Some remarks about the weak containment property for groupoids and semigroups

Claire Anantharaman-Delaroche

SOME REMARKS ABOUT THE WEAK CONTAINMENT PROPERTY FOR GROUPOIDS AND SEMIGROUPS

CLAIRE ANANTHARAMAN-DELAROCHE

Abstract. A locally compact groupoid is said to have the weak containment property if its full C^*-algebra coincide with its reduced one. Although it is now known that this property is strictly weaker than amenability, we show that the two properties are the same under a mild exactness assumption. Then we apply our result to get informations about the corresponding weak containment property for some semigroups.

Introduction

The notion of amenability for locally compact groups takes many forms and is well understood (see [34] for instance). Amenability was introduced in the measured setting for discrete group actions and countable equivalence relations by Zimmer [48, 46, 47] at the end of the seventies. Soon after, Renault extended this notion to general measured groupoids and to locally compact groupoids [37]. This was followed by further studies, for example in [2, 3] for group actions. A detailed general study is provided in the monograph [1]. In particular it has long been known [37, 39, 1] that every amenable groupoid has the weak containment property, in the sense that its full and reduced C^*-algebras coincide. Yet, at that time the converse was left open: *is a locally compact groupoid amenable when it has the weak containment property* ? For locally compact groups, this is well known to be true, due to a theorem of Hulanicki [14]. More generally, this is true for any transitive locally compact groupoid [7] (that is, a groupoid acting transitively on its set of units). It was proved recently [24] that the weak containment property for an action of an exact discrete group on a compact space implies the amenability of the action, leading to believe in a positive answer to the question in full generality. However, in 2015 Willett exhibited [45] a nice simple example of an étale groupoid having the weak containment property without being amenable.

In this paper we show that, nevertheless, the answer to the above question is quite often positive. It suffices that the groupoid satisfies a weak form of exactness that we call *inner exactness* (see Definition 2.6). This covers many examples. Every minimal locally compact groupoid (*i.e.* such that the only invariant closed subsets of the unit space are the empty set and the whole set of units) is inner exact. In particular all locally compact groups are inner exact. For every continuous action

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of an exact locally compact group on a locally compact space, the corresponding transformation groupoid is inner exact. However it has long been known that there exist groupoids that are not inner exact. The first example (for a non Hausdorff locally compact groupoid) was given by Skandalis in [39]. Later, in order to provide counterexamples to the Baum-Connes conjecture for groupoids, Higson, Lafforgue and Skandalis [12] have exhibited other (Hausdorff) locally compact groupoids, in fact bundles of groups, that are not inner exact. The above mentioned example of Willet is of the same kind.

The relations between weak containment and amenability are partially clarified in this paper as follows (see Theorem 2.10).

Theorem A. *Let $\mathcal{G}$ be an inner exact locally compact groupoid. Then $\mathcal{G}$ has the weak containment property if and only if it is measurewise amenable.*

Corollary. *Let $\mathcal{G}$ be a locally compact groupoid. The two following conditions are equivalent:*

(i) *$\mathcal{G}$ is measurewise amenable;*
(ii) *for every $\mathcal{G}$-C^*-algebra A, the full crossed product $A \rtimes \mathcal{G}$ coincide with the reduced crossed product $A \rtimes_r \mathcal{G}$.*

Measurewise amenability is a notion which is slightly weaker than (topological) amenability but is the same in many usual cases, for instance for étale groupoids. The difference is explained in the main part of the text (see Definition 2.3 and Remark 2.4). So, for an étale inner exact groupoid, the weak containment property is equivalent to amenability. This answers a question raised by Willett in [45, §3].

The previous theorem has the following generalization that covers interesting examples.

Theorem B. *Let $\mathcal{G}$ be a locally compact groupoid which is equivalent to an inner exact groupoid. Then $\mathcal{G}$ has the weak containment property if and only if it is measurewise amenable.*

After having established these results in Section 2, we turn in Section 3 to the case of discrete semigroups. We limit ourself to semigroups not too far from the case of discrete groups, namely inverse semigroups (defined in Section 3.1) and sub-semigroups of groups. We give partial answers to a recurrent question concerning semigroups: *what is the right definition of amenability for a semigroup*? To the semigroups that we consider are attached a full C^*-algebra and a reduced C^*-algebra, generalizing the classical case of groups. There are three obvious candidates for the notion of amenability, and it is natural to wonder what are the relations between them:

(1) *left amenability*, that is, there exists a left invariant mean on the semigroup;
(2) *weak containment property*, that is, the full and reduced C^*-algebras of the semigroup are the same;
(3) *nuclearity* of the reduced C^*-algebra of the semigroup.

This problem has been addressed in many papers (see [33], [9], [29], [18], [8], [25], [22], [23], [11], to cite a few of them). Of course, these three properties are equivalent for a discrete group.

Let us consider first the case of an inverse semigroup, that we denote by S. A very useful feature of such a semigroup is that its full and reduced C^*-algebras are described via the groupoid $\mathcal{G}_S$

canonically associated to it [35]. As a consequence, we see that (3) $\Rightarrow$ (2) in this case: *if the reduced C^*-algebra $C_r^*(S)$ is nuclear, then S has the weak containment property*, since $\mathcal{G}_S$ is amenable.

It is the only general fact that can be stated. *The example given by Willett*, once reinterpreted in the setting of inverse semigroups, *allows us to show that (2) $\not\Rightarrow$ (3) in general, for inverse semigroups* (see Example 3.14). This answers a question raised in [11, Remark 3.7]. This example is a Clifford inverse semigroup, that is an inverse semigroup which is a disjoint union $S = \sqcup_{e\in E} S_e$ of groups where the set E of idempotents is contained in the center of S. This gives an example of Clifford semigroup which has the weak containment property, although not all groups S_e, $e \in E$ are amenable. This answers a question raised in [33].

We observe that the notion of left amenability is not interesting, except when S has not a zero element, *i.e.* an element 0 such that $0s = 0 = s0$ for every $s \in S$. Indeed, an inverse semigroup with a zero is left amenable, since the Dirac measure at zero is a left invariant mean. Even *if S has no zero, the left amenability of S does not imply the weak containment property, and a fortiori the nuclearity of $C_r^*(S)$* (see Example 3.14).

However, using Theorem B we present a class of inverse semigroups for which Conditions (2) and (3) are equivalent:

Theorem. *Let S be a G-strong F^*-inverse semigroup with respect to a morphism into an exact group G* (see Definition 3.7). *Then S has the weak containment property if and only if the reduced C^*-algebra $C_r^*(S)$ is nuclear* (Theorem 3.15 (2)).

Next, we consider the case of a pair (P, G) where P is a sub-semigroup of a group G containing the unit e. As pointed out in [22, 23, 31], a handy tool in order to study the C^*-algebras of P is its left inverse hull $S(P)$. It is an inverse semigroup with nice properties (Propositions 3.1 and 3.6). Following Xin Li [22, 23], we define the full C^*-algebra of P to be the full C^*-algebra of $\mathcal{G}_{S(P)}$. This extends the definition given by Nica in [29] for quasi-lattice ordered groups (Definition 3.19). On the other hand, $C_r^*(P)$ is a quotient of the reduced C^*-algebra of $\mathcal{G}_{S(P)}$.

We first observe that *the left amenability of P always implies the nuclearity of $C_r^*(P)$* (see Proposition 3.21). *It is not true in general that the weak containment property implies the left amenability of P* as shown by Nica in [29]. He considered the free group $G = \mathbb{F}_n$ on n generators $a_1, \dots, a_n$ and $P = \mathbb{P}_n$ is the semigroup generated by $a_1, \dots, a_n$. Using the uniqueness property of the Cuntz algebra $\mathcal{O}_n$, Nica proved that $\mathbb{P}_n$ has the weak containment property although it is not left amenable. Note that $C_r^*(\mathbb{P}_n)$ is the Cuntz-Toeplitz C^*-algebra, that is, the C^*-algebra generated by n isometries $s_1, \dots s_n$ such that $\sum_{1\leq i\leq n} s_i s_i^* \lneq 1$. The weak containment property is equivalent to the uniqueness of the Cuntz-Toeplitz C^*-algebra. Moreover, $C_r^*(\mathbb{P}_n)$ is an extension of $\mathcal{O}_n$ by the algebra of compact operators and therefore is nuclear.

The pair $(\mathbb{P}_n, \mathbb{F}_n)$ is a quasi-lattice ordered group (Definition 3.4) and $\mathbb{P}_n\mathbb{P}_n^{-1}$ is strictly contained in $\mathbb{F}_n$. The case $G = PP^{-1}$ is simpler to handle. For the sake of completeness we have reviewed some results in this setting, most of them being already known (see Proposition 3.22 and Corollary 3.23).

Whether the weak containment property for P implies the nuclearity of $C_r^*(P)$ is an old problem that was raised by several authors, for instance by Laca and Raeburn [18, Remark 6.9], and more recently by Xin Li [23, §9]. Using Theorem A, we give the following partial answer (Theorem 3.24).

Theorem. *Let (P, G) which satisfies the Toeplitz condition* (Definition 3.5). *If G is an exact group, then the weak containment property implies that $C_r^*(P)$ is nuclear.*

This result follows from the fact that $S(P)$ is a G-strong F^*-inverse semigroup when the Toeplitz condition is satisfied. It applies in particular for quasi-lattice ordered groups and when $G = P^{-1}P$.

In [22], Xin Li introduced the independence property for P, which can be rephrased by saying that the quotient map from $C_r^*(\mathcal{G}_{S(P)})$ onto $C_r^*(P)$ is injective. In this case (which occurs for instance for quasi-lattice ordered groups), the nuclearity of $C_r^*(P)$ implies the weak containment property for $\mathcal{G}_{S(P)}$ and thus for P .

In order to facilitate the reading of the paper, in the first section we recall the main facts about groupoids and their C^*-algebras that will be used in the sequel, and we fix the notation. We emphasize that *the locally compact spaces will always be Hausdorff* (unless explicitly mentioned) *and second countable. Locally compact groupoids will always come equipped with a Haar system and Hilbert spaces will be separable.*

1. Preliminaries

1.1. **Groupoids.** We assume that the reader is familiar with the basic definitions about groupoids. For details we refer to [37], [35]. Let us recall some notation and terminology. A *groupoid* consists of a set $\mathcal{G}$, a subset $\mathcal{G}^{(0)}$ called the set of *units*, two maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ called respectively the *range* and *source* maps, a *composition law* $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)} \mapsto \gamma_1\gamma_2 \in \mathcal{G}$, where

$$\mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} : s(\gamma_1) = r(\gamma_2)\},$$

and an *inverse* map $\gamma \mapsto \gamma^{-1}$. These operations satisfy obvious rules, such as the facts that the composition law (*i.e.*, product) is associative, that the elements of $\mathcal{G}^{(0)}$ act as units (*i.e.*, $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$), that $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$, and so on (see [37, Definition 1.1]). For $x \in \mathcal{G}^{(0)}$ we set $\mathcal{G}^x = r^{-1}(x)$ and $\mathcal{G}_x = s^{-1}(x)$. Usually, X will denote the set of units of $\mathcal{G}$.

A *locally compact groupoid* is a groupoid $\mathcal{G}$ equipped with a locally compact topology such that the structure maps are continuous, where $\mathcal{G}^{(2)}$ has the topology induced by $\mathcal{G} \times \mathcal{G}$ and $\mathcal{G}^{(0)}$ has the topology induced by $\mathcal{G}$. We assume that the range (and therefore the source) map is open, which is a necessary condition for the existence of a Haar system. We denote by $\mathcal{C}_c(\mathcal{G})$ the algebra of continuous complex valued functions with compact support on $\mathcal{G}$.

Definition 1.1. Let $\mathcal{G}$ be a locally compact groupoid. A *Haar system* on $\mathcal{G}$ is a family $\lambda = (\lambda^x)_{x \in X}$ of measures on $\mathcal{G}$, indexed by the set $X = \mathcal{G}^{(0)}$ of units, satisfying the following conditions:

- *Support*: λ^x has exactly $\mathcal{G}^x$ as support, for every $x \in X$;

- *Continuity*: for every $f \in \mathcal{C}_c(\mathcal{G})$, the function $x \mapsto \lambda(f)(x) = \int_{\mathcal{G}^x} f \, \mathrm{d}\lambda^x$ is continuous;
- *Invariance*: for $\gamma \in \mathcal{G}$ and $f \in \mathcal{C}_c(\mathcal{G})$, we have

$$\int_{\mathcal{G}^{s(\gamma)}} f(\gamma\gamma_1) \, \mathrm{d}\lambda^{s(\gamma)}(\gamma_1) = \int_{\mathcal{G}^{r(\gamma)}} f(\gamma_1) \, \mathrm{d}\lambda^{r(\gamma)}(\gamma_1).$$

Examples 1.2. (a) *Transformation groupoid.* Let G be a locally compact group acting continuously to the right on a locally compact space X. The topological product space $X \times G$ has a natural groupoid structure with X as space of units. The range and source maps are given respectively by $r(x, g) = x$ and $s(x, g) = xg$. The product is given by $(x, g)(xg, h) = (x, gh)$ and the inverse by $(x, g)^{-1} = (xg, g^{-1})$. We denote by $X \rtimes G$ this groupoid. A Haar system λ is given by $\lambda^x = \delta_x \times \tilde{\lambda}$ where $\tilde{\lambda}$ is a left Haar measure on G. Similarly, one defines $G \ltimes X$ for a left action of G.

(b) *Groupoid group bundle.* It is a locally compact groupoid such that the range and source maps are equal. By [39, Lemma 1.3], one can choose, for $x \in \mathcal{G}^{(0)}$, a left Haar measure λ^x on the group $\mathcal{G}^x = \mathcal{G}_x$ in such a way that $(\lambda^x)_{x \in X}$ forms a Haar system on $\mathcal{G}$. An explicit example will be given in Section 1.3.

(c) *Etale groupoids.* A locally compact groupoid is called *étale* when its range (and therefore its source) map is a local homeomorphism from $\mathcal{G}$ onto $\mathcal{G}^{(0)}$. Then $\mathcal{G}^x$ and $\mathcal{G}_x$ are discrete and $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$. Moreover the family of counting measures λ^x on $\mathcal{G}^x$ forms a Haar system (see [37, Proposition 2.8]). Groupoids associated with actions (or more generally with partial actions) of discrete groups are étale.

As already said, *in the sequel, the locally compact groupoids are implicitly supposed to be Hausdorff, second countable, and equipped with a Haar system λ. In the three above examples, λ will be the mentioned Haar system.*

1.2. **Representations of a locally compact groupoid.** Let $(\mathcal{G}, \lambda)$ be a locally compact groupoid with a Haar system λ. We set $X = \mathcal{G}^{(0)}$. The space $\mathcal{C}_c(\mathcal{G})$ is an involutive algebra with respect to the following operations for $f, g \in \mathcal{C}_c(\mathcal{G})$:

$$(f * g)(\gamma) = \int f(\gamma_1) g(\gamma_1^{-1}\gamma) d\lambda^{r(\gamma)}(\gamma_1) \tag{1}$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}. \tag{2}$$

We define a norm on $\mathcal{C}_c(\mathcal{G})$ by

$$\|f\|_I = \max\left\{ \sup_{x \in X} \int |f(\gamma)| \, \mathrm{d}\lambda^x(\gamma), \ \sup_{x \in X} \int \left|f(\gamma^{-1})\right| \mathrm{d}\lambda^x(\gamma) \right\}.$$

Definition 1.3. A *representation* of $\mathcal{C}_c(\mathcal{G})$ is a $*$-homomorphism π from $\mathcal{C}_c(\mathcal{G})$ into the C^*-algebra $\mathcal{B}(H)$ of bounded operators of a Hilbert space H such that $\|\pi(f)\| \leq \|f\|_I$ for every $f \in \mathcal{C}_c(\mathcal{G})$.

Example 1.4. Let $x \in X = \mathcal{G}^{(0)}$. We denote by λ_x the image of λ^x be the inverse map $\gamma \mapsto \gamma^{-1}$. Let $\pi_x : \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}\big(L^2(\mathcal{G}_x, \lambda_x)\big)$ be defined by

$$(\pi_x(f)\xi)(\gamma) = \int_{\mathcal{G}_x} f(\gamma\gamma_1^{-1})\xi(\gamma_1)\,\mathrm{d}\lambda_x(\gamma)$$

for $f \in \mathcal{C}_c(\mathcal{G})$ and $\xi \in L^2(\mathcal{G}_x, \lambda_x)$. Then π_x is a representation of $\mathcal{C}_c(\mathcal{G})$.

More generally, let μ be a (Radon) measure on X. We denote by $\nu = \mu \circ \lambda$ the measure on $\mathcal{G}$ defined by the formula

$$\int_{\mathcal{G}} f\,\mathrm{d}\nu = \int_X \big(\int_{\mathcal{G}^x} f(\gamma)\,\mathrm{d}\lambda^x(\gamma)\big)\,\mathrm{d}\mu(x).$$

Let ν^{-1} be the image of ν under the inverse map. For $f \in \mathcal{C}_c(\mathcal{G})$ and $\xi \in L^2(\mathcal{G}, \nu^{-1})$ we define the operator $\mathrm{Ind}_\mu(f)$ by the formula

$$\big(\mathrm{Ind}_\mu(f)\xi\big)(\gamma) = \int_{\mathcal{G}^{r(\gamma)}} f(\gamma_1)\xi(\gamma_1^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma_1).$$

Then Ind_μ is a representation of $\mathcal{C}_c(\mathcal{G})$, called the *induced representation associated with* μ. We have $\mathrm{Ind}_{\delta_x} = \pi_x$.

The *full* C^**-algebra* $C^*(\mathcal{G})$ of $\mathcal{G}$ is the completion of $\mathcal{C}_c(\mathcal{G})$ with respect to the norm

$$\|f\| = \sup \|\pi(f)\|$$

where π runs over all representations of $\mathcal{C}_c(\mathcal{G})$. The *reduced* C^**-algebra* $C_r^*(\mathcal{G})$ is the completion of $\mathcal{C}_c(\mathcal{G})$ with respect to the norm

$$\|f\| = \sup_{x \in X} \|\pi_x(f)\|.$$

Obviously, the identity map of $\mathcal{C}_c(\mathcal{G})$ extends to a surjective homomorphism from $C^*(\mathcal{G})$ onto $C_r^*(\mathcal{G})$.

Remark 1.5. Assume that $\mathcal{G}$ is a locally compact group. Let us observe that the involution on $\mathcal{C}_c(\mathcal{G})$ that we introduced is not the usual one in group theory. If Δ denotes the modular function of $\mathcal{G}$, usually the involution is defined by $f^\star(\gamma) = \overline{f(\gamma^{-1})}\Delta(\gamma^{-1})$. The map $f \mapsto \tilde{f}$ where $\tilde{f}(\gamma) = f(\gamma)\Delta(\gamma)^{-1/2}$ is an isomorphism of involutive algebra between the $*$-algebra $\mathcal{C}_c(\mathcal{G})$ with the involution * introduced in (2) and the usual one with the involution *. The full and reduced C^*-algebras defined above are then canonically identified respectively with the classical full and reduced C^*-algebras of the group $\mathcal{G}$.

Similarly, the full and reduced C^*-algebras of a transformation groupoid $X \rtimes G$ are identified with the full crossed product $\mathcal{C}_0(X) \rtimes G$ and the reduced crossed product $\mathcal{C}_0(X) \rtimes_r G$ respectively, where $\mathcal{C}_0(X)$ is the C^*-algebra of complex valued functions on X vanishing to 0 at infinity.

A familiar result in group theory relates in a bijective and natural way the non-degenerate representations of the full group C^*-algebra and the unitary representations of the group. A similar result holds for groupoids. Its statement requires some preparation.

Let $(\mathcal{G}, \lambda)$ be a locally compact groupoid and let μ be a (Radon) measure on $X = \mathcal{G}^{(0)}$. We set $\nu = \mu \circ \lambda$. We say that μ is *quasi-invariant* if ν is equivalent to ν^{-1}. In this case, we denote by Δ the Radon-Nikodým derivative $\mathrm{d}\nu / \mathrm{d}\nu^{-1}$. A groupoid $(\mathcal{G}, \lambda)$ equipped with a quasi-invariant measure μ is called a *measured groupoid*.

Definition 1.6. A *unitary representation* of $\mathcal{G}$ is a triple (μ, H, U) where

(i) μ is a quasi-invariant measure on X;
(ii) $H = (\{H_x : x \in X\}, \mathcal{E})$ is a measurable field of Hilbert spaces over X (where $\mathcal{E}$ is a fundamental sequence of measurable vector fields);
(iii) U is a measurable action of $\mathcal{G}$ on H by isometries, that is, for every $\gamma \in \mathcal{G}$ we have an isometric isomorphism $U(\gamma) : H_{s(\gamma)} \to H_{r(\gamma)}$ such that
  (a) for $x \in X$, $U(x)$ is the identity map of $\mathcal{H}_x$;
  (b) for $(\gamma, \gamma_1) \in \mathcal{G}^{(2)}$, $U(\gamma\gamma_1) = U(\gamma)U(\gamma_1)$;
  (c) for $\xi, \eta \in \mathcal{E}$, the function $\gamma \mapsto \langle \xi \circ r(\gamma), U(\gamma)\eta \circ s(\gamma) \rangle_{r(\gamma)}$ is measurable.

We denote by $\mathcal{H} = L^2(X, H, \mu)$ the Hilbert space of square integrable sections of H, and for $f \in \mathcal{C}_c(\mathcal{G})$ we define the operator $\pi_U(f)$ on $\mathcal{H}$ by the formula

$$\langle \xi, \pi_U(f)\eta \rangle = \int_X \Big(\int_{\mathcal{G}^x} f(\gamma)\Delta(\gamma)^{-1/2} \langle \xi \circ r(\gamma), U(\gamma)\eta \circ s(\gamma) \rangle_{r(\gamma)} \, \mathrm{d}\lambda^x(\gamma) \Big) \, \mathrm{d}\mu(x).$$

Then $f \mapsto \pi_U(f)$ is a representation of $\mathcal{C}_c(\mathcal{G})$, called the *integrated form of* (μ, H, U) (or simply U). A crucial result, due to J. Renault, asserts that every representation of $\mathcal{C}_c(\mathcal{G})$ can be disintegrated. Here, the fact that the groupoid is assumed to be second countable is needed.

Theorem 1.7. *(*[38, Proposition 4.2] *Let π be a non-degenerate representation of $\mathcal{C}_c(\mathcal{G})$ on a Hilbert space $\mathcal{H}$. There is a unitary representation (μ, H, U) of $\mathcal{G}$ such that π is unitary equivalent to the integrated form π_U of (μ, H, U). We say that π disintegrates over μ.*

Example 1.8. The *left regular representation of $\mathcal{G}$ over a quasi-invariant measure μ* is $(\mu, H = L^2(\lambda), L)$ where $L^2(\lambda) = \big(\{L^2(\mathcal{G}^x, \lambda^x) : x \in X\}, \mathcal{E} = \mathcal{C}_c(\mathcal{G})\big)$, and

$$L(\gamma) : L^2(\mathcal{G}^{s(\gamma)}, \lambda^{s(\gamma)}) \to L^2(\mathcal{G}^{r(\gamma)}, \lambda^{r(\gamma)})$$

is given, for $\xi \in L^2(\mathcal{G}^{s(\gamma)})$, $\gamma_1 \in \mathcal{G}^{r(\gamma)}$, by

$$\big(L(\gamma)\xi\big)(\gamma_1) = \xi(\gamma^{-1}\gamma_1).$$

Note that $L^2(X, H, \mu) = L^2(\mathcal{G}, \mu \circ \lambda)$ and that, for $f \in \mathcal{C}_c(\mathcal{G})$, $\xi \in L^2(\mathcal{G}, \mu \circ \lambda)$ we have

$$\big(\pi_L(f)\xi\big)_x = \int_{\mathcal{G}^x} f(\gamma)\Delta(\gamma)^{-1/2} L(\gamma)\xi_{s(\gamma)} \, \mathrm{d}\lambda^x(\gamma).$$

It is well known (and easy to see) that the map $W : L^2(\mathcal{G}, \nu) \to L^2(\mathcal{G}, \nu^{-1})$ defined by the formula $W\xi = \Delta^{1/2}\xi$ is an isometric isomorphism which implements a unitary equivalence between π_L and Ind_μ.

We denote by $C^*_r(\mathcal{G},\mu)$ the norm closure of $\pi_L(\mathcal{C}_c(\mathcal{G}))$ in $\mathcal{B}(L^2(\mathcal{G},\mu\circ\lambda))$.

For $f\in\mathcal{C}_c(\mathcal{G})$ we define the seminorm $\sup\|\pi(f)\|$ where π ranges over all representations of $\mathcal{C}_c(\mathcal{G})$ that disintegrate over μ. We denote by $C^*(\mathcal{G},\mu)$ the C^*-algebra obtained by separation and completion of $\mathcal{C}_c(\mathcal{G})$ with respect to this seminorm[a]. Note that we have canonical surjective homomorphisms from $C^*(\mathcal{G})$ onto $C^*(\mathcal{G},\mu)$ and from $C^*(\mathcal{G},\mu)$ onto $C^*_r(\mathcal{G},\mu)$.

These C^*-algebras will play a crucial role in the rest of the paper and are related to reductions of the groupoid $\mathcal{G}$. We give some details in the next section.

1.3. **About reductions of a groupoid.** Let $\mathcal{G}$ be a groupoid and Y a subset of $X=\mathcal{G}^{(0)}$. We set $\mathcal{G}(Y)=r^{-1}(Y)\cap s^{-1}(Y)$. Then $\mathcal{G}(Y)$ is a subgroupoid of $\mathcal{G}$ called the *reduction of $\mathcal{G}$ by Y*. When Y is reduced to a single element x, then $\mathcal{G}(x)=r^{-1}(x)\cap s^{-1}(x)$ is a group called the *isotropy group of $\mathcal{G}$ at x*.

Let now $\mathcal{G}$ be a locally compact groupoid and let Y be a locally compact subset of X which is $\mathcal{G}$-invariant, meaning that for $\gamma\in\mathcal{G}$, we have $r(\gamma)\in Y$ if and only if $s(\gamma)\in Y$. Then $\mathcal{G}(Y)$ is a locally compact groupoid whose Haar system is obtained by restriction of the Haar system of $\mathcal{G}$.

Let μ be a quasi-invariant measure on X and let F be its support. It is a closed $\mathcal{G}$-invariant subset of X. Then Ind_μ is a faithful representation of $C^*_r(\mathcal{G}(F))$ (see [15, Corollary 2.4] for instance). It follows that $C^*_r(\mathcal{G},\mu)$ is canonically identified with $C^*_r(\mathcal{G}(F))$.

Besides, the representations of $\mathcal{C}_c(\mathcal{G}(F))$ contains all the representations of $\mathcal{C}_c(\mathcal{G})$ that disintegrate over μ. It follows there is a canonical surjective map q' from $C^*(\mathcal{G}(F))$ onto $C^*(\mathcal{G},\mu)$.

Let us consider the general situation where a closed $\mathcal{G}$-invariant subset F of X is given and set $U=X\setminus F$. It is well known that the inclusion $\iota_U:\mathcal{C}_c(\mathcal{G}(U))\to\mathcal{C}_c(\mathcal{G})$ extends to injective homomorphisms from $C^*(\mathcal{G}(U))$ into $C^*(\mathcal{G})$ and from $C^*_r(\mathcal{G}(U))$ into $C^*_r(\mathcal{G})$. Similarly, the restriction map $p_F:\mathcal{C}_c(\mathcal{G})\to\mathcal{C}_c(\mathcal{G}(F))$ extends to surjective homomorphisms from $C^*(\mathcal{G})$ onto $C^*(\mathcal{G}(F))$ and from $C^*_r(\mathcal{G})$ onto $C^*_r(\mathcal{G}(F))$. Moreover the sequence

$$0\to C^*(\mathcal{G}(U))\to C^*(\mathcal{G})\to C^*(\mathcal{G}(F))\to 0$$

is exact. For these facts, we refer to [37, page 102], [13, Section 2.4], or to [36, Proposition 2.4.2] for a detailed proof.

On the other hand, the corresponding sequence with respect to the reduced C^*-algebras is not always exact, as shown for a non-Hausdorff groupoid by Skandalis in the Appendix of [39].

Example 1.9. Another interesting class of examples was provided in [12] by Higson, Lafforgue and Skandalis. There, the authors consider a residually finite group Γ and an decreasing sequence $\Gamma\supset N_0\supset\Gamma_1\cdots\supset N_k\supset\cdots$ of finite index normal subgroups with $\cap_k N_k=\{e\}$. Let $\widehat{\mathbb{N}}=\mathbb{N}\cup\{\infty\}$ be the Alexandroff compactification of $\mathbb{N}$. We set $N_\infty=\{e\}$ and we denote by $q_k:\Gamma\to\Gamma/N_k$ the quotient homomorphism. Let $\mathcal{G}$ be the quotient of $\widehat{\mathbb{N}}\times\Gamma$ with respect to the equivalence relation

$$(k,t)\sim(l,u)\ \text{ if }\ k=l\ \text{ and }\ q_k(t)=q_k(u).$$

[a]For simplicity, we do not include λ in the notation, since the Haar system is always implicitely given.

Equipped with the quotient topology, $\mathcal{G}$ has a natural structure of (Hausdorff) étale locally compact groupoid group bundle: its space of units is $\widehat{\mathbb{N}}$, the range and source maps are given by $r([k,t]) = s([k,t]) = q_k(t)$, where $[k,t] = (k, q_k(t))$ is the equivalence class of (k,t). The fibre $\mathcal{G}(k)$ of the bundle is the quotient group Γ/N_k if $k \in \mathbb{N}$ and Γ if $k = \infty$. We call this groupoid an HLS-*groupoid*. A basic result of [12] is that the sequence

$$0 \longrightarrow C_r^*(\mathcal{G}(\mathbb{N})) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathcal{G}(\infty)) \longrightarrow 0$$

is not exact whenever Γ has Kazdhan's property (T) (it is not even exact in K-theory!).

1.4. **Crossed products.** For the definition of actions of groupoids on C^*-algebras we refer to [16]. Let us recall a few facts.

Definition 1.10. Let X be a locally compact space. A $\mathcal{C}_0(X)$-*algebra* is a C^*-algebra A equipped with a homomorphism ρ from $\mathcal{C}_0(X)$ into the centre of the multiplier algebra of A, which is non-degenerated in the sense that there exists an approximate unit (u_λ) of $\mathcal{C}_0(X)$ such that $\lim_\lambda \rho(u_\lambda)a = a$ for every $a \in A$.

Given $f \in \mathcal{C}_0(X)$ and $a \in A$, for simplicity we will write fa instead of $\rho(f)a$.

Let U be an open subset of X and $F = X \backslash U$. We view $\mathcal{C}_0(U)$ as an ideal of $\mathcal{C}_0(X)$ and we denote by $\mathcal{C}_0(U)A$ the closed linear span of $\{fa : f \in \mathcal{C}_0(U), a \in A\}$. It is a closed ideal of A and in fact, we have $\mathcal{C}_0(U)A = \{fa : f \in \mathcal{C}_0(U), a \in A\}$ (see [5, Corollaire 1.9]). We set $A_F = A/\mathcal{C}_0(U)A$ and whenever $F = \{x\}$ we write $\mathcal{C}_x(X)$ instead of $\mathcal{C}_0(X \setminus \{x\})$ and A_x instead of $A_{\{x\}}$. We denote by $e_x : A \to A_x$ the quotient map and for $a \in A$ we set $a(x) = e_x(a)$. Recall that $\|a\| = \sup_{x\in X} \|a(x)\|$ (so that $a \mapsto (a(x))_{x\in X}$ from A into $\prod_{x\in X} A_x$ is injective) and that $x \mapsto \|a(x)\|$ is upper semi-continuous (see [42, 5]). Then, $(A, \{e_x : A \to A_x\}_{x\in X}, X)$ is an upper semi-continuous field of C^*-algebras.

Let A and B be two $\mathcal{C}_0(X)$-algebras. A *morphism* $\alpha : A \to B$ *of* $\mathcal{C}_0(X)$-*algebras* is a morphism of C^*-algebras which is $\mathcal{C}_0(X)$-linear, that is, $\alpha(fa) = f\alpha(a)$ for $f \in \mathcal{C}_0(X)$ and $a \in A$. For $x \in X$, in this case α factors through a morphism $\alpha_x : A_x \to B_x$ such that $\alpha_x(a(x)) = \alpha(a)(x)$.

Let Y, X be locally compact spaces and $f : Y \to X$ a continuous map. To any $\mathcal{C}_0(X)$-algebra A is associated a $\mathcal{C}_0(Y)$-algebra $f^*A = \big(\mathcal{C}_0(Y) \otimes A\big)_F$ where $F = \{(y, f(y)) : y \in Y\} \subset Y \times X$. For $y \in Y$, we have $\big(f^*A\big)_y = A_{f(y)}$ (see [21, 16]).

Definition 1.11. ([21, 16]) Let $(\mathcal{G}, \lambda)$ be a locally compact groupoid with a Haar system and $X = \mathcal{G}^{(0)}$. An *action* of $\mathcal{G}$ on a C^*-algebra A is given by a structure of $\mathcal{C}_0(X)$-algebra on A and an isomorphism $\alpha : s^*A \to r^*A$ of $\mathcal{C}_0(\mathcal{G})$-algebras such that for every $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$ we have $\alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1}\alpha_{\gamma_2}$, where $\alpha_\gamma : A_{s(\gamma)} \to A_{r(\gamma)}$ is the isomorphism deduced from α by factorization.

When A is equipped with such an action, we say that A is a $\mathcal{G}$-C^*-*algebra*.

Let A be a $\mathcal{G}$-C^*-algebra. We set $\mathcal{C}_c(r^*(A)) = \mathcal{C}_c(\mathcal{G})r^*(A)$. It is the space of the continuous sections with compact support of the upper semi-continuous field of C^*-algebras defined by the

$\mathcal{C}_0(\mathcal{G})$-algebra r^*A. Then, $\mathcal{C}_c(\mathcal{G})r^*(A)$ is a $*$-algebra with respect to the following operations:

$$(f*g)(\gamma)=\int_{\mathcal{G}^{r(\gamma)}} f(\gamma_1)\alpha_{\gamma_1}\big(g(\gamma_1^{-1}\gamma)\big)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma_1)$$

and

$$f^*(\gamma)=\alpha_\gamma\big(f(\gamma^{-1})^*\big)$$

(see [27, Proposition 4.4]). We define a norm on $\mathcal{C}_c(r^*(A))$ by

$$\|f\|_I=\max\left\{\sup_{x\in X}\int_{\mathcal{G}^x}\|f(\gamma)\|\,\mathrm{d}\lambda^x(\gamma),\quad \sup_{x\in X}\int_{\mathcal{G}^x}\big\|f(\gamma^{-1})\big\|\,\mathrm{d}\lambda^x(\gamma)\right\}.$$

The *full crossed product* $A\rtimes_\alpha\mathcal{G}$ is the enveloping C^*-algebra of the Banach $*$-algebra obtained by completion of $\mathcal{C}_c(r^*(A))$ with respect to $\|\cdot\|_I$.

Let us now define the reduced crossed product. For $x\in X$ let us consider the Hilbert A_x-module $L^2(\mathcal{G}_x,\lambda_x)\otimes A_x$, defined as the completion of the space $\mathcal{C}_c(\mathcal{G}_x;A_x)$ of continuous compactly supported functions from $\mathcal{G}_x$ into A_x, with respect to the A_x-valued inner product

$$\langle\xi,\eta\rangle=\int_{\mathcal{G}_x}\xi(\gamma)^*\eta(\gamma)\,\mathrm{d}\lambda_x(\gamma).$$

For $f\in\mathcal{C}_c(r^*(A))$, $\xi\in\mathcal{C}_c(\mathcal{G}_x;A_x)$ and $\gamma\in\mathcal{G}_x$, we set

$$\big(\pi_x(f)\xi\big)(\gamma)=\int_{\mathcal{G}_x}\alpha_\gamma^{-1}\big(f(\gamma\gamma_1^{-1})\big)\xi(\gamma_1)\,\mathrm{d}\lambda_x(\gamma_1).$$

Then $\pi_x(f)$ extends to a bounded operator with adjoint acting on the Hilbert A_x-module $L^2(\mathcal{G}_x,\lambda_x)\otimes A_x$. In this way we get a representation of the $*$-algebra $\mathcal{C}_c(r^*(A))$. The reduced crossed product $A\rtimes_{\alpha,r}\mathcal{G}$ is the completion of $\mathcal{C}_c(r^*(A))$ with respect to the norm $\|f\|=\sup_{x\in X}\|\pi_x(f)\|$ (see [16])[b].

Remarks 1.12. (a) We note that if Y is a locally compact $\mathcal{G}$-invariant subset of $X=\mathcal{G}^{(0)}$, then $\mathcal{C}_0(Y)$ has a natural structure of $\mathcal{G}$-C^*-algebra. Moreover, $\mathcal{C}_c(r^*(\mathcal{C}_0(Y)))=\mathcal{C}_c(\mathcal{G}(Y))$ and $\mathcal{C}_0(Y)\rtimes\mathcal{G}$ and $\mathcal{C}_0(Y)\rtimes_r\mathcal{G}$ are canonically isomorphic to $C^*(\mathcal{G}(Y))$ and $C_r^*(\mathcal{G}(Y))$ respectively.

(b) Let B be a C^*-algebra and set $A=\mathcal{C}_0(X)\otimes B$. Since $\mathcal{C}_0(X)$ is a $\mathcal{G}$-C^*-algebra, we see that $A=\mathcal{C}_0(X)\otimes B$ is a $\mathcal{G}$-C^*-algebra, the action being trivial on B. Moreover, $A\rtimes\mathcal{G}$ and $A\rtimes_r\mathcal{G}$ are canonically isomorphic to $B\otimes_{max}C^*(\mathcal{G})$ and $B\otimes C_r^*(\mathcal{G})$ respectively.

2. Amenability and weak containment

The reference for this section is [1]. The notion of amenable locally compact groupoid has many equivalent definitions. We shall recall two of them. Before, let us recall a notation: given a locally compact groupoid $\mathcal{G}$, $\gamma\in\mathcal{G}$ and μ a measure on $\mathcal{G}^{s(\gamma)}$, then $\gamma\mu$ is the measure on $\mathcal{G}^{r(\gamma)}$ defined by $\int_{\mathcal{G}^{r(\gamma)}}f\,\mathrm{d}\gamma\mu=\int_{\mathcal{G}^{s(\gamma)}}f(\gamma\gamma_1)\,\mathrm{d}\mu(\gamma_1)$.

[b] π_x is what is denoted Λ_x in [16] except that the authors consider $\mathcal{C}_c(s^*(A))$ instead of $\mathcal{C}_c(r^*(A))$. This explains why our formula is not exactly the same.

Definition 2.1. ([1, Definitions 2.2.2, 2.2.8]) We say that $\mathcal{G}$ is *amenable* if there exists a net (m_i), where $m_i = (m_i^x)_{x\in\mathcal{G}^{(0)}}$ is a family of probability measures m_i^x on $\mathcal{G}^x$, such that

(i) each m_i is continuous in the sense that for all $f \in \mathcal{C}_c(\mathcal{G})$, the function $x \mapsto \int f \,\mathrm{d}m_i^x$ is continuous;
(ii) $\lim_i \left\|\gamma m_i^{s(\gamma)} - m_i^{r(\gamma)}\right\|_1 = 0$ uniformly on the compact subsets of $\mathcal{G}$.

We say that $(m_i)_i$ is an *approximate invariant continuous mean* on $\mathcal{G}$. Note that if $\mathcal{G}$ is amenable and if Y is a locally compact $\mathcal{G}$-invariant subset of X, then the groupoid $\mathcal{G}(Y)$ is amenable.

Proposition 2.2. ([1, Proposition 2.2.13]) *Let $(\mathcal{G},\lambda)$ be a locally compact groupoid with Haar system. Then $\mathcal{G}$ is amenable if and only if there exists a net (g_i) of non-negative functions in $\mathcal{C}_c(\mathcal{G})$ such that*

(a) $\int g_i \,\mathrm{d}\lambda^x \leq 1$ *for every* $x \in \mathcal{G}^{(0)}$;
(b) $\lim_i \int g_i \,\mathrm{d}\lambda^x = 1$ *uniformly on the compact subsets of* $\mathcal{G}^{(0)}$;
(c) $\lim_i \int \left|g_i(\gamma^{-1}\gamma_1) - g_i(\gamma_1)\right| \mathrm{d}\lambda^{r(\gamma)}(\gamma_1) = 0$ *uniformly on the compact subsets of* $\mathcal{G}$.

Definition 2.3. ([1, Proposition 3.2.14]) Let $(\mathcal{G},\lambda)$ be a locally compact groupoid.

(i) Let μ be a quasi-invariant measure on X. We say that the measured groupoid $(\mathcal{G},\lambda,\mu)$ is amenable if there exists a net (g_i) of $(\mu\circ\lambda)$-measurable non-negative functions on $\mathcal{G}$ such that
  (a) $\int g_i \,\mathrm{d}\lambda^x = 1$ for *a.e.* $x \in \mathcal{G}^{(0)}$;
  (b) $\lim_i \int \left|g_i(\gamma^{-1}\gamma_1) - g_i(\gamma_1)\right| \mathrm{d}\lambda^{r(\gamma)}(\gamma_1) = 0$ in the weak*-topology of $L^\infty(\mathcal{G},\mu\circ\lambda)$.
(ii) We say that $(\mathcal{G},\lambda)$ is *measurewise amenable* if $(\mathcal{G},\lambda,\mu)$ is an amenable measured groupoid for every quasi-invariant measure μ.

Remark 2.4. An amenable groupoid is measurewise amenable. The converse is true for groupoids that have countable orbits (see [1, Theorem 3.3.7]). This is in particular the case for étale groupoids and for locally compact groups.

Theorem 2.5. *Let $(\mathcal{G},\lambda)$ be a locally compact groupoid. Consider the following conditions:*

(a) $(\mathcal{G},\lambda)$ *is measurewise amenable;*
(b) *for every quasi-invariant measure μ, the canonical surjection from $C^*(\mathcal{G},\mu)$ onto $C_r^*(\mathcal{G},\mu)$ is injective;*
(c) $C_r^*(\mathcal{G})$ *is nuclear;*
(d) *for every $\mathcal{G}$-C^*-algebra A, the canonical surjection from $A\rtimes\mathcal{G}$ onto $A\rtimes_r\mathcal{G}$ is injective;*
(e) *the canonical surjection from $C^*(\mathcal{G})$ onto $C_r^*(\mathcal{G})$ is injective.*

Then, we have (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (d) $\Rightarrow$ (e). Moreover, if the isotropy groups $\mathcal{G}(x)$ are discrete for every $x \in X$, then (c) $\Rightarrow$ (a).

The equivalence between (a) and (b) is proved in [1, Theorem 6.1.4]. That (a) $\Rightarrow$ (c) is contained in [1, Corollary 6.2.14] as well as the fact that (c) $\Rightarrow$ (a) when the isotropy is discrete. For the proof of (a) $\Rightarrow$ (d) see [39, Theorem 3.6] or [1, Proposition 6.1.10].

Definition 2.6. We say that an action of a locally compact groupoid $(\mathcal{G}, \lambda)$ on a C^*-algebra A is *inner exact* if for every $\mathcal{G}$-invariant closed ideal I of A the sequence

$$0 \longrightarrow I \rtimes_r \mathcal{G} \longrightarrow A \rtimes_r \mathcal{G} \longrightarrow (A/I) \rtimes_r \mathcal{G} \longrightarrow 0$$

is exact.

We say that $\mathcal{G}$ is *inner exact* if the canonical action of $\mathcal{G}$ on $\mathcal{C}_0(\mathcal{G}^{(0)})$ is inner exact, *i.e.,* if for every invariant closed subset F of $\mathcal{G}^{(0)}$, the sequence

$$0 \longrightarrow C_r^*(\mathcal{G}(U)) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathcal{G}(F)) \longrightarrow 0$$

is exact.

The term "inner" in the above definitions aims to highlight that we only consider short sequences with respect to the specific given action of the groupoid. A possible definition of exactness for a groupoid is the following one. Other candidates are considered in [4].

Definition 2.7. We say that a groupoid $(\mathcal{G}, \lambda)$ is *exact in the sense of Kirchberg-Wassermann* (or KW-*exact* in short) if every action $(\mathcal{G}, \lambda)$ on a C^*-algebra A is inner exact.

This notion was studied by Kirchberg and Wassermann for locally compact groups. They proved in particular that this property is equivalent to the exactness of $C_r^*(G)$ for a discrete group G.

Examples 2.8. (a) Every minimal groupoid is inner exact. In particular, every locally compact group is inner exact.

(b) Every KW-exact groupoid is inner exact.

(c) Let G be a locally compact KW-exact group acting to the right on a locally compact space X. Then the transformation groupoid $\mathcal{G} = X \rtimes G$ is KW-exact. Indeed let α be an action of $\mathcal{G}$ on a $\mathcal{C}_0(X)$-algebra A. Then G acts on A by $(\beta_g a)(x) = \alpha_{(x,g)}(a(xg))$ and it is straightforward to check that $A \rtimes_{\beta,r} G$ is canonically isomorphic to $A \rtimes_{\alpha,r} \mathcal{G}$. Moreover, this identification is functorial. In fact the groupoid is exact in a very strong sense. Indeed, G acts amenably on a compact space Y[c] therefore it acts amenably on $Y \rtimes X$ by $(y, x)g = (yg, xg)$ (see [1, Proposition 2.2.9]). Then $\mathcal{G} = X \rtimes G$ acts amenably on $Y \times X$ by $(y, x)(x, g) = (yg, xg)$. Since the momentum map $(y, x) \mapsto x \in \mathcal{G}^{(0)}$ is proper it is easily seen that $\mathcal{G}$ is KW-exact. The proof is the same as the proof showing that a group acting amenably on a compact space is KW-exact. More details on the notion of exactness for groupoids are given in [4].

(d) The case of an étale bundle of groups is described in the next proposition.

Proposition 2.9. *Let $\mathcal{G}$ be an étale groupoid group bundle over a locally compact space X. The following conditions are equivalent:*

(i) *$\mathcal{G}$ is inner exact;*

[c] In the non discrete case this fact is proved in a recent preprint of Brodzki, Cave and Li [6].

(ii) *for every* $x \in X$ *the sequence*

$$0 \longrightarrow C_r^*(\mathcal{G}(X \setminus \{x\})) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathcal{G}(x)) \longrightarrow 0$$

is exact;

(iii) $C_r^*(\mathcal{G})$ *is a continuous field of* C^**-algebras over* X *with fibres* $C_r^*(\mathcal{G}(x))$.

Proof. (i) $\Rightarrow$ (ii) is obvious. Let us show that (ii) $\Rightarrow$ (i). Let F be a closed $\mathcal{G}$-invariant subset of X, let $x \in F$ and assume that (ii) holds. Then in the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_r^*(\mathcal{G}(X \setminus F)) & \longrightarrow & C_r^*(\mathcal{G}) & \xrightarrow{p_F} & C_r^*(\mathcal{G}(F)) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & C_r^*(\mathcal{G}(X \setminus \{x\})) & \longrightarrow & C_r^*(\mathcal{G}) & \xrightarrow{\pi_x} & C_r^*(\mathcal{G}(x)) & \longrightarrow & 0 \end{array}$$

the bottom line is exact, the left vertical line is injective and the right one is surjective. If $a \in C_r^*(\mathcal{G})$ is such that $p_F(a) = 0$ then $a \in C_r^*(\mathcal{G}(X \setminus \{x\}))$. Since $\mathcal{G}$ is étale, a is an element of $\mathcal{C}_0(\mathcal{G})$, and we see that $a(\gamma) = 0$ for $\gamma \in \mathcal{G}(x)$. But this holds for every $x \in X$ and therefore $a = 0$.

Let us prove the equivalence between (ii) and (iii). For $x \in X$ let π_x be the canonical surjective map from $C_r^*(\mathcal{G})$ onto $C_r^*(\mathcal{G}(x))$. Then $(C_r^*(\mathcal{G}), \{\pi_x : C_r^*(\mathcal{G}) \to C_r^*(\mathcal{G}(x))\}_{x\in X}, X)$ is a field of C^*-algebras on X, which is lower semi-continuous in the sense that $x \mapsto \|\pi_x(a)\|$ is lower semi-continuous for every $a \in C_r^*(\mathcal{G})$ (see for instance [19, Theorem 5.5]). On the other hand, $C_r^*(\mathcal{G})$ is a $\mathcal{C}_0(X)$-algebra. Indeed, for $f \in \mathcal{C}_0(X)$, $g \in \mathcal{C}_c(\mathcal{G})$ and $\gamma \in \mathcal{G}$, we set $(fg)(\gamma) = f \circ r(\gamma)g(\gamma)$. The map $g \mapsto fg$ extends continuously in order to define a structure of $\mathcal{C}_0(X)$-algebra on $C_r^*(\mathcal{G})$. We have $\mathcal{C}_c(\mathcal{G}(X \setminus \{x\})) = \mathcal{C}_x(X)\mathcal{C}_c(\mathcal{G})$ and by continuity we get $C_r^*(\mathcal{G}(X \setminus \{x\})) = \mathcal{C}_x(X)C_r^*(\mathcal{G})$. Note that for $f \in \mathcal{C}_0(X)$, $a \in C_r^*(\mathcal{G})$ and $x \in X$, we have $\pi_x(fa) = f(x)\pi_x(a)$. It follows from [17, Lemma 2.3] that the function $x \mapsto \|\pi_x(a)\|$ is upper semi-continuous at x_0 for all $a \in C_r^*(\mathcal{G})$ if and only if the kernel of π_{x_0} is $C_r^*(\mathcal{G}(X \setminus \{x_0\}))$. □

Theorem 2.10. *Let* $(\mathcal{G}, \lambda)$ *be a locally compact groupoid. The two following conditions are equivalent:*

(1) $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$ *and* $\mathcal{G}$ *is inner exact;*
(2) *the groupoid* $\mathcal{G}$ *is measurewise amenable.*

Proof. Theorem 2.5 gives us (2) $\Rightarrow$ (1). For the converse we have to show that, for every quasi-invariant measure μ, the canonical surjective map $q_\mu : C^*(\mathcal{G}, \mu) \to C_r^*(\mathcal{G}, \mu)$ is injective, still using Theorem 2.5. Let F be the support of μ and set $U = X \setminus F$. The following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(\mathcal{G}(U)) & \longrightarrow & C^*(\mathcal{G}) & \longrightarrow & C^*(\mathcal{G}(F)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow q & & \downarrow q_F & & \\ 0 & \longrightarrow & C_r^*(\mathcal{G}(U)) & \longrightarrow & C_r^*(\mathcal{G}) & \longrightarrow & C_r^*(\mathcal{G}(F)) & \longrightarrow & 0 \end{array}$$

where the two lines are exact and q is injective. Then an elementary diagram chasing shows that q_F is injective. Recall that $C_r^*(\mathcal{G}(F))$ is canonically identified to $C_r^*(\mathcal{G}, \mu)$ (see Section 1.3).

Observe that q_F factorizes through $C^*(\mathcal{G},\mu)$ as $q_F = q_\mu \circ q'$ where $q' : C^*(\mathcal{G}(F)) \to C^*(\mathcal{G},\mu)$ is surjective. It follows that q_μ is injective. □

In [45], Willett considered an HLS-groupoid constructed from a well-chosen sequence of normal subgroups with finite index in the free group $\mathbb{F}_2$ with two generators. This groupoid has the weak containment property although it is not amenable.

The following theorem is a converse to [1, Proposition 6.1.10]. It is already known when $\mathcal{G}$ has discrete isotropy since Condition (b) implies that $C^*(\mathcal{G}) = C^*_r(\mathcal{G})$ and $C^*_r(\mathcal{G})$ nuclear.

Corollary 2.11. *Let $(\mathcal{G},\lambda)$ be a locally compact groupoid. The following conditions are equivalent:*

(a) *$\mathcal{G}$ is measurewise amenable;*
(b) *$A \rtimes \mathcal{G} = A \rtimes_r \mathcal{G}$ for every $\mathcal{G}$-C^*-algebra A.*

Proof. We recalled in Theorem 2.5 that (a) ⇒ (b). Conversely, (b) immediately implies that $\mathcal{G}$ has the weak containment property and is KW-exact and so (a) holds. □

The following corollary is immediate since amenability and measurewise amenability coincide for an étale groupoid.

Corollary 2.12. *Let $(\mathcal{G},\lambda)$ be an étale groupoid which is inner exact (for instance KW-exact or minimal). Then $\mathcal{G}$ has the weak containment property if and only if it is amenable.*

For the notion of (topological) equivalence between locally compact groupoids we refer to [1, Definition 2.2.15].

Corollary 2.13. *Let $(\mathcal{G},\lambda)$ be a locally compact groupoid which is equivalent to an inner exact groupoid. Then $\mathcal{G}$ has the weak containment property if and only if it is measurewise amenable.*

Proof. Assume that $\mathcal{G}$ has the weak containment property and is equivalent to an inner exact groupoid $\mathcal{G}'$. Then by [44, Theorem 17], the groupoid $\mathcal{G}'$ has the weak containment property and therefore is measurewise amenable. To conclude, we use the fact that measurewise amenability is preserved under equivalence [1, Theorem 3.2.16]. □

3. Applications to semigroup C^*-algebras

3.1. **Semigroups.** We will consider two kinds of semigroups: inverse semigroups and sub-semigroups of a group.

An *inverse semigroup* S is a semigroup such that for every $s \in S$ there exists a unique element s^* such that $ss^*s = s$ and $s^*ss^* = s^*$. Note that groups are inverse semigroups with exactly one idempotent. Our references for this notion are [35, 20]. The set E_S of idempotents of S plays a crucial role. It is an abelian sub-semigroup of S. On S one defines the equivalence relation $s \underset{\sigma}{\sim} t$ if there exists an idempotent e such that $se = te$. The quotient S/σ is a group, called the *maximal group homomorphism image* of S, since every homomorphism from S into a group G

factors through S/σ. This group S/σ is trivial when S has a zero element 0, which is a frequent situation.

By an abuse of notation, σ will also denote the quotient map from S onto S/σ. We denote by $S^\times$ the set $S \setminus \{0\}$.

Given a set X, we denote by $\mathrm{IS}(X)$ the inverse semigroup of partial bijections of X. Its zero element 0 is the application with empty domain. The Wagner-Preston theorem [35, Proposition 2.1.3] identifies any inverse semigroup S with a sub-semigroup of $\mathrm{IS}(S)$.

Let G be a group and P a sub-semigroup of G containing the unit e. The *left inverse hull* $S(P)$ of P is the inverse sub-semigroup of $\mathrm{IS}(P)$ generated by the injection $\ell_p : x \mapsto px$. It has a unit, namely ℓ_e. Observe that ℓ_p^* is the map $px \mapsto x$ defined on pP. Every element of $S(P)$ is of the form $s = \ell_{p_1}^* \ell_{q_1} \cdots \ell_{p_n}^* \ell_{q_n}$ with $p_i, q_i \in P$ and $n \geq 1$. Let us recall some important properties of $S(P)$.

Proposition 3.1. *Let (P, G) as above. Then*

(1) *$0 \notin S(P)$ if and only if PP^{-1} is a subgroup de G;*
(2) *The application $\psi : S(P)^\times \to G$ sending $s = \ell_{p_1}^* \ell_{q_1} \cdots \ell_{p_n}^* \ell_{q_n}$ to $p_1^{-1} q_1 \cdots p_n^{-1} q_n$ is well defined. It satisfies $\psi(st) = \psi(s)\psi(t)$ if $st \neq 0$ and we have $\psi^{-1}(e) = E_{S(P)}^\times$.*

Proof. PP^{-1} is a sub-group de G if and only if $pP \cap qP \neq \emptyset$ for every $p, q \in P$ (*i.e.,* P is *left reversible*). Then, assertion (1) is Lemma 3.4.1 of [31].

(2) is proved in [31, Proposition 3.2.11]. □

Recall that on an inverse semigroup S, a partial order is defined as follows: $s \leq t$ if there exists an idempotent e such that $s = te$ (see [20, page 21] for instance).

Definition 3.2. An inverse semigroup S is said to be *E-unitary* if E_S is the kernel of $\sigma : S \to S/\sigma$ (equivalently, every element greater than an idempotent is an idempotent. When S has a zero, this means that $S = E_S$.

Definition 3.3. Let S be an inverse semigroup. A *morphism* (or *grading*) is an application ψ from $S^\times$ into a group G such that $\psi(st) = \psi(s)\psi(t)$ if $st \neq 0$. If in addition $\psi^{-1}(e) = E_S^\times$, we say that ψ is an *idempotent pure morphism*. When such an application ψ from $S^\times$ into a group G exists, the inverse semigroup S is called *strongly E^*-unitary*.

Note that when S is without zero, S is strongly E^*-unitary if and only if it is E-unitary.

Remark 3.4. Let (P, G) with $G = PP^{-1}$. Then the map $\tau : S(P)/\sigma \to G$ such that $\tau \circ \sigma = \psi$ is an isomorphism. Indeed ψ is surjective, so τ is also surjective. Assume that $\tau(\sigma(x)) = e$, with $x \in S(P)$. Since ψ is idempotent pure, we see that x is an idempotent and therefore $\sigma(x)$ is the unit of $S(P)/\sigma$.

In [30], Nica has introduced the Toeplitz inverse semigroup $S(G, P)$ which is the inverse sub-semigroup of $\mathrm{IS}(P)$ generated by the maps $\alpha_g : g^{-1}P \cap P \to P \cap gP$, $g \in G$, where $\alpha_g(x) = gx$ if $x \in g^{-1}P \cap P$. For $p \in P$ we have $\alpha_p = \ell_p$. Therefore we have $S(P) \subset S(G, P)$.

Definition 3.5. We say that (P, G) satisfies the *Toeplitz condition* if $S(P) = S(G, P)$.

We will give in Section 3.4 another characterization of the Toeplitz condition, along with examples.

Proposition 3.6. *Assume that (P, G) satisfies the Toeplitz condition. Let $\psi : S(P)^\times \to G$ as defined in Proposition 3.1. Then, $\alpha_g \neq 0$ if and only if g is in the image of ψ. In this case we have $\psi(\alpha_g) = g$, and α_g is the greatest element of $\psi^{-1}(g)$.*

For the proof, see [32, Proposition 4.1] or [30, Lemma 3.2].

Definition 3.7. Let S be an inverse semigroup. If there exists an idempotent pure morphism $\psi : S^\times \to G$ such that for every $g \neq e$ in the image of ψ then $\psi^{-1}(g)$ contains a greatest element, we will say that S is a *G-strong F^*-inverse semigroup* with respect to $\psi : S^\times \to G$.

Remark 3.8. Assume that S is a G-strong F^*-inverse monoid. Then every element of $S^\times$ is majored by a unique maximal element. Therefore, S is an F^*-inverse semigroup, a class of inverse semigroups that was introduced by Nica in [30] when he studied the Toeplitz inverse semigroup $S(G, P)$. If S has no zero, then a G-strong F^*-inverse monoid with respect to $\psi : S^\times \to G$ is an F-inverse semigroup (see [20, page 202] for the definition), and we may take $G = S/\sigma$ and for ψ the quotient map. Conversely, every F-inverse semigroup is (S/σ)-strongly F^*-inverse with respect to σ.

3.2. **Groupoid associated with an inverse semigroup.** Let S be an inverse semigroup. We recall the construction of the associated groupoid $\mathcal{G}_S$ that is described in detail in [35]. We denote by X the space of non-zero maps χ from E_S into $\{0, 1\}$ such that $\chi(ef) = \chi(e)\chi(f)$ and $\chi(0) = 0$ whenever S has a zero. Equipped with the topology induced from the product space $\{0, 1\}^E$, the space X, called the *spectrum* of S, is locally compact and totally disconnected. Note that when S is a monoid (*i.e.,* has a unit element 1) then χ is nonzero if and only if $\chi(1) = 1$, and therefore X is compact.

The semigroup S acts on X as follows. The domain (open and compact) of $t \in S$ is $D_{t^*t} = \{\chi \in X : \chi(t^*t) = 1\}$ and we set $\theta_t(\chi)(e) = \chi(t^*et)$. We define on $\Xi = \{(t, \chi) \in S \times X : \chi \in D_{t^*t}\}$ the equivalence relation $(t, \chi) \sim (t_1, \chi_1)$ if $\chi = \chi_1$ and there exists $e \in E_S$ with $\chi(e) = 1$ and $te = t_1e$. Then $\mathcal{G}_S$ is the quotient of Ξ with respect to this equivalence relation, equipped with the quotient topology. The range of the class $[t, \chi]$ of (t, χ) is $\theta_t(\chi)$ and its source is χ. The composition law is given by $[u, \chi][v, \chi'] = [uv, \chi']$ if $\theta_v(\chi') = \chi$ (see [35] or [10] for details). In general, $\mathcal{G}_S$ is not Hausdorff. But for the inverse semigroups we are interested in, like $S(P)$, we will see that the quotient topology is Hausdorff.

Proposition 3.9. *Let S be a strongly E^*-unitary inverse semigroup, and let $\psi : S^\times \to G$ be an idempotent pure morphism. Then there is a partial action of G on the spectrum X of S such that the groupoid $\mathcal{G}_S$ is topologically isomorphic to the groupoid $G \ltimes X$ associated with the partial action. In particular, $\mathcal{G}_S$ is Hausdorff and étale. Moreover, X is compact when S has a unit.*

Proof. This result is described in [26]. Let us recall first that a *partial action of a group* G on a locally compact space X is a morphism β from G into the inverse semigroup of partial homeomorphisms of X between open subsets, such that $\beta_e = \mathrm{Id}_X$ and $\beta_g\beta_h \leq \beta_{gh}$ for $g, h \in G$, meaning that β_{gh} extends $\beta_g\beta_h$. Then

$$G \ltimes X = \left\{(g, x) : g \in G, x \in X_{g^{-1}}\right\} \subset G \times X$$

with the topology induced from the product topology, where $X_{g^{-1}}$ is the domain of β_g. The range and source maps of the groupoid $G \rtimes X$ are given respectively by $r(g, x) = \beta_g(x)$ and $s(g, x) = x$. The product is defined by $(g, x)(h, y) = (gh, y)$ when $x = \beta_h(y)$, and the inverse is given by $(g, x)^{-1} = (g^{-1}, \beta_g(x))$.

Assume that S is strongly E^*-unitary. The partial action of G on the spectrum X of S is defined by setting $X_{g^{-1}} = \cup_{t\in\psi^{-1}(g)} D_{t^*t}$ (which can be empty). For $\chi \in X_{g^{-1}}$ we set $\beta_g(\chi) = \theta_t(\chi)$ where $t \in \psi^{-1}(g)$ is such that $\chi \in D_{t^*t}$. This does not depend on the choice of t as shown in [26, Lemma 3.1]. Moreover, by [26, Theorem 3.2], the groupoid $\mathcal{G}_S$ is canonically isomorphic to $G \ltimes X$.[d] □

Proposition 3.10. *Let S be a strongly E^*-unitary inverse semigroup. We assume that there is an idempotent pure morphism $\psi : S^\times \to G$ such that if $g \neq e$ is in the image of ψ, then $\psi^{-1}(g)$ has a greatest element α_g. Then the groupoid $\mathcal{G}_S$ is equivalent[e] to a transformation groupoid $G \ltimes Y$ for an action of G on an Hausdorff locally compact space Y.*

Proof. Let $t \in S^\times$ such that $\psi(t) = g$. Since $t \leq \alpha_g$ we have $D_{t^*t} \subset D_{\alpha_g^*\alpha_g}$ and therefore $X_{g^{-1}} = D_{\alpha_g^*\alpha_g}$ is a closed subset of X. Moreover, for $\chi \in X_{g^{-1}}$ we have $\beta_g(\chi) = \theta_{\alpha_g}(\chi)$. It follows that the cocycle $c : G \ltimes X \to G$ sending (g, x) to g is injective and closed. Injectivity means that the map $\gamma \in G \ltimes X \mapsto (c(\gamma), s(\gamma))$ is injective. The cocycle is said to be closed if $\gamma \mapsto (r(\gamma), c(\gamma), s(\gamma))$ from $G \ltimes X$ into $X \times G \times X$ is closed. Since the cocycle c is injective and closed, there exists a locally compact space Y endowed with a continuous action of G such that the transformation groupoid $G \ltimes Y$ is equivalent to $G \ltimes X = \mathcal{G}_S$. When S has a unit, X is compact and the equivalence is given by a groupoid isomorphism j from $G \ltimes X$ onto a reduction of $G \ltimes Y$) (see [15, Theorem 1.8] and [40, Theorem 6.2]). □

Corollary 3.11. *Let P a sub-semigroup of a group G containing the unit.*

(i) *The groupoid $\mathcal{G}_{S(P)}$ is defined by a partial action of G on a compact space.*
(ii) *If (P, G) satisfies the Toeplitz condition, then $\mathcal{G}_{S(P)}$ is equivalent to a transformation groupoid defined by an action of G on a locally compact space.*

Proof. (i) By Proposition 3.1 there is an idempotent pure morphism $\psi : S(P)^\times \to G$ and we use Proposition 3.9.

(ii) follows from Propositions 3.6 and 3.10. □

[d] In [26], the proofs are carried out assuming that S is E-unitary (*i.e.,* without zero) but they immediately extend to our setting.

[e] For this notion of equivalence of groupoids we refer to [1, Definition 2.2.15].

3.3. **Weak containment for inverse semigroups.** Let S be an inverse semigroup. Let us recall the definition of the full and reduced C^*-algebras of S (for more details, see [35, §2.1]). Given $f, g \in \ell^1(S)$, we set

$$(f \star g)(t) = \sum_{uv=t} f(u)g(v), \quad f^*(t) = \overline{f(t^*)}.$$

Then $\ell^1(S)$ is Banach $*$-algebra, and the *full C^*-algebra* $C^*(S)$ of S is defined as the enveloping C^*-algebra of $\ell^1(S)$. It is the universal C^*-algebra for the representations of S by partial isometries. The *left regular representation* $\pi_2 : S \to \mathcal{B}(\ell^2(S))$ is defined by

$$\pi_2(t)\delta_u = \delta_{tu} \text{ if } (t^*t)u = u, \quad \pi_2(t)\delta_u = 0 \text{ otherwise.}$$

The extension of π_2 to $\ell^1(S)$ is faithful. The *reduced C^*-algebra* $C^*_r(S)$ of S is the sub-C^*-algebra of $\mathcal{B}(\ell^2(S))$ generated by $\pi_2(S)$. We still denote by $\pi_2 : C^*(S) \to C^*_r(S)$ the extension of the left regular representation to $C^*(S)$.

When S has a zero, we have $\pi_2(0)\delta_0 = \delta_0$ and $\pi_2(0)\delta_t = 0$ if $t \neq 0$. It follows that $\mathbb{C}\delta_0$ is an ideal in $C^*(S)$ that it is preferable to get rid of. So we set $C^*_0(S) = C^*(S)/\mathbb{C}\delta_0$ and similarly $C^*_{r,0}(S) = C^*_r(S)/\pi_2(\mathbb{C}\delta_0)$. We denote by $\pi_{2,0}$ the canonical surjective homomorphism from $C^*_0(S)$ onto $C^*_{r,0}(S)$ (see [31]).

As shown in [35] and [15], the C^*-algebras $C^*_0(S)$ and $C^*_{r,0}(S)$ are canonically isomorphic to $C^*(\mathcal{G}_S)$ and $C^*_r(\mathcal{G}_S)$ respectively.[f] This is the reason for having introduced $C^*_0(S)$ and $C^*_{r,0}(S)$. Note that $C^*_{r,0}(S)$ is nuclear if and only $C^*_r(S)$ is so, and that π_2 is injective if and only if it is the case for $\pi_{2,0}$.

Definition 3.12. We say that S has the *weak containment property* if π_2 (or equivalently $\pi_{2,0}$) is an isomorphism.

Observe that S has the weak containment property if and only if the groupoid $\mathcal{G}_S$ has this property.

Recall that a semigroup S is *left amenable* if there exists a left invariant mean on $\ell^\infty(S)$. The following result was proved by Paterson in [33].

Proposition 3.13. *Assume that S has the weak containment property. Then S is left amenable.*

An inverse semigroup with zero is of course left amenable since the Dirac measure at zero is a left invariant mean. So the above result is not very instructive in general.

If $C^*_r(S)$ is nuclear, the groupoid $\mathcal{G}_S$ is amenable and therefore S has the weak containment property. What about the converse?

[f] More precisely in [35, 15], the authors consider the C^*-algebras $C^*(S)$ and $C^*_r(S)$, but their definition of $\mathcal{G}_S$ is also slightly different because for the space $X = \mathcal{G}_S^{(0)}$ they do not require that the maps χ from E_S into $\{0, 1\}$ satisfy $\chi(0) = 0$. Their proof also works in our setting.

The following example shows that left amenability, even in the absence of zero, does not imply the weak containment property. It also shows that the weak containment property is strictly weaker than the nuclearity of $C_r^*(S)$.

Example 3.14. Let Γ be a residually finite group and $(N_k)_{k\geq 0}$ a decreasing sequence as in Example 1.9, whose notation we keep. Let $S = \left\{q_k(t) : k \in \widehat{\mathbb{N}}, t \in \Gamma\right\}$. Formally, $S = \mathcal{G}$, the groupoid defined in Example 1.9 but we view S as an inverse semigroup in the following way. The product is given by $q_m(t).q_n(u) = q_{m\wedge n}(tu)$ where $m \wedge n$ is the smallest of the two elements m, n. We set $q_m(t)^* = q_m(t^{-1})$. The set E_S of idempotents is $\left\{q_m(e) : m \in \widehat{\mathbb{N}}\right\}$ that we identify with $\widehat{\mathbb{N}}$. The product of two idempotents is given by $m.n = m \wedge n$. The spectrum X is the set $\left\{\chi_m : m \in \widehat{\mathbb{N}}\right\}$ where $\chi_m(n) = 1$ if and only if $m \leq n$. It is homeomorphic to the compact space $\widehat{\mathbb{N}}$. The groupoid $\mathcal{G}_S$ associated with S is the space of equivalence classes of pairs $\big(q_m(t), \chi_k\big)$ with $k \leq m$, where $\big(q_m(t), \chi_k\big) \sim \big(q_n(t), \chi'_k\big)$ if and only if $k = k'$ and $q_k(t) = q_k(u)$. The map sending the class of $\big(q_m(t), \chi_k\big)$ to $q_k(t)$ is an isomorphism of topological groupoids from $\mathcal{G}_S$ onto the HLS-groupoid $\mathcal{G}$. It is easily checked that S has a zero if and only if $N_0 = \Gamma$, the zero being then $q_0(e)$. The maximal group homomorphism image is the finite group Γ/N_0 and $\sigma : S \to S/\sigma = \Gamma/N_0$ is $q_m(t) \mapsto q_0(t)$. It is not idempotent pure.

Let us observe that $S = \sqcup_{k\in\widehat{\mathbb{N}}}\Gamma/N_k$ is a Clifford semigroup.

Since S/σ is amenable, this semigroup S is left amenable by a result of Duncan and Namioka (see [35, Proposition A.0.5]). If $\Gamma = \mathbb{F}_2$ and the sequence $(N_k)_k$ is the one defined by Willett in [45], then S has the weak containment property but $C_r^*(S)$ is not nuclear. Moreover, S is a Clifford semigroup for which not every subgroup is amenable, although it has the weak containment property.

On the other hand, if we realize $\mathbb{F}_2$ as a finite index subgroup of $SL(2,\mathbb{Z})$, and choose N_k to be the intersection of the kernel of the reduction map $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/2^k\mathbb{Z})$ and $\mathbb{F}_2$, then the corresponding HLS-groupoid has not the weak containment property, as observed in [45, Remarks 2.9]. Hence, the left amenability of S does not imply its weak containment property in general.

The next theorem gives in particular a sufficient condition for the equivalence between the weak containment property and the nuclearity of the reduced C^*-algebra.

Theorem 3.15. *Let S be a strongly E^*-unitary inverse semigroup, and let $\psi : S^\times \to G$ be an idempotent pure morphism.*

1. *Assume that G is amenable. Then we have $C^*(S) = C_r^*(S)$ and $C_r^*(S)$ is nuclear.*
2. *Assume that G is exact and that for every $g \neq e$ in the image of ψ, the set $\psi^{-1}(g)$ has a greatest element. Then, the weak containment property of S is equivalent to the nuclearity of $C_r^*(S)$.*

Proof. By Proposition 3.9, the groupoid $\mathcal{G}_S$ is associated to a partial action of G on the spectrum of S. If G is amenable, then $\mathcal{G}_S$ is amenable (see [41]) and therefore the statement of (1) holds.

Under the assumptions of (2), the groupoid $\mathcal{G}_S$ is equivalent to some transformation groupoid $G \ltimes Y$ by Proposition 3.10. Then, by Corollary 2.13 , $\mathcal{G}_S$ has the weak containment property if and only if it is amenable, and therefore if and only if $C^*_{r,0}(S) = C^*_r(\mathcal{G}_S)$ is nuclear, since $\mathcal{G}_S$ is étale. □

Example 3.16. Let $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ be a directed graph and $S_\mathcal{E}$ the associated graph inverse semigroup (see [25, Section 4]). Let $\mathbb{F}$ be the free group generated by the set $\mathcal{E}^1$ of edges. Then there is an idempotent pure morphism $\psi : S^\times_\mathcal{E} \to \mathbb{F}$ satisfying the assumption of Proposition 3.9 (2). It has the weak containment property (see [25, Theorem 4.3]) although $\mathbb{F}$ is not amenable if the cardinal of $\mathcal{E}^1$ is ≥ 2.

For completeness, we end this section by a known result for E-unitary inverse semigroups (see [33, 25]).

Proposition 3.17. *Let S be a E-unitary inverse semigroup The following conditions are equivalent:*

(1) *S is left amenable;*
(2) *$C^*_r(S)$ is nuclear;*
(3) *$C^*(S) = C^*_r(S)$.*

Proof. We may assume that S is without 0, otherwise $S = E_S$ and the three properties are satisfied.

(1) ⇒ (2). The left amenability of S implies that the maximal group homomorphism image $G = S/\sigma$ is an amenable group by a result of Duncan and Namioka. By Proposition 3.9, the groupoid $\mathcal{G}_S$ is associated to a partial action of S/σ on the spectrum of S and therefore is amenable. It follows that $C^*_r(S)$ nuclear.

That (2) ⇒ (3) and (3) ⇒ (1) have already be mentioned and do not require that S is E-unitary. □

3.4. **Weak containment for semigroups embedded in groups.** In this section we consider a discrete group G and a sub-semigroup P which contains the unit e. We denote by λ the left regular representation of G, and for $p \in P$ we denote by $V_p : \ell^2(P) \to \ell^2(P)$ the isometry given by $V_p\delta_q = \delta_{pq}$. The *reduced C^*-algebra* $C^*_r(P)$ of P is the C^*-algebra generated by the isometries V_p, $p \in P$.

The right definition of the full C^*-algebra of P is more speculative. The universal C^*-algebra generated by elements v_p, $p \in P$, such that $v_p^* v_p = 1$ and $v_p v_q = v_{pq}$ for every $p, q \in P$, is too big. For instance Murphy proved that, for the commutative semigroup $\mathbb{N}^2$, this universal C^*-algebra is not nuclear [28]. A reasonable definition for the *full C^*-algebra* of P was introduced by Xin Li in [22, Definition 2.2] and a variant in [22, Definition 3.2]. It is this variant (denoted $C^*_s(P)$ in [22]) that we adopt as the definition of the full C^*-algebra of P in the sequel, and we denote it $C^*(P)$. By [31, Proposition 3.3.1], $C^*(P)$ can be defined as $C^*_0(S(P))$. Let us recall (see [31,

Lemma 3.2.2]) that the inverse semigroup $S(P)$ is canonically isomorphic to the inverse semigroup of partial isometries

$$V(P) = \big\{V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n} : n \in \mathbb{N}, p_i, q_i \in P\big\}.$$

Let us also recall that there is a surjective homomorphism $h : C_{r,0}^*(S(P)) \to C_r^*(P)$ such that $h(\pi_2(\ell_p)) = V_p$ for $p \in P$ (see [31, Lemma 3.2.12]). Therefore we have the following situation

$$C^*(P) = C_0^*(S(P)) \equiv C^*(\mathcal{G}_{S(P)}) \overset{\pi_{2,0}}{\longrightarrow\!\!\!\!\!\rightarrow} C_{r,0}^*(S(P)) \equiv C_r^*(\mathcal{G}_{S(P)}) \overset{h}{\longrightarrow\!\!\!\!\!\rightarrow} C_r^*(P).$$

Definition 3.18. We say that P has the *weak containment property* if $C^*(P) = C_r^*(P)$.

Note that the weak containment property of P implies the weak containment property of $S(P)$.

Definition 3.19. Let (P, G) as above and assume in addition that $P \cap P^{-1} = \{e\}$. Then we define on G a partial order by setting $x \leq y$ if $x^{-1}y \in P$.

(a) We say that (P, G) is a *quasi-lattice ordered group* if for every $g \in G$, we have either $P \cap gP = \emptyset$ or $P \cap gP = rP$ for some $r \in P$ (equivalently, every pair of elements in G having a common upper bound has a least common upper bound (see [8, Lemma 7] for more on this)).
(b) We say that (P, G) is a *lattice ordered group* if every pair of elements of G has a least upper bound (equivalently (P, G) is quasi-lattice ordered and $G = PP^{-1}$ (see [8, Lemma 27])).

The Toeplitz condition for (P, G) is equivalent to the following property: for every $g \in G$ such that $E_P \lambda_g E_P \neq 0$, there exist $p_1, \ldots, p_n, q_1, \ldots, q_n \in P$ such that $E_P \lambda_g E_P = V_{q_1}^* V_{p_1} \ldots V_{q_n}^* V_{p_n}$ (see for instance the proof of [32, Proposition 4.1]).

Quasi-lattice ordered semigroups and semigroups (P, G) such that $G = P^{-1}P$ satisfy the Toeplitz condition (see [23, §8]). Xin Li has also introduced an important condition for P, he called *independence* ([22, Definition 2.26]). We will not describe it here. We only note that when P is contained in a group, this condition is equivalent to the injectivity of h (see [31, Theorem 3.2.14]) and that it is satisfied for quasi-lattice ordered groups (see [22, Lemma 28]).

Proposition 3.20. *Let P be a sub-semigroup of a group G with $e \in P$. Assume that P satisfies the independence condition. Then the nuclearity of $C_r^*(P)$ implies the weak containment property for P, i.e., $C^*(P) = C_r^*(P)$.*

Proof. Assume that $C_r^*(P)$ is nuclear. Since $C_r^*(P) = C_r^*(\mathcal{G}_{S(P)})$, we see that the groupoid $\mathcal{G}_{S(P)}$ is amenable. It follows that $S(P)$ (and thus P) has the weak containment property. □

Proposition 3.21. *Let (P, G) as in the above proposition.*

(1) *If G is amenable, then $C_r^*(P)$ is nuclear.*
(2) *If P is left amenable, then PP^{-1} is an amenable subgroup of G and $C_r^*(P)$ is nuclear.*

Proof. Assume that G is amenable. By Proposition 3.1 and Proposition 3.9 the groupoid $\mathcal{G}_{S(P)}$ is amenable since it is isomorphic to the groupoid defined be a partial action of G. It follows that $C_r^*(\mathcal{G}_{S(P)})$ is nuclear as well as its quotient $C_r^*(P)$.

Suppose now that P is left amenable. Then it is left reversible (*i.e.,* $pP \cap qP \neq \emptyset$ for all $p, q \in P$) and therefore $G' = PP^{-1}$ is an amenable subgroup of G (see [34, Propositions 1.23, 1.27]). To see that $C_r^*(\mathcal{G}_{S(P)})$ is nuclear, we apply the first part of the proof. □

The following proposition extends the classical case where $G = P$.

Proposition 3.22. *Let (P, G) such that $G = PP^{-1}$ (right Ore semigroup). Consider the following conditions:*

(1) *P has the weak containment property.*
(2) *G is amenable.*
(3) *P is left amenable.*
(4) *$C_r^*(P)$ is nuclear.*

Then we have (1) ⇒ (2) ⇔ (3) ⇒ (4).

Proof. (3) ⇒ (2) and (3) ⇒ (4) are proved in the previous proposition. Although is is not always true that a sub-semigroup P of an amenable group G is left amenable, this is the case when $G = PP^{-1}$ (see [34, Proposition 1.28]) and so (2) ⇒ (3).

Assume that P has the weak containment property. Then $S(P)$ has the weak containment property. On the other hand, using Proposition 3.1 and Remark 3.4, we see that $S(P)$ is E-unitary without zero, and that there is an idempotent pure morphism from $S(P)$ into $G = S(P)/\sigma$. Then, by Proposition 3.17, we get that $S(P)$ is amenable and thus G is amenable. □

Corollary 3.23. *Assume that $G = PP^{-1}$ and that the independence condition is satisfied (for instance assume that (P, G) is a lattice ordered group). The following conditions are equivalent:*

(1) *P has the weak containment property.*
(2) *G is amenable.*
(3) *P is left amenable.*
(4) *$C_r^*(P)$ is nuclear.*

Proof. Use Propositions 3.20 and 3.22. □

The equivalence between (1) and (2) in the previous corollary was obtained by Nica [29, Remark 2, Section 5.1] when (P, G) is a lattice ordered group (see also [8, Proposition 28]). For related results, see [22, §4.1].

Theorem 3.24. *Let P be a sub-semigroup of a group G, containing the unit e. Assume that (P, G) satisfies the Toeplitz condition (for instance assume that (P, G) is a quasi-lattice ordered group or that $G = P^{-1}P$) and that G is an exact group. The two following conditions are equivalent.*

(1) *$C^*(P) = C_r^*(P)$;*

(2) *C^*-algebra $C_r^*(P)$ is nuclear and P satisfies the independence condition.*

Proof. The weak containment property for P implies the weak containment property for $\mathcal{G}_{S(P)}$ and also the independence. So the assertion (1) $\Rightarrow$ (2) follows from Corollaries 3.11 and 2.13. The converse is immediate. □

4. Some questions

(1) The only example of an étale groupoid which satisfies the weak containment property although it is not amenable was provided by Willett. It is a bundle of groups. At the opposite, are there principal groupoids (*i.e.,* such that the units are the only elements with the same source and range) which are amenable without the weak containment property? It would also be interesting to know whether there are examples which are transformation groupoids $G \ltimes X$ where G is a discrete group acting on a locally compact space X. To this respect, is it true that G is exact when a groupoid $G \ltimes X$ has the weak containment property with X compact? This question which seems quite difficult was already raised in [45].

For the boundary compact set $X = \partial G = \beta G \setminus G$, equipped with the natural action of G the answer is positive. Indeed, the weak containment property for $G \ltimes \partial G$ implies that the sequence

$$0 \longrightarrow C_r^*(G \ltimes G) \longrightarrow C_r^*(G \ltimes \beta G) \longrightarrow C_r^*(G \ltimes \partial G) \longrightarrow 0$$

is exact. Roe and Willett proved in [43] that this exactness property implies that G has Yu's property A and thus is exact.

(2) If G is exact and acts partially on a locally compact space X, is it true that the groupoid $G \ltimes X$ is KW-exact, or at least inner exact? If yes, this would imply that for any sub-semigroup P of an exact group, the weak containment property for P would imply that $C_r^*(P)$ is nuclear.

(3) Find an example of a pair (P, G) such that $C_r^*(P)$ is nuclear but P has not the weak containment property.

(4) Find a characterization of the weak containment property for a locally compact groupoid.

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Université d'Orléans et CNRS (UMR 7349 et FR2964),
B. P. 6759, F-45067 Orléans Cedex 2

E-mail address: `claire.anantharaman@univ-orleans.fr`