume that the target distribution has a density with respect to the  $d$ -dimensional Lebesgue measure given by:

$$\pi^d(x^d) = \prod_{i=1}^d \pi(x_i^d). \quad (1)$$

The Random Walk Metropolis-Hastings (RWM) updating scheme was first applied in [4] and proceeds as follows. Given the current state  $X_k^d$ , a new value  $Y_{k+1}^d = (Y_{k+1,i}^d)_{i=1}^d$  is obtained by moving independently each coordinate, i.e.  $Y_{k+1,i}^d = X_{k,i}^d + \ell d^{-1/2} Z_{k+1}^d$  where  $\ell > 0$  is a scaling factor and  $(Z_k)_{k \geq 1}$  is a sequence of independent and identically distributed (i. i. d.) Gaussian random variables. Here  $\ell$  governs the overall size of the proposed jump and plays a crucial role in determining the efficiency of the algorithm. The proposal is then accepted or rejected according to the acceptance probability  $\alpha(X_k^d, Y_{k+1}^d)$  where  $\alpha(x^d, y^d) = 1 \wedge \pi^d(y^d)/\pi^d(x^d)$ . If the proposed value is accepted it becomes the next current value, otherwise the current value is left unchanged:

$$X_{k+1}^d = X_k^d + \ell d^{-1/2} Z_{k+1}^d \mathbb{1}_{\mathcal{A}_{k+1}^d}, \quad (2)$$

$$\mathcal{A}_{k+1}^d = \left\{ U_{k+1} \leq \prod_{i=1}^d \pi(X_{k,i}^d + \ell d^{-1/2} Z_{k+1,i}^d) / \pi(X_{k,i}^d) \right\}, \quad (3)$$

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where  $(U_k)_{k \geq 1}$  of i. i. d. uniform random variables on  $[0, 1]$  independent of  $(Z_k)_{k \geq 1}$ .

Under certain regularity assumptions on  $\pi$ , it has been proved in [6] that if the  $X_0^d$  is distributed under the stationary distribution  $\pi^d$ , then each component of  $(X_k^d)_{k \geq 0}$  appropriately rescaled in time converges weakly to a Langevin diffusion process with invariant distribution  $\pi$  as  $d \rightarrow +\infty$ .

This result allows to compute the asymptotic mean acceptance rate and to derive a practical rule to tune the factor  $\ell$ . It is shown in [6] that the speed of the limiting diffusion has a function of  $\ell$  has a unique maximum. The corresponding mean acceptance rate in stationarity is equal to 0.234.

These results have been derived for target distributions of the form (1) where  $\pi(x) \propto \exp(-V(x))$  where  $V$  is three-times continuously differentiable. Therefore, they do not cover the cases where the target density is continuous but not smooth, for example the Laplace distribution which plays a key role as a sparsity-inducing prior in high-dimensional Bayesian inference.

The aim of this paper is to extend the scaling results for the RWM algorithm introduced in the seminal paper [6, Theorem 3] to densities which are absolutely continuous densities differentiable in  $L^p$  mean (DLM) for some  $p \geq 2$  but can be either non-differentiable at some points or are supported on an interval. As shown in [3, Section 17.3], differentiability of the square root of the density in  $L^2$  norm implies a quadratic approximation property for the log-likelihood known as local asymptotic normality. As shown below, the DLM permits the quadratic expansion of the log-likelihood without paying the twice-differentiability price usually demanded by such a Taylor expansion (such expansion of the log-likelihood plays a key role in [6]).

The paper is organised as follows. In Section 2 the target density  $\pi$  is assumed to be positive on  $\mathbb{R}$ . Theorem 2 proves that under the DLM assumption of this paper, the average acceptance rate and the expected square jump distance are the same as in [6]. Theorem 3 shows that under the same assumptions the rescaled in time Markov chain produced by the RWM algorithm converges weakly to a Langevin diffusion. We show that these results may be applied to a density of the form  $\pi(x) \propto \exp(-\lambda|x| + U(x))$ , where  $\lambda \geq 0$  and  $U$  is a smooth function. In Section 3, we focus on the case where  $\pi$  is supported only on an open interval of  $\mathbb{R}$ . Under appropriate assumptions, Theorem 4 and Theorem 5 show that the same asymptotic results (limiting average acceptance rate and limiting Langevin diffusion associated with  $\pi$ ) hold. We apply our results to Gamma and Beta distributions. The proofs are postponed to Section 4 and Section 5.

## 2 Positive Target density on $\mathbb{R}$

The key of the proof of our main result will be to show that the acceptance ratio and the expected square jump distance converge to a finite and non trivial limit. In the original proof of [6], the density of the product form (1) with

$$\pi(x) \propto \exp(-V(x)) \quad (4)$$

is three-times continuously differentiable and the acceptance ratio is expanded using the usual pointwise Taylor formula. More precisely, the log-ratio of the density evaluated at the proposed value and at the current state is given by  $\sum_{i=1}^d \Delta V_i^d$  where

$$\Delta V_i^d = V(X_i^d) - V\left(X_i^d + \ell d^{-1/2} Z_i^d\right), \quad (5)$$

and where  $X^d$  is distributed according to  $\pi^d$  and  $Z^d$  is a  $d$ -dimensional standard Gaussian random variable independent of  $X$ . Heuristically, the two leading terms are  $\ell d^{-1/2} \sum_{i=1}^d \dot{V}(X_i^d) Z_i^d$

and  $\ell^2 d^{-1} \sum_{i=1}^d \dot{V}(X_i^d)(Z_i^d)^2/2$ , where  $\dot{V}$  and  $\ddot{V}$  are the first and second derivatives of  $V$ , respectively. By the central limit theorem, this expression converges in distribution to a zero-mean Gaussian random variable with variance  $\ell^2 I$  where

$$I = \int_{\mathbb{R}} \dot{V}^2(x) \pi(x) dx. \quad (6)$$

Note that  $I$  is the Fisher information associated with the translation model  $\theta \mapsto \pi(x + \theta)$  evaluated at  $\theta = 0$ . Under appropriate technical conditions, the second term converges almost surely to  $-\ell^2 I/2$ . Assuming that these limits exist, the acceptance ratio in the RWM algorithm converges to  $\mathbb{E}[1 \wedge \exp(Z)]$  where  $Z$  is a Gaussian random variable with mean  $-\ell^2 I/2$  and variance  $\ell^2 I$ ; elementary computations show that  $\mathbb{E}[1 \wedge \exp(Z)] = 2\Phi(-\ell/2 \sqrt{I})$ , where  $\Phi$  stands for the cumulative distribution function of a standard normal distribution.

For  $t \geq 0$ , denote by  $Y_t^d$  the linear interpolation of the Markov chain  $(X_k^d)_{k \geq 0}$  after time rescaling:

$$Y_t^d = (\lceil dt \rceil - dt) X_{\lceil dt \rceil}^d + (dt - \lfloor dt \rfloor) X_{\lfloor dt \rfloor}^d \quad (7)$$

$$= X_{\lfloor dt \rfloor}^d + (dt - \lfloor dt \rfloor) \ell d^{-1/2} Z_{\lceil dt \rceil}^d \mathbf{1}_{\mathcal{A}_{\lceil dt \rceil}^d}, \quad (8)$$

where  $\lfloor \cdot \rfloor$  and  $\lceil \cdot \rceil$  denote the lower and the upper integer part functions. Note that for all  $k \geq 0$ ,  $Y_{k/d}^d = X_k^d$ . Denote by  $(B_t, t \geq 0)$  the standard Brownian motion.

**Theorem 1** ([6]). *Suppose that the target  $\pi^d$  and the proposal distribution are given by (1)-(4) and (2) respectively. Assume that*

- (i)  $V$  is twice continuously differentiable and  $\dot{V}$  is Lipschitz continuous ;
- (ii)  $\mathbb{E}[(\dot{V}(X))^8] < \infty$  and  $\mathbb{E}[(\dot{V}(X))^4] < \infty$  where  $X$  is distributed according to  $\pi$ .

Then  $(Y_{t,1}^d, t \geq 0)$ , where  $Y_{t,1}^d$  is the first component of the vector  $Y_t^d$  defined in (7), converges weakly in the Wiener space (equipped with the uniform topology) to the Langevin diffusion

$$dY_t = \sqrt{h(\ell)} dB_t - \frac{1}{2} h(\ell) \dot{V}(Y_t) dt, \quad (9)$$

where  $Y_0$  is distributed according to  $\pi$ ,  $h(\ell)$  is given by

$$h(\ell) = 2\ell^2 \Phi\left(-\frac{\ell}{2} \sqrt{I}\right), \quad (10)$$

and  $I$  is defined in (6).

Whereas  $V$  is assumed to be twice continuously differentiable, the dual representation of the Fisher information  $-\mathbb{E}[\ddot{V}(X)] = \mathbb{E}[(\dot{V}(X))^2] = I$  allows us to remove in the statement of the theorem all mention to the second derivative of  $V$ , which hints that two derivatives might not really be required. For all  $\theta, x \in \mathbb{R}$ , define

$$\xi_\theta(x) = \sqrt{\pi(x + \theta)}, \quad (11)$$

For  $p \geq 1$ , denote  $\|f\|_{\pi,p}^p = \int |f(x)|^p \pi(x) dx$ . Consider the following assumptions:

**H1.** *There exists a measurable function  $\dot{V} : \mathbb{R} \rightarrow \mathbb{R}$  such that:*

(i) There exist  $p > 4$ ,  $C > 0$  and  $\beta > 1$  such that for all  $\theta \in \mathbb{R}$ ,

$$\left\| V(\cdot + \theta) - V(\cdot) - \theta \dot{V}(\cdot) \right\|_{\pi, p} \leq C|\theta|^\beta.$$

(ii) The function  $\dot{V}$  satisfies  $\left\| \dot{V} \right\|_{\pi, 6} < +\infty$ .

**Lemma 1.** Assume **H1**. Then, the family of densities  $\theta \rightarrow \pi(\cdot + \theta)$  is Differentiable in Quadratic Mean (DQM) at  $\theta = 0$  with derivative  $\dot{V}$ , i.e. there exists  $C > 0$  such that for all  $\theta \in \mathbb{R}$ ,

$$\left( \int_{\mathbb{R}} \left( \xi_\theta(x) - \xi_0(x) + \theta \dot{V}(x) \xi_0(x) / 2 \right)^2 dx \right)^{1/2} \leq C|\theta|^\beta,$$

where  $\xi_\theta$  is given by (11).

*Proof.* The proof is postponed to Section 4.1.  $\square$

The first step in the proof is to show that the acceptance ratio  $\mathbb{P}(\mathcal{A}_1^d) = \mathbb{E}(1 \wedge \exp\{\sum_{i=1}^d \Delta V_i^d\})$ , and the expected square jump distance  $\mathbb{E}[(Z_1^d)^2 \{1 \wedge \exp(\sum_{i=1}^d \Delta V_i^d)\}]$  both converge to a finite value. To that purpose, we consider

$$\mathbb{E}^d(q) = \mathbb{E} \left[ (Z_1^d)^q \left| 1 \wedge \exp \left( \sum_{i=1}^d \Delta V_i^d \right) - 1 \wedge \exp(v^d) \right| \right],$$

where  $\Delta V_i^d$  is given by (5),

$$v^d = -\ell d^{-1/2} Z_1^d \dot{V}(X_1^d) + \sum_{i=2}^d b^d(X_i^d, Z_i^d) \quad (12)$$

$$b^d(x, z) = -\frac{\ell z}{\sqrt{d}} \dot{V}(x) + \mathbb{E} [2\zeta^d(X_1^d, Z_1^d)] - \frac{\ell^2}{4d} \dot{V}^2(x), \quad (13)$$

$$\zeta^d(x, z) = \exp \left\{ \left( V(x) - V(x + \ell d^{-1/2} z) \right) / 2 \right\} - 1. \quad (14)$$

**Proposition 1.** Assume **H1** holds. Let  $X^d$  be a random variable distributed according to  $\pi^d$  and  $Z^d$  be a zero-mean standard Gaussian random variable, independent of  $X^d$ . Then, for any  $q \geq 0$ ,  $\lim_{d \rightarrow +\infty} \mathbb{E}^d(q) = 0$ .

*Proof.* The proof is postponed to Section 4.2.  $\square$

Proposition 1 shows that it is enough to consider  $v^d$  to analyse the asymptotic behaviour of the acceptance ratio and the expected square jump distance as  $d \rightarrow +\infty$ . By the central limit theorem, the term  $-\ell \sum_{i=2}^d (Z_i^d / \sqrt{d}) \dot{V}(X_i^d)$  in (12) converges in distribution to a zero-mean Gaussian random variable with variance  $\ell^2 I$ , where  $I$  is defined in (6). By Lemma 4 (Section 4.3), the second term, which is  $d \mathbb{E} [2\zeta^d(X_1^d, Z_1^d)] = -d \mathbb{E} [(\zeta^d(X_1^d, Z_1^d))^2]$  converges to  $-\ell^2 I/4$ . The last term converges in probability to  $-\ell^2 I/4$ . Therefore, the two last terms plays a similar role in the expansion of the acceptance ratio as the second derivative of  $V$  in the regular case.

**Theorem 2.** Assume **H1** holds. Then,  $\lim_{d \rightarrow +\infty} \mathbb{P}[\mathcal{A}_1^d] = a(\ell)$ , where  $a(\ell) = 2\Phi(-\sqrt{I}\ell/2)$ .

*Proof.* The proof is postponed to Section 4.3.  $\square$

The second result of this paper is that the sequence  $\{(Y_{t,1}^d)_{t \geq 0}, d \in \mathbb{N}^*\}$  defined by (7) converges weakly to a Langevin diffusion. Let  $(\mu_d)_{d \geq 1}$  be the sequence of distributions of  $\{(Y_{t,1}^d)_{t \geq 0}, d \in \mathbb{N}^*\}$ .

**Proposition 2.** *Assume **H1** holds. Then, the sequence  $(\mu_d)_{d \geq 1}$  is tight in  $\mathbf{W}$ .*

*Proof.* The proof is adapted from [2]; it is postponed to Section 4.4.  $\square$

By the Prohorov theorem, the tightness of  $(\mu_d)_{d \geq 1}$  implies that this sequence has a weak limit point. We now prove that any limit point is the law of a solution to (9). For that purpose, we use the equivalence between the weak formulation of stochastic differential equations and martingale problems. The generator  $L$  of the Langevin diffusion (9) is given, for all  $\phi \in C_c^2(\mathbb{R}, \mathbb{R})$ , by

$$L\phi(x) = \frac{h(\ell)}{2} \left( -\dot{V}(x)\dot{\phi}(x) + \ddot{\phi}(x) \right), \quad (15)$$

where for  $k \in \mathbb{N}$  and  $I$  an open subset of  $\mathbb{R}$ ,  $C_c^k(I, \mathbb{R})$  is the space of  $k$ -times differentiable functions with compact support, endowed with the topology of uniform convergence of all derivatives up to order  $k$ . We set  $C_c^\infty(I, \mathbb{R}) = \bigcap_{k=0}^\infty C_c^k(I, \mathbb{R})$  and  $\mathbf{W} = C(\mathbb{R}_+, \mathbb{R})$ . The canonical process is denoted by  $(W_t)_{t \geq 0}$  and  $(\mathcal{B}_t)_{t \geq 0}$  is the associated filtration. For any probability measure  $\mu$  on  $\mathbf{W}$ , the expectation with respect to  $\mu$  is denoted by  $\mathbb{E}^\mu$ . A probability measure  $\mu$  on  $\mathbf{W}$  is said to solve the martingale problem associated with (9) if the pushforward of  $\mu$  by  $W_0$  is  $\pi$  and if for all  $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ , the process

$$\left( \phi(W_t) - \phi(W_0) - \int_0^t L\phi(W_u) du \right)_{t \geq 0}$$

is a martingale with respect to  $\mu$  and the filtration  $(\mathcal{B}_t)_{t \geq 0}$ , i.e. if for all  $s, t \in \mathbb{R}_+, s \leq t$ ,  $\mu$ -a.s.

$$\mathbb{E}^\mu \left[ \phi(W_t) - \phi(W_0) - \int_0^t L\phi(W_u) du \middle| \mathcal{B}_s \right] = \phi(W_s) - \phi(W_0) - \int_0^s L\phi(W_u) ds.$$

**H2.** *The function  $\dot{V}$  is continuous on  $\mathbb{R}$  except on a Lebesgue-negligible set  $\mathcal{D}_{\dot{V}}$  and is bounded on all compact sets of  $\mathbb{R}$ .*

If  $\dot{V}$  satisfies **H2**, [7, Lemma 1.9, Theorem 20.1 Chapter 5] show that any solution to the martingale problem associated with (9) coincides with the law of a solution to the SDE (9), and conversely. Therefore, uniqueness in law of weak solutions to (9) implies uniqueness of the solution of the martingale problem.

**Proposition 3.** *Assume **H2** holds. Assume also that for all  $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ ,  $m \in \mathbb{N}^*$ ,  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  bounded and continuous, and  $0 \leq t_1 \leq \dots \leq t_m \leq s \leq t$ :*

$$\lim_{d \rightarrow +\infty} \mathbb{E}^{\mu^d} \left[ \left( \phi(W_t) - \phi(W_s) - \int_s^t L\phi(W_u) du \right) g(W_{t_1}, \dots, W_{t_m}) \right] = 0. \quad (16)$$

*Then, every limit point of the sequence of probability measures  $(\mu_d)_{d \geq 1}$  on  $\mathbf{W}$  is a solution to the martingale problem associated with (9).*

*Proof.* The proof is postponed to Section 4.5.  $\square$

**Theorem 3.** *Assume **H1** and **H2** hold. Assume also that (9) has a unique weak solution. Then,  $\{(Y_{t,1}^d)_{t \geq 0}, d \in \mathbb{N}^*\}$  converges weakly to the solution  $(Y_t)_{t \geq 0}$  of the Langevin equation defined by (9). Furthermore,  $h(\ell)$  is maximized at the unique value of  $\ell$  for which  $a(\ell) = 0.234$ , where  $a$  is defined in Theorem 2.*

*Proof.* The proof is postponed to Section 4.6.  $\square$

**Example 1** (Bayesian Lasso). *We apply the results obtained above to a target density  $\pi$  on  $\mathbb{R}$  given by  $x \mapsto e^{-V(x)} / \int_{\mathbb{R}} e^{-V(y)} dy$  where  $V$  is given by*

$$V : x \mapsto U(x) + \lambda |x| ,$$

where  $\lambda \geq 0$  and  $U$  is twice continuously differentiable with bounded second derivative. Furthermore,  $\int_{\mathbb{R}} |x|^6 e^{-V(x)} dx < +\infty$ . Define  $\dot{V} : x \mapsto U'(x) + \lambda \text{sign}(x)$ , with  $\text{sign}(x) = -1$  if  $x \leq 0$  and  $\text{sign}(x) = 1$  otherwise. We first check that **H1**(i) holds. Note that for all  $x, y \in \mathbb{R}$ ,

$$||x + y| - |x| - \text{sign}(x)y| \leq 2|y| \mathbf{1}_{\mathbb{R}_+}(|y| - |x|), \quad (17)$$

which implies that, for any  $p \geq 1$ , there exists  $C_p$  such that

$$\begin{aligned} \left\| V(\cdot + \theta) - V(\cdot) - \theta \dot{V}(\cdot) \right\|_{\pi, p} &\leq \|U(\cdot + \theta) - U(\cdot) - \theta U'(\cdot)\|_{\pi, p} + \lambda \| |\cdot + \theta| - |\cdot| - \theta \text{sign}(\cdot) \|_{\pi, p} \\ &\leq \|U''\|_{\infty} \theta^2 + 2 |\theta| \lambda \{\pi([- \theta, \theta])\}^{1/p} \leq C |\theta|^{p+1/p} \vee |\theta|^2 . \end{aligned}$$

Assumptions **H1**(ii) and **H2** are easy to check. The uniqueness in law of (9) is established in [1, Theorem 4.5 (i)]. Therefore, Theorem 3 can be applied.

### 3 Target density supported on an interval of $\mathbb{R}$

We now extend our results to densities supported by a open interval  $\mathcal{I} \subset \mathbb{R}$  :

$$\pi(x) \propto \exp(-V(x)) \mathbf{1}_{\mathcal{I}}(x) ,$$

where  $V : \mathcal{I} \rightarrow \mathbb{R}$  is a measurable function. Note that by convention  $V(x) = -\infty$  for all  $x \notin \mathcal{I}$ . Denote by  $\bar{\mathcal{I}}$  the closure of  $\mathcal{I}$  in  $\mathbb{R}$ . The results of Section 2 cannot be directly used in such a case, as  $\pi$  is no longer positive on  $\mathbb{R}$ . Consider the following assumption.

**G1.** *There exists a measurable function  $\dot{V} : \mathcal{I} \rightarrow \mathbb{R}$  and  $r > 1$  such that:*

(i) *There exist  $p > 4$ ,  $C > 0$  and  $\beta > 1$  such that for all  $\theta \in \mathbb{R}$ ,*

$$\left\| \{V(\cdot + \theta) - V(\cdot)\} \mathbf{1}_{\mathcal{I}}(\cdot + r\theta) \mathbf{1}_{\mathcal{I}}(\cdot + (1-r)\theta) - \theta \dot{V}(\cdot) \right\|_{\pi, p} \leq C |\theta|^\beta ,$$

with the convention  $0 \times \infty = 0$ .

(ii) *The function  $\dot{V}$  satisfies  $\|\dot{V}\|_{\pi, 6} < +\infty$ .*

(iii) *There exist  $\gamma \geq 6$  and  $C > 0$  such that, for all  $\theta \in \mathbb{R}$ ,*

$$\int_{\mathbb{R}} \mathbf{1}_{\mathcal{I}^c}(x + \theta) \pi(x) dx \leq C |\theta|^\gamma .$$

As an important consequence of **G1**(iii), if  $X$  is distributed according to  $\pi$  and is independent of the standard random variable  $Z$ , there exists a constant  $C$  such that

$$\mathbb{P} \left( X + \ell d^{-1/2} Z \in \mathcal{I}^c \right) \leq C d^{-\gamma/2} . \quad (18)$$

**Theorem 4.** *Assume **G1** holds. Then,  $\lim_{d \rightarrow +\infty} \mathbb{P}[\mathcal{A}_1^d] = a(\ell)$ , where  $a(\ell) = 2\Phi(-\sqrt{\ell}/2)$ .*

*Proof.* The proof is postponed to Section 5.1.  $\square$

We now established the weak convergence of the sequence  $\{(Y_{t,1}^d)_{t \geq 0}, d \in \mathbb{N}^*\}$ , following the same steps as for the proof of Theorem 3. Denote for all  $d \geq 1$ ,  $\mu_d$  the law of the process  $(Y_{t,1}^d)_{t \geq 0}$ .

**Proposition 4.** *Assume **G1** holds. Then, the sequence  $(\mu_d)_{d \geq 1}$  is tight in  $\mathbf{W}$ .*

*Proof.* The proof is postponed to Section 5.2.  $\square$

Contrary to the case where  $\pi$  is positive on  $\mathbb{R}$ , we do not assume that  $\dot{V}$  is bounded on all compact sets of  $\mathbb{R}$ . Therefore, we consider the local martingale problem associated with (9): with the notations of Section 2, a probability measure  $\mu$  on  $\mathbf{W}$  is said to solve the local martingale problem associated with (9) if the pushforward of  $\mu$  by  $W_0$  is  $\pi$  and if for all  $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ , the process

$$\left( \psi(W_t) - \psi(W_0) - \int_0^t L\psi(W_u) du \right)_{t \geq 0}$$

is a local martingale with respect to  $\mu$  and the filtration  $(\mathcal{B}_t)_{t \geq 0}$ . By [1, Theorem 1.27], any solution to the local martingale problem associated with (9) coincides with the law of a solution to the SDE (9) and conversely. If (9) admits a unique solution in law, this law is the unique solution to the local martingale problem associated with (9). We first prove that any limit point  $\mu$  of  $(\mu_d)_{d \geq 1}$  is a solution to the local martingale problem associated with (9).

**G2.** *The function  $\dot{V}$  is continuous on  $\mathcal{I}$  except on a null-set  $\mathcal{D}_{\dot{V}}$ , with respect to the Lebesgue measure, and is bounded on all compact sets of  $\mathcal{I}$ .*

This condition does not preclude that  $\dot{V}$  is unbounded at the boundary of  $\mathcal{I}$ .

**Proposition 5.** *Assume **G1** and **G2** hold. Assume also that for all  $\phi \in C_c^\infty(\mathcal{I}, \mathbb{R})$ ,  $m \in \mathbb{N}^*$ ,  $g: \mathbb{R}^m \rightarrow \mathbb{R}$  bounded and continuous, and  $0 \leq t_1 \leq \dots \leq t_m \leq s \leq t$ :*

$$\lim_{d \rightarrow +\infty} \mathbb{E}^{\mu_d} \left[ \left( \phi(W_t) - \phi(W_s) - \int_s^t L\phi(W_u) du \right) g(W_{t_1}, \dots, W_{t_m}) \right] = 0. \quad (19)$$

*Then, every limit point of the sequence of probability measures  $(\mu_d)_{d \geq 1}$  on  $\mathbf{W}$  is a solution to the local martingale problem associated with (9).*

*Proof.* The proof is postponed to Section 5.3.  $\square$

**Theorem 5.** *Assume **G1** and **G2** hold. Assume also that (9) has a unique weak solution. Then,  $\{(Y_{t,1}^d)_{t \geq 0}, d \in \mathbb{N}^*\}$  converges weakly to the solution  $(Y_t)_{t \geq 0}$  of the Langevin equation defined by (9). Furthermore,  $h(\ell)$  is maximized at the unique value of  $\ell$  for which  $a(\ell) = 0.234$ , where  $a$  is defined in Theorem 2.*

*Proof.* The proof is along the same lines as the proof of Theorem 3 and is postponed to Section 5.4.  $\square$

The conditions for uniqueness in law of singular one-dimensional stochastic differential equations are given in [1]. These conditions are rather involved and difficult to summarize in full generality. We rather illustrate Theorem 5 by two examples.



**Example 2** (Application to the Gamma distribution). *Define the class of the generalized Gamma distributions as the family of densities on  $\mathbb{R}$  given by*

$$\pi_\gamma : x \mapsto x^{a_1-1} \exp(-x^{a_2}) \mathbf{1}_{\mathbb{R}_+^*}(x) / \int_{\mathbb{R}_+^*} y^{a_1-1} \exp(-y^{a_2}) dy,$$

with two parameters  $a_1 > 6$  and  $a_2 > 0$ . Note that in this case  $\mathcal{I} = \mathbb{R}_+^*$ , for all  $x \in \mathcal{I}$ ,  $V_\gamma : x \mapsto x^{a_2} - (a_1 - 1) \log x$  and  $\dot{V}_\gamma : x \mapsto a_2 x^{a_2-1} - (a_1 - 1)/x$ . We check that **G1** holds with  $r = 3/2$ . First, we show that **G1(i)** holds with  $p = 5$ . Write for all  $\theta \in \mathbb{R}$  and  $x \in \mathcal{I}$ ,

$$\{V_\gamma(x + \theta) - V_\gamma(x)\} \mathbf{1}_{\mathcal{I}}(x + (1-r)\theta) \mathbf{1}_{\mathcal{I}}(x + r\theta) - \theta \dot{V}_\gamma(x) = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3,$$

where

$$\begin{aligned} \mathcal{E}_1 &= \theta \dot{V}_\gamma(x) \{ \mathbf{1}_{\mathcal{I}}(x - \theta/2) \mathbf{1}_{\mathcal{I}}(x + 3\theta/2) - 1 \}, \\ \mathcal{E}_2 &= (1 - a_1) \{ \log(1 + \theta/x) - \theta/x \} \mathbf{1}_{\mathcal{I}}(x - \theta/2) \mathbf{1}_{\mathcal{I}}(x + 3\theta/2), \\ \mathcal{E}_3 &= ((x + \theta)^{a_2} - x^{a_2} - a_2 \theta x^{a_2-1}) \mathbf{1}_{\mathcal{I}}(x - \theta/2) \mathbf{1}_{\mathcal{I}}(x + 3\theta/2). \end{aligned}$$

It is enough to prove that there exists  $q > 5$  such that for all  $i \in \{1, 2, 3\}$ ,  $\int_{\mathcal{I}} |\mathcal{E}_i|^5 \pi_\gamma(x) dx \leq C|\theta|^q$ . The result is proved for  $\theta < 0$  (the proof for  $\theta > 0$  follows the same lines). For all  $\theta \in \mathbb{R}$  using  $a_1 > 6$ ,

$$\begin{aligned} \int_{\mathbb{R}_+^*} |\mathcal{E}_1|^5 \pi_\gamma(x) dx &\leq C|\theta|^5 \int_0^{3|\theta|/2} \left\{ 1/x^5 + x^{5(a_2-1)} \right\} x^{a_1-1} e^{-x^{a_2}} dx, \\ &\leq C|\theta|^{a_1} \left( \int_0^{3/2} x^{a_1-6} e^{-(|\theta|x)^{a_2}} dx + |\theta|^{5a_2} \int_0^{3/2} x^{5(a_2-1)+a_1-1} e^{-(|\theta|x)^{a_2}} dx \right), \\ &\leq C(|\theta|^{a_1} + |\theta|^{5a_2+a_1}). \end{aligned} \quad (20)$$

On the other hand, as for all  $x > -1$ ,  $x/(x+1) \leq \log(1+x) \leq x$ , for all  $\theta < 0$ , and  $x \geq 3|\theta|/2$ ,

$$|\log(1 + \theta/x) - \theta/x| \leq \frac{|\theta|^2}{x^2(1 + \theta/x)} \leq 3|\theta|^2/x^2,$$

where the last inequality come from  $|\theta|/x \leq 2/3$ . Then, it yields

$$\begin{aligned} \int_{\mathbb{R}_+^*} |\mathcal{E}_2(x)|^5 \pi_\gamma(x) dx &\leq C|\theta|^{10} \left( \int_{3|\theta|/2}^1 x^{a_1-11} e^{-x^{a_2}} dx + \int_1^{+\infty} x^{a_1-11} e^{-x^{a_2}} dx \right), \\ &\leq C(|\theta|^{a_1} + |\theta|^{10}). \end{aligned} \quad (21)$$

For the last term, for all  $\theta < 0$  and all  $x \geq 3|\theta|/2$ , using a Taylor expansion of  $x \mapsto x^{a_2}$ , there exists  $\zeta \in [x + \theta, x]$  such that

$$|(x + \theta)^{a_2} - x^{a_2} - a_2 \theta x^{a_2-1}| \leq C|\theta|^2 |\zeta|^{a_2-2} \leq C|\theta|^2 |x|^{a_2-2}.$$

Then,

$$\int_{\mathbb{R}_+^*} |\mathcal{E}_3(x)|^5 \pi_\gamma(x) dx \leq C|\theta|^{10} \int_{3|\theta|/2}^{+\infty} x^{5(a_2-2)+a_1-1} e^{-x^{a_2}} dx \leq C(|\theta|^{5a_2+a_1} + |\theta|^{10}). \quad (22)$$

Combining (20), (21),(22) and using that  $a_1 > 6$  concludes the proof of **G1(i)** for  $p = 5$ . The proof of **G1(ii)** follows from

$$\int_{\mathbb{R}_+^*} |\dot{V}_\gamma(x)|^6 \pi_\gamma(x) dx \leq C \left( \int_{\mathbb{R}_+^*} x^{a_1-1+6(a_2-1)} e^{-x^{a_2}} dx + \int_{\mathbb{R}_+^*} x^{a_1-7} e^{-x^{a_2}} dx \right) < \infty$$

and **G1(iii)** follows from  $\int_{\mathbb{R}} \mathbb{1}_{\mathcal{I}^c}(x + \theta) \pi_\gamma(x) dx \leq C|\theta|^{a_1}$ . Now consider the Langevin equation associated with  $\pi_\gamma$  given by  $dY_t = -\dot{V}_\gamma(Y_t)dt + \sqrt{2}dB_t$  with initial distribution  $\pi_\gamma$ . This stochastic differential equation has 0 as singular point, which has right type 3 according to the terminology of [1]. On the other hand  $\infty$  has type A and the existence and uniqueness in law for the SDE follows from [1, Theorem 4.6 (viii)]. Since **G2** is straightforward, Theorem 5 can be applied.

**Example 3** (Application to the Beta distribution). Consider now the case of the Beta distributions  $\pi_\beta$  with density  $x \mapsto x^{a_1-1}(1-x)^{a_2-1} \mathbb{1}_{(0,1)}(x)$  with  $a_1, a_2 > 6$ . Here  $\mathcal{I} = (0, 1)$  and the log-density  $V_\beta$  and its derivative on  $\mathcal{I}$  are defined by  $V_\beta(x) = -(a_1 - 1) \log x - (a_2 - 1) \log(1 - x)$  and  $\dot{V}_\beta(x) = -(a_1 - 1)/x - (a_2 - 1)/(1 - x)$ . Along the same lines as above,  $\pi_\beta$  satisfies **G1** and **G2**. Hence Theorem 4 can be applied if we establish the uniqueness in law for the Langevin equation associated with  $\pi_\beta$  defined by  $dY_t = -\dot{V}_\beta(Y_t)dt + \sqrt{2}dB_t$  with initial distribution  $\pi_\beta$ . In the terminology of [1], 0 has right type 3 and 1 has left type 3. Therefore by [1, Theorem 2.16 (i), (ii)], the SDE has a global unique weak solution. To illustrate our findings, consider the Beta distribution with parameters  $a_1 = 10$  and  $a_2 = 10$ . Define the expected square distance by  $\text{ESJD}^d(\ell) = \mathbb{E} \left[ \|X_1^d - X_0^d\|^2 \right]$  where  $X_0^d$  has distribution  $\pi_\beta^d$  and  $X_1^d$  is the first iterate of the Markov chain defined by the Random Walk Metropolis algorithm given in (2). By Theorem 4 and Theorem 5, we have  $\lim_{d \rightarrow +\infty} \text{ESJD}^d(\ell) = h(\ell) = \ell^2 a(\ell)$ . Figure 1 displays an empirical estimation for the  $\text{ESJD}^d$  for dimensions  $d = 10, 50, 100$  as a function of the empirical mean acceptance rate. We can observe that as expected, the  $\text{ESJD}^d$  converges to some limit function as  $d$  goes infinity, and this function has a maximum for a mean acceptance probability around 0.23.

## 4 Proofs of Section 2

For any real random variable  $Y$  and any  $p \geq 1$ , let  $\|Y\|_p := \mathbb{E}[|Y|^p]^{1/p}$ .

### 4.1 Proof of Lemma 1

Let  $\Delta_\theta V(x) = V(x) - V(x + \theta)$ . By definition of  $\xi_\theta$  and  $\pi$ ,

$$\left( \xi_\theta(x) - \xi_0(x) + \theta \dot{V}(x) \xi_0(x) / 2 \right)^2 \leq 2 \{A_\theta(x) + B_\theta(x)\} \pi(x),$$

where

$$\begin{aligned} A_\theta(x) &= (\exp(\Delta_\theta V(x)/2) - 1 - \Delta_\theta V(x)/2)^2, \\ B_\theta(x) &= \left( \Delta_\theta V(x) + \theta \dot{V}(x) \right)^2 / 4. \end{aligned}$$

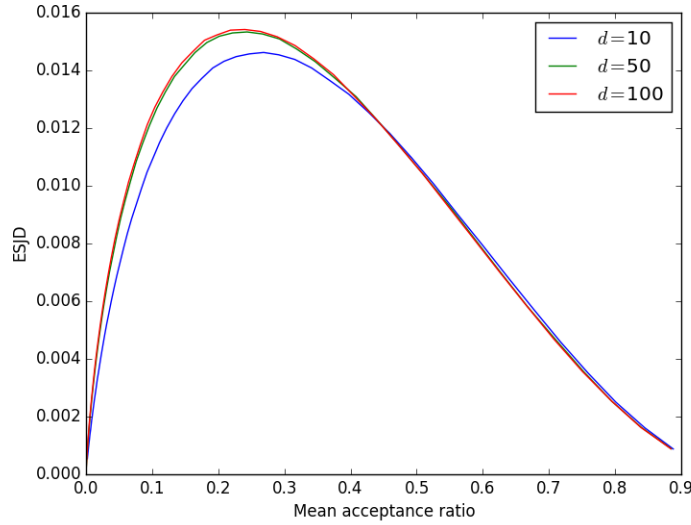


Figure 1: Expected square jumped distance for the beta distribution with parameters  $a_1 = 10$  and  $a_2 = 10$  as a function of the mean acceptance rate for  $d = 10, 50, 100$ .

By **H1**(i),  $\|B_\theta\|_{\pi,p} \leq C|\theta|^\beta$ . For  $A_\theta$ , note that for all  $x \in \mathbb{R}$ ,  $(\exp(x) - 1 - x)^2 \leq 2x^4(\exp(2x) + 1)$ . Then,

$$\begin{aligned} \int_{\mathbb{R}} A_\theta(x)\pi(x)dx &\leq C \int_{\mathbb{R}} \Delta_\theta V(x)^4 \left(1 + e^{\Delta_\theta V(x)}\right) \pi(x)dx \\ &\leq C \int_{\mathbb{R}} (\Delta_\theta V(x)^4 + \Delta_{-\theta} V(x)^4) \pi(x)dx. \end{aligned}$$

The proof is completed writing (the same inequality holds for  $\Delta_{-\theta}V$ ):

$$\int_{\mathbb{R}} \Delta_\theta V(x)^4 \pi(x)dx \leq C \left[ \int_{\mathbb{R}} (\Delta_\theta V(x) - \theta \dot{V}(x))^4 \pi(x)dx + \theta^4 \int_{\mathbb{R}} \dot{V}^4(x) \pi(x)dx \right]$$

and using **H1**(i)-(ii).

## 4.2 Proof of Proposition 1

Define

$$R(x) = \int_0^x \frac{(x-u)^2}{(1+u)^3} du. \quad (23)$$

$R$  is the remainder term of the Taylor expansion of  $x \mapsto \log(1+x)$ :

$$\log(1+x) = x - x^2/2 + R(x). \quad (24)$$

We preface the proof by the following Lemma.

**Lemma 2.** *Assume **H1** holds. Then, if  $X$  is a random variable distributed according to  $\pi$  and  $Z$  is a standard Gaussian random variable independent of  $X$ ,*

- (i)  $\lim_{d \rightarrow +\infty} d \left\| \zeta^d(X, Z) + \ell Z \dot{V}(X)/(2\sqrt{d}) \right\|_2^2 = 0.$
- (ii)  $\lim_{d \rightarrow +\infty} \sqrt{d} \left\| V(X) - V(X + \ell Z/\sqrt{d}) + \ell Z \dot{V}(X)/\sqrt{d} \right\|_p = 0.$
- (iii)  $\lim_{d \rightarrow \infty} d \left\| R(\zeta^d(X, Z)) \right\|_1 = 0,$

where  $\zeta^d$  is given by (14).

*Proof.* Using the definitions (11) and (14) of  $\zeta^d$  and  $\xi_\theta$ ,

$$\zeta^d(x, z) = \xi_{\ell z d^{-1/2}}(x)/\xi_0(x) - 1. \quad (25)$$

- (i) The proof follows from Lemma 1 using that  $\beta > 1$ :

$$\left\| \zeta^d(X, Z) + \ell Z \dot{V}(X)/(2\sqrt{d}) \right\|_2^2 \leq C \ell^{2\beta} d^{-\beta} \mathbb{E} [|Z|^{2\beta}].$$

- (ii) Using **H1**(i), we get that

$$\left\| V(X) - V(X + \ell Z/\sqrt{d}) + \ell Z \dot{V}(X)/\sqrt{d} \right\|_p^p \leq C \ell^{\beta p} d^{-\beta p/2} \mathbb{E} [|Z|^{\beta p}]$$

and the proof follows since  $\beta > 1$ .

- (iii) Note that for all  $x > 0$ ,  $u \in [0, x]$ ,  $|(x-u)(1+u)^{-1}| \leq |x|$ , and the same inequality holds for  $x \in (-1, 0]$  and  $u \in [x, 0]$ . Then by (23) and (24), for all  $x > -1$ ,  $|R(x)| \leq x^2 |\log(1+x)|$ .

Then by (50), setting  $\Psi_d(x, z) = R(\zeta^d(x, z))$

$$|\Psi_d(x, z)| \leq (\xi_{\ell z d^{-1/2}}(x)/\xi_0(x) - 1)^2 |V(x + \ell z d^{-1/2}) - V(x)|/2.$$

Since for all  $x \in \mathbb{R}$ ,  $|\exp(x) - 1| \leq |x|(\exp(x) + 1)$ , this yields,

$$|\Psi_d(x, z)| \leq 4^{-1} |V(x + \ell z d^{-1/2}) - V(x)|^3 \left( \exp(V(x) - V(x + \ell z d^{-1/2})) + 1 \right),$$

which implies that

$$\int_{\mathbb{R}} |\Psi_d(x, z)| \pi(x) dx \leq 4^{-1} \int_{\mathbb{R}} |V(x + \ell z d^{-1/2}) - V(x)|^3 \{\pi(x) + \pi(x + \ell z d^{-1/2})\} dx.$$

By Hölder's inequality and using **H1**(i),

$$\int_{\mathbb{R}} |\Psi_d(x, z)| \pi(x) dx \leq C \left( |\ell z d^{-1/2}|^3 \left( \int_{\mathbb{R}} |\dot{V}(x)|^4 \pi(x) dx \right)^{3/4} + |\ell z d^{-1/2}|^{3\beta} \right).$$

The proof follows from **H1**(ii) since  $\beta > 1$ . □

For all  $d \geq 1$ , let  $X^d$  be distributed according to  $\pi^d$ , and  $Z^d$  be  $d$ -dimensional Gaussian random variable independent of  $X^d$ , set

$$J^d = \left\| \sum_{i=2}^d \{ \Delta V_i^d - b^d(X_i^d, Z_i^d) \} \right\|_1,$$

where  $\Delta V_i^d$  and  $b^d$  are defined in (5) and (13), respectively.

**Lemma 3.**  $\lim_{d \rightarrow +\infty} J^d = 0$ .

*Proof.* Noting that  $\Delta V_i^d = 2 \log(1 + \zeta^d(X_i^d, Z_i^d))$  and using (24), we get

$$J^d \leq \sum_{i=1}^3 J_i^d = \left\| \sum_{i=2}^d 2\zeta^d(X_i^d, Z_i^d) + \frac{\ell Z_i^d}{\sqrt{d}} \dot{V}(X_i^d) - \mathbb{E}[2\zeta^d(X_i^d, Z_i^d)] \right\|_1 \\ + \left\| \sum_{i=2}^d \zeta^d(X_i^d, Z_i^d)^2 - \frac{\ell^2}{4d} \dot{V}^2(X_i^d) \right\|_1 + 2 \left\| \sum_{i=2}^d R(\zeta^d(X_i^d, Z_i^d)) \right\|_1,$$

where  $R$  is defined by (23). By Lemma 2(i), the first term goes to 0 as  $d$  goes to  $+\infty$  since

$$J_1^d \leq \sqrt{d} \left\| 2\zeta^d(X_1^d, Z_1^d) + \frac{\ell Z_1^d}{\sqrt{d}} \dot{V}(X_1^d) \right\|_2.$$

Consider now  $J_2^d$ . We use the following decomposition for all  $2 \leq i \leq d$ ,

$$\zeta^d(X_i^d, Z_i^d)^2 - \frac{\ell^2}{4d} \dot{V}^2(X_i^d) = \left( \zeta^d(X_i^d, Z_i^d) + \frac{\ell}{2\sqrt{d}} Z_i^d \dot{V}(X_i^d) \right)^2 \\ - \frac{\ell}{\sqrt{d}} Z_i^d \dot{V}(X_i^d) \left( \zeta^d(X_i^d, Z_i^d) + \frac{\ell}{2\sqrt{d}} Z_i^d \dot{V}(X_i^d) \right) + \frac{\ell^2}{4d} \{(Z_i^d)^2 - 1\} \dot{V}^2(X_i^d).$$

Then,

$$J_2^d \leq d \left\| \zeta^d(X_1^d, Z_1^d) + \frac{\ell}{2\sqrt{d}} Z_1^d \dot{V}(X_1^d) \right\|_2^2 + \frac{\ell^2}{4d} \left\| \sum_{i=2}^d \dot{V}^2(X_i^d) \{(Z_i^d)^2 - 1\} \right\|_1 \\ + \ell\sqrt{d} \left\| \dot{V}(X_1^d) Z_1^d \left( \zeta^d(X_1^d, Z_1^d) + \frac{\ell}{2\sqrt{d}} Z_1^d \dot{V}(X_1^d) \right) \right\|_1.$$

Using **H1**(ii), Lemma 2(i) and the Cauchy-Schwarz inequality show that the first and the last term converge to zero. For the second term note that  $\mathbb{E}[(Z_i^d)^2 - 1] = 0$  so that

$$d^{-1} \left\| \sum_{i=2}^d \dot{V}^2(X_i^d) \{(Z_i^d)^2 - 1\} \right\|_1 \leq d^{-1/2} \text{Var} \left[ \dot{V}^2(X_1^d) \{(Z_1^d)^2 - 1\} \right]^{1/2} \rightarrow 0.$$

Finally,  $\lim_{d \rightarrow \infty} J_3^d = 0$  by (24) and Lemma 2(iii).  $\square$

*Proof of Proposition 1.* Let  $q > 0$  and  $\Lambda^d = -\ell d^{-1/2} Z_1^d \dot{V}(X_1^d) + \sum_{i=2}^d \Delta V_i^d$ . By the triangle inequality,  $E^d(q) \leq E_1^d(q) + E_2^d(q)$  where

$$E_1^d(q) = \mathbb{E} \left[ (Z_1^d)^q \left| 1 \wedge \exp \left\{ \sum_{i=1}^d \Delta V_i^d \right\} - 1 \wedge \exp \{ \Lambda^d \} \right| \right], \\ E_2^d(q) = \mathbb{E} \left[ (Z_1^d)^q \left| 1 \wedge \exp \{ \Lambda^d \} - 1 \wedge \exp \{ v^d \} \right| \right].$$

Since  $t \mapsto 1 \wedge e^t$  is 1-Lipschitz, by the Cauchy-Schwarz inequality we get

$$E_1^d(q) \leq \|Z_1^d\|_{2q}^q \left\| \Delta V_1^d + \ell d^{-1/2} Z_1^d \dot{V}(X_1^d) \right\|_2.$$

By Lemma 2(ii),  $E_1^d(q)$  goes to 0 as  $d$  goes to  $+\infty$ . Consider now  $E_2^d(q)$ . Using again that  $t \mapsto 1 \wedge e^t$  is 1-Lipschitz and Lemma 3,  $E_2^d(q)$  goes to 0.  $\square$

### 4.3 Proof of Theorem 2

Following [2], we introduce the function  $\mathcal{G}$  defined on  $\overline{\mathbb{R}}_+ \times \mathbb{R}$  by:

$$\mathcal{G}(a, b) = \begin{cases} \exp\left(\frac{a-b}{2}\right) \Phi\left(\frac{b}{2\sqrt{a}} - \sqrt{a}\right) & \text{if } a \in (0, +\infty), \\ 0 & \text{if } a = +\infty, \\ \exp\left(-\frac{b}{2}\right) \mathbf{1}_{\{b>0\}} & \text{if } a = 0, \end{cases} \quad (26)$$

where  $\Phi$  is the cumulative distribution function of a standard normal variable, and  $\Gamma$ :

$$\Gamma(a, b) = \begin{cases} \Phi\left(-\frac{b}{2\sqrt{a}}\right) + \exp\left(\frac{a-b}{2}\right) \Phi\left(\frac{b}{2\sqrt{a}} - \sqrt{a}\right) & \text{if } a \in (0, +\infty), \\ \frac{1}{2} & \text{if } a = +\infty, \\ \exp\left(-\frac{b}{2}\right) & \text{if } a = 0. \end{cases} \quad (27)$$

Note that  $\mathcal{G}$  and  $\Gamma$  are bounded on  $\overline{\mathbb{R}}_+ \times \mathbb{R}$ .  $\mathcal{G}$  and  $\Gamma$  are used throughout Section 4.

**Lemma 4.** *Assume **H1** holds. For all  $d \in \mathbb{N}^*$ , let  $X^d$  be a random variable distributed according to  $\pi^d$  and  $Z^d$  be a standard Gaussian random variable in  $\mathbb{R}^d$ , independent of  $X$ . Then,*

$$\lim_{d \rightarrow +\infty} d \mathbb{E} [2\zeta^d(X_1^d, Z_1^d)] = -\frac{\ell^2}{4} I,$$

where  $I$  is defined in (6) and  $\zeta^d$  in (14).

*Proof.* By (14),

$$\begin{aligned} d \mathbb{E} [2\zeta^d(X_1^d, Z_1^d)] &= 2d \mathbb{E} \left[ \int_{\mathbb{R}} \sqrt{\pi(x + \ell d^{-1/2} Z_1^d)} \sqrt{\pi(x)} dx - 1 \right], \\ &= -d \mathbb{E} \left[ \int_{\mathbb{R}} \left( \sqrt{\pi(x + \ell d^{-1/2} Z_1^d)} - \sqrt{\pi(x)} \right)^2 dx \right] = -d \mathbb{E} [\{\zeta^d(X_1^d, Z_1^d)\}^2]. \end{aligned}$$

The proof is then completed by Lemma 2(i). □

*Proof of Theorem 2.* By definition of  $\mathcal{A}_1^d$ , see (3),

$$\mathbb{P} [\mathcal{A}_1^d] = \mathbb{E} \left[ 1 \wedge \exp \left\{ \sum_{i=1}^d \Delta V_i^d \right\} \right],$$

where  $\Delta V_i^d = V(X_{0,i}^d) - V(X_{0,i}^d + \ell d^{-1/2} Z_{1,i}^d)$  and where  $X_0^d$  is distributed according to  $\pi^d$  and independent of the standard  $d$ -dimensional Gaussian random variable  $Z_1^d$ . Following the same steps as in the proof of Proposition 1 yields:

$$\lim_{d \rightarrow +\infty} |\mathbb{P} [\mathcal{A}_1^d] - \mathbb{E} [1 \wedge \exp \{\Theta^d\}]| = 0, \quad (28)$$

where

$$\Theta^d = -\ell d^{-1/2} \sum_{i=1}^d Z_{1,i}^d \dot{V}(X_{0,i}^d) - \ell^2 \sum_{i=2}^d \dot{V}(X_{0,i}^d)^2 / (4d) + 2(d-1) \mathbb{E} [\zeta^d(X_{0,1}^d, Z_{1,1}^d)].$$

Conditional on  $X_0^d$ ,  $\Theta^d$  is a one dimensional Gaussian random variable with mean  $\mu_d$  and variance  $\sigma_d^2$ , defined by

$$\begin{aligned}\mu_d &= -\ell^2 \sum_{i=2}^d \dot{V}(X_{0,i}^d)^2 / (4d) + 2(d-1) \mathbb{E} [\zeta^d(X_{0,1}^d, Z_{1,1}^d)] \\ \sigma_d^2 &= \ell^2 d^{-1} \sum_{i=1}^d \dot{V}(X_{0,i}^d)^2.\end{aligned}$$

Therefore, since for any  $G \sim \mathcal{N}(\mu, \sigma^2)$ ,  $\mathbb{E}[1 \wedge \exp(G)] = \Phi(\mu/\sigma) + \exp(\mu + \sigma^2/2)\Phi(-\sigma - \mu/\sigma)$ , taking the expectation conditional on  $X_0^d$ , we have

$$\begin{aligned}\mathbb{E} [1 \wedge \exp \{\Theta^d\}] &= \mathbb{E} [\Phi(\mu_d/\sigma_d) + \exp(\mu_d + \sigma_d^2/2)\Phi(-\sigma_d - \mu_d/\sigma_d)] \\ &= \mathbb{E} [\Gamma(\sigma_d^2, -2\mu_d)],\end{aligned}$$

where the function  $\Gamma$  is defined in (27). By Lemma 4 and the law of large numbers, almost surely,  $\lim_{d \rightarrow +\infty} \mu_d = -\ell^2 I/2$  and  $\lim_{d \rightarrow +\infty} \sigma_d^2 = \ell^2 I$ . Thus, as  $\Gamma$  is bounded, by Lebesgue's dominated convergence theorem:

$$\lim_{d \rightarrow +\infty} \mathbb{E} [1 \wedge \exp \{\Theta^d\}] = 2\Phi(-\ell\sqrt{I}/2).$$

The proof is then completed by (28).  $\square$

#### 4.4 Proof of Proposition 2

By Kolmogorov's criterion it is enough to prove that there exists a non-decreasing function  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for all  $d \geq 1$  and all  $0 \leq s \leq t$ ,

$$\mathbb{E} \left[ (Y_{t,1}^d - Y_{s,1}^d)^4 \right] \leq \gamma(t)(t-s)^2.$$

The inequality is straightforward for all  $0 \leq s \leq t$  such that  $\lfloor ds \rfloor = \lfloor dt \rfloor$ . For all  $0 \leq s \leq t$  such that  $\lfloor ds \rfloor \leq \lfloor dt \rfloor$ ,

$$Y_{t,1}^d - Y_{s,1}^d = X_{\lfloor dt \rfloor, 1}^d - X_{\lfloor ds \rfloor, 1}^d + \frac{dt - \lfloor dt \rfloor}{\sqrt{d}} \ell Z_{\lfloor dt \rfloor, 1}^d \mathbb{1}_{\mathcal{A}_{\lfloor dt \rfloor}^d} + \frac{\lfloor ds \rfloor - ds}{\sqrt{d}} \ell Z_{\lfloor ds \rfloor, 1}^d \mathbb{1}_{\mathcal{A}_{\lfloor ds \rfloor}^d}.$$

Then by the Hölder inequality,

$$\mathbb{E} \left[ (Y_{t,1}^d - Y_{s,1}^d)^4 \right] \leq C \left( (t-s)^2 + \mathbb{E} \left[ (X_{\lfloor dt \rfloor, 1}^d - X_{\lfloor ds \rfloor, 1}^d)^4 \right] \right),$$

where we have used

$$\frac{(dt - \lfloor dt \rfloor)^2}{d^2} + \frac{(\lfloor ds \rfloor - ds)^2}{d^2} \leq \frac{(dt - ds)^2 + (\lfloor ds \rfloor - \lfloor dt \rfloor)^2}{d^2} \leq 2(t-s)^2.$$

The proof is completed using Lemma 5.

**Lemma 5.** *Assume H1. Then, there exists  $C > 0$  such that, for all  $0 \leq k_1 < k_2$ ,*

$$\mathbb{E} \left[ (X_{k_2, 1}^d - X_{k_1, 1}^d)^4 \right] \leq C \sum_{p=2}^4 \frac{(k_2 - k_1)^p}{d^p}.$$

*Proof.* For all  $0 \leq k_1 < k_2$ ,

$$\mathbb{E} \left[ (X_{k_2,1}^d - X_{k_1,1}^d)^4 \right] = \frac{\ell^4}{d^2} \mathbb{E} \left[ \left( \sum_{k=k_1+1}^{k_2} Z_{k,1}^d - \sum_{k=k_1+1}^{k_2} Z_{k,1}^d \mathbb{1}_{(\mathcal{A}_k^d)^c} \right)^4 \right].$$

Therefore by the Hölder inequality,

$$\mathbb{E} \left[ (X_{k_2,1}^d - X_{k_1,1}^d)^4 \right] \leq \frac{24\ell^4}{d^2} (k_2 - k_1)^2 + \frac{8\ell^4}{d^2} \mathbb{E} \left[ \left( \sum_{k=k_1+1}^{k_2} Z_{k,1}^d \mathbb{1}_{(\mathcal{A}_k^d)^c} \right)^4 \right]. \quad (29)$$

The second term can be written:

$$\mathbb{E} \left[ \left( \sum_{k=k_1+1}^{k_2} Z_{k,1}^d \mathbb{1}_{(\mathcal{A}_k^d)^c} \right)^4 \right] = \sum \mathbb{E} \left[ \prod_{i=1}^4 Z_{m_i,1}^d \mathbb{1}_{(\mathcal{A}_{m_i}^d)^c} \right],$$

where the sum is over all the quadruplets  $(m_i)_{i=1}^4$  satisfying  $m_i \in \{k_1 + 1, \dots, k_2\}$ ,  $i = 1, \dots, 4$ . The expectation on the right hand side can be upper bounded depending on the cardinality of  $\{m_1, \dots, m_4\}$ . For all  $1 \leq j \leq 4$ , define

$$\mathcal{I}_j = \{(m_1, \dots, m_4) \in \{k_1 + 1, \dots, k_2\}^4 ; \#\{m_1, \dots, m_4\} = j\}. \quad (30)$$

Let  $(m_1, m_2, m_3, m_4) \in \{k_1 + 1, \dots, k_2\}^4$  and  $(\tilde{X}_k^d)_{k \geq 0}$  be defined as:

$$\tilde{X}_0^d = X_0^d \quad \text{and} \quad \tilde{X}_{k+1}^d = \tilde{X}_k^d + \mathbb{1}_{k \notin \{m_1-1, m_2-1, m_3-1, m_4-1\}} \frac{\ell}{\sqrt{d}} Z_{k+1}^d \mathbb{1}_{\tilde{\mathcal{A}}_{k+1}^d},$$

with  $\tilde{\mathcal{A}}_{k+1}^d = \left\{ U_{k+1} \leq \exp \left( \sum_{i=1}^d \Delta \tilde{V}_{k,i}^d \right) \right\}$ , where for all  $k \geq 0$  and all  $1 \leq i \leq d$ ,  $\Delta \tilde{V}_{k,i}^d$  is defined by

$$\Delta \tilde{V}_{k,i}^d = V \left( \tilde{X}_{k,i}^d \right) - V \left( \tilde{X}_{k,i}^d + \frac{\ell}{\sqrt{d}} Z_{k+1,i}^d \right).$$

Note that on the event  $\bigcap_{j=1}^4 \left\{ \mathcal{A}_{m_j}^d \right\}^c$ , the two processes  $(X_k)_{k \geq 0}$  and  $(\tilde{X}_k)_{k \geq 0}$  are equal. Let  $\mathcal{F}$  be the  $\sigma$ -field generated by  $(\tilde{X}_k^d)_{k \geq 0}$ .

(a)  $\#\{m_1, \dots, m_4\} = 4$ , as the  $\left\{ (U_{m_j}, Z_{m_j,1}^d, \dots, Z_{m_j,d}^d) \right\}_{1 \leq j \leq 4}$  are independent conditionally to  $\mathcal{F}$ ,

$$\begin{aligned} \mathbb{E} \left[ \prod_{j=1}^4 Z_{m_j,1}^d \mathbb{1}_{(\mathcal{A}_{m_j}^d)^c} \middle| \mathcal{F} \right] &= \prod_{j=1}^4 \mathbb{E} \left[ Z_{m_j,1}^d \mathbb{1}_{(\tilde{\mathcal{A}}_{m_j}^d)^c} \middle| \mathcal{F} \right], \\ &= \prod_{j=1}^4 \mathbb{E} \left[ Z_{m_j,1}^d \varphi \left( \sum_{i=1}^d \Delta \tilde{V}_{m_j-1,i}^d \right) \middle| \mathcal{F} \right]. \end{aligned}$$

where  $\varphi(x) = (1 - e^x)_+$ . Since the function  $\varphi$  is 1-Lipschitz, we get

$$\begin{aligned} \left| \varphi \left( \sum_{i=1}^d \Delta \tilde{V}_{m_j-1,i}^d \right) - \varphi \left( -\frac{\ell}{\sqrt{d}} \dot{V}(\tilde{X}_{m_j-1,1}^d) Z_{m_j,1}^d + \sum_{i=2}^d \Delta \tilde{V}_{m_j-1,i}^d \right) \right| \\ \leq \left| \Delta \tilde{V}_{m_j-1,1}^d + \frac{\ell}{\sqrt{d}} \dot{V}(\tilde{X}_{m_j-1,1}^d) Z_{m_j,1}^d \right|. \end{aligned}$$



Then,

$$\left| \mathbb{E} \left[ \prod_{j=1}^4 Z_{m_j,1}^d \mathbf{1}_{(\mathcal{A}_{m_j}^d)^c} \right] \right| \leq \mathbb{E} \left[ \prod_{j=1}^4 \{A_{m_j}^d + B_{m_j}^d\} \right],$$

where

$$\begin{aligned} A_{m_j}^d &= \mathbb{E} \left[ \left| Z_{m_j,1}^d \right| \left| \Delta \tilde{V}_{m_j-1,1}^d + \frac{\ell}{\sqrt{d}} \dot{V}(\tilde{X}_{m_j-1,1}^d) Z_{m_j,1}^d \right| \middle| \mathcal{F} \right], \\ B_{m_j}^d &= \left| \mathbb{E} \left[ Z_{m_j,1}^d \left( 1 - \exp \left\{ -\frac{\ell}{\sqrt{d}} \dot{V}(\tilde{X}_{m_j-1,1}^d) Z_{m_j,1}^d + \sum_{i=2}^d \Delta \tilde{V}_{m_j-1,i}^d \right\} \right) \middle| \mathcal{F} \right] \right|. \end{aligned}$$

By the inequality of arithmetic and geometric means and convex inequalities,

$$\left| \mathbb{E} \left[ \prod_{j=1}^4 Z_{m_j,1}^d \mathbf{1}_{(\mathcal{A}_{m_j}^d)^c} \right] \right| \leq 8 \mathbb{E} \left[ \sum_{j=1}^4 \left( A_{m_j}^d \right)^4 + \left( B_{m_j}^d \right)^4 \right].$$

By Lemma 2(ii) and the Hölder inequality, there exists  $C > 0$  such that  $\mathbb{E} \left[ \left( A_{m_j}^d \right)^4 \right] \leq Cd^{-2}$ .

On the other hand, by [2, Lemma 6] since  $Z_{m_j,1}^d$  is independent of  $\mathcal{F}$ ,

$$B_{m_j}^d = \left| \mathbb{E} \left[ \frac{\ell}{\sqrt{d}} \dot{V}(\tilde{X}_{m_j-1,1}^d) \mathcal{G} \left( \frac{\ell^2}{d} \dot{V}(\tilde{X}_{m_j-1,1}^d)^2, -2 \sum_{i=2}^d \Delta \tilde{V}_{m_j-1,i}^d \right) \middle| \mathcal{F} \right] \right|,$$

where the function  $\mathcal{G}$  is defined in (26). By **H1**(ii) and since  $\mathcal{G}$  is bounded,  $\mathbb{E}[(B_{m_j}^d)^4] \leq Cd^{-2}$ .

Therefore  $|\mathbb{E}[\prod_{j=1}^4 Z_{m_j,1}^d \mathbf{1}_{(\mathcal{A}_{m_j}^d)^c}]| \leq Cd^{-2}$ , showing that

$$\sum_{(m_1, m_2, m_3, m_4) \in \mathcal{I}_4} \left| \mathbb{E} \left[ \prod_{i=1}^4 Z_{m_i,1}^d \mathbf{1}_{(\mathcal{A}_{m_i}^d)^c} \right] \right| \leq \frac{C}{d^2} \binom{k_2 - k_1}{4}. \quad (31)$$

(b)  $\#\{m_1, \dots, m_4\} = 3$ , as the  $\left\{ \left( U_{m_j}, Z_{m_j,1}^d, \dots, Z_{m_j,d}^d \right) \right\}_{1 \leq j \leq 3}$  are independent conditionally to  $\mathcal{F}$ ,

$$\begin{aligned} & \left| \mathbb{E} \left[ \left( Z_{m_1,1}^d \right)^2 \mathbf{1}_{(\mathcal{A}_{m_1}^d)^c} \prod_{j=2}^3 Z_{m_j,1}^d \mathbf{1}_{(\mathcal{A}_{m_j}^d)^c} \middle| \mathcal{F} \right] \right| \\ & \leq \mathbb{E} \left[ \left( Z_{m_1,1}^d \right)^2 \middle| \mathcal{F} \right] \left| \prod_{j=2}^3 \mathbb{E} \left[ Z_{m_j,1}^d \mathbf{1}_{(\mathcal{A}_{m_j}^d)^c} \middle| \mathcal{F} \right] \right| \leq \left| \prod_{j=2}^3 \mathbb{E} \left[ Z_{m_j,1}^d \mathbf{1}_{(\mathcal{A}_{m_j}^d)^c} \middle| \mathcal{F} \right] \right|. \end{aligned}$$

Then, following the same steps as above, and using Holder's inequality yields

$$\left| \mathbb{E} \left[ \prod_{j=2}^3 Z_{m_j,1}^d \mathbf{1}_{(\mathcal{A}_{m_j}^d)^c} \right] \right| \leq C \mathbb{E} \left[ \sum_{j=2}^3 \left( A_{m_j}^d \right)^2 + \left( B_{m_j}^d \right)^2 \right] \leq Cd^{-1}$$

and

$$\sum_{(m_1, m_2, m_3, m_4) \in \mathcal{I}_3} \left| \mathbb{E} \left[ \prod_{i=1}^4 Z_{m_i,1}^d \mathbf{1}_{(\mathcal{A}_{m_i}^d)^c} \right] \right| \leq \frac{C}{d} \binom{k_2 - k_1}{3} \leq \frac{C}{d} (k_2 - k_1)^3. \quad (32)$$

(c) If  $\#\{m_1, \dots, m_4\} = 2$  two cases have to be considered:

$$\begin{aligned} \mathbb{E} \left[ (Z_{m_1,1}^d)^2 \mathbf{1}_{(\tilde{\mathcal{A}}_{m_1}^d)^c} (Z_{m_2,1}^d)^2 \mathbf{1}_{(\mathcal{A}_{m_2}^d)^c} \right] &\leq \mathbb{E} \left[ (Z_{m_1,1}^d)^2 \right] \mathbb{E} \left[ (Z_{m_2,1}^d)^2 \right] \leq 1, \\ \mathbb{E} \left[ (Z_{m_1,1}^d)^3 \mathbf{1}_{(\mathcal{A}_{m_1}^d)^c} Z_{m_2,1}^d \mathbf{1}_{(\mathcal{A}_{m_2}^d)^c} \right] &\leq \mathbb{E} \left[ |Z_{m_1,1}^d|^3 \right] \mathbb{E} [|Z_{m_2,1}^d|] \leq \frac{4}{\pi}. \end{aligned}$$

This yields

$$\begin{aligned} \sum_{(m_1, m_2, m_3, m_4) \in \mathcal{I}_2} \left| \mathbb{E} \left[ \prod_{i=1}^4 Z_{m_i,1}^d \mathbf{1}_{(\mathcal{A}_{m_i}^d)^c} \right] \right| \\ \leq \left( 3 + 4 \cdot \frac{4}{\pi} \right) (k_2 - k_1)(k_2 - k_1 - 1) \leq C(k_2 - k_1)^2. \end{aligned} \quad (33)$$

(d) If  $\#\{m_1, \dots, m_4\} = 1$ :  $\mathbb{E} \left[ (Z_{m_i,1}^d \mathbf{1}_{(\mathcal{A}_{m_i}^d)^c})^4 \right] \leq \mathbb{E} \left[ (Z_{m_1,1}^d)^4 \right] \leq 3$ , then

$$\sum_{(m_1, m_2, m_3, m_4) \in \mathcal{I}_1} \left| \mathbb{E} \left[ \prod_{i=1}^4 Z_{m_i,1}^d \mathbf{1}_{(\mathcal{A}_{m_i}^d)^c} \right] \right| \leq 3(k_2 - k_1). \quad (34)$$

The proof is completed by combining (29) with (53), (32), (33) and (34).  $\square$

## 4.5 Proof of Proposition 3

We preface the proof by a preliminary lemma.

**Lemma 6.** *Assume that **H1** holds. Let  $\mu$  be a limit point of the sequence of laws  $(\mu_d)_{d \geq 1}$  of  $\{(Y_{t,1}^d)_{t \geq 0}, d \in \mathbb{N}^*\}$ . Then for all  $t \geq 0$ , the pushforward measure of  $\mu$  by  $W_t$  is  $\pi$ .*

*Proof.* By (7),

$$\lim_{d \rightarrow +\infty} \mathbb{E} \left[ \left| Y_{t,1}^d - X_{[dt],1}^d \right| \right] = 0.$$

Since  $(\mu_d)_{d \geq 1}$  converges weakly to  $\mu$ , for all bounded Lipschitz function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\mathbb{E}^\mu[\psi(W_t)] = \lim_{d \rightarrow +\infty} \mathbb{E}[\psi(Y_{t,1}^d)] = \lim_{d \rightarrow +\infty} \mathbb{E}[\psi(X_{[dt],1}^d)]$ . The proof is completed upon noting that for all  $d \in \mathbb{N}^*$  and all  $t \geq 0$ ,  $X_{[dt],1}^d$  is distributed according to  $\pi$ .  $\square$

*Proof of Proposition 3.* Let  $\mu$  be a limit point of  $(\mu_d)_{d \geq 1}$ . It is straightforward to show that  $\mu$  is a solution to the martingale problem associated with  $L$  if for all  $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ ,  $m \in \mathbb{N}^*$ ,  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  bounded and continuous, and  $0 \leq t_1 \leq \dots \leq t_m \leq s \leq t$ :

$$\mathbb{E}^\mu \left[ \left( \phi(W_t) - \phi(W_s) - \int_s^t L\phi(W_u) du \right) g(W_{t_1}, \dots, W_{t_m}) \right] = 0. \quad (35)$$

Let  $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ ,  $m \in \mathbb{N}^*$ ,  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  continuous and bounded,  $0 \leq t_1 \leq \dots \leq t_m \leq s \leq t$  and  $\mathbf{W}_{\dot{V}} = \{w \in \mathbf{W} | w_u \notin \mathcal{D}_{\dot{V}} \text{ for almost every } u \in [s, t]\}$ . Note first that  $w \in \mathbf{W}_{\dot{V}}^c$  if and only if  $\int_s^t \mathbf{1}_{\mathcal{D}_{\dot{V}}}(w_u) du > 0$ . Therefore, by **H2** and Fubini's theorem:

$$\mathbb{E}^\mu \left[ \int_s^t \mathbf{1}_{\mathcal{D}_{\dot{V}}}(W_u) du \right] = \int_s^t \mathbb{E}^\mu [\mathbf{1}_{\mathcal{D}_{\dot{V}}}(W_u)] du = 0,$$

showing that  $\mu(\mathbf{W}_{\dot{V}}^c) = 0$ . We now prove that on  $\mathbf{W}_{\dot{V}}$ ,

$$\Psi_{s,t} : w \mapsto \left\{ \phi(w_t) - \phi(w_s) - \int_s^t L\phi(w_u) du \right\} g(w_{t_1}, \dots, w_{t_m}) \quad (36)$$

is continuous. It is clear that it is enough to show that  $w \mapsto \int_s^t L\phi(w_u) du$  is continuous on  $\mathbf{W}_{\dot{V}}$ . So let  $w \in \mathbf{W}_{\dot{V}}$  and  $(w^n)_{n \geq 0}$  be a sequence in  $\mathbf{W}$  which converges to  $w$  in the uniform topology on compact sets. Then by **H2**, for any  $u$  such that  $w_u \notin \mathcal{D}_{\dot{V}}$ ,  $L\phi(w_u^n)$  converges to  $L\phi(w_u)$  when  $n$  goes to infinity and  $L\phi$  is bounded. Therefore by Lebesgue's dominated convergence theorem,  $\int_s^t L\phi(w_u^n) du$  converges to  $\int_s^t L\phi(w_u) du$ . Hence, the map defined by (36) is continuous on  $\mathbf{W}_{\dot{V}}$ . Since  $(\mu_d)_{d \geq 1}$  converges weakly to  $\mu$ , by (16):

$$\mu(\Psi_{s,t}) = \lim_{d \rightarrow +\infty} \mu^d(\Psi_{s,t}) = 0,$$

which is precisely (35).  $\square$

## 4.6 Proof of Theorem 3

By Proposition 3, it is enough to check (16) to prove that  $\mu$  is a solution to the martingale problem. The core of the proof of Theorem 3 is Proposition 6, for which we need two technical lemmata.

**Lemma 7.** *Let  $X, Y$  and  $U$  be  $\mathbb{R}$ -valued random variables and  $\epsilon > 0$ . Assume that  $U$  is non-negative and bounded by 1. Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded function on  $\mathbb{R}$  such that for all  $(x, y) \in (-\infty, -\epsilon]^2 \cup [\epsilon, +\infty)^2$ ,  $|g(x) - g(y)| \leq C_g |x - y|$ .*

(i) *For all  $a > 0$ ,*

$$\begin{aligned} \mathbb{E}[U |g(X) - g(Y)|] &\leq C_g \mathbb{E}[U |X - Y|] \\ &\quad + \text{osc}(g) \left\{ \mathbb{P}[|X| \leq \epsilon] + a^{-1} \mathbb{E}[U |X - Y|] + \mathbb{P}[\epsilon < |X| < \epsilon + a] \right\}, \end{aligned}$$

where  $\text{osc}(g) = \sup(g) - \inf(g)$ .

(ii) *If there exist  $\mu \in \mathbb{R}$  and  $\sigma, C_X \in \mathbb{R}_+$  such that*

$$\sup_{x \in \mathbb{R}} |\mathbb{P}[X \leq x] - \Phi((x - \mu)/\sigma)| \leq C_X,$$

then

$$\begin{aligned} \mathbb{E}[U |g(X) - g(Y)|] &\leq C_g \mathbb{E}[U |X - Y|] \\ &\quad + 2 \text{osc}(g) \left\{ C_X + \sqrt{2 \mathbb{E}[U |X - Y|] (2\pi\sigma^2)^{-1/2} + \epsilon (2\pi\sigma^2)^{-1/2}} \right\}. \end{aligned}$$

*Proof.* (i) Consider the following decomposition

$$\begin{aligned} \mathbb{E}[U |g(X) - g(Y)|] &= \mathbb{E} \left[ U |(g(X) - g(Y))| \mathbf{1}_{\{(X,Y) \in (-\infty, -\epsilon]^2 \cup \{(X,Y) \in [\epsilon, +\infty)^2\}} \right] \\ &\quad + \mathbb{E} \left[ U |g(X) - g(Y)| \left( \mathbf{1}_{\{X \in [-\epsilon, \epsilon]\}} + \mathbf{1}_{\{X < -\epsilon\} \cap \{Y \geq -\epsilon\}} \cup \{X > \epsilon\} \cap \{Y \leq \epsilon\} \right) \right]. \end{aligned}$$

In addition, for all  $a > 0$ ,

$$\begin{aligned} & (\{X < -\epsilon\} \cap \{Y \geq -\epsilon\}) \cup (\{X > \epsilon\} \cap \{Y \leq \epsilon\}) \\ & \subset \{\epsilon < |X| < \epsilon + a\} \cup (\{|X| \geq \epsilon + a\} \cap \{|X - Y| \geq a\}) . \end{aligned}$$

Then using that  $U \in [0, 1)$ , we get

$$\mathbb{E}[U|g(X) - g(Y)|] \leq C_g \mathbb{E}[U|X - Y|] + \text{osc}(g) (\mathbb{P}[|X| < \epsilon + a] + a^{-1} \mathbb{E}[U|X - Y|]) .$$

(ii) The result is straightforward if  $\mathbb{E}[U|X - Y|] = 0$ . Assume  $\mathbb{E}[U|X - Y|] > 0$ . Combining the additional assumption and the previous result,

$$\begin{aligned} \mathbb{E}[U|g(X) - g(Y)|] & \leq C_g \mathbb{E}[U|X - Y|] \\ & \quad + \text{osc}(g) \left\{ 2C_X + 2(\epsilon + a)(2\pi\sigma^2)^{-1/2} + a^{-1} \mathbb{E}[U|X - Y|] \right\} . \end{aligned}$$

As this result holds for all  $a > 0$ , the proof is concluded by setting  $a = \sqrt{\mathbb{E}[U|X - Y|] (2\pi\sigma^2)^{1/2}/2}$ .  $\square$

**Lemma 8.** *Assume **H1** holds. Let  $X^d$  be distributed according to  $\pi^d$  and  $Z^d$  be a  $d$ -dimensional standard Gaussian random variable, independent of  $X^d$ . Then,  $\lim_{d \rightarrow +\infty} E^d = 0$ , where*

$$E^d = \mathbb{E} \left[ \left| \dot{V}(X_1^d) \left\{ \mathcal{G} \left( \frac{\ell^2}{d} \dot{V}(X_1^d)^2, 2 \sum_{i=2}^d \Delta V_i^d \right) - \mathcal{G} \left( \frac{\ell^2}{d} \dot{V}(X_1^d)^2, 2 \sum_{i=2}^d b_i^d \right) \right\} \right| \right] ,$$

$\Delta V_i^d$  and  $b_i^d$  are resp. given by (5) and (13).

*Proof.* Set for all  $d \geq 1$ ,  $\bar{Y}_d = \sum_{i=2}^d \Delta V_i^d$  and  $\bar{X}_d = \sum_{i=2}^d b_i^d$ . By (26),  $\partial_b \mathcal{G}(a, b) = -\mathcal{G}(a, b)/2 + \exp(-b^2/8a)/(2\sqrt{2\pi a})$ . As  $\mathcal{G}$  is bounded and  $x \mapsto x \exp(-x)$  is bounded on  $\mathbb{R}_+$ , we get  $\sup_{a \in \mathbb{R}_+, |b| \geq a^{1/4}} \partial_b \mathcal{G}(a, b) < +\infty$ . Therefore, there exists  $C \geq 0$  such that, for all  $a \in \mathbb{R}_+$  and  $(b_1, b_2) \in (-\infty, -a^{1/4})^2 \cup (a^{1/4}, +\infty)^2$ ,

$$|\mathcal{G}(a, b_1) - \mathcal{G}(a, b_2)| \leq C |b_1 - b_2| . \quad (37)$$

By definition of  $b_i^d$  (13),  $\bar{X}_d$  may be expressed as  $\bar{X}_d = \sigma_d \bar{S}_d + \mu_d$ , where

$$\begin{aligned} \mu_d &= 2(d-1) \mathbb{E}[\zeta^d(X_1^d, Z_1^d)] - \frac{\ell^2(d-1)}{4d} \mathbb{E}[\dot{V}(X_1^d)^2] , \\ \sigma_d^2 &= \ell^2 \mathbb{E}[\dot{V}(X_1^d)^2] + \frac{\ell^4}{16d} \mathbb{E} \left[ \left( \dot{V}(X_1^d)^2 - \mathbb{E}[\dot{V}(X_1^d)^2] \right)^2 \right] , \\ \bar{S}_d &= (\sqrt{d}\sigma_d)^{-1} \sum_{i=2}^d \beta_i^d , \\ \beta_i^d &= -\ell Z_i^d \dot{V}(X_i^d) - \frac{\ell^2}{4\sqrt{d}} \left( \dot{V}(X_i^d)^2 - \mathbb{E}[\dot{V}(X_i^d)^2] \right) . \end{aligned}$$

By **H1(ii)** the Berry-Essen Theorem [5, Theorem 5.7] can be applied to  $\bar{S}_d$ . Then, there exists a universal constant  $C$  such that for all  $d > 0$ ,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left[ \left( \frac{d}{d-1} \right)^{1/2} \bar{S}_d \leq x \right] - \Phi(x) \right| \leq C/\sqrt{d} .$$

It follows that

$$\sup_{x \in \mathbb{R}} |\mathbb{P} [\bar{X}_d \leq x] - \Phi((x - \mu_d)/\tilde{\sigma}_d)| \leq C/\sqrt{d},$$

where  $\tilde{\sigma}_d^2 = (d-1)\sigma_d^2/d$ . By this result and (37), Lemma 7 can be applied to obtain a constant  $C \geq 0$ , independent of  $d$ , such that:

$$\begin{aligned} \mathbb{E} \left[ \left| \mathcal{G} \left( \ell^2 \dot{V}(X_1^d)^2/d, 2\bar{Y}_d \right) - \mathcal{G} \left( \ell^2 \dot{V}(X_1^d)^2/d, 2\bar{X}_d \right) \right| \middle| X_1^d \right] \\ \leq C \left( \varepsilon_d + d^{-1/2} + \sqrt{2\varepsilon_d(2\pi\tilde{\sigma}_d^2)^{-1/2}} + \sqrt{\ell|\dot{V}(X_1^d)|/(2\pi d^{1/2}\tilde{\sigma}_d^2)} \right), \end{aligned}$$

where  $\varepsilon_d = \mathbb{E} [|\bar{X}_d - \bar{Y}_d|]$ . Using this result, we have

$$\begin{aligned} \mathbb{E}^d \leq C \left\{ \left( \varepsilon_d + d^{-1/2} + \sqrt{2\varepsilon_d(2\pi\tilde{\sigma}_d^2)^{-1/2}} \right) \mathbb{E} [|\dot{V}(X_1^d)|] \right. \\ \left. + \ell^{1/2} \mathbb{E} [|\dot{V}(X_1^d)|^{3/2}] (2\pi d^{1/2}\tilde{\sigma}_d^2)^{-1/2} \right\}. \quad (38) \end{aligned}$$

By Lemma 3,  $\varepsilon_d$  goes to 0 as  $d$  goes to infinity, and by **H1(ii)**  $\lim_{d \rightarrow +\infty} \sigma_d^2 = \ell^2 \mathbb{E} [\dot{V}(X)^2]$ . Combining these results with (38), it follows that  $\mathbb{E}^d$  goes to 0 when  $d$  goes to infinity.  $\square$

For all  $n \geq 0$ , define  $\mathcal{F}_n^d = \sigma(\{X_k^d, k \leq n\})$  and for all  $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ ,

$$\begin{aligned} M_n^d(\phi) = \frac{\ell}{\sqrt{d}} \sum_{k=0}^{n-1} \phi'(X_{k,1}^d) \left\{ Z_{k+1,1}^d \mathbb{1}_{\mathcal{A}_{k+1}^d} - \mathbb{E} \left[ Z_{k+1,1}^d \mathbb{1}_{\mathcal{A}_{k+1}^d} \middle| \mathcal{F}_k^d \right] \right\} \\ + \frac{\ell^2}{2d} \sum_{k=0}^{n-1} \phi''(X_{k,1}^d) \left\{ (Z_{k+1,1}^d)^2 \mathbb{1}_{\mathcal{A}_{k+1}^d} - \mathbb{E} \left[ (Z_{k+1,1}^d)^2 \mathbb{1}_{\mathcal{A}_{k+1}^d} \middle| \mathcal{F}_k^d \right] \right\}. \quad (39) \end{aligned}$$

**Proposition 6.** *Assume **H1** and **H2** hold. Then, for all  $s \leq t$  and all  $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ ,*

$$\lim_{d \rightarrow +\infty} \mathbb{E} \left[ \left| \phi(Y_{t,1}^d) - \phi(Y_{s,1}^d) - \int_s^t L\phi(Y_{r,1}^d) dr - \left( M_{[dt]}^d(\phi) - M_{[ds]}^d(\phi) \right) \right| \right] = 0.$$

*Proof.* First, since  $dY_{r,1}^d = \ell\sqrt{d}Z_{[dr],1}^d \mathbb{1}_{\mathcal{A}_{[dr]}^d} dr$ ,

$$\phi(Y_{t,1}^d) - \phi(Y_{s,1}^d) = \ell\sqrt{d} \int_s^t \phi'(Y_{r,1}^d) Z_{[dr],1}^d \mathbb{1}_{\mathcal{A}_{[dr]}^d} dr. \quad (40)$$

As  $\phi$  is  $C^3$ , using (7) and a Taylor expansion, for all  $r \in [s, t]$  there exists  $\chi_r \in [X_{[dr],1}^d, Y_{r,1}^d]$  such that:

$$\begin{aligned} \phi'(Y_{r,1}^d) = \phi'(X_{[dr],1}^d) + \frac{\ell}{\sqrt{d}}(dr - [dr])\phi''(X_{[dr],1}^d)Z_{[dr],1}^d \mathbb{1}_{\mathcal{A}_{[dr]}^d} \\ + \frac{\ell^2}{2d}(dr - [dr])^2\phi^{(3)}(\chi_r) \left( Z_{[dr],1}^d \right)^2 \mathbb{1}_{\mathcal{A}_{[dr]}^d}. \end{aligned}$$

Plugging this expression into (40) yields:

$$\begin{aligned}\phi(Y_{t,1}^d) - \phi(Y_{s,1}^d) &= \ell\sqrt{d} \int_s^t \phi'(X_{[dr],1}^d) Z_{[dr],1}^d \mathbf{1}_{\mathcal{A}_{[dr]}^d} dr \\ &\quad + \ell^2 \int_s^t (dr - [dr]) \phi''(X_{[dr],1}^d) (Z_{[dr],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[dr]}^d} dr \\ &\quad + \frac{\ell^3}{2\sqrt{d}} \int_s^t (dr - [dr])^2 \phi^{(3)}(X_r) (Z_{[dr],1}^d)^3 \mathbf{1}_{\mathcal{A}_{[dr]}^d} dr.\end{aligned}$$

As  $\phi^{(3)}$  is bounded,

$$\lim_{d \rightarrow +\infty} \mathbb{E} \left[ \left| d^{-1/2} \int_s^t (dr - [dr])^2 \phi^{(3)}(X_r) (Z_{[dr],1}^d)^3 \mathbf{1}_{\mathcal{A}_{[dr]}^d} dr \right| \right] = 0.$$

On the other hand,  $I = \int_s^t \phi''(X_{[dr],1}^d) (dr - [dr]) (Z_{[dr],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[dr]}^d} dr = I_1 + I_2$  with

$$\begin{aligned}I_1 &= \int_s^{\lceil ds \rceil / d} + \int_{\lfloor dt \rfloor / d}^t \phi''(X_{[dr],1}^d) (dr - [dr] - 1/2) (Z_{[dr],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[dr]}^d} dr \\ I_2 &= \frac{1}{2} \int_s^t \phi''(X_{[dr],1}^d) (Z_{[dr],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[dr]}^d} dr.\end{aligned}$$

Note that

$$\begin{aligned}I_1 &= \frac{1}{2d} (\lceil ds \rceil - ds) (ds - \lfloor ds \rfloor) \phi''(X_{[ds],1}^d) (Z_{[ds],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[ds]}^d} \\ &\quad + \frac{1}{2d} (\lfloor dt \rfloor - dt) (dt - \lceil dt \rceil) \phi''(X_{[dt],1}^d) (Z_{[dt],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[dt]}^d}\end{aligned}$$

showing, as  $\phi''$  is bounded, that  $\lim_{d \rightarrow +\infty} \mathbb{E}[|I_1|] = 0$ . Therefore,

$$\lim_{d \rightarrow +\infty} \mathbb{E} [|\phi(Y_{t,1}^d) - \phi(Y_{s,1}^d) - I_{s,t}|] = 0,$$

where

$$I_{s,t} = \int_s^t \left\{ \ell\sqrt{d} \phi'(X_{[dr],1}^d) Z_{[dr],1}^d + \ell^2 \phi''(X_{[dr],1}^d) (Z_{[dr],1}^d)^2 / 2 \right\} \mathbf{1}_{\mathcal{A}_{[dr]}^d} dr.$$

Write

$$I_{s,t} - \int_s^t L\phi(Y_{r,1}^d) dr - \left( M_{[dt]}^d(\phi) - M_{[ds]}^d(\phi) \right) = T_1^d + T_2^d + T_3^d - T_4^d + T_5^d,$$

where

$$\begin{aligned}
T_1^d &= \int_s^t \phi'(X_{[dr],1}^d) \left( \ell\sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \mathbb{1}_{\mathcal{A}_{[dr]}^d} \mid \mathcal{F}_{[dr]}^d \right] + \frac{h(\ell)}{2} \dot{V}(X_{[dr],1}^d) \right) dr, \\
T_2^d &= \int_s^t \phi''(X_{[dr],1}^d) \left( \frac{\ell^2}{2} \mathbb{E} \left[ (Z_{[dr],1}^d)^2 \mathbb{1}_{\mathcal{A}_{[dr]}^d} \mid \mathcal{F}_{[dr]}^d \right] - \frac{h(\ell)}{2} \right) dr, \\
T_3^d &= \int_s^t \left( L\phi(Y_{[dr]/d,1}^d) - L\phi(Y_{r,1}^d) \right) dr, \\
T_4^d &= \frac{\ell([\!dt] - dt)}{\sqrt{d}} \phi'(X_{[dt],1}^d) \left( Z_{[dt],1}^d \mathbb{1}_{\mathcal{A}_{[dt]}^d} - \mathbb{E} \left[ Z_{[dt],1}^d \mathbb{1}_{\mathcal{A}_{[dt]}^d} \mid \mathcal{F}_{[dt]}^d \right] \right) \\
&\quad + \frac{\ell^2([\!dt] - dt)}{2d} \phi''(X_{[dt],1}^d) \left( (Z_{[dt],1}^d)^2 \mathbb{1}_{\mathcal{A}_{[dt]}^d} - \mathbb{E} \left[ (Z_{[dt],1}^d)^2 \mathbb{1}_{\mathcal{A}_{[dt]}^d} \mid \mathcal{F}_{[dt]}^d \right] \right), \\
T_5^d &= \frac{\ell([\!ds] - ds)}{\sqrt{d}} \phi'(X_{[ds],1}^d) \left( Z_{[ds],1}^d \mathbb{1}_{\mathcal{A}_{[ds]}^d} - \mathbb{E} \left[ Z_{[ds],1}^d \mathbb{1}_{\mathcal{A}_{[ds]}^d} \mid \mathcal{F}_{[ds]}^d \right] \right) \\
&\quad + \frac{\ell^2([\!ds] - ds)}{2d} \phi''(X_{[ds],1}^d) \left( (Z_{[ds],1}^d)^2 \mathbb{1}_{\mathcal{A}_{[ds]}^d} - \mathbb{E} \left[ (Z_{[ds],1}^d)^2 \mathbb{1}_{\mathcal{A}_{[ds]}^d} \mid \mathcal{F}_{[ds]}^d \right] \right).
\end{aligned}$$

It is now proved that for all  $1 \leq i \leq 5$ ,  $\lim_{d \rightarrow +\infty} \mathbb{E}[|T_i^d|] = 0$ . First, as  $\phi'$  and  $\phi''$  are bounded,

$$\mathbb{E} [|T_4^d| + |T_5^d|] \leq Cd^{-1/2}. \quad (41)$$

Denote for all  $r \in [s, t]$  and  $d \geq 1$ ,

$$\begin{aligned}
\Delta V_{r,i}^d &= V(X_{[dr],i}^d) - V(X_{[dr],i}^d + \ell d^{-1/2} Z_{[dr],i}^d) \\
\Xi_r^d &= 1 \wedge \exp \left\{ -\ell Z_{[dr],1}^d \dot{V}(X_{[dr],1}^d) / \sqrt{d} + \sum_{i=2}^d b_{[dr],i}^d \right\}, \\
\Upsilon_r^d &= 1 \wedge \exp \left\{ -\ell Z_{[dr],1}^d \dot{V}(X_{[dr],1}^d) / \sqrt{d} + \sum_{i=2}^d \Delta V_{r,i}^d \right\},
\end{aligned}$$

where for all  $k, i \geq 0$ ,  $b_{k,i}^d = b^d(X_{k,i}^d, Z_{k+1,i}^d)$ , and for all  $x, z \in \mathbb{R}$ ,  $b^d(x, y)$  is given by (13). By the triangle inequality,

$$|T_1^d| \leq \int_s^t \left| \phi'(X_{[dr],1}^d) \right| (A_{1,r} + A_{2,r} + A_{3,r}) dr, \quad (42)$$

where

$$\begin{aligned}
A_{1,r} &= \left| \ell\sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \left( \mathbb{1}_{\mathcal{A}_{[dr]}^d} - \Upsilon_r^d \right) \mid \mathcal{F}_{[dr]}^d \right] \right|, \\
A_{2,r} &= \left| \ell\sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \left( \Upsilon_r^d - \Xi_r^d \right) \mid \mathcal{F}_{[dr]}^d \right] \right|, \\
A_{3,r} &= \left| \ell\sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \Xi_r^d \mid \mathcal{F}_{[dr]}^d \right] + \dot{V}(X_{[dr],1}^d) h(\ell) / 2 \right|.
\end{aligned}$$

Since  $t \mapsto 1 \wedge \exp(t)$  is 1-Lipschitz, by Lemma 2(ii)  $\mathbb{E}[|A_{1,r}^d|]$  goes to 0 as  $d \rightarrow +\infty$  for almost all  $r$ . So by the Fubini theorem, the first term in (42) goes to 0 as  $d \rightarrow +\infty$ . For  $A_{2,r}^d$ , by [2,

Lemma 6],

$$\mathbb{E} [ |A_{2,r}^d| ] \leq \mathbb{E} \left[ \left| \ell^2 \dot{V}(X_{[dr],1}^d) \left\{ \mathcal{G} \left( \frac{\ell^2 \dot{V}(X_{[dr],1}^d)^2}{d}, 2 \sum_{i=2}^d \Delta V_{r,i}^d \right) - \mathcal{G} \left( \frac{\ell^2 \dot{V}(X_{[dr],1}^d)^2}{d}, 2 \sum_{i=2}^d b_{[dr],i}^d \right) \right\} \right| \right],$$

where  $\mathcal{G}$  is defined in (26). By Lemma 8, this expectation goes to zero when  $d$  goes to infinity. Then by the Fubini theorem and the Lebesgue dominated convergence theorem, the second term of (42) goes 0 as  $d \rightarrow +\infty$ . For the last term, by [2, Lemma 6] again:

$$\begin{aligned} \ell\sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \Xi_r^d \middle| \mathcal{F}_{[dr]}^d \right] &= -\ell^2 \dot{V}(X_{[dr],1}^d) \\ &\times \mathcal{G} \left( \frac{\ell^2}{d} \sum_{i=1}^d \dot{V}(X_{[dr],i}^d)^2, \frac{\ell^2}{2d} \sum_{i=2}^d \dot{V}(X_{[dr],i}^d)^2 - 4(d-1) \mathbb{E} [\zeta^d(X, Z)] \right), \end{aligned} \quad (43)$$

where  $X$  is distributed according to  $\pi$  and  $Z$  is a standard Gaussian random variable independent of  $X$ . As  $\mathcal{G}$  is continuous on  $\mathbb{R}_+ \times \mathbb{R} \setminus \{0, 0\}$  (see [2, Lemma 2]), by **H1(ii)**, Lemma 4 and the law of large numbers, almost surely,

$$\begin{aligned} \lim_{d \rightarrow +\infty} \ell^2 \mathcal{G} \left( \frac{\ell^2}{d} \sum_{i=1}^d \dot{V}(X_{[dr],i}^d)^2, \frac{\ell^2}{2d} \sum_{i=2}^d \dot{V}(X_{[dr],i}^d)^2 - 4(d-1) \mathbb{E} [\zeta^d(X, Z)] \right) \\ = \ell^2 \mathcal{G} \left( \ell^2 \mathbb{E} [\dot{V}(X)^2], \ell^2 \mathbb{E} [\dot{V}(X)^2] \right) = h(\ell)/2, \end{aligned} \quad (44)$$

where  $h(\ell)$  is defined in (10). Therefore by Fubini's Theorem, (43) and Lebesgue's dominated convergence theorem, the last term of (42) goes to 0 as  $d$  goes to infinity. The proof for  $T_2^d$  follows the same lines. By the triangle inequality,

$$\begin{aligned} |T_2^d| &\leq \left| \int_s^t \phi''(X_{[dr],1}^d) (\ell^2/2) \mathbb{E} \left[ (Z_{[dr],1}^d)^2 \left( \mathbf{1}_{\mathcal{A}_{[dr]}^d} - \Xi_r^d \right) \middle| \mathcal{F}_{[dr]}^d \right] dr \right| \\ &\quad + \left| \int_s^t \phi''(X_{[dr],1}^d) \left( (\ell^2/2) \mathbb{E} \left[ (Z_{[dr],1}^d)^2 \Xi_r^d \middle| \mathcal{F}_{[dr]}^d \right] - h(\ell)/2 \right) dr \right|. \end{aligned} \quad (45)$$

By Fubini's Theorem, Lebesgue's dominated convergence theorem and Proposition 1, the expectation of the first term goes to zero when  $d$  goes to infinity. For the second term, by [2, Lemma 6 (A.5)],

$$\begin{aligned} (\ell^2/2) \mathbb{E} \left[ (Z_{[dr],1}^d)^2 \mathbf{1} \wedge \exp \left\{ -\frac{\ell Z_{[dr],1}^d}{\sqrt{d}} \dot{V}(X_{[dr],1}^d) + \sum_{i=2}^d b_{[dr],i}^d \right\} \middle| \mathcal{F}_{[dr]}^d \right] \\ = (B_1 + B_2 - B_3)/2, \end{aligned} \quad (46)$$



where

$$\begin{aligned}
B_1 &= \ell^2 \Gamma \left( \frac{\ell^2}{d} \sum_{i=1}^d \dot{V}(X_{[dr],i}^d)^2, \frac{\ell^2}{2d} \sum_{i=2}^d \dot{V}(X_{[dr],i}^d)^2 - 4(d-1) \mathbb{E} [\zeta^d(X, Z)] \right), \\
B_2 &= \frac{\ell^4 \dot{V}(X_{[dr],1}^d)^2}{d} \mathcal{G} \left( \frac{\ell^2}{d} \sum_{i=1}^d \dot{V}(X_{[dr],i}^d)^2, \frac{\ell^2}{2d} \sum_{i=2}^d \dot{V}(X_{[dr],i}^d)^2 - 4(d-1) \mathbb{E} [\zeta^d(X, Z)] \right), \\
B_3 &= \frac{\ell^4 \dot{V}(X_{[dr],1}^d)^2}{d} \left( 2\pi \ell^2 \sum_{i=1}^d \dot{V}(X_{[dr],i}^d)^2 / d \right)^{-1/2} \\
&\quad \times \exp \left\{ - \frac{\left[ -(d-1) \mathbb{E} [2\zeta^d(X, Z)] + (\ell^2 / (4d)) \sum_{i=2}^d \dot{V}(X_{[dr],i}^d)^2 \right]^2}{2\ell^2 \sum_{i=1}^d \dot{V}(X_{[dr],i}^d)^2 / d} \right\},
\end{aligned}$$

where  $\Gamma$  is defined in (27). As  $\Gamma$  is continuous on  $\mathbb{R}_+ \times \mathbb{R} \setminus \{0, 0\}$  (see [2, Lemma 2]), by **H1(ii)**, Lemma 4 and the law of large numbers, almost surely,

$$\begin{aligned}
\lim_{d \rightarrow +\infty} \ell^2 \Gamma \left( \frac{\ell^2}{d} \sum_{i=1}^d \dot{V}(X_{[dr],i}^d)^2, \frac{\ell^2}{2d} \sum_{i=2}^d \dot{V}(X_{[dr],i}^d)^2 - 4(d-1) \mathbb{E} [\zeta^d(X, Z)] \right) \\
= \ell^2 \Gamma \left( \ell^2 \mathbb{E} [\dot{V}(X)^2], \ell^2 \mathbb{E} [\dot{V}(X)^2] \right) = h(\ell). \quad (47)
\end{aligned}$$

By Lemma 4, by **H1(ii)** and the law of large numbers, almost surely,

$$\begin{aligned}
\lim_{d \rightarrow +\infty} \exp \left\{ - \frac{\left[ -(d-1) \mathbb{E} [2\zeta^d(X, Z)] + (\ell^2 / (4d)) \sum_{i=2}^d \dot{V}(X_{[dr],i}^d)^2 \right]^2}{2\ell^2 \sum_{i=1}^d \dot{V}(X_{[dr],i}^d)^2 / d} \right\} \\
= \exp \left\{ - \frac{\ell^2}{8} \mathbb{E} [\dot{V}(X)^2] \right\}.
\end{aligned}$$

Then, as  $\mathcal{G}$  is bounded on  $\mathbb{R}_+ \times \mathbb{R}$ ,

$$\lim_{d \rightarrow +\infty} \mathbb{E} \left[ \left| \int_s^t \phi''(X_{[dr],1}^d) (B_2 - B_3) dr \right| \right] = 0. \quad (48)$$

Therefore, by Fubini's Theorem, (46), (47), (48) and Lebesgue's dominated convergence theorem, the second term of (45) goes to 0 as  $d$  goes to infinity. Write  $T_3^d = (h(\ell)/2)(T_{3,1}^d - T_{3,2}^d)$  where

$$\begin{aligned}
T_{3,1}^d &= \int_s^t \left\{ \phi''(X_{[dr],1}^d) - \phi''(Y_{r,1}^d) \right\} dr, \\
T_{3,2}^d &= \int_s^t \left\{ \dot{V}(X_{[dr],1}^d) \phi'(X_{[dr],1}^d) - \dot{V}(Y_{r,1}^d) \phi'(Y_{r,1}^d) \right\} dr.
\end{aligned}$$

It is enough to show that  $\mathbb{E}[|T_{3,1}^d|]$  and  $\mathbb{E}[|T_{3,2}^d|]$  go to 0 when  $d$  goes to infinity to conclude the proof. By (7) and the mean value theorem, for all  $r \in [s, t]$  there exists  $\chi_r \in [X_{[dr],1}^d, Y_{r,1}^d]$  such that

$$\phi''(X_{[dr],1}^d) - \phi''(Y_{r,1}^d) = \phi^{(3)}(\chi_r) (dr - [dr]) (\ell / \sqrt{d}) Z_{[dr],1}^d \mathbf{1}_{\mathcal{A}_{[dr]}^d}.$$

Since  $\phi^{(3)}$  is bounded, it follows that  $\lim_{d \rightarrow +\infty} \mathbb{E}[|T_{3,1}^d|] = 0$ . On the other hand,

$$\begin{aligned} T_{3,2}^d &= \int_s^t \left\{ \dot{V} \left( X_{[dr],1}^d \right) - \dot{V} \left( Y_{r,1}^d \right) \right\} \phi' \left( X_{[dr],1}^d \right) dr \\ &\quad + \int_s^t \left\{ \phi' \left( X_{[dr],1}^d \right) - \phi' \left( Y_{r,1}^d \right) \right\} \dot{V} \left( Y_{r,1}^d \right) dr. \end{aligned}$$

Since  $\phi'$  has a bounded support, by **H2**, Fubini's theorem, and Lebesgue's dominated convergence theorem, the expectation of the absolute value of the first term goes to 0 as  $d$  goes to infinity. The second term is dealt with following the same steps as for  $T_{3,1}^d$  and using **H1(ii)**.  $\square$

*Proof of Theorem 3.* By Proposition 2, Proposition 3 and Proposition 6, it is enough to prove that for all  $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ ,  $p \geq 1$ , all  $0 \leq t_1 \leq \dots \leq t_p \leq s \leq t$  and  $g : \mathbb{R}^p \rightarrow \mathbb{R}$  bounded and continuous function,

$$\lim_{d \rightarrow +\infty} \mathbb{E} \left[ \left( M_{[dt]}^d(\phi) - M_{[ds]}^d(\phi) \right) g(Y_{t_1}^d, \dots, Y_{t_p}^d) \right] = 0,$$

where for  $n \geq 1$ ,  $M_n^d(\phi)$  is defined in (39). But this result is straightforward taking successively the conditional expectations with respect to  $\mathcal{F}_k$ , for  $k = [dt], \dots, [ds]$ .  $\square$

## 5 Proofs of Section 3

### 5.1 Proof of Theorem 4

The proof of this theorem follows the same steps as the the proof of Theorem 2. Note that  $\xi_\theta$  and  $\xi_0$ , given by (11), are well defined on  $\mathcal{I} \cap \{x \in \mathbb{R} \mid x + r\theta \in \mathcal{I}\}$ . Let the function  $v : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined for  $x, \theta \in \mathbb{R}$  by

$$v(x, \theta) = \mathbf{1}_{\mathcal{I}}(x + r\theta) \mathbf{1}_{\mathcal{I}}(x + (1-r)\theta). \quad (49)$$

**Lemma 9.** *Assume **G1** holds. Then, there exists  $C > 0$  such that for all  $\theta \in \mathbb{R}$ ,*

$$\left( \int_{\mathcal{I}} \left( \{\xi_\theta(x) - \xi_0(x)\} v(x, \theta) + \theta \dot{V}(x) \xi_0(x) / 2 \right)^2 dx \right)^{1/2} \leq C |\theta|^\beta.$$

*Proof.* The proof follows as Lemma 1 and is omitted.  $\square$

**Lemma 10.** *Assume that **G1** holds. Let  $X$  be a random variable distributed according to  $\pi$  and  $Z$  be a standard Gaussian random variable independent of  $X$ . Define*

$$\mathcal{D}_{\mathcal{I}} = \{X + rld^{-1/2}Z \in \mathcal{I}\} \cap \{X + (1-r)ld^{-1/2}Z \in \mathcal{I}\}.$$

*Then,*

$$(i) \lim_{d \rightarrow +\infty} d \left\| \mathbf{1}_{\mathcal{D}_{\mathcal{I}}} \zeta^d(X, Z) + \ell Z \dot{V}(X) / (2\sqrt{d}) \right\|_2^2 = 0.$$

(ii) *Let  $p$  be given by **G1(i)**. Then,*

$$\lim_{d \rightarrow +\infty} \sqrt{d} \left\| \mathbf{1}_{\mathcal{D}_{\mathcal{I}}} \left\{ V(X) - V(X + \ell Z / \sqrt{d}) \right\} + \ell Z \dot{V}(X) / \sqrt{d} \right\|_p = 0.$$

(iii)  $\lim_{d \rightarrow \infty} d \left\| \mathbb{1}_{\mathcal{D}_x} (\log(1 + \zeta_d(X, Z)) - \zeta^d(X, Z) + [\zeta^d]^2(X, Z)/2) \right\|_1 = 0$ ,

where  $\zeta^d$  is given by (14).

*Proof.* Note by definition of  $\zeta^d$  and  $\xi_\theta$  (11), for  $x \in \mathcal{I}$  and  $x + \ell d^{-1/2}z \in \mathcal{I}$ ,

$$\zeta^d(x, z) = \xi_{\ell z d^{-1/2}}(x) / \xi_0(x) - 1. \quad (50)$$

Using Lemma 9,

$$\begin{aligned} & \left\| \mathbb{1}_{\mathcal{D}_x} \zeta^d(X, Z) + \ell Z \dot{V}(X) / (2\sqrt{d}) \right\|_2^2 \\ &= \mathbb{E} \left[ \int_{\mathcal{I}} \left( v(x, \ell Z d^{-1/2}) \{ \xi_{\ell Z d^{-1/2}}(x) - \xi_0(x) \} + \ell Z \dot{V}(x) \xi_0(x) / (2\sqrt{d}) \right)^2 dx \right] \\ &\leq C \ell^{2\beta} d^{-\beta} \mathbb{E} [ |Z|^{2\beta} ]. \end{aligned}$$

The proof of (i) is completed using  $\beta > 1$ . For (ii), write for all  $x \in \mathcal{I}$  and  $x + \ell z d^{-1/2}z \in \mathcal{I}$ ,  $\Delta V(x, z) = V(x) - V(x + \ell z d^{-1/2})$ . By **G1**(i)

$$\begin{aligned} \left\| \mathbb{1}_{\mathcal{D}_x} \Delta V(X, Z) + \ell Z \dot{V}(X) / \sqrt{d} \right\|_p^p &= \mathbb{E} \left[ \int_{\mathcal{I}} \left( v(x, \ell Z d^{-1/2}) \Delta V(X, Z) + \ell Z \dot{V}(x) / \sqrt{d} \right)^p \pi(x) dx \right] \\ &\leq C \ell^{\beta p} d^{-\beta p/2} \mathbb{E} [ |Z|^{\beta p} ] \end{aligned}$$

and the proof of (ii) follows from  $\beta > 1$ . For (iii), note that for all  $x > 0$ ,  $u \in [0, x]$ ,  $|(x - u)(1 + u)^{-1}| \leq |x|$ , and the same inequality holds for  $x \in (-1, 0]$  and  $u \in [x, 0]$ . Then by (23) and (24), for all  $x > -1$ ,

$$\left| \log(1 + x) - x + x^2/2 \right| = |R(x)| \leq x^2 |\log(1 + x)|.$$

Then by (50), for  $x \in \mathcal{I}$  and  $x + \ell d^{-1/2}z \in \mathcal{I}$ ,

$$\begin{aligned} & \left| \log(1 + \zeta_d(x, z)) - \zeta^d(x, z) + [\zeta^d]^2(x, z)/2 \right| \\ &\leq (\xi_{\ell z d^{-1/2}}(x) / \xi_0(x) - 1)^2 |\log(\xi_{\ell z d^{-1/2}}(x) / \xi_0(x))|, \\ &\leq (\xi_{\ell z d^{-1/2}}(x) / \xi_0(x) - 1)^2 \left| V(x + \ell z d^{-1/2}) - V(x) \right| / 2. \end{aligned}$$

Since for all  $x \in \mathbb{R}$ ,  $|\exp(x) - 1| \leq |x|(\exp(x) + 1)$ , this yields,

$$\begin{aligned} & \left| \log(1 + \zeta_d(x, z)) - \zeta^d(x, z) + [\zeta^d]^2(x, z)/2 \right| \\ &\leq \left| V(x + \ell z d^{-1/2}) - V(x) \right|^3 \left( \exp(V(x) - V(x + \ell z d^{-1/2})) + 1 \right) / 4. \end{aligned}$$

Therefore,

$$\int_{\mathcal{I}} v(x, \ell z d^{-1/2}) \left| \log(1 + \zeta_d(x, z)) - \zeta^d(x, z) + [\zeta^d]^2(x, z)/2 \right| \pi(x) dx \leq (I_1 + I_2) / 4,$$

where

$$\begin{aligned} I_1 &= \int_{\mathcal{I}} v(x, \ell z d^{-1/2}) \left| V(x + \ell z d^{-1/2}) - V(x) \right|^3 \pi(x) dx \\ I_2 &= \int_{\mathcal{I}} v(x, \ell z d^{-1/2}) \left| V(x + \ell z d^{-1/2}) - V(x) \right|^3 \pi(x + \ell z d^{-1/2}) dx. \end{aligned}$$

By Hölder's inequality, a change of variable and using **G1**(i),

$$I_1 + I_2 \leq C \left( \left| \ell z d^{-1/2} \right|^3 \left( \int_{\mathcal{I}} \left| \dot{V}(x) \right|^4 \pi(x) dx \right)^{3/4} + \left| \ell z d^{-1/2} \right|^{3\beta} \right).$$

The proof follows from **G1**(ii) and  $\beta > 1$ .  $\square$

For ease of notation, write for all  $d \geq 1$  and  $i, j \in \{1, \dots, d\}$ ,

$$\begin{aligned} \mathcal{D}_{\mathcal{I},j}^d &= \left\{ X_j^d + r \ell d^{-1/2} Z_j^d \in \mathcal{I} \right\} \cap \left\{ X_j^d + (1-r) \ell d^{-1/2} Z_j^d \in \mathcal{I} \right\}, \\ \mathcal{D}_{\mathcal{I},i;j}^d &= \bigcap_{k=i}^j \mathcal{D}_{\mathcal{I},k}^d. \end{aligned} \quad (51)$$

**Lemma 11.** *Assume that **G1** holds. For all  $d \geq 1$ , let  $X^d$  be distributed according to  $\pi^d$ , and  $Z^d$  be  $d$ -dimensional Gaussian random variable independent of  $X^d$ . Then,  $\lim_{d \rightarrow +\infty} \mathbb{J}_{\mathcal{I}}^d = 0$  where*

$$\mathbb{J}_{\mathcal{I}}^d = \left\| \mathbb{1}_{\mathcal{D}_{\mathcal{I},2;d}^d} \sum_{i=2}^d \left\{ \left( \Delta V_i^d + \frac{\ell Z_i^d}{\sqrt{d}} \dot{V}(X_i^d) \right) - 2\mathbb{E} \left[ \mathbb{1}_{\mathcal{D}_{\mathcal{I},i}^d} \zeta^d(X_i^d, Z_i^d) \right] + \frac{\ell^2}{4d} \dot{V}^2(X_i^d) \right\} \right\|_1.$$

*Proof.* The proof follows the same lines as the proof of Lemma 3 and is omitted.  $\square$

Define for all  $d \geq 1$ ,

$$\begin{aligned} \mathbb{E}_{\mathcal{I}}^d &= \mathbb{E} \left[ (Z_1^d)^2 \mathbb{1}_{\mathcal{D}_{\mathcal{I},1;d}^d} 1 \wedge \exp \left\{ \sum_{i=1}^d \Delta V_i^d \right\} \right. \\ &\quad \left. - 1 \wedge \exp \left\{ -\ell d^{-1/2} Z_1^d \dot{V}(X_1^d) + \sum_{i=2}^d b_{\mathcal{I}}^d(X_i^d, Z_i^d) \right\} \right], \end{aligned}$$

where  $\Delta V_i^d$  is given by (5), for all  $x \in \mathcal{I}$ ,  $z \in \mathbb{R}$ ,

$$b_{\mathcal{I}}^d(x, z) = -\frac{\ell z}{\sqrt{d}} \dot{V}(x) + 2\mathbb{E} \left[ \mathbb{1}_{\mathcal{D}_{\mathcal{I},1}^d} \zeta^d(X_1^d, Z_1^d) \right] - \frac{\ell^2}{4d} \dot{V}^2(x), \quad (52)$$

and  $\zeta^d$  is given by (14).

**Proposition 7.** *Assume **G1** holds. Let  $X^d$  be a random variable distributed according to  $\pi^d$  and  $Z^d$  be a zero-mean standard Gaussian random variable, independent of  $X$ . Then  $\lim_{d \rightarrow +\infty} \mathbb{E}_{\mathcal{I}}^d = 0$ .*

*Proof.* Let  $\Lambda^d = -\ell d^{-1/2} Z_1^d \dot{V}(X_1^d) + \sum_{i=2}^d \Delta V_i^d$ . By the triangle inequality,  $\mathbb{E}^d \leq \mathbb{E}_1^d + \mathbb{E}_2^d + \mathbb{E}_3^d$  where

$$\begin{aligned} \mathbb{E}_{1,\mathcal{I}}^d &= \mathbb{E} \left[ (Z_1^d)^2 \mathbb{1}_{\mathcal{D}_{\mathcal{I},1;d}^d} \left| 1 \wedge \exp \left\{ \sum_{i=1}^d \Delta V_i^d \right\} - 1 \wedge \exp \{ \Lambda^d \} \right| \right], \\ \mathbb{E}_{2,\mathcal{I}}^d &= \mathbb{E} \left[ (Z_1^d)^2 \mathbb{1}_{\mathcal{D}_{\mathcal{I},2;d}^d} \left| 1 \wedge \exp \{ \Lambda^d \} - 1 \wedge \exp \left\{ -\ell d^{-1/2} Z_1^d \dot{V}(X_1^d) + \sum_{i=2}^d b^d(X_i^d, Z_i^d) \right\} \right| \right], \\ \mathbb{E}_{3,\mathcal{I}}^d &= \mathbb{E} \left[ (Z_1^d)^2 \mathbb{1}_{(\mathcal{D}_{\mathcal{I},2;d}^d)^c} \left| 1 \wedge \exp \left\{ -\ell d^{-1/2} Z_1^d \dot{V}(X_1^d) + \sum_{i=2}^d b^d(X_i^d, Z_i^d) \right\} \right| \right], \end{aligned}$$

Since  $t \mapsto 1 \wedge e^t$  is 1-Lipschitz, by the Cauchy-Schwarz inequality we get

$$\mathbb{E}_{1,\mathcal{I}}^d \leq \mathbb{E} \left[ (Z_1^d)^2 \mathbb{1}_{\mathcal{D}_{\mathcal{I},1}^d} \left| \Delta V_1^d + \ell d^{-1/2} Z_1^d \dot{V}(X_1^d) \right| \right] \leq \|Z_1^d\|_4^2 \left\| \mathbb{1}_{\mathcal{D}_{\mathcal{I},1}^d} \Delta V_1^d + \ell d^{-1/2} Z_1^d \dot{V}(X_1^d) \right\|_2.$$

By Lemma 2(ii),  $\mathbb{E}_{1,\mathcal{I}}^d$  goes to 0 as  $d$  goes to  $+\infty$ . Using again that  $t \mapsto 1 \wedge e^t$  is 1-Lipschitz and Lemma 11,  $\mathbb{E}_{2,\mathcal{I}}^d$  goes to 0 as well. Note that, as  $Z_1^d$  and  $\mathbb{1}_{(\mathcal{D}_{\mathcal{I},2;d}^d)^c}$  are independent, by (18),

$$\mathbb{E}_{3,\mathcal{I}}^d \leq d\mathbb{P} \left( \{\mathcal{D}_{\mathcal{I},1}^d\}^c \right) \leq Cd^{1-\gamma/2}.$$

Therefore,  $\mathbb{E}_{3,\mathcal{I}}^d$  goes to 0 as  $d$  goes to  $+\infty$  by **G1**(iii).  $\square$

**Lemma 12.** *Assume **G1** holds. For all  $d \in \mathbb{N}^*$ , let  $X^d$  be a random variable distributed according to  $\pi^d$  and  $Z^d$  be a standard Gaussian random variable in  $\mathbb{R}^d$ , independent of  $X$ . Then,*

$$\lim_{d \rightarrow +\infty} 2d \mathbb{E} \left[ \mathbb{1}_{\mathcal{D}_{\mathcal{I},1}^d} \zeta^d(X_1^d, Z_1^d) \right] = -\frac{\ell^2}{4} I,$$

where  $I$  is defined in (6) and  $\zeta^d$  in (14).

*Proof.* Noting that for all  $\theta \in \mathbb{R}$ ,

$$\int_{\mathcal{I}} \mathbb{1}_{\mathcal{I}}(x+r\theta) \mathbb{1}_{\mathcal{I}}(x+(1-r)\theta) \pi(x+\theta) dx = \int_{\mathcal{I}} \mathbb{1}_{\mathcal{I}}(x+(r-1)\theta) \mathbb{1}_{\mathcal{I}}(x-r\theta) \pi(x) dx.$$

the proof follows the same steps as the the proof of Lemma 4 and is omitted.  $\square$

*Proof of Theorem 4.* The proof follows the same lines as the proof of Theorem 2 and is therefore omitted.  $\square$

## 5.2 Proof of Proposition 4

As for the proof of Proposition 2, the proof follows from Lemma 13.

**Lemma 13.** *Assume **G1**. Then, there exists  $C > 0$  such that, for all  $0 \leq k_1 < k_2$ ,*

$$\mathbb{E} \left[ (X_{k_2,1}^d - X_{k_1,1}^d)^4 \right] \leq C \sum_{p=2}^4 \frac{(k_2 - k_1)^p}{d^p}.$$

*Proof.* We use the same decomposition of  $\mathbb{E}[(X_{k_2,1}^d - X_{k_1,1}^d)^4]$  as in the proof of Lemma 5 so that we only need to upper bound the following term:

$$d^{-2} \mathbb{E} \left[ \left( \sum_{k=k_1+1}^{k_2} Z_{k,1}^d \mathbb{1}_{(\mathcal{A}_k^d)^c} \right)^4 \right] = d^{-2} \sum \mathbb{E} \left[ \prod_{i=1}^4 Z_{m_i,1}^d \mathbb{1}_{(\mathcal{A}_{m_i}^d)^c} \right],$$

where the sum is over all the quadruplets  $(m_p)_{p=1}^4$  satisfying  $m_p \in \{k_1+1, \dots, k_2\}$ ,  $p = 1, \dots, 4$ . Let  $(m_1, m_2, m_3, m_4) \in \{k_1+1, \dots, k_2\}^4$  and  $(\tilde{X}_k^d)_{k \geq 0}$  be defined as:

$$\tilde{X}_0^d = X_0^d \quad \text{and} \quad \tilde{X}_{k+1}^d = \tilde{X}_k^d + \mathbb{1}_{k \notin \{m_1-1, m_2-1, m_3-1, m_4-1\}} \ell d^{-1/2} Z_{k+1}^d \mathbb{1}_{\tilde{\mathcal{A}}_{k+1}^d},$$

where for all  $k \geq 0$  and all  $1 \leq i \leq d$ ,

$$\begin{aligned}\tilde{\mathcal{A}}_{k+1}^d &= \left\{ U_{k+1} \leq \exp \left( \sum_{i=1}^d \Delta \tilde{V}_{k,i}^d \right) \right\} \\ \Delta \tilde{V}_{k,i}^d &= V \left( \tilde{X}_{k,i}^d \right) - V \left( \tilde{X}_{k,i}^d + \ell d^{-1/2} Z_{k+1,i}^d \right).\end{aligned}$$

Define, for all  $k_1 + 1 \leq k \leq k_2$ ,  $1 \leq i, j \leq d$ ,

$$\begin{aligned}\tilde{\mathcal{D}}_{\mathcal{I},j}^{d,k} &= \left\{ \tilde{X}_{k,j}^d + r \ell d^{-1/2} Z_{k+1,j}^d \in \mathcal{I} \right\} \cap \left\{ \tilde{X}_{k,j}^d + (1-r) \ell d^{-1/2} Z_{k+1,j}^d \in \mathcal{I} \right\}, \\ \tilde{\mathcal{D}}_{\mathcal{I},i;j}^{d,k} &= \bigcap_{\ell=i}^j \tilde{\mathcal{D}}_{\mathcal{I},\ell}^{d,k}.\end{aligned}$$

Note that by convention  $V(x) = -\infty$  for all  $x \notin \mathcal{I}$ ,  $\tilde{\mathcal{A}}_{k+1}^d \subset \tilde{\mathcal{D}}_{\mathcal{I},1:d}^{d,k}$  so that  $(\tilde{\mathcal{A}}_{k+1}^d)^c$  may be written  $(\tilde{\mathcal{A}}_{k+1}^d)^c = (\tilde{\mathcal{D}}_{\mathcal{I},1:d}^{d,k})^c \cup \left( (\tilde{\mathcal{A}}_{k+1}^d)^c \cap \tilde{\mathcal{D}}_{\mathcal{I},1:d}^{d,k} \right)$ . Let  $\mathcal{F}$  be the  $\sigma$ -field generated by  $(\tilde{X}_k^d)_{k \geq 0}$ . Consider the case  $\#\{m_1, \dots, m_4\} = 4$ . The case  $\#\{m_1, \dots, m_4\} = 3$  is dealt with similarly and the two other cases follow the same lines as the proof of Lemma 13. As  $\left\{ (U_{m_j}, Z_{m_j,1}^d, \dots, Z_{m_j,d}^d) \right\}_{1 \leq j \leq 4}$  are independent conditionally to  $\mathcal{F}$ ,

$$\mathbb{E} \left[ \prod_{j=1}^4 Z_{m_j,1}^d \mathbb{1}_{(\mathcal{A}_{m_j}^d)^c} \middle| \mathcal{F} \right] = \prod_{j=1}^4 \left\{ \mathbb{E} \left[ \mathbb{1}_{(\tilde{\mathcal{D}}_{\mathcal{I},1:d}^{d,m_j-1})^c} Z_{m_j,1}^d \middle| \mathcal{F} \right] + \mathbb{E} \left[ \mathbb{1}_{\tilde{\mathcal{D}}_{\mathcal{I},1:d}^{d,m_j-1}} \mathbb{1}_{(\tilde{\mathcal{A}}_{m_j}^d)^c} Z_{m_j,1}^d \middle| \mathcal{F} \right] \right\}.$$

As  $U_{m_j}$  is independent of  $(Z_{m_j,1}^d, \dots, Z_{m_j,d}^d)$  conditionally to  $\mathcal{F}$ , the second term may be written:

$$\mathbb{E} \left[ \mathbb{1}_{\tilde{\mathcal{D}}_{\mathcal{I},1:d}^{d,m_j-1}} \mathbb{1}_{(\tilde{\mathcal{A}}_{m_j}^d)^c} Z_{m_j,1}^d \middle| \mathcal{F} \right] = \mathbb{E} \left[ \mathbb{1}_{\tilde{\mathcal{D}}_{\mathcal{I},1:d}^{d,m_j-1}} Z_{m_j,1}^d \left( 1 - \exp \left\{ \sum_{i=1}^d \Delta \tilde{V}_{m_j-1,i}^d \right\} \right)_+ \middle| \mathcal{F} \right].$$

Since the function  $x \mapsto (1 - e^x)_+$  is 1-Lipschitz, on  $\tilde{\mathcal{D}}_{\mathcal{I},1:d}^{d,m_j-1}$

$$\left| \left( 1 - \exp \left\{ \sum_{i=1}^d \Delta \tilde{V}_{m_j-1,i}^d \right\} \right)_+ - \Theta_{m_j} \right| \leq \left| \Delta \tilde{V}_{m_j-1,1}^d + \ell d^{-1/2} \dot{V}(\tilde{X}_{m_j-1,1}^d) Z_{m_j,1}^d \right|,$$

where  $\Theta_{m_j} = (1 - \exp\{-\ell d^{-1/2} \dot{V}(\tilde{X}_{m_j-1,1}^d) Z_{m_j,1}^d + \sum_{i=2}^d \Delta \tilde{V}_{m_j-1,i}^d\})_+$ . Then,

$$\left| \mathbb{E} \left[ \mathbb{1}_{\tilde{\mathcal{D}}_{\mathcal{I},1:d}^{d,m_j-1}} Z_{m_j,1}^d \left( 1 - \exp \left\{ \sum_{i=1}^d \Delta \tilde{V}_{m_j-1,i}^d \right\} \right)_+ \middle| \mathcal{F} \right] \right| \leq A_{m_j}^d + B_{m_j}^d,$$

where

$$\begin{aligned}A_{m_j}^d &= \mathbb{E} \left[ \left| Z_{m_j,1}^d \right| \left| \mathbb{1}_{\tilde{\mathcal{D}}_{\mathcal{I},1}^{d,m_j-1}} \Delta \tilde{V}_{m_j-1,1}^d + \ell d^{-1/2} \dot{V}(\tilde{X}_{m_j-1,1}^d) Z_{m_j,1}^d \right| \middle| \mathcal{F} \right], \\ B_{m_j}^d &= \left| \mathbb{E} \left[ \mathbb{1}_{\tilde{\mathcal{D}}_{\mathcal{I},2:d}^{d,m_j-1}} Z_{m_j,1}^d \Theta_{m_j} \middle| \mathcal{F} \right] \right|.\end{aligned}$$

By Jensen inequality,

$$\begin{aligned} \left| \mathbb{E} \left[ \prod_{j=1}^4 Z_{m_j,1}^d \mathbf{1}_{(\mathcal{A}_{m_j}^d)^c} \right] \right| &\leq \mathbb{E} \left[ \prod_{j=1}^4 \left\{ \mathbb{E} \left[ \mathbf{1}_{(\tilde{\mathcal{D}}_{\mathcal{I},1;d}^{d,m_{j-1}})^c} |Z_{m_j,1}^d| \middle| \mathcal{F} \right] + A_{m_j}^d + B_{m_j}^d \right\} \right], \\ &\leq C \mathbb{E} \left[ \sum_{j=1}^4 \mathbb{E} \left[ \mathbf{1}_{(\tilde{\mathcal{D}}_{\mathcal{I},1;d}^{d,m_{j-1}})^c} |Z_{m_j,1}^d|^4 \middle| \mathcal{F} \right] + (A_{m_j}^d)^4 + (B_{m_j}^d)^4 \right], \end{aligned}$$

By **G1**(iii) and Holder's inequality applied with  $\alpha = 1/(1 - 2/\gamma) > 1$ , for all  $1 \leq j \leq 4$ ,

$$\begin{aligned} \mathbb{E} \left[ \mathbf{1}_{(\tilde{\mathcal{D}}_{\mathcal{I},1;d}^{d,m_{j-1}})^c} |Z_{m_j,1}^d|^4 \right] &\leq \mathbb{E} \left[ \mathbf{1}_{(\tilde{\mathcal{D}}_{\mathcal{I},1}^{d,m_{j-1}})^c} |Z_{m_j,1}^d|^4 \right] + \sum_{i=2}^d \mathbb{E} \left[ \mathbf{1}_{(\tilde{\mathcal{D}}_{\mathcal{I},i}^{d,m_{j-1}})^c} \right], \\ &\leq \mathbb{E} \left[ |Z_{m_j,1}^d|^{4\alpha/(\alpha-1)} \right]^{(\alpha-1)/\alpha} d^{-\gamma/(2\alpha)} + d^{1-\gamma/2}, \\ &\leq C d^{1-\gamma/2}. \end{aligned}$$

By Lemma 10(ii) and the Holder's inequality, there exists  $C > 0$  such that  $\mathbb{E} \left[ (A_{m_j}^d)^4 \right] \leq C d^{-2}$ .

On the other hand, by [2, Lemma 6] since  $Z_{m_j,1}^d$  is independent of  $\mathcal{F}$ ,

$$B_{m_j}^d = \left| \mathbb{E} \left[ \mathbf{1}_{\tilde{\mathcal{D}}_{\mathcal{I},2;d}^{d,m_{j-1}}} \ell d^{-1/2} \dot{V}(\tilde{X}_{m_{j-1},1}^d) \mathcal{G} \left( \ell^2 d^{-1} \dot{V}(\tilde{X}_{m_{j-1},1}^d)^2, -2 \sum_{i=2}^d \Delta \tilde{V}_{m_{j-1},i}^d \right) \middle| \mathcal{F} \right] \right|,$$

where the function  $\mathcal{G}$  is defined in (26). By **G1**(ii) and since  $\mathcal{G}$  is bounded,  $\mathbb{E}[(B_{m_j}^d)^4] \leq C d^{-2}$ .

Since  $\gamma \geq 6$  in **G1**(iii),  $|\mathbb{E}[\prod_{j=1}^4 Z_{m_j,1}^d \mathbf{1}_{(\mathcal{A}_{m_j}^d)^c}]| \leq C d^{-2}$ , showing that

$$\sum_{(m_1, m_2, m_3, m_4) \in \mathcal{I}_4} \left| \mathbb{E} \left[ \prod_{i=1}^4 Z_{m_i,1}^d \mathbf{1}_{(\mathcal{A}_{m_i}^d)^c} \right] \right| \leq C d^{-2} \binom{k_2 - k_1}{4}. \quad (53)$$

□

### 5.3 Proof of Proposition 5

**Lemma 14.** *Assume that **G1** holds. Let  $\mu$  be a limit point of the sequence of laws  $(\mu_d)_{d \geq 1}$  of  $\{(Y_{t,1}^d)_{t \geq 0}, d \in \mathbb{N}^*\}$ . Then for all  $t \geq 0$ , the pushforward measure of  $\mu$  by  $W_t$  is  $\pi$ .*

*Proof.* The proof is the same as in Lemma 6 and is omitted. □

We preface the proof by a lemma which provides a condition to verify that any limit point  $\mu$  of  $(\mu_d)_{d \geq 1}$  is a solution to the local martingale problem associated with (9).

**Lemma 15.** *Assume **G1**. Let  $\mu$  be a limit point of the sequence  $(\mu_d)_{d \geq 1}$ . If for all  $\phi \in C_c^\infty(\mathcal{I}, \mathbb{R})$ , the process  $(\phi(W_t) - \phi(W_0) - \int_0^t L\phi(W_u) du)_{t \geq 0}$  is a martingale with respect to  $\mu$  and the filtration  $(\mathcal{B}_t)_{t \geq 0}$ , then  $\mu$  solves the local martingale problem associated with (9).*

*Proof.* As for all  $t \geq 0$  and  $d \geq 1$ ,  $Y_{t,1}^d \in \mathcal{I}$ , for all  $d \geq 1$   $\mu^d(C(\mathbb{R}_+, \bar{\mathcal{I}})) = 1$ . Since  $C(\mathbb{R}_+, \bar{\mathcal{I}})$  is closed in  $\mathbf{W}$ , we have by the Portmanteau theorem,  $\mu(C(\mathbb{R}_+, \bar{\mathcal{I}})) = 1$ . Therefore, we only need to prove that for all  $\psi \in C^\infty(\bar{\mathcal{I}}, \mathbb{R})$ , the process  $(\psi(W_t) - \psi(W_0) - \int_0^t L\psi(W_u)du)_{t \geq 0}$  is a local martingale with respect to  $\mu$  and the filtration  $(\mathcal{B}_t)_{t \geq 0}$ . Let  $\psi \in C^\infty(\bar{\mathcal{I}}, \mathbb{R})$ .

Suppose first that for all  $\varpi \in C_c^\infty(\bar{\mathcal{I}}, \mathbb{R})$ ,  $(\varpi(W_t) - \varpi(W_0) - \int_0^t L\varpi(W_u)du)_{t \geq 0}$  is a martingale. Then, consider the sequence of stopping time defined for  $k \geq 1$  by  $\tau_k = \inf\{t \geq 0 \mid |W_t| \geq k\}$  and a sequence  $(\varpi_k)_{k \geq 0}$  in  $C_c^\infty(\bar{\mathcal{I}}, \mathbb{R})$  satisfying:

1. for all  $k \geq 1$  and all  $x \in \bar{\mathcal{I}} \cap [-k, k]$ ,  $\varpi_k(x) = \psi(x)$ ,
2.  $\lim_{k \rightarrow +\infty} \varpi_k = \psi$  in  $C^\infty(\bar{\mathcal{I}}, \mathbb{R})$ .

Since for all  $k \geq 1$ ,

$$\begin{aligned} \left( \psi(W_{t \wedge \tau_k}) - \psi(W_0) - \int_0^{t \wedge \tau_k} L\psi(W_u)du \right)_{t \geq 0} \\ = \left( \varpi_k(W_{t \wedge \tau_k}) - \varpi_k(W_0) - \int_0^{t \wedge \tau_k} L\varpi_k(W_u)du \right)_{t \geq 0} \end{aligned}$$

and the sequence  $(\tau_k)_{k \geq 1}$  goes to  $+\infty$  as  $k$  goes to  $+\infty$  almost surely, it follows that  $(\psi(W_t) - \psi(W_0) - \int_0^t L\psi(W_u)du)_{t \geq 0}$  is a local martingale with respect to  $\mu$  and the filtration  $(\mathcal{B}_t)_{t \geq 0}$ . It remains to show that for all  $\varpi \in C_c^\infty(\bar{\mathcal{I}}, \mathbb{R})$ ,  $(\varpi(W_t) - \varpi(W_0) - \int_0^t L\varpi(W_u)du)_{t \geq 0}$  is a martingale under the assumption of the proposition. We only need to prove that for all  $\varpi \in C_c^\infty(\bar{\mathcal{I}}, \mathbb{R})$ ,  $0 \leq s \leq t$ ,  $m \in \mathbb{N}^*$ ,  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  bounded and continuous, and  $0 \leq t_1 \leq \dots \leq t_m \leq s \leq t$ :

$$\mathbb{E}^\mu \left[ \left( \varpi(W_t) - \varpi(W_s) - \int_s^t L\varpi(W_u)du \right) g(W_{t_1}, \dots, W_{t_m}) \right] = 0. \quad (54)$$

Let  $(\phi_k)_{k \geq 0}$  be a sequence of functions in  $C_c^\infty(\mathcal{I}, \mathbb{R})$  and converging to  $\varpi$  in  $C_c^\infty(\bar{\mathcal{I}}, \mathbb{R})$ . First note that for all  $u \in [s, t]$ ,  $\mu$ -almost everywhere,

$$\lim_{k \rightarrow +\infty} \phi_k(W_u) = \varpi(W_u). \quad (55)$$

By Lemma 14, for all  $u \in [s, t]$  the pushforward measure of  $\mu$  by  $W_u$  has density  $\pi$  with respect to the Lebesgue measure and  $\mu$ -almost everywhere,  $\lim_{k \rightarrow +\infty} L\phi_k(W_u) = L\varpi(W_u)$ . On the other hand, there exists  $C \geq 0$  such that for all  $k \geq 0$ ,  $|L\phi_k(W_u)| \leq C(1 + |\dot{V}(W_u)|)$ . Then,

$$\begin{aligned} \mathbb{E}^\mu \left[ \int_s^t \left( 1 + |\dot{V}(W_u)| \right) du \right] &\leq (t-s) + \int_s^t \mathbb{E}^\mu \left[ |\dot{V}(W_u)| \right] du \\ &\leq (t-s) \left( 1 + \int_{\mathcal{I}} |\dot{V}(x)| \pi(x) dx \right). \end{aligned}$$

Therefore,  $\mu$ -almost everywhere by **G1(ii)** and the Lebesgue dominated convergence theorem, we get

$$\lim_{k \rightarrow +\infty} \int_s^t L\phi_k(W_u)du = \int_s^t L\varpi(W_u)du. \quad (56)$$

Therefore, (54) follows from (55) and (56), using again the Lebesgue dominated convergence theorem and **G1(ii)**.  $\square$



*Proof of Proposition 5.* Let  $\mu$  be a limit point of  $(\mu_d)_{d \geq 1}$ . By Lemma 15, we only need to prove that for all  $\phi \in C_c^\infty(\mathcal{I}, \mathbb{R})$ , the process  $(\phi(W_t) - \phi(W_0) - \int_0^t L\phi(W_u)du)_{t \geq 0}$  is a martingale with respect to  $\mu$  and the filtration  $(\mathcal{B}_t)_{t \geq 0}$ . Then, the proof follows the same line as the proof of Proposition 3 and is omitted.  $\square$

## 5.4 Proof of Theorem 5

**Lemma 16.** *Assume **G1** holds. Let  $X^d$  be distributed according to  $\pi^d$  and  $Z^d$  be a  $d$ -dimensional standard Gaussian random variable, independent of  $X^d$ . Then,  $\lim_{d \rightarrow +\infty} E^d = 0$ , where*

$$E^d = \mathbb{E} \left[ \left| \dot{V}(X_1^d) \mathbf{1}_{\mathcal{D}_{\bar{X}, 2, d}^d} \left\{ \mathcal{G} \left( \ell^2 \dot{V}(X_1^d)^2 / d, 2\bar{Y}_d \right) - \mathcal{G} \left( \ell^2 \dot{V}(X_1^d)^2 / d, 2\bar{X}_d \right) \right\} \right| \right],$$

where  $\bar{Y}_d = \sum_{i=2}^d \Delta V_i^d$ ,  $\Delta V_i^d$  and  $\mathcal{D}_{\bar{X}, 2, d}^d$  are given by (5) and (51) and  $\bar{X}_d = \sum_{i=2}^d b_{\bar{X}, i}^d$ ,  $b_{\bar{X}, i}^d = b_{\bar{X}}^d(X_i^d, Z_i^d)$  with  $b_{\bar{X}}^d$  given by (52).

*Proof.* Set for all  $d \geq 1$ ,  $\bar{Y}_d = \sum_{i=2}^d \Delta V_i^d$  and  $\bar{X}_d = \sum_{i=2}^d b_{\bar{X}, i}^d$ . By definition of  $b_{\bar{X}}^d$  (52),  $\bar{X}_d$  may be expressed as  $\bar{X}_d = \sigma_d \bar{S}_d + \mu_d$ , where

$$\begin{aligned} \mu_d &= 2(d-1) \mathbb{E} \left[ \mathbf{1}_{\mathcal{D}_{\bar{X}, 1}^d} \zeta^d(X_1^d, Z_1^d) \right] - \frac{\ell^2(d-1)}{4d} \mathbb{E} \left[ \dot{V}(X_1^d)^2 \right], \\ \sigma_d^2 &= \ell^2 \mathbb{E} \left[ \dot{V}(X_1^d)^2 \right] + \frac{\ell^4}{16d} \mathbb{E} \left[ \left( \dot{V}(X_1^d)^2 - \mathbb{E} \left[ \dot{V}(X_1^d)^2 \right] \right)^2 \right], \\ \bar{S}_d &= (\sqrt{d}\sigma_d)^{-1} \sum_{i=2}^d \beta_i^d, \\ \beta_i^d &= -\ell Z_i^d \dot{V}(X_i^d) - \frac{\ell^2}{4\sqrt{d}} \left( \dot{V}(X_i^d)^2 - \mathbb{E} \left[ \dot{V}(X_i^d)^2 \right] \right). \end{aligned}$$

By **G1(ii)** the Berry-Essen Theorem [5, Theorem 5.7] can be applied to  $\bar{S}_d$ . Then, there exists a universal constant  $C$  such that for all  $d > 0$ ,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left[ \sqrt{\frac{d}{d-1}} \bar{S}_d \leq x \right] - \Phi(x) \right| \leq C/\sqrt{d}.$$

It follows, with  $\tilde{\sigma}_d^2 = (d-1)\sigma_d^2/d$ , that

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} [\bar{X}_d \leq x] - \Phi((x - \mu_d)/\tilde{\sigma}_d) \right| \leq C/\sqrt{d}.$$

By this result and (37), Lemma 7 can be applied to obtain a constant  $C \geq 0$ , independent of  $d$ , such that:

$$\begin{aligned} & \mathbb{E} \left[ \mathbf{1}_{\mathcal{D}_{\bar{X}, 2, d}^d} \left| \mathcal{G} \left( \ell^2 \dot{V}(X_1^d)^2 / d, 2\bar{Y}_d \right) - \mathcal{G} \left( \ell^2 \dot{V}(X_1^d)^2 / d, 2\bar{X}_d \right) \right| \middle| X_1^d \right] \\ & \leq C \left( \mathbb{E} \left[ \mathbf{1}_{\mathcal{D}_{\bar{X}, 2, d}^d} |\bar{X}_d - \bar{Y}_d| \right] + d^{-1/2} + \sqrt{2\mathbb{E} \left[ \mathbf{1}_{\mathcal{D}_{\bar{X}, 2, d}^d} |\bar{X}_d - \bar{Y}_d| \right] (2\pi\tilde{\sigma}_d^2)^{-1/2}} \right. \\ & \quad \left. + \sqrt{\ell |\dot{V}(X_1^d)| / (2\pi d^{1/2} \tilde{\sigma}_d^2)} \right). \end{aligned}$$

Using this result, we have

$$\begin{aligned} E^d \leq C & \left\{ \ell^{1/2} \mathbb{E} \left[ |\dot{V}(X_1^d)|^{3/2} \right] (2\pi d^{1/2} \tilde{\sigma}_d^2)^{-1/2} + \mathbb{E} \left[ |\dot{V}(X_1^d)| \right] \right. \\ & \left. \times \left( \mathbb{E} \left[ \mathbf{1}_{\mathcal{D}_{\bar{X}, 2; d}^d} |\bar{X}_d - \bar{Y}_d| \right] + d^{-1/2} + \sqrt{2 \mathbb{E} \left[ \mathbf{1}_{\mathcal{D}_{\bar{X}, 2; d}^d} |\bar{X}_d - \bar{Y}_d| \right] (2\pi \tilde{\sigma}_d^2)^{-1/2}} \right) \right\}. \end{aligned} \quad (57)$$

By Lemma 11,  $\mathbb{E}[\mathbf{1}_{\mathcal{D}_{\bar{X}, 2; d}^d} |\bar{X}_d - \bar{Y}_d|]$  goes to 0 as  $d$  goes to infinity, and by **G1(ii)**  $\lim_{d \rightarrow +\infty} \tilde{\sigma}_d^2 = \ell^2 \mathbb{E}[\dot{V}(X)^2]$ . Combining these results with (57), it follows that  $E^d$  goes to 0 when  $d$  goes to infinity.  $\square$

For all  $n \geq 0$ , define  $\mathcal{F}_n^d = \sigma(\{X_k^d, k \leq n\})$  and for all  $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ ,

$$\begin{aligned} M_n^d(\phi) &= \frac{\ell}{\sqrt{d}} \sum_{k=0}^{n-1} \phi'(X_{k,1}^d) \left\{ Z_{k+1,1}^d \mathbf{1}_{\mathcal{A}_{k+1}^d} - \mathbb{E} \left[ Z_{k+1,1}^d \mathbf{1}_{\mathcal{A}_{k+1}^d} \mid \mathcal{F}_k^d \right] \right\} \\ &\quad + \frac{\ell^2}{2d} \sum_{k=0}^{n-1} \phi''(X_{k,1}^d) \left\{ (Z_{k+1,1}^d)^2 \mathbf{1}_{\mathcal{A}_{k+1}^d} - \mathbb{E} \left[ (Z_{k+1,1}^d)^2 \mathbf{1}_{\mathcal{A}_{k+1}^d} \mid \mathcal{F}_k^d \right] \right\}. \end{aligned} \quad (58)$$

**Proposition 8.** *Assume **G1** and **G2** hold. Then, for all  $s \leq t$  and all  $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ ,*

$$\lim_{d \rightarrow +\infty} \mathbb{E} \left[ \left| \phi(Y_{t,1}^d) - \phi(Y_{s,1}^d) - \int_s^t L\phi(Y_{r,1}^d) dr - \left( M_{[dt]}^d(\phi) - M_{[ds]}^d(\phi) \right) \right| \right] = 0.$$

*Proof.* Using the same decomposition as in the proof of Proposition 6, we only need to prove that for all  $1 \leq i \leq 5$ ,  $\lim_{d \rightarrow +\infty} \mathbb{E}[|T_i^d|] = 0$ , where

$$\begin{aligned} T_1^d &= \int_s^t \phi'(X_{[dr],1}^d) \left( \ell \sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \mathbf{1}_{\mathcal{A}_{[dr]}^d} \mid \mathcal{F}_{[dr]}^d \right] + \frac{h(\ell)}{2} \dot{V}(X_{[dr],1}^d) \right) dr, \\ T_2^d &= \int_s^t \phi''(X_{[dr],1}^d) \left( \frac{\ell^2}{2} \mathbb{E} \left[ (Z_{[dr],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[dr]}^d} \mid \mathcal{F}_{[dr]}^d \right] - \frac{h(\ell)}{2} \right) dr, \\ T_3^d &= \int_s^t \left( L\phi(Y_{[dr]/d,1}^d) - L\phi(Y_{r,1}^d) \right) dr, \\ T_4^d &= \frac{\ell([\!dt] - dt)}{\sqrt{d}} \phi'(X_{[dt],1}^d) \left( Z_{[dt],1}^d \mathbf{1}_{\mathcal{A}_{[dt]}^d} - \mathbb{E} \left[ Z_{[dt],1}^d \mathbf{1}_{\mathcal{A}_{[dt]}^d} \mid \mathcal{F}_{[dt]}^d \right] \right) \\ &\quad + \frac{\ell^2([\!dt] - dt)}{2d} \phi''(X_{[dt],1}^d) \left( (Z_{[dt],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[dt]}^d} - \mathbb{E} \left[ (Z_{[dt],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[dt]}^d} \mid \mathcal{F}_{[dt]}^d \right] \right), \\ T_5^d &= \frac{\ell([\!ds] - ds)}{\sqrt{d}} \phi'(X_{[ds],1}^d) \left( Z_{[ds],1}^d \mathbf{1}_{\mathcal{A}_{[ds]}^d} - \mathbb{E} \left[ Z_{[ds],1}^d \mathbf{1}_{\mathcal{A}_{[ds]}^d} \mid \mathcal{F}_{[ds]}^d \right] \right) \\ &\quad + \frac{\ell^2([\!ds] - ds)}{2d} \phi''(X_{[ds],1}^d) \left( (Z_{[ds],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[ds]}^d} - \mathbb{E} \left[ (Z_{[ds],1}^d)^2 \mathbf{1}_{\mathcal{A}_{[ds]}^d} \mid \mathcal{F}_{[ds]}^d \right] \right). \end{aligned}$$

First, as  $\phi'$  and  $\phi''$  are bounded,  $\mathbb{E}[|T_4^d| + |T_5^d|] \leq Cd^{-1/2}$ . Denote for all  $r \in [s, t]$  and  $d \geq 1$ ,

$$\begin{aligned} \Delta V_{r,i}^d &= V \left( X_{[dr],i}^d \right) - V \left( X_{[dr],i}^d + \ell d^{-1/2} Z_{[dr],i}^d \right) \\ \Xi_r^d &= 1 \wedge \exp \left\{ -\ell Z_{[dr],1}^d \dot{V}(X_{[dr],1}^d) / \sqrt{d} + \sum_{i=2}^d b_{\mathcal{X},i}^{d,[dr]} \right\}, \end{aligned}$$

where for all  $k, i \geq 0$ ,  $b_{\mathcal{I},i}^{d,k} = b_{\mathcal{I}}^d(X_{k,i}^d, Z_{k+1,i}^d)$ , and for all  $x, z \in \mathbb{R}$ ,  $b_{\mathcal{I}}^d(x, y)$  is given by (52). For all  $k \geq 0$ ,  $1 \leq i, j \leq d$ , define

$$\begin{aligned} \mathcal{D}_{\mathcal{I},j}^{d,k} &= \left\{ X_{k,j}^d + r\ell d^{-1/2} Z_{k+1,j}^d \in \mathcal{I} \right\} \cap \left\{ X_{k,j}^d + (1-r)\ell d^{-1/2} Z_{k+1,j}^d \in \mathcal{I} \right\} \\ \mathcal{D}_{\mathcal{I},i;j}^{d,k} &= \bigcap_{\ell=i}^j \mathcal{D}_{\mathcal{I},\ell}^{d,k}. \end{aligned}$$

By the triangle inequality,

$$|T_1| \leq \int_s^t \left| \phi'(X_{[dr],1}^d) \right| (A_{1,r} + A_{2,r} + A_{3,r} + A_{4,r}) dr, \quad (59)$$

where

$$\begin{aligned} \Pi_r^d &= 1 \wedge \exp \left\{ -\ell d^{-1/2} Z_{[dr],1}^d \dot{V}(X_{[dr],1}^d) + \sum_{i=2}^d \Delta V_{r,i}^d \right\}, \\ A_{1,r} &= \left| \ell \sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \left( \mathbb{1}_{\mathcal{A}_{[dr]}^d} - \mathbb{1}_{\mathcal{D}_{\mathcal{I},1;d}^{d,[dr]}} \Pi_r^d \right) \middle| \mathcal{F}_{[dr]}^d \right] \right|, \\ A_{2,r} &= \left| \ell \sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \mathbb{1}_{\mathcal{D}_{\mathcal{I},1;d}^{d,[dr]}} \left( \Pi_r^d - \Xi_r^d \right) \middle| \mathcal{F}_{[dr]}^d \right] \right|, \\ A_{3,r} &= \left| \ell \sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \mathbb{1}_{\left( \mathcal{D}_{\mathcal{I},1;d}^{d,[dr]} \right)^c} \Xi_r^d \middle| \mathcal{F}_{[dr]}^d \right] \right|, \\ A_{4,r} &= \left| \ell \sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \Xi_r^d \middle| \mathcal{F}_{[dr]}^d \right] + \dot{V}(X_{[dr],1}^d) h(\ell) / 2 \right|. \end{aligned}$$

Since  $t \mapsto 1 \wedge \exp(t)$  is 1-Lipschitz,

$$\begin{aligned} \mathbb{E} [|A_{1,r}^d|] &\leq \ell \sqrt{d} \mathbb{E} \left[ \mathbb{1}_{\mathcal{D}_{\mathcal{I},1;d}^{d,[dr]}} \left| Z_{[dr],1}^d \right| \left| \Delta V_{r,1}^d - \ell d^{-1/2} Z_{[dr],1}^d \dot{V}(X_{[dr],1}^d) \right| \right], \\ &\leq \ell \sqrt{d} \mathbb{E} \left[ \mathbb{1}_{\mathcal{D}_{\mathcal{I},1;d}^{d,[dr]}} \left| Z_{[dr],1}^d \right| \left| \Delta V_{r,1}^d - \ell d^{-1/2} Z_{[dr],1}^d \dot{V}(X_{[dr],1}^d) \right| \right], \\ &\leq \ell \sqrt{d} \mathbb{E} \left[ \left| Z_{[dr],1}^d \right| \left| \mathbb{1}_{\mathcal{D}_{\mathcal{I},1;d}^{d,[dr]}} \Delta V_{r,1}^d - \ell d^{-1/2} Z_{[dr],1}^d \dot{V}(X_{[dr],1}^d) \right| \right] \end{aligned}$$

and  $\mathbb{E}[|A_{1,r}^d|]$  goes to 0 as  $d \rightarrow +\infty$  for almost all  $r$  by Lemma 10(ii). So by the Fubini theorem, the first term in (59) goes to 0 as  $d \rightarrow +\infty$ . For  $A_{2,r}^d$ , note that

$$A_{2,r} \leq \left| \ell \sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \mathbb{1}_{\mathcal{D}_{\mathcal{I},2;d}^{d,[dr]}} \left( \Pi_r^d - \Xi_r^d \right) \middle| \mathcal{F}_{[dr]}^d \right] \right|.$$

Then, by [2, Lemma 6],

$$\begin{aligned} \mathbb{E} [|A_{2,r}^d|] &\leq \mathbb{E} \left[ \left[ \ell^2 \dot{V}(X_{[dr],1}^d) \mathbb{1}_{\mathcal{D}_{\mathcal{I},2;d}^{d,[dr]}} \left\{ \mathcal{G} \left( \frac{\ell^2 \dot{V}(X_{[dr],1}^d)^2}{d}, 2 \sum_{i=2}^d \Delta V_{r,i}^d \right) \right. \right. \right. \\ &\quad \left. \left. \left. - \mathcal{G} \left( \frac{\ell^2 \dot{V}(X_{[dr],1}^d)^2}{d}, 2 \sum_{i=2}^d b_{\mathcal{I},i}^{d,[dr]} \right) \right\} \right] \right], \end{aligned}$$

where  $\mathcal{G}$  is defined in (26). By Lemma 16, this expectation goes to zero when  $d$  goes to infinity. Then by the Fubini theorem and the Lebesgue dominated convergence theorem, the second term

of (59) goes 0 as  $d \rightarrow +\infty$ . On the other hand, by **G1**(iii) and Holder's inequality applied with  $\alpha = 1/(1 - 2/\gamma) > 1$ , for all  $1 \leq j \leq 4$ ,

$$\begin{aligned} \mathbb{E}[|A_{3,r}^d|] &\leq \ell\sqrt{d} \left( \mathbb{E} \left[ |Z_{[dr],1}^d| \mathbf{1}_{(\mathcal{D}_{\mathcal{I},1}^{d,[dr]})^c} \right] + \sum_{i=2}^d \mathbb{E} \left[ \mathbf{1}_{(\mathcal{D}_{\mathcal{I},i}^{d,[dr]})^c} \right] \right), \\ &\leq \ell\sqrt{d} \left( \mathbb{E} \left[ |Z_{m_j,1}^d|^{\alpha/(\alpha-1)} \right]^{(\alpha-1)/\alpha} d^{-\gamma/(2\alpha)} + d^{1-\gamma/2} \right) \leq Cd^{3/2-\gamma/2} \end{aligned}$$

and  $\mathbb{E}[|A_{3,r}^d|]$  goes to 0 as  $d \rightarrow +\infty$  for almost all  $r$ . Define

$$\bar{V}_{d,1} = \sum_{i=1}^d \dot{V}(X_{[dr],i}^d) \quad \text{and} \quad \bar{V}_{d,2} = \bar{V}_{d,1} - \dot{V}(X_{[dr],1}^d)^2.$$

For the last term, by [2, Lemma 6]:

$$\begin{aligned} \ell\sqrt{d} \mathbb{E} \left[ Z_{[dr],1}^d \Xi_r^d \middle| \mathcal{F}_{[dr]}^d \right] &= -\ell^2 \dot{V}(X_{[dr],1}^d) \\ &\quad \times \mathcal{G} \left( \frac{\ell^2}{d} \bar{V}_{d,1}, \left\{ \frac{\ell^2}{2d} \bar{V}_{d,2} - 4(d-1) \mathbb{E} \left[ \mathbf{1}_{\mathcal{D}_{\mathcal{I}}} \zeta^d(X, Z) \right] \right\} \right), \quad (60) \end{aligned}$$

where  $\mathcal{D}_{\mathcal{I}} = \{X + \ell d^{-1/2} Z \in \mathcal{I}\}$ ,  $X$  is distributed according to  $\pi$  and  $Z$  is a standard Gaussian random variable independent of  $X$ . As  $\mathcal{G}$  is continuous on  $\mathbb{R}_+ \times \mathbb{R} \setminus \{0, 0\}$  (see [2, Lemma 2]), by **G1**(ii), Lemma 12 and the law of large numbers, almost surely,

$$\begin{aligned} \lim_{d \rightarrow +\infty} \ell^2 \mathcal{G} \left( \ell^2 \bar{V}_{d,1}/d, \ell^2 \bar{V}_{d,2}/(2d) - 4(d-1) \mathbb{E} \left[ \mathbf{1}_{\mathcal{D}_{\mathcal{I}}} \zeta^d(X, Z) \right] \right) \\ = \ell^2 \mathcal{G} \left( \ell^2 \mathbb{E}[\dot{V}(X)^2], \ell^2 \mathbb{E}[\dot{V}(X)^2] \right) = h(\ell)/2, \quad (61) \end{aligned}$$

where  $h(\ell)$  is defined in (10). Therefore by Fubini's Theorem, (60) and Lebesgue's dominated convergence theorem, the last term of (59) goes to 0 as  $d$  goes to infinity. The proof for  $T_2^d$  follows the same lines. By the triangle inequality,

$$\begin{aligned} |T_2^d| &\leq \left| \int_s^t \phi''(X_{[dr],1}^d) (\ell^2/2) \mathbb{E} \left[ (Z_{[dr],1}^d)^2 \left( \mathbf{1}_{\mathcal{A}_{[dr]}^d} - \Xi_r^d \right) \middle| \mathcal{F}_{[dr]}^d \right] dr \right| \\ &\quad + \left| \int_s^t \phi''(X_{[dr],1}^d) \left( (\ell^2/2) \mathbb{E} \left[ (Z_{[dr],1}^d)^2 \Xi_r^d \middle| \mathcal{F}_{[dr]}^d \right] - h(\ell)/2 \right) dr \right|. \quad (62) \end{aligned}$$

By Fubini's Theorem, Lebesgue's dominated convergence theorem and Proposition 7, the expectation of the first term goes to zero when  $d$  goes to infinity. For the second term, by [2, Lemma 6 (A.5)],

$$\begin{aligned} (\ell^2/2) \mathbb{E} \left[ (Z_{[dr],1}^d)^2 \mathbf{1} \wedge \exp \left\{ -\frac{\ell Z_{[dr],1}^d}{\sqrt{d}} \dot{V}(X_{[dr],1}^d) + \sum_{i=2}^d b_{\mathcal{I},i}^{d,[dr]} \right\} \middle| \mathcal{F}_{[dr]}^d \right] \\ = (B_1 + B_2 - B_3)/2, \quad (63) \end{aligned}$$

where

$$\begin{aligned} B_1 &= \ell^2 \Gamma \left( \ell^2 \bar{V}_{d,1}/d, \ell^2 \bar{V}_{d,2}/(2d) - 4(d-1) \mathbb{E} [\mathbb{1}_{\mathcal{D}_X} \zeta^d(X, Z)] \right), \\ B_2 &= \frac{\ell^4 \dot{V}(X_{[dr],1}^d)^2}{d} \mathcal{G} \left( \ell^2 \bar{V}_{d,1}/d, \ell^2 \bar{V}_{d,2}/(2d) - 4(d-1) \mathbb{E} [\mathbb{1}_{\mathcal{D}_X} \zeta^d(X, Z)] \right), \\ B_3 &= \frac{\ell^4 \dot{V}(X_{[dr],1}^d)^2}{d} (2\pi \ell^2 \bar{V}_{d,1}/d)^{-1/2} \\ &\quad \times \exp \left\{ -\frac{[-2(d-1) \mathbb{E} [\mathbb{1}_{\mathcal{D}_X} \zeta^d(X, Z)] + (\ell^2/(4d)) \bar{V}_{d,2}]^2}{2\ell^2 \bar{V}_{d,1}/d} \right\}, \end{aligned}$$

where  $\Gamma$  is defined in (27). As  $\Gamma$  is continuous on  $\mathbb{R}_+ \times \mathbb{R} \setminus \{0, 0\}$  (see [2, Lemma 2]), by **G1(ii)**, Lemma 12 and the law of large numbers, almost surely,

$$\begin{aligned} \lim_{d \rightarrow +\infty} \ell^2 \Gamma \left( \ell^2 \bar{V}_{d,1}/d, \{ \ell^2 \bar{V}_{d,2}/(2d) - 4(d-1) \mathbb{E} [\mathbb{1}_{\mathcal{D}_X} \zeta^d(X, Z)] \} \right) \\ = \ell^2 \Gamma \left( \ell^2 \mathbb{E}[\dot{V}(X)^2], \ell^2 \mathbb{E}[\dot{V}(X)^2] \right) = h(\ell). \end{aligned} \quad (64)$$

By Lemma 12, by **G1(ii)** and the law of large numbers, almost surely,

$$\lim_{d \rightarrow +\infty} \exp \left\{ -\frac{[-2(d-1) \mathbb{E} [\mathbb{1}_{\mathcal{D}_X} \zeta^d(X, Z)] + (\ell^2/(4d)) \bar{V}_{d,2}]^2}{2\ell^2 \bar{V}_{d,1}/d} \right\} = \exp \left\{ -\frac{\ell^2}{8} \mathbb{E}[\dot{V}(X)^2] \right\}.$$

Then, as  $\mathcal{G}$  is bounded on  $\mathbb{R}_+ \times \mathbb{R}$ ,

$$\lim_{d \rightarrow +\infty} \mathbb{E} \left[ \left| \int_s^t \phi''(X_{[dr],1}^d) (B_2 - B_3) dr \right| \right] = 0. \quad (65)$$

Therefore, by Fubini's Theorem, (63), (64), (65) and Lebesgue's dominated convergence theorem, the second term of (62) goes to 0 as  $d$  goes to infinity. The proof for  $T_3^d$  follows exactly the same lines as the proof of Proposition 6.  $\square$

*Proof of Theorem 5.* Using Proposition 4, Proposition 5 and Proposition 8, the proof follows the same lines as the proof of Theorem 3.  $\square$

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