On a Class of Oscillations in the Finite-Deformation Theory of Elasticity

James Knowles

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J. K. KNOWLES
Associate Professor of Applied Mechanics, California Institute of Technology, Pasadena, Calif.

This paper treats the large-amplitude radial oscillations of a perfectly elastic, incompressible cylindrical tube of infinite length due to suddenly applied pressures on its lateral surfaces. The motion is studied for materials with essentially arbitrary strain-energy density. Sufficient conditions for periodic motions and a formula for the period of oscillation are given in terms of the strain energy. The results are specialized to the case of a rubberlike material of Mooney type, and asymptotic formulas are given for the case of a thin shell and for the case of small applied pressure.

In recent years there has been considerable interest in problems involving large strains in perfectly elastic solids. A special class of materials for which such problems are of importance are those capable of sustaining only locally volume-preserving deformations. Included among the practically important materials which possess this property are various types of rubber. It is well known from the work of Rivlin, A. E. Green, and others that the condition of incompressibility makes it possible to obtain relatively explicit exact solutions of many special static problems in the finite-deformation theory of elasticity without first specializing the strain-energy function characteristic of the material.[1]

This paper is concerned with arbitrarily large deformation dynamic response of perfectly elastic incompressible materials. Specifically, we discuss the problem of radial oscillations of an infinitely long, hollow circular cylinder of an arbitrary material of this type. The motion results from the sudden application of uniform normal pressures over the inner and outer surfaces of the cylinder. Such a discussion has two purposes: (*a*) It indicates by example that certain information can be obtained in dynamic problems involving large strains for essentially the entire class of incompressible materials, regardless of the special form of the strain-energy function. (*b*) It provides explicit results for the amplitude and period of oscillation for a particular rubberlike material when the strain-energy function characteristic of the material is suitably specialized.

In a recent paper[2] the problem of radial oscillations of an infinitely long hollow cylinder of isotropically elastic incompressible material subject on its inner and outer surfaces to spatially uniform time-dependent pressures was formulated and reduced to a single nonlinear ordinary differential equation for the inner radius as a function of time. In that paper, however, the only case treated in detail was one in which the tube is set in motion by imparting to its inner surface an initial displacement and an initial velocity and subsequently oscillates freely in the absence of external lateral surface pressures. In the present paper we treat the "forced" oscillations of the hollow cylinder which follow the sudden application of uniform pressures to the inner and outer surfaces. The applied pressures are maintained constant in time and the cylinder is taken to be initially at rest and undeformed.

Results similar to those in this paper can also be obtained for an oscillating hollow sphere.

Equation of Motion

Let the inner and outer radii of the tube in the undeformed state be r_1 and r_2, respectively. Let r, θ, z denote the co-ordinates of a point in the tube in the undeformed state with reference to a fixed cylindrical co-ordinate system coaxial with the cylinder. A particle which was at r, θ, z in the undeformed state is assumed to have cylindrical co-ordinates $R(r, t), \theta, z$ at time t. The motion completely determined once $R = R(r, t)$ is known.

Denote by $R_i(t)$ and $R_2(t)$, respectively, the inner and outer radii of the cylinder at time t; i.e., $R_i(t) = R(r_i, t), i = 1, 2$. The assumption that the material is incompressible clearly requires

$$\pi R^2 - \pi R_1^2 = \pi r^2 - \pi r_1^2, \quad r_1 \leq r \leq r_2, \quad t \geq 0 \qquad (1)$$

Thus if the function $R_1(t)$ is known, $R(r, t)$ may be determined by (1), so that the problem is reduced to the determination of the inner radius R_1 as a function of time.

Since the material of the tube is taken to be homogeneous, elastic, and isotropic, it is characterized by a strain energy W per unit undeformed volume which is a function of two of the three principal strain invariants[3] I_1, I_2, and I_3: $W = W(I_1, I_2)$. Incompressibility requires that the third strain invariant I_3 must satisfy $I_3 = 1$, a condition which can be shown to be equivalent to (1). For the motion considered here it is found that

$$I_1 = I_2 = 1 + (r/R)^2 + (R/r)^2 \qquad (2)$$

We introduce a dimensionless inner radius

$$x(t) = R_1(t)/r_1,$$

and a positive parameter

$$\mu = (r_2/r_1)^2 - 1$$

which is small for the special case of the thin-walled cylindrical tube. Let ρ denote the mass density of the material of the tube, and $P_1(t)$ and $P_2(t)$ stand for the external pressures acting on the inner and outer surfaces at time t. It was shown by Knowles[2] that the exact equations of motion, stress-strain relations, and boundary conditions of the finite deformation theory of elasticity

[1] An extensive discussion of finite deformations of incompressible elastic solids is contained in "Theoretical Elasticity," by A. E. Green and W. Zerna, Clarendon Press, Oxford, England, 1954. This work contains additional references.

[2] J. K. Knowles, "Large Amplitude Oscillations of a Tube of Incompressible Elastic Materials," *Quarterly of Applied Mathematics*, vol. 18, 1960, pp. 71–77.

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[3] The terminology and notation in what follows are essentially those of Green and Zerna.[1]

for incompressible materials (as presented, for example, by Green and Zerna[1]) reduce to the following single nonlinear ordinary differential equation for the nondimensional inner radius $x(t)$:

$$[x \log (1 + \mu/x^2)] \frac{d^2x}{dt^2} + \left[\log (1 + \mu/x^2) - \frac{\mu}{\mu + x^2}\right] \left(\frac{dx}{dt}\right)^2 + f(x, \mu) = \frac{P_1(t) - P_2(t)}{\rho r_1^2/2}, \quad (3)$$

where the function f is given in terms of the strain-energy density W by

$$f(x, \mu) = \int_{(\mu+x^2)/(\mu+1)}^{x^2} (u - 1)^{-1} \frac{dW_0}{du} du, \quad (4)$$

where

$$W_0(u) = \frac{2}{r_1^2 \rho} W(I_1, I_2)_{I_1=I_2=1+u+1/u} \quad (5)$$

The first two terms on the left side of the differential equation (3) arise from expressing the radial acceleration $\partial^2 R/\partial t^2$ of an element of the tube in terms of the acceleration d^2R_1/dt^2 and velocity dR_1/dt of the inner surface of the tube by means of the incompressibility condition (1). The function $f(x, \mu)$ in (3) represents the elastic characteristics of the material as manifested by the strain energy W.

In the paper it will be assumed that $W(I_1, I_2)$ possesses continuous first partial derivatives with respect to I_1 and I_2 and that

$$\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)_{I_1=I_2} \geq 0 \quad (6)$$

This inequality is implied by inequalities discussed by Truesdell[4] in connection with restrictions which must be applied to a strain-energy density in order that it be physically admissible. From this inequality and (5) it follows that

$$\frac{dW_0}{du} = \frac{2}{r_1^2 \rho} \left(1 - \frac{1}{u^2}\right) \left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) \quad (7)$$

has the sign of $u - 1$, and hence from (4) that

$$f(x, \mu) \begin{cases} < 0 & \text{if } 0 < x < 1 \\ = 0 & \text{if } x = 1 \\ > 0 & \text{if } x > 1 \end{cases} \quad (8)$$

In the paper by Knowles the differential equation (3) was considered for the special case $P_1(t) = P_2(t) = 0$ and subject to the initial conditions $x(0) = x_0$ and $(dx/dt)(0) = v_0$. Here we shall consider the case in which

$$\frac{P_1(t) - P_2(t)}{\frac{1}{2} r_1^2 \rho} = \begin{cases} 0 & t \leq 0 \\ p & t > 0 \end{cases} \quad (9)$$

where p is constant. We assume that the tube is initially undeformed and at rest so that $x(0) = 1$, $\dot{x}(0) = 0$. Thus we are considering the motion which follows the sudden application of constant pressures to the lateral surfaces of the tube. While a compressible material would respond to such a loading by propagating cylindrical dilation waves at a finite speed away from its lateral surfaces, this response does not occur in the incompressible case since such a material cannot propagate dilatation waves at a finite speed. It will be shown that the cylindrical tube responds by oscillating about a static equilibrium state in accordance with (1) and (3).

[4] C. Truesdell, "The Mechanical Foundations of Elasticity and Fluid Mechanics," *Journal of Rational Mechanics and Analysis*, vol. 1, 1952, pp. 125–300.

Periodic Motion for a General Strain-Energy Density

The class of nonlinear oscillations which we wish to consider is thus described by the following initial value problem:

$$x \log (1 + \mu/x^2)\ddot{x} + \left[\log (1 + \mu/x^2) - \frac{\mu}{\mu + x^2}\right] \dot{x}^2 + f(x, \mu) = p, \quad t > 0 \quad (10)$$

$$x(0) = 1, \quad \dot{x}(0) = 0 \quad (11)$$

where p is a constant and a dot indicates differentiation with respect to t. With the notation $v = dx/dt$, we may write $\ddot{x} = v\, dv/dx$, so that (10) becomes

$$\frac{d}{dx} [\tfrac{1}{2} v^2 x^2 \log (1 + \mu/x^2) + F(x, \mu)] = px, \quad (12)$$

where

$$F(x, \mu) = \int_1^x \xi f(\xi, \mu) d\xi \quad (13)$$

Making use of (11) we thus have a first integral in the form

$$\tfrac{1}{2} v^2 x^2 \log (1 + \mu/x^2) + F(x, \mu) = \tfrac{1}{2} p(x^2 - 1) \quad (14)$$

Equation (14) describes the trajectory C, Fig. 1, in the x-v plane associated with the motion. C is symmetric about the x-axis. The motion is periodic if C is closed and if the period of oscillation

$$T = \oint_C \frac{dx}{v} \quad (15)$$

is finite. The curve C given by (14) starts at the initial point $x = 1$, $v = 0$ at time $t = 0$. The original differential equation (10) may easily be used to show that if p is positive (negative), so that the net pressure on the lateral surface is outward (inward), the point x, v then moves into the region $x > 1, v > 0$ ($x < 1, v < 0$) as t increases from zero. If for p positive (negative) v passes through a maximum (minimum) and returns to zero as x increases (decreases) from unity, the curve C will be closed According to (14) this will happen if there is a root (different from unity) of the equation

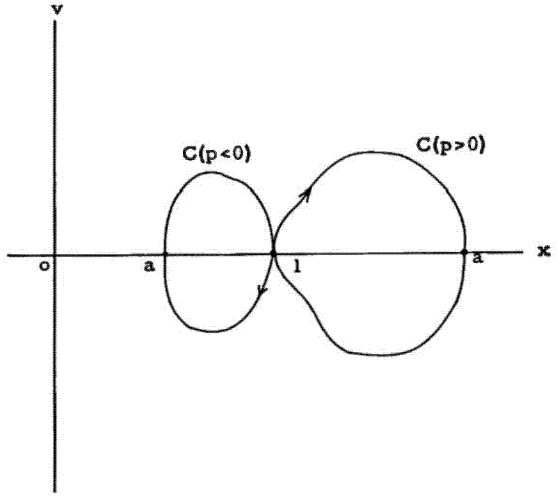

Fig. 1 Possible trajectories for general periodic motions

$$F(x, \mu) = \tfrac{1}{2} p(x^2 - 1), \tag{16}$$

so that $v = 0$ for some $x \neq 1$.

Let us assume that, for a given p, that (16) possesses such a root.[5] Let $x = a \neq 1$ be the positive root of (16) which is nearest unity. Thus when $x = a, v = 0$. Since according to the definition (13) we have $F'(x, \mu) = xf(x, \mu)$ (where prime denotes differentiation with respect to x), it follows from (8) that F decreases as x increases for $0 < x < 1$, vanishes at $x = 1$, and increases as x increases for $x > 1$. Therefore the root $x = a$ of (16) is greater than 1 if $p > 0$ and less than 1 if $p < 0$. In the former case a represents the greatest value of the (dimensionless) inner radius of the tube during the oscillation under net outward pressure, while in the latter case a represents the smallest value of the inner radius oscillating under net inward pressure.

Since

$$F(x, \mu) - \tfrac{1}{2} p(x^2 - 1)$$

vanishes at $x = 1$ and $x = a$, and since

$$F'(x, \mu) = xf(x, \mu),$$

it is clear that the existence of a root $x = a$ of (16) implies the existence of a root $x = b$ between $x = 1$ and $x = a$ of the equation

$$f(x) = p \tag{17}$$

A solution $x = b$ of (17) represents a static equilibrium state of the tube under the net pressure $^1/_2(r_1^2 \rho p)$, since it is a time-independent solution of the differential equation (10). Thus if (16) possesses a solution $x = a$, the inner radius of the tube will oscillate about a static equilibrium value between $x = 1$ and $x = a$.

The period of oscillation for the tube will be given by (15); using (14) to obtain v in terms of x, we find

$$T = \oint_C \frac{dx}{v} = 2 \operatorname{sgn}(a - 1) \int_1^a \left[\frac{\log(1 + \mu/x^2)}{p(x^2 - 1) - 2F(x, \mu)}\right]^{1/2} x dx \tag{18}$$

On the assumption that (16) possesses a root $x = a$, equation (18) expresses the period of oscillation of the tube of arbitrary incompressible material in terms of the strain-energy density characteristic of the material by means of (13) and (4). The period T will be finite provided that the root $x = a$ of (16) is *not* a root of (17).

We now turn to the question of a sufficient condition on the strain-energy function $W_0(u)$ of (5) which will guarantee that (16) has a root. For positive p (outward net pressure) it can be shown that

$$F(x, \mu) - \tfrac{1}{2} p(x^2 - 1)$$

is negative when $x - 1$ is small and positive; thus if F grows fast enough to insure that

$$F(x, \mu) - \tfrac{1}{2} p(x^2 - 1)$$

is positive for large x, we must indeed have a root $x = a > 1$. Similar remarks apply when $p < 0$. A detailed analysis of F, using the definitions (13) and (4) and the fact that $W_0(u) = W_0(1/u)$, leads to the following sufficient conditions:

(*a*) If $W_0(u) \sim Ku^k$ as $u \to \infty$ for some constants $K > 0$ and $k > 1$, then Equation (16) will have a root for all values of the pressure p, and the trajectory C in the x-v plane will be a closed curve.

(*b*) If $W_0(u) \sim Ku$ as $u \to \infty$ for some constant $K > 0$, then equation (16) will have a root *for all p satisfying* $p < K \log(1 + \mu)$, and the trajectory C in the x-v plane will be a closed curve.

The sufficient condition (*b*) is included since the physically important case of the Mooney material considered in the following section leads to a strain-energy density whose asymptotic form for large u is of the type covered in condition (*b*).

A Special Strain Energy

In this section we specialize the results of the foregoing to the case of a rubberlike material whose strain-energy density $W(I_1, I_2)$ is given by

$$W(I_1, I_2) = \tfrac{1}{2} \alpha(I_1 - 3) + \tfrac{1}{2} \beta(I_2 - 3), \tag{19}$$

This expression describes the so-called Mooney material[6]; α and β are positive constants.

In this case the functions defined in (5), (4), and (13) are easily calculated to be

$$W_0(u) = K(u + u^{-1} - 2), \tag{20}$$

$$f(x, \mu) = \left[K\left(1 - \frac{1}{x^2}\right)\frac{\mu}{\mu + x^2} - \log\left(\frac{1 + \mu/x^2}{1 + \mu}\right)\right], \tag{21}$$

and

$$F(x, \mu) = \tfrac{1}{2} K(1 - x^2) \log \frac{1 + \mu/x^2}{1 + \mu}, \tag{22}$$

where the constant K is given by $K = (\alpha + \beta)/r_1^2 \rho$. Inspection of (20) indicates that $W_0(u) \sim Ku$ as $u \to \infty$, so that part (*b*) of the sufficient condition for a closed trajectory is relevant.

The condition (16) which determines the greatest (or least) value of the inner radius of the tube during the oscillation is, in this case, given by

$$K \log\left(\frac{1 + \mu}{1 + \mu/x^2}\right) = p \tag{23}$$

The left side of equation (23) is a monotone increasing function of x for $x > 0$. It increases from minus infinity, when $x = 0+$, through negative values, vanishes at $x = 1$, and tends to K log $(1 + \mu)$ as x tends to plus infinity. Thus for all values of the applied pressure satisfying

$$p < p_{cr} = K \log(1 + \mu) \tag{24}$$

a unique positive root of (23) exists, and a tube of material characterized by the strain-energy density (19) will undergo periodic oscillations in accordance with our earlier discussion. For pressures

$$p \geq p_{cr} = K \log(1 + \mu)$$

periodic motions will not exist.

The critical value p_{cr} of p defined by (24) has another property in the present case. Examination of the expression (21) for f for the material presently under consideration shows that f is also monotone increasing from minus infinity at $x = 0+$ and approaches p_{cr} as x tends to plus infinity. Thus for $p > p_{cr}$, there is no root $x = b$ of equation (17) determining the static equilibrium state of the tube for a given applied pressure difference.

The restriction (24) which is necessary for periodic motions can be written as follows in terms of the original physical quantities pertaining to the material, geometry, and loading of the hollow cylinder:

[5] A sufficient condition on the strain-energy density which guarantees the existence of this root is given in the sequel.

[6] Refer to Green and Zerna,[1] p. 76.

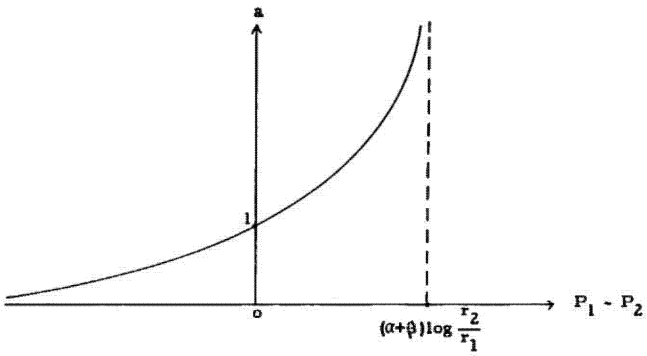

Fig. 2 Ratio of maximum ($P_1 - P_2 > 0$) or minimum ($P_1 - P_2 < 0$) inner radius to undeformed inner radius versus net outward pressure $P_1 - P$

$$P_1 - P_2 < (\alpha + \beta) \log \frac{r_2}{r_1} \tag{25}$$

Here $P_1 - P_2$ is the net outward (constant) pressure, α and β are the material constants from (19), and r_1 and r_2 are the inner and outer radii of the tube in its undeformed state. We observe that (25) is always satisfied if $P_1 - P_2 < 0$, so that the net pressure is inward. Moreover, it may be observed that the critical pressure on the right side of (25) tends to zero for a given material as the wall thickness $r_2 - r_1$ tends to zero.

The root $x = a$ of equation (23) can be computed explicitly as

$$a = \left[\frac{\mu}{(1 + \mu)e^{-p/K} - 1}\right]^{1/2} \tag{26}$$

In terms of physical variables,

$$a = \left\{\frac{r_2^2 - r_1^2}{r_2^2 \exp\left[-2(P_1 - P_2)/(\alpha + \beta)\right] - r_1^2}\right\}^{1/2} \tag{27}$$

If $P_1 - P_2 > 0$, so that the net pressure is outward, a is greater than 1 and represents the ratio of the largest value of the inner radius during the motion to the undeformed inner radius r_1. If $P_1 - P_2 < 0$, a is less than 1 and represents the ratio of the smallest value of the inner radius to r_1. Note that $a \to \infty$ as

$$P_1 - P_2 \to (\alpha + \beta) \log \frac{r_2}{r_1}$$

The general character of a as a function of $P_1 - P_2$ is shown in Fig. 2.

The period of oscillation T is found from (18) to be

$$T = 2(\operatorname{sgn} p) \int_1^a \left\{\frac{x^2 \log (1 + \mu/x^2)}{(x^2 - 1)[p - K \log (1 + \mu) + K \log (1 + \mu/x^2)]}\right\}^{1/2} dx \tag{28}$$

We rewrite (28) by using (26) to express p in terms of a and by introducing the change of variables $z = x^2$. There follows

$$TK^{1/2} = [\operatorname{sgn} (a^2 - 1)] \int_1^{a^2} \left\{\frac{\log (1 + \mu/z)}{(z - 1) \log [(1 + \mu/z)(1 + \mu/a^2)]}\right\}^{1/2} dz \tag{29}$$

The integral (29) is always finite if the condition (25) guaranteeing periodic motion is satisfied.

While the expression (28) or (29) for the period of oscillation does not appear to be capable of evaluation in closed form in terms of elementary functions, it is possible to obtain asymptotic approximations in two important special cases. The first is that in which the parameter $\mu = (r_2/r_1)^2 - 1$ is small compared with unity, so that the tube is a thin-walled cylindrical shell. The second special case is that in which the (nondimensional) net pressure p/K is small compared with unity.

We consider the thin-shell approximation $\mu \ll 1$ first. Suppose first that the net pressure is outward, so that $p > 0$. In this case, by using the inequalities

$$\frac{x - y}{1 + x} \leqq \log \frac{1 + x}{1 + y} \leqq \frac{x - y}{1 + y}, \quad x \geq y, \tag{30}$$

and

$$\frac{x}{1 + x} \leqq \log (1 + x) \leqq x \quad x \geq 0, \tag{31}$$

it is possible to obtain the following bounds on the period of oscillation T:

$$\frac{\pi a}{K^{1/2}} (1 + \mu)^{-1/2} \leqq T \leqq \frac{\pi a}{K^{1/2}} (1 + \mu)^{1/2} \tag{32}$$

where $a = a(\mu, p)$ is the maximum nondimensional radius during the oscillation. It is important to note that the bounds (32) hold for all $p \geq 0$. From (32) it follows immediately that the asymptotic relation

$$T \sim \frac{\pi a}{K^{1/2}} \quad \text{as} \quad \mu \to 0 \tag{33}$$

holds, in the sense that

$$\lim_{\mu \to 0} \left(\frac{TK^{1/2}}{\pi a} - 1\right) = 0 \tag{34}$$

The asymptotic relation (33) is uniformly valid for $p \geqq 0$; i.e., the limit (34) holds uniformly in p for $p \geqq 0$.

A similar argument shows that the asymptotic relation also holds when p is negative but only in the sense that the limit (34) holds uniformly for $p \geqq -c\mu$ for any arbitrary but fixed positive constant c.

In terms of physical quantities, with a given by (27), the thin-shell approximation (33) to the period becomes

$$T \sim \pi r_1 \left(\frac{\rho}{\alpha + \beta}\right)^{1/2} \left\{\frac{r_2^2 - r_1^2}{r_2^2 \exp\left[-2(P_1 - P_2)/(\alpha + \beta)\right] - r_1^2}\right\}^{1/2} \tag{35}$$

In the case where the shell is not necessarily thin but the applied pressure is small, considerations again based on the inequalities (30) lead to the asymptotic relation

$$T \sim \frac{\pi}{K^{1/2}} \left\{\frac{(1 + \mu) \log (1 + \mu)}{\mu}\right\}^{1/2} \quad \text{as} \quad p \to 0 \tag{36}$$

The relation (36) holds for every $\mu > 0$; moreover, it is uniform in μ as long as μ is bounded away from zero; i.e., for $\mu \geq \mu_0 > 0$, where μ_0 is an arbitrary positive number, no matter how small.

In terms of physical quantities, the small pressure approximation takes the form

$$T \sim \pi r_1 \left(\frac{\rho}{\alpha + \beta}\right)^{1/2} \left[\frac{2r_2^2 \log (r_2/r_1)}{r_2^2 - r_1^2}\right]^{1/2} \tag{37}$$

It may be remarked that (37) is precisely the formula obtained for the period of oscillation of the hollow cylinder when the problem is analyzed on the basis of the classical linear theory of elasticity specialized to the case of an incompressible material.