



THE SYMMETRIC INVARIANTS OF CENTRALIZERS AND SLODOWY GRADING II

Jean-Yves Charbonnel, Anne Moreau

► To cite this version:

Jean-Yves Charbonnel, Anne Moreau. THE SYMMETRIC INVARIANTS OF CENTRALIZERS AND SLODOWY GRADING II. 2016. hal-01298516

HAL Id: hal-01298516

<https://hal.science/hal-01298516>

Preprint submitted on 6 Apr 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE SYMMETRIC INVARIANTS OF CENTRALIZERS AND SŁODOWY GRADING II

JEAN-YVES CHARBONNEL AND ANNE MOREAU

ABSTRACT. Let \mathfrak{g} be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field \mathbb{k} of characteristic zero, and let (e, h, f) be an \mathfrak{sl}_2 -triple of \mathfrak{g} . Denote by \mathfrak{g}^e the centralizer of e in \mathfrak{g} and by $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ the algebra of symmetric invariants of \mathfrak{g}^e . We say that e is good if the nullvariety of some ℓ homogenous elements of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ in $(\mathfrak{g}^e)^*$ has codimension ℓ . If e is good then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. In this paper, we prove that the converse of the main result of [CM16] is true. Namely, we prove that e is good if and only if for some homogenous generating sequence q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the initial homogenous components of their restrictions to $e + \mathfrak{g}^f$ are algebraically independent over \mathbb{k} .

CONTENTS

1. Introduction	1
2. Proof of Theorem 1.5	4
3. Proof of Theorem 1.4	12
Appendix A. General facts on commutative algebra	13
References	24

1. INTRODUCTION

1.1. Let \mathfrak{g} be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field \mathbb{k} of characteristic zero, let $\langle \cdot, \cdot \rangle$ be the Killing form of \mathfrak{g} and let G be the adjoint group of \mathfrak{g} . If \mathfrak{a} is a subalgebra of \mathfrak{g} , we denote by $S(\mathfrak{a})$ the symmetric algebra of \mathfrak{a} . For $x \in \mathfrak{g}$, we denote by \mathfrak{g}^x the centralizer of x in \mathfrak{g} and by G^x the stabilizer of x in G . Then $\text{Lie}(G^x) = \text{Lie}(G_0^x) = \mathfrak{g}^x$ where G_0^x is the identity component of G^x . Moreover, $S(\mathfrak{g}^x)$ is a \mathfrak{g}^x -module and $S(\mathfrak{g}^x)^{\mathfrak{g}^x} = S(\mathfrak{g}^x)^{G_0^x}$.

In [CM16], we continued the investigations of [PPY07] and we studied the question on whether the algebra $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial in ℓ variables; see [Y07, CM10, JS10] for other references related to the topic.

Date: April 6, 2016.

1991 Mathematics Subject Classification. 17B35, 17B20, 13A50, 14L24.

Key words and phrases. symmetric invariant, centralizer, polynomial algebra, Slodowy grading.

1.2. Let us first summarize the main results of [CM16].

Definition 1.1 ([CM16, Definition 1.3]). An element $x \in \mathfrak{g}$ is called a *good element* of \mathfrak{g} if for some homogenous sequence (p_1, \dots, p_ℓ) in $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$, the nullvariety of p_1, \dots, p_ℓ in $(\mathfrak{g}^x)^*$ has codimension ℓ in $(\mathfrak{g}^x)^*$.

Thus an element $x \in \mathfrak{g}$ is good if the nullcone of $S(\mathfrak{g}^x)$, that is, the nullvariety in $(\mathfrak{g}^x)^*$ of the augmentation ideal $S(\mathfrak{g}^x)_+^{\mathfrak{g}^x}$ of $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$, is a complete intersection in $(\mathfrak{g}^x)^*$ since the transcendence degree over \mathbb{k} of the fraction field of $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is ℓ by the main result of [CM10].

For example, regular nilpotent elements are good; see the introduction of [CM16] for more details and other examples.

Theorem 1.2 ([CM16, Theorem 3.3]). *Let x be a good element of \mathfrak{g} . Then $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $S(\mathfrak{g}^x)$ is a free extension of $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$.*

Furthermore, x is good if and only if so is its nilpotent component in the Jordan decomposition [CM16, Proposition 3.5]. As a consequence, we can restrict the study to the case of nilpotent elements.

Let e be a nilpotent element of \mathfrak{g} . By the Jacobson-Morosov Theorem, e is embedded into a \mathfrak{sl}_2 -triple (e, h, f) of \mathfrak{g} . Denote by $S_e := e + \mathfrak{g}^f$ the *Slodowy slice associated with e* . Identify \mathfrak{g}^* with \mathfrak{g} , and $(\mathfrak{g}^e)^*$ with \mathfrak{g}^f , through the Killing form $\langle \cdot, \cdot \rangle$ of \mathfrak{g} . For p in $S(\mathfrak{g}) \simeq \mathbb{k}[\mathfrak{g}^*] \simeq \mathbb{k}[\mathfrak{g}]$, denote by ${}^e p$ the initial homogenous component of its restriction to S_e . According to [PPY07, Proposition 0.1], if p is in $S(\mathfrak{g})^{\mathfrak{g}}$, then ${}^e p$ is in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

Theorem 1.3 ([CM16, Theorem 1.5]). *Suppose that for some homogenous generators q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} . Then e is a good element of \mathfrak{g} . In particular, $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $S(\mathfrak{g}^e)$ is a free extension of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Moreover, ${}^e q_1, \dots, {}^e q_\ell$ is a regular sequence in $S(\mathfrak{g}^e)$.*

In other words, Theorem 1.3 provides a sufficient condition for that $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial. By [PPY07], one knows that for homogenous elements q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent if and only if

$$(1) \quad \sum_{i=1}^{\ell} \deg {}^e q_i = \frac{\dim \mathfrak{g}^e + \ell}{2}.$$

So we have a practical criterion to verify the sufficient condition of Theorem 1.3. However, note that even if the condition of Theorem 1.3 holds, that is, if (1) holds, $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not necessarily generated by the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$. As a matter of fact, there are nilpotent elements e satisfying this condition and for which $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not generated by some ${}^e q_1, \dots, {}^e q_\ell$, for any choice of homogenous generators q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$ (cf. [CM16, Remark 2.25]).

Theorem 1.3 can be applied to a great number of nilpotent orbits in the simple classical Lie algebras, and for some nilpotent orbits in the exceptional Lie algebras, see [CM16, Sections 5 and 6]. We also provided in [CM16, Example 7.8] an example of a nilpotent element e for which $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not polynomial.

1.3. In this note, we prove that the converse of Theorem 1.3 also holds. Namely, we prove the following result.

Theorem 1.4. *The nilpotent element e of \mathfrak{g} is good if and only if for some homogenous generating sequence q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} .*

Theorem 1.4 was conjectured in [CM16, Conjecture 7.11]. Notice that it may happen that for some r_1, \dots, r_ℓ in $S(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^{e} r_1, \dots, {}^{e} r_\ell$ are algebraically independent over \mathbb{k} , and that however e is not good. This is the case for instance for the nilpotent elements in $\mathfrak{so}(\mathbb{k}^{12})$ associated with the partition $(5, 3, 2, 2)$, cf. [CM16, Example 7.6]. In fact, according to [PPY07, Corollary 2.3], for any nilpotent element e of \mathfrak{g} , there exist r_1, \dots, r_ℓ in $S(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^{e} r_1, \dots, {}^{e} r_\ell$ are algebraically independent over \mathbb{k} .

1.4. We introduce in this subsection the main notations of the paper and explain our approach to prove Theorem 1.4.

Let x_1, \dots, x_r be a basis of \mathfrak{g}^e such that for $i = 1, \dots, r$, $[h, x_i] = n_i x_i$ with n_i a nonnegative integer. For $\mathbf{j} = (j_1, \dots, j_r)$ in \mathbb{N}^r , set:

$$|\mathbf{j}| := j_1 + \dots + j_r, \quad |\mathbf{j}|_e := j_1(n_1 + 2) + \dots + j_r(n_r + 2), \quad x^{\mathbf{j}} := x_1^{j_1} \dots x_r^{j_r}.$$

There are two gradations on $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, the standard one and the Slodowy gradation. For all \mathbf{j} in \mathbb{N}^r , $x^{\mathbf{j}}$ has standard degree $|\mathbf{j}|$ and Slodowy degree $|\mathbf{j}|_e$. Denoting by $t \mapsto h(t)$ the one parameter subgroup of G generated by $\text{ad} h$, the Slodowy slice $e + \mathfrak{g}^f$ is invariant under the one parameter subgroup $t \mapsto t^{-2}h(t)$ and for all \mathbf{j} in \mathbb{N}^r and for all y in \mathfrak{g}^f ,

$$x^{\mathbf{j}}(t^{-2}h(t)(e + y)) = t^{-|\mathbf{j}|_e} x^{\mathbf{j}}(e + y).$$

Let t be an indeterminate and let R be the polynomial algebra $\mathbb{k}[t]$. The polynomial algebra

$$S(\mathfrak{g}^e)[t] := \mathbb{k}[t] \otimes_{\mathbb{k}} S(\mathfrak{g}^e)$$

identifies with the algebra of polynomial functions on $\mathfrak{g}^f \times \mathbb{k}$. The gradation of $S(\mathfrak{g}^e)$ induces a gradation of $S(\mathfrak{g}^e)[t]$ such that t has degree 0.

Denote by ε the evaluation map at $t = 0$ so that ε is a graded morphism from $S(\mathfrak{g}^e)[t]$ onto $S(\mathfrak{g}^e)$. For p in $S(\mathfrak{g})$, denote by $\kappa(p)$ the polynomial function $x \mapsto p(e + x)$ on \mathfrak{g}^f so that $\kappa(p) \in S(\mathfrak{g}^e)$. Then ${}^e p$ is the homogeneous component of $\kappa(p)$ of minimal degree with respect to the standard gradation on $S(\mathfrak{g}^e)$. Let τ be the embedding of $S(\mathfrak{g}^e)$ into $S(\mathfrak{g}^e)[t]$ such that $\tau(x_i) := tx_i$ for $i = 1, \dots, r$.

Denote by A the intersection of $S(\mathfrak{g}^e)[t]$ with the sub- $\mathbb{k}[t, t^{-1}]$ -module of

$$S(\mathfrak{g}^e)[t, t^{-1}] := \mathbb{k}[t, t^{-1}] \otimes_{\mathbb{k}} S(\mathfrak{g}^e)$$

generated by $\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})$. Let \mathcal{V} be the nullvariety of A_+ in $\mathfrak{g}^f \times \mathbb{k}$ where A_+ is the augmentation ideal of A . Let \mathcal{V}_* be the union of the irreducible components of \mathcal{V} which are not contained in $\mathfrak{g}^f \times \{0\}$ and let \mathcal{N} be the nullvariety of $\varepsilon(A)_+$ in \mathfrak{g}^f , with $\varepsilon(A)_+$ the augmentation ideal of $\varepsilon(A)$. Then \mathcal{V} is the union of \mathcal{V}_* and $\mathcal{N} \times \{0\}$. Using properties of the varieties \mathcal{V} and \mathcal{V}_* , we prove the following result (see Section 2).

Theorem 1.5. *Suppose that \mathcal{N} has dimension $r - \ell$. Then for some homogeneous generating sequence q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} .*

The key point is to show that, under the hypothesis of Theorem 1.5, $\varepsilon(A)$ is the subalgebra of $S(\mathfrak{g}^e)$ generated by the family ${}^e p$, $p \in S(\mathfrak{g})^{\mathfrak{g}}$, and hence that \mathcal{N} coincides with the nullvariety in \mathfrak{g}^f of ${}^e q_1, \dots, {}^e q_\ell$. So, if \mathcal{N} has dimension $r - \ell$, then the elements ${}^e q_1, \dots, {}^e q_\ell$ must be algebraically independent over \mathbb{k} .

The proof of Theorem 1.5 is done in Section 2. Then Theorem 1.4 will be a consequence of Theorem 1.5 and it will be proven in Section 3. In Appendix A, we state general facts on commutative algebra that are used throughout the paper.

Acknowledgments. The second author is partially supported by the ANR Project GeoLie Grant number ANR-15-CE40-0012.

2. PROOF OF THEOREM 1.5

In this section, unless otherwise specified, the gradation on $S(\mathfrak{g}^e)$ is the Slodowy gradation.

For m a nonnegative integer, $S(\mathfrak{g}^e)^{[m]}$ denotes the space of degree m of $S(\mathfrak{g}^e)$. We retain the notations of the introduction, in particular of Subsection 1.4.

2.1. Let R be the ring $\mathbb{k}[t]$. As in Appendix A, for M a graded subspace of $S(\mathfrak{g}^e)[t] = R \otimes_{\mathbb{k}} S(\mathfrak{g}^e)$, its subspace of degree m is denoted by $M^{[m]}$. In particular, $S(\mathfrak{g}^e)[t]^{[m]}$ is equal to $S(\mathfrak{g}^e)^{[m]}[t]$ and it is a free R -module of finite rank. As a result, for all graded R -submodule M of $S(\mathfrak{g}^e)[t]$, its Hilbert series is well defined.

For m a nonnegative integer, denote by F_m the space of elements of $\kappa(S(\mathfrak{g})^{\mathfrak{g}})$ whose component of minimal standard degree is at least m . Then F_0, F_1, \dots is a decreasing filtration of the algebra $\kappa(S(\mathfrak{g})^{\mathfrak{g}})$. Let d_1, \dots, d_ℓ be the standard degrees of a homogeneous generating sequence of $S(\mathfrak{g})^{\mathfrak{g}}$. We assume that the sequence d_1, \dots, d_ℓ is increasing.

Recall that A is the intersection of $S(\mathfrak{g}^e)[t]$ with the sub- $\mathbb{k}[t, t^{-1}]$ -module of $S(\mathfrak{g}^e)[t, t^{-1}]$ generated by $\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})$, and that A_+ is the augmentation ideal of A .

Lemma 2.1. (i) *For p a homogeneous element of standard degree d in $S(\mathfrak{g})^{\mathfrak{g}}$, the element $\kappa(p)$ and ${}^e p$ have degree $2d$.*

- (ii) For some homogeneous sequence a_1, \dots, a_ℓ in A_+ , the elements t, a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} , and A is the intersection of $S(\mathfrak{g}^e)[t]$ with $\mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell]$.
- (iii) The Hilbert series of the R -algebra A is equal to

$$P_{A,R}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}.$$

- (iv) The Hilbert series of the \mathbb{k} -algebra $\varepsilon(A)$ is equal to

$$P_{\varepsilon(A),\mathbb{k}}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}.$$

- (v) The subalgebra $\varepsilon(A)$ is the graded algebra associated with the filtration F_0, F_1, \dots

Proof. (i) For y in \mathfrak{g}^f and s in \mathbb{k}^* ,

$$p(s^{-2}h(s)(e + y)) = s^{-2d}p(h(s)(e + y)) = s^{-2d}p(e + y)$$

since p is invariant under the one parameter subgroup $s \mapsto h(s)$. Hence $\kappa(p)$ is homogeneous of degree $2d$. Since the monomials $x^{\mathbf{j}}$ are homogeneous, ep has degree $2d$.

(ii) Let q_1, \dots, q_ℓ be a homogeneous generating sequence of $S(\mathfrak{g})^{\mathfrak{g}}$. By a well known fact [CM16, Lemma 4.4,(i)], the morphism

$$G \times (e + \mathfrak{g}^f) \longrightarrow \mathfrak{g}, \quad (g, x) \longmapsto g(x)$$

is dominant. Then $\kappa(S(\mathfrak{g})^{\mathfrak{g}})$ is a polynomial algebra generated by $\kappa(q_1), \dots, \kappa(q_\ell)$. So, setting $a_i := \tau \circ \kappa(q_i)$ for $i = 1, \dots, \ell$, the sequence a_1, \dots, a_ℓ is a homogeneous sequence in A_+ such that

$$\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}] = \mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell].$$

Let $\bar{\tau}$ be the automorphism of $S(\mathfrak{g}^e)[t, t^{-1}]$ extending τ and such that $\bar{\tau}(t) = t$. Then

$$\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}] = \bar{\tau}(\kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]).$$

Since $\kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]$ has dimension $\ell + 1$, $\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]$ has dimension $\ell + 1$ too, and t, a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} . By definition, $A = S(\mathfrak{g}^e)[t] \cap \tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]$. Hence

$$A[t^{-1}] = \mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell] \quad \text{and} \quad A = S(\mathfrak{g}^e)[t] \cap \mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell].$$

(iii) Since t has degree 0, the gradation of $S(\mathfrak{g}^e)[t]$ extends to a gradation of $S(\mathfrak{g}^e)[t, t^{-1}]$ such that for all m , its space of degree m is equal to $S(\mathfrak{g}^e)^{[m]}[t, t^{-1}]$. Then for all $\mathbb{k}[t, t^{-1}]$ -submodule M of $S(\mathfrak{g}^e)[t, t^{-1}]$, M has a Hilbert series:

$$P_{M, \mathbb{k}[t, t^{-1}]}(T) := \sum_{m \in \mathbb{N}} \text{rk } M^{[m]} T^m$$

with $M^{[m]}$ the subspace of degree m of M . From the equality $A[t^{-1}] = \mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell]$, we deduce

$$P_{A[t^{-1}], \mathbb{k}[t, t^{-1}]}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}$$

since for $i = 1, \dots, \ell$, the element a_i has degree $2d_i$ by (i). For all m , the rank of the R -module $A^{[m]}$ is equal to the rank of the $\mathbb{k}[t, t^{-1}]$ -module $A[t^{-1}]^{[m]}$, whence

$$P_{A, R}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}.$$

(iv) Let m be a nonnegative integer. The R -module $A^{[m]}$ is free of finite rank and for (v_1, \dots, v_n) a basis of this module, (tv_1, \dots, tv_n) is a basis of the R -module $tA^{[m]}$. Since $\varepsilon(A)^{[m]}$ is the quotient of $A^{[m]}$ by $tA^{[m]}$,

$$\dim \varepsilon(A)^{[m]} = n = \text{rk } A^{[m]},$$

whence the assertion by (iii).

(v) Let $\text{gr}_F A$ be the graded algebra associated with the filtration F_0, F_1, \dots of $\kappa(S(\mathfrak{g})^{\mathfrak{g}})$. Denote by $a \mapsto a(1)$ the evaluation map at $t = 1$ from $S(\mathfrak{g}^e)[t]$ to $S(\mathfrak{g}^e)$. For a in A such that $\varepsilon(a) \neq 0$, $a(1)$ is in $\kappa(S(\mathfrak{g})^{\mathfrak{g}})$ and $\varepsilon(a)$ is the component of minimal degree of $a(1)$ with respect to the standard gradation, whence $\varepsilon(A) \subset \text{gr}_F A$. Conversely, let \bar{a} be a homogeneous element of degree m of $\text{gr}_F A$ and let a be a representative of \bar{a} in F_m . Then $\tau(a) = t^m b$ with b in A such that $\varepsilon(b) = \bar{a}$, whence $\text{gr}_F A \subset \varepsilon(A)$ and the assertion. \square

Let R_* be the localization of R at the prime ideal tR and set

$$\widehat{R} := \mathbb{k}[[t]], \quad A_* := R_* \otimes_R A, \quad \widehat{A} := \widehat{R} \otimes_R A.$$

The gradation of A extends to gradations on A_* and \widehat{A} such that $A_*^{[0]} = R_*$ and $\widehat{A}^{[0]} = \widehat{R}$.

Proposition 2.2. (i) *The algebra $\varepsilon(A)$ is polynomial if and only if for some standard homogeneous generating sequence q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} . Moreover, in this case, A is a polynomial algebra.*

(ii) *If A_* is a polynomial algebra, then for some homogeneous sequence p_1, \dots, p_ℓ in A_+ , we have $A_* = R_*[p_1, \dots, p_\ell]$, the elements t, p_1, \dots, p_ℓ are algebraically independent over \mathbb{k} and p_1, \dots, p_ℓ have degree $2d_1, \dots, 2d_\ell$ respectively.*

Proof. (i) Let q_1, \dots, q_ℓ be a homogeneous generating sequence of $S(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} . We can assume that for $i = 1, \dots, \ell$, q_i has standard degree d_i . For $i = 1, \dots, \ell$, ${}^e q_i$ has degree $2d_i$ by Lemma 2.1(i), and we set

$$Q_i := t^{-2d_i} \tau \circ \kappa(q_i).$$

For $\mathbf{i} = (i_1, \dots, i_\ell)$ in \mathbb{N}^ℓ , set:

$$q^{\mathbf{i}} := q_1^{i_1} \cdots q_\ell^{i_\ell}, \quad Q^{\mathbf{i}} := Q_1^{i_1} \cdots Q_\ell^{i_\ell}, \quad {}^e q^{\mathbf{i}} := {}^e q_1^{i_1} \cdots {}^e q_\ell^{i_\ell},$$

$$|\mathbf{i}|_{\min} := 2i_1 d_1 + \cdots + 2i_\ell d_\ell.$$

Then, for all \mathbf{i} in \mathbb{N}^ℓ ,

$$\tau \circ \kappa(q^{\mathbf{i}}) = t^{|\mathbf{i}|_{\min}} Q^{\mathbf{i}}.$$

Moreover,

$$\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}] = \mathbb{k}[t, t^{-1}, Q_1, \dots, Q_\ell].$$

Let a be in A . For some l in \mathbb{N} and for some sequence $c_{\mathbf{i}, m}$, $(\mathbf{i}, m) \in \mathbb{N}^\ell \times \mathbb{N}$ in \mathbb{k} , of finite support,

$$t^l a = \sum_{(\mathbf{i}, m) \in \mathbb{N}^\ell \times \mathbb{N}} c_{\mathbf{i}, m} t^m Q^{\mathbf{i}} \quad \text{whence} \quad \sum_{\mathbf{i} \in \mathbb{N}^\ell} c_{\mathbf{i}, m} {}^e q^{\mathbf{i}} = 0$$

for $m < l$. Hence a is in $R[Q_1, \dots, Q_\ell]$ since the sequence ${}^e q^{\mathbf{i}}$, $\mathbf{i} \in \mathbb{N}^\ell$, is linearly free over \mathbb{k} . As a result,

$$A = R[Q_1, \dots, Q_\ell] \quad \text{and} \quad \varepsilon(A) = \mathbb{k}[{}^e q_1, \dots, {}^e q_\ell]$$

so that A and $\varepsilon(A)$ are polynomial algebras over \mathbb{k} since ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} .

Conversely, suppose that $\varepsilon(A)$ is a polynomial algebra. By Lemma 2.1, (i) and (iv), the algebra $\varepsilon(A)$ is Slodowy graded and standard graded. Let d be the dimension of $\varepsilon(A)$. As $\varepsilon(A)$ is a polynomial algebra, it is regular so that the \mathbb{k} -space $\varepsilon(A)_+ / \varepsilon(A)_+^2$ has dimension d . Moreover, the two gradations on $\varepsilon(A)$ induce gradations on $\varepsilon(A)_+ / \varepsilon(A)_+^2$. Hence $\varepsilon(A)_+ / \varepsilon(A)_+^2$ has a bihomogeneous basis. Then some bihomogeneous sequence u_1, \dots, u_d in $\varepsilon(A)_+$ represents a basis of $\varepsilon(A)_+ / \varepsilon(A)_+^2$. As a result, the \mathbb{k} -algebra $\varepsilon(A)$ is generated by the bihomogeneous sequence u_1, \dots, u_d . For $i = 1, \dots, d$, denote by δ_i the Slodowy degree of u_i . As ε is homogeneous with respect to the Slodowy gradation, $u_i = \varepsilon(r_i)$ for some homogeneous element r_i of degree δ_i of A . Let m_i be the smallest nonnegative integer such that $t^{m_i} r_i$ is in $\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})$. According to Lemma 2.1, (i), δ_i is even and for some standard homogeneous element p_i of standard degree $\delta_i/2$ of $S(\mathfrak{g})^{\mathfrak{g}}$, $t^{m_i} r_i = \tau \circ \kappa(p_i)$. Then $u_i = {}^e p_i$ since p_i is standard homogeneous.

Let \mathfrak{P} be the subalgebra of $S(\mathfrak{g})$ generated by p_1, \dots, p_d . Suppose that \mathfrak{P} is strictly contained in $S(\mathfrak{g})^{\mathfrak{g}}$. A contradiction is expected. For some positive integer m , the space $S(\mathfrak{g})_m^{\mathfrak{g}}$ of standard degree m of $S(\mathfrak{g})^{\mathfrak{g}}$ is not contained in \mathfrak{P} . Let q be in $(S(\mathfrak{g})^{\mathfrak{g}})_m \setminus \mathfrak{P}$ such that ${}^e q$ has maximal standard degree. By Lemma 2.1, (i), ${}^e q$ is a polynomial in u_1, \dots, u_d , of degree $2m$. So, for some polynomial q' of degree m in \mathfrak{P} , ${}^e(q - q')$ has standard degree bigger than the standard degree of ${}^e q$. So, by maximality of the standard degree of ${}^e q$, the elements $q - q'$ and q are in \mathfrak{P} , whence the contradiction. As a result, $\mathfrak{P} = S(\mathfrak{g})^{\mathfrak{g}}$ and $d = \ell$.

(ii) Suppose that A_* is a polynomial algebra. Denoting by J the ideal of A_* generated by t and A_+ , the \mathbb{k} -space J/J^2 is a graded space of dimension ℓ since A_* is a regular algebra of dimension $\ell + 1$. Then for some homogeneous sequence p_1, \dots, p_ℓ in A_+ , (t, p_1, \dots, p_ℓ) is a basis of J modulo J^2 . Since p_1, \dots, p_ℓ have positive degree, we prove by induction on d that

$$A_*^{[d]} \subset R_*[p_1, \dots, p_\ell]^{[d]} + tA_*^{[d]}.$$

Then by induction on m , we get

$$A_*^{[d]} \subset R_*[p_1, \dots, p_\ell] + t^m A_*^{[d]}.$$

So, since the R_* -module $A_*^{[d]}$ is finitely generated,

$$A_*^{[d]} \subset \widehat{R}[p_1, \dots, p_\ell]^{[d]}$$

whence $\widehat{A} = \widehat{R}[p_1, \dots, p_\ell]$ and $A_* = R_*[p_1, \dots, p_\ell]$ by Lemma A.3.

Denote by $\delta_1, \dots, \delta_\ell$ the respective degrees of p_1, \dots, p_ℓ . We can suppose that p_1, \dots, p_ℓ is ordered so that $\delta_1 \leq \dots \leq \delta_\ell$. Prove by induction on i that $\delta_j = 2d_j$ for $j = 1, \dots, i$. By Lemma 2.1, (iii), $2d_1$ is the smallest positive degree of the elements of A . Moreover, δ_1 is the smallest positive degree of the elements of $R[p_1, \dots, p_\ell]$, whence $\delta_1 = 2d_1$. Suppose $\delta_j = 2d_j$ for $j = 1, \dots, i-1$. Set $A_i := R[p_i, \dots, p_\ell]$. Then, by induction hypothesis and Lemma 2.1, (iii),

$$P_{A_i, R}(T) = \prod_{j=i}^{\ell} \frac{1}{1 - T^{\delta_j}} = \prod_{j=i}^{\ell} \frac{1}{1 - T^{2d_j}}.$$

By the first equality, δ_i is the smallest positive degree of the elements of A_i and by the second equality, $2d_i$ is the smallest positive degree of the elements of A_i too, whence $\delta_i = 2d_i$. Then with $i = \ell$, we get that $\delta_j = 2d_j$ for $j = 1, \dots, \ell$. \square

Recall that $\widehat{R} = \mathbb{k}[[t]]$.

Corollary 2.3. *Suppose that A_* is a polynomial algebra. Then for some standard homogeneous generating sequence q_1, \dots, q_ℓ in $S(\mathfrak{g})^{\mathfrak{g}}$,*

$$A_* = R_*[t^{-2d_1} \tau \circ \kappa(q_1), \dots, t^{-2d_\ell} \tau \circ \kappa(q_\ell)].$$

Proof. For m nonnegative integer, denote by $S(\mathfrak{g})_m^{\mathfrak{g}}$ the space of standard degree m of $S(\mathfrak{g})^{\mathfrak{g}}$. By Proposition 2.2, (ii), for some homogeneous sequence p_1, \dots, p_ℓ in A_+ such that p_1, \dots, p_ℓ have degree $2d_1, \dots, 2d_\ell$ respectively,

$$A_* = R_*[p_1, \dots, p_\ell].$$

For $i = 1, \dots, \ell$, let m_i be the smallest integer such that $t^{m_i} p_i$ is in $\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})$. By Lemma 2.1, (i), $t^{m_i} p_i$ has an expansion

$$t^{m_i} p_i = \sum_{j \in \mathbb{N}} t^j \tau \circ \kappa(q_{i,j})$$

with $q_{i,j}$, $j \in \mathbb{N}$, in $S(\mathfrak{g})_{d_i}^{\mathfrak{g}}$ of finite support. Denoting by $\delta_{i,j}$ the standard degree of ${}^e q_{i,j}$, set:

$$J'_i := \{j \in \mathbb{N} ; m_i = j + \delta_{i,j}\}, \quad \delta_i := \inf\{\delta_{i,j} ; j \in J'_i\},$$

$$j_i := m_i - 2d_i, \quad Q_i := t^{-2d_i} \tau \circ \kappa(q_{i,j_i}).$$

For $i = 1, \dots, \ell$, since p_i is not divisible by t in A ,

$$p_i - Q_i \in tA,$$

whence

$$A_* \subset R[Q_1, \dots, Q_\ell] + tA_*.$$

Then, by induction m ,

$$A_* \subset R_*[Q_1, \dots, Q_m] + t^m A_*$$

for all m . As a result,

$$\widehat{A} = \widehat{R}[Q_1, \dots, Q_\ell],$$

since for all d , the R_* -module $A_*^{[d]}$ is finitely generated. Then, by Lemma A.3,

$$A_* = R_*[Q_1, \dots, Q_\ell].$$

As a result, since A has dimension $\ell + 1$, the elements t, Q_1, \dots, Q_ℓ are algebraically independent over \mathbb{k} and so are $q_{1,j_1}, \dots, q_{\ell,j_\ell}$. Moreover the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ is generated by $q_{1,j_1}, \dots, q_{\ell,j_\ell}$ since they have degree d_1, \dots, d_ℓ respectively. \square

2.2. Denote by \mathcal{V} the nullvariety of A_+ in $\mathfrak{g}^f \times \mathbb{k}$. Let \mathcal{V}_* be the union of the irreducible components of \mathcal{V} which are not contained in $\mathfrak{g}^f \times \{0\}$. The following result is proven in [CM16, Corollary 4.4,(i)]. Indeed, the proof of this result does not use the assumption of [CM16, Section 4] that for some homogeneous generators q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent.

Lemma 2.4 ([CM16, Corollary 4.4,(i)]). (i) *The variety \mathcal{V}_* is equidimensional of dimension $r + 1 - \ell$.*

(ii) *For all irreducible component X of \mathcal{V}_* and for all z in \mathbb{k} , X is not contained in $\mathfrak{g}^f \times \{z\}$.*

Let \mathcal{N} be the nullvariety of $\varepsilon(A)_+$ in \mathfrak{g}^f . Then \mathcal{V} is the union of \mathcal{V}_* and $\mathcal{N} \times \{0\}$.

Lemma 2.5. (i) *All irreducible component of \mathcal{N} has dimension at least $r - \ell$ and all irreducible component of \mathcal{V} has dimension at least $r + 1 - \ell$.*

(ii) *Assume that \mathcal{N} has dimension $r - \ell$. Then for some homogeneous sequence $p_1, \dots, p_{r-\ell}$ in $S(\mathfrak{g}^e)_+$, the nullvariety of $t, p_1, \dots, p_{r-\ell}$ in \mathcal{V} is equal to $\{0\}$.*

Proof. (i) By Lemma 2.1(ii), for some homogeneous sequence a_1, \dots, a_ℓ in A_+ , the elements t, a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} . Let b_1, \dots, b_m be a homogeneous sequence in A_+ , generating the ideal $S(g^e)[t]A_+$ of $S(g^e)[t]$. Set:

$$B := \mathbb{k}[a_1, \dots, a_\ell, b_1, \dots, b_m], \quad B_+ := Ba_1 + \dots + Ba_\ell + Bb_1 + \dots + Bb_m,$$

$$C := B[t], \quad C_{++} := B_+[t] + Ct.$$

Then B and C are graded subalgebras of A and B_+ and C_{++} are maximal ideals of B and C respectively. Moreover, C has dimension $\ell + 1$. We have a commutative diagram

$$\begin{array}{ccc} & g^f \times \mathbb{k} & \\ \alpha \swarrow & & \searrow \beta \\ \text{Specm}(C) & \xrightarrow{\pi} & \text{Specm}(B) \end{array}$$

with α, β, π the morphisms whose comorphisms are the canonical injections

$$C \longrightarrow S(g^e)[t], \quad B \longrightarrow S(g^e)[t], \quad B \longrightarrow C$$

respectively. Since C has dimension $\ell + 1$, the irreducible components of the fibers of α have dimension at least $r - \ell$, whence the result for \mathcal{N} since $\mathcal{N} \times \{0\} = \alpha^{-1}(C_{++})$. Moreover, $\mathcal{V} = \beta^{-1}(B_+)$ and $\pi^{-1}(B_+)$ is a subvariety of dimension 1 of $\text{Specm}(C)$. Hence all irreducible component of \mathcal{V} has dimension at least $r + 1 - \ell$.

(ii) Prove by induction on i that there exists a homogeneous sequence p_1, \dots, p_i in $S(g^e)_+$ such that the minimal prime ideals of $S(g^e)$ containing $\varepsilon(A)_+$ and p_1, \dots, p_i have height $\ell + i$. First of all, $S(g^e)\varepsilon(A)_+$ is graded. Then the minimal prime ideals of $S(g^e)$ containing $\varepsilon(A)_+$ are graded too. By (i), $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ have height ℓ since \mathcal{N} has dimension $r - \ell$ by hypothesis. In particular, they are strictly contained in $S(g^e)_+$. Hence, by Lemma A.1(ii), for some homogeneous element p_1 in $S(g^e)$, p_1 is not in the union of these ideals so that the statement is true for $i = 1$ by [Ma86, Ch. 5, Theorem 13.5]. Suppose that it is true for $i - 1$. Then the minimal prime ideals containing $\varepsilon(A)_+$ and p_1, \dots, p_{i-1} are graded and strictly contained in $S(g^e)_+$ by the induction hypothesis. So, by Lemma A.1(ii), for some homogeneous element p_i in $S(g^e)$, p_i is not in the union of these ideals and the sequence p_1, \dots, p_i verify the condition of the statement by [Ma86, Ch. 5, Theorem 13.5]. For $i = r - \ell$, the nullvariety of $p_1, \dots, p_{r-\ell}$ in \mathcal{N} has dimension 0. Then it is equal to $\{0\}$ as the nullvariety of a graded ideal, whence the assertion since $\mathcal{N} \times \{0\}$ is the nullvariety of t in \mathcal{V} . \square

2.3. We assume in this subsection that \mathcal{N} has dimension $r - \ell$. Let $p_1, \dots, p_{r-\ell}$ be as in Lemma 2.5(ii), and set

$$C := A[p_1, \dots, p_{r-\ell}].$$

Then $p_1, \dots, p_{r-\ell}$ are algebraically independent over A since \mathcal{N} has dimension $r - \ell$.

Lemma 2.6. *The ideal $S(\mathfrak{g}^e)[t]_+$ of $S(\mathfrak{g}^e)[t]$ is the radical of $S(\mathfrak{g}^e)[t]C_+$.*

Proof. Let Y be an irreducible component of the nullvariety of C_+ in $\mathfrak{g}^f \times \mathbb{k}$. Then Y has dimension at least 1. By definition the nullvariety of t in Y is equal to $\{0\}$. Hence Y has dimension 1. The gradation on $S(\mathfrak{g}^e)[t]$ induces an action of the one-dimensional multiplicative group G_m on $\mathfrak{g}^f \times \mathbb{k}$ such that for all (x, z) in $\mathfrak{g}^f \times \mathbb{k}$, $(0, z)$ is in the closure of the orbit of (x, z) under G_m . Since C_+ is graded, Y is invariant under G_m . As a result, $Y = \{0\} \times \mathbb{k}$ or for some x in $\mathfrak{g}^f \times \mathbb{k}$, Y is the closure of the orbit of $(x, 0)$ under G_m since 0 is the nullvariety of t in Y . In the last case, x is a zero of $p_1, \dots, p_{r-\ell}$ in \mathcal{N} , that is $x = 0$. Hence $Y = \{0\} \times \mathbb{k}$. As a result, the nullvariety of C_+ in $\mathfrak{g}^f \times \mathbb{k}$ is equal to $\{0\} \times \mathbb{k}$ that is the nullvariety of $S(\mathfrak{g}^e)[t]_+$, whence the assertion since $S(\mathfrak{g}^e)[t]_+$ is a prime ideal of $S(\mathfrak{g}^e)[t]$. \square

For \mathfrak{p} a prime ideal of A , denote by $A_{\mathfrak{p}}$ the localization of A at \mathfrak{p} and by $\bar{\mathfrak{p}}$ the ideal of C generated by \mathfrak{p} . Since C is a polynomial algebra over A , $\bar{\mathfrak{p}}$ is a prime ideal of C and the localization $C_{\bar{\mathfrak{p}}}$ of C at $\bar{\mathfrak{p}}$ is a localization of the polynomial algebra $A_{\mathfrak{p}}[p_1, \dots, p_{r-\ell}]$. According to [Ma86, Ch. 6, Theorem 17.4], if $C_{\bar{\mathfrak{p}}}$ is Cohen-Macaulay so is $A_{\mathfrak{p}}$ as a quotient of $C_{\bar{\mathfrak{p}}}$ by an ideal generated by a regular sequence.

Proof of Theorem 1.5. By Lemma 2.6 and Proposition A.2, the algebra C is finitely generated. Then A is finitely generated as a quotient of C . So, by Lemma A.7, (ii), A is a factorial ring and so is C as a polynomial ring over A . As a result, C is normal so that $S(\mathfrak{g}^e)[t]$ and C verify Conditions (1), (2), (3) of Proposition A.6. Hence by Proposition A.6, for all prime ideal \mathfrak{p} of A , containing t , $C_{\bar{\mathfrak{p}}}$ is Cohen-Macaulay, whence $A_{\mathfrak{p}}$ is Cohen-Macaulay. By Lemma 2.1, (ii), for \mathfrak{p} a prime ideal of A , not containing t , $A_{\mathfrak{p}}$ is the localization of $\mathbb{k}[t, t^{-1}, a_1, \dots, a_{\ell}]$ at the prime ideal generated by \mathfrak{p} . Hence $A_{\mathfrak{p}}$ is Cohen-Macaulay since the algebra $\mathbb{k}[t, t^{-1}, a_1, \dots, a_{\ell}]$ is regular. As a result A is Cohen-Macaulay. In particular, A verifies Conditions (1), (2), (3) of Subsection A.2. So, by Proposition A.10, A_* is a polynomial algebra. Then by Corollary 2.3, for some homogeneous generating sequence q_1, \dots, q_{ℓ} in $S(\mathfrak{g})^{\mathfrak{g}}$,

$$A_* = R_*[t^{-2d_1}\tau \circ \kappa(q_1), \dots, t^{-2d_{\ell}}\tau \circ \kappa(q_{\ell})].$$

Form the above equality, we deduce that any element of A is the product of an element of the algebra $R[t^{-2d_1}\tau \circ \kappa(q_1), \dots, t^{-2d_{\ell}}\tau \circ \kappa(q_{\ell})]$ by a polynomial in t with nonzero constant term, whence

$$A = R[t^{-2d_1}\tau \circ \kappa(q_1), \dots, t^{-2d_{\ell}}\tau \circ \kappa(q_{\ell})] \quad \text{and so} \quad \varepsilon(A) = \mathbb{k}[{}^e q_1, \dots, {}^e q_{\ell}]$$

since for $i = 1, \dots, \ell$,

$${}^e q_i := \varepsilon(t^{-2d_i}\tau \circ \kappa(q_i)).$$

Since $\mathcal{N} \times \{0\}$ is the nullvariety of t and A_+ in $\mathfrak{g}^f \times \mathbb{k}$, \mathcal{N} is the nullvariety in \mathfrak{g}^f of ${}^e q_1, \dots, {}^e q_\ell$. Hence ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} since \mathcal{N} has dimension $r - \ell$. \square

3. PROOF OF THEOREM 1.4

Let (e, h, f) be an \mathfrak{sl}_2 -triple in \mathfrak{g} . We use the notations κ and ${}^e p, p \in S(\mathfrak{g})^{\mathfrak{g}}$, as in Introduction. In this section, we use the standard gradations on $S(\mathfrak{g})$ and $S(\mathfrak{g}^e)$. Let A_0 be the subalgebra of $S(\mathfrak{g}^e)$ generated by the family ${}^e p, p \in S(\mathfrak{g})^{\mathfrak{g}}$, and let \mathcal{N}_0 be the nullvariety of $A_{0,+}$ in \mathfrak{g}^f where $A_{0,+}$ denotes the augmentation ideal of A_0 .

Let a_1, \dots, a_m be a homogeneous sequence in $A_{0,+}$ generating the ideal of $S(\mathfrak{g}^e)$ generated by $A_{0,+}$. According to [PPY07, Corollary 2.3], A_0 contains a homogeneous sequence b_1, \dots, b_ℓ algebraically free over \mathbb{k} .

Lemma 3.1. *Let \mathfrak{A} be the integral closure of $\mathbb{k}[a_1, \dots, a_m, b_1, \dots, b_\ell]$ in the fraction field of $S(\mathfrak{g}^e)$.*

- (i) *The algebra \mathfrak{A} is contained in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ and its fraction field is the fraction field of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$.*
- (ii) *Let a in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}_+$. If a is equal to 0 on \mathcal{N}_0 , then a is in \mathfrak{A}_+ .*
- (iii) *The ideal $S(\mathfrak{g}^e)^{\mathfrak{g}^e} \mathfrak{A}_+$ of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is equal to \mathfrak{A}_+ .*
- (iv) *The algebra \mathfrak{A} is the integral closure of A_0 in the fraction field of $S(\mathfrak{g}^e)$.*

Proof. (i) Let K_0 be the field of invariant elements under the adjoint action of \mathfrak{g}^e in the fraction field of $S(\mathfrak{g}^e)$. According to [CM16, Lemma 3.1], K_0 is the fraction field of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Since $a_1, \dots, a_m, b_1, \dots, b_\ell$ are in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, \mathfrak{A} is contained in K_0 . Moreover, \mathfrak{A} is contained in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ since $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is integrally closed in K_0 . Since K_0 has transcendence degree ℓ over \mathbb{k} and since b_1, \dots, b_ℓ are algebraically independent over \mathbb{k} , K_0 is the fraction field of \mathfrak{A} .

(ii) Since \mathcal{N}_0 is the nullvariety of $a_1, \dots, a_m, b_1, \dots, b_\ell$ in \mathfrak{g}^f , \mathcal{N}_0 is the nullvariety of \mathfrak{A}_+ in \mathfrak{g}^f . Let a be in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}_+$ such that a is equal to 0 on \mathcal{N}_0 . Since \mathcal{N}_0 is a cone, all homogeneous components of a is equal to 0 on \mathcal{N}_0 . So it suffices to prove the assertion for a homogeneous. We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{g}^f & \xrightarrow{\pi} & \mathrm{Specm}(\mathfrak{A}[a]) \\ & \searrow \alpha \quad \swarrow \beta & \\ & \mathrm{Specm}(\mathfrak{A}) & \end{array}$$

with π, α, β the comorphisms of the canonical injections

$$\mathfrak{A}[a] \longrightarrow S(\mathfrak{g}^e), \quad \mathfrak{A} \longrightarrow S(\mathfrak{g}^e), \quad \mathfrak{A} \longrightarrow \mathfrak{A}[a].$$

Since \mathcal{N}_0 is the nullvariety of $\mathfrak{A}[a]_+$ and \mathfrak{A}_+ in \mathfrak{g}^f , $\beta^{-1}(\mathfrak{A}_+) = \mathfrak{A}[a]_+$. The gradations of \mathfrak{A} and $\mathfrak{A}[a]$ induce actions of G_m on $\mathrm{Specm}(\mathfrak{A})$ and $\mathrm{Specm}(\mathfrak{A}[a])$ such that β is equivariant. Moreover, \mathfrak{A}_+ is in the closure of all orbit under G_m in $\mathrm{Specm}(\mathfrak{A})$.

Hence β is a quasi finite morphism. Moreover, β is a birational since \mathfrak{A} and $\mathfrak{A}[a]$ have the same fraction field by (i). Hence, by Zariski's main theorem [Mu88], β is an open immersion from $\text{Specm}(\mathfrak{A}[a])$ into $\text{Specm}(\mathfrak{A})$. So, β is surjective since \mathfrak{A}_+ is in the image of β and since it is in the closure of all G_m -orbit in $\text{Specm}(\mathfrak{A})$. As a result, β is an isomorphism and a is in \mathfrak{A} , whence the assertion.

(iii) For a in \mathfrak{A}_+ and b in $S(g^e)$, ab is equal to 0 on \mathcal{N}_0 . Hence ab is in \mathfrak{A}_+ by (ii).

(iv) By (ii), A_0 is contained in \mathfrak{A} . Moreover, since $a_1, \dots, a_m, b_1, \dots, b_\ell$ are in A_0 , \mathfrak{A} is contained in the integral closure of A_0 in the fraction field of $S(g^e)$, whence the assertion. \square

Corollary 3.2. *Suppose that the algebra $S(g^e)^{g^e}$ is finitely generated. Then \mathfrak{A} is equal to $S(g^e)^{g^e}$.*

Proof. Let C be the quotient of $S(g^e)^{g^e}$ by the ideal $S(g^e)^{g^e}\mathfrak{A}_+$. By hypothesis, C is finitely generated. Then it has finitely many minimal prime ideals. Denote them by $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. For a in the radical of $S(g^e)^{g^e}\mathfrak{A}_+$, a is equal to 0 on \mathcal{N}_0 . Moreover, it is in $S(g^e)_+^{g^e}$. Then, by Lemma 3.1, (ii), a is in \mathfrak{A}_+ . As a result, C is a reduced algebra and the canonical map

$$A_0 \longrightarrow A_0/\mathfrak{p}_1 \times \dots \times A_0/\mathfrak{p}_m$$

is injective. Since \mathfrak{A} and $S(g^e)^{g^e}$ have the same fraction field, they have the same Krull dimension. Then, by Lemma 3.1, (iii), $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ are maximal ideals of C . As a result, the \mathbb{k} -algebra C is finite dimensional. Let V be a graded complement to $S(g^e)^{g^e}\mathfrak{A}_+$ in $S(g^e)$. According to Lemma 3.1, (iii), $S(g^e)^{g^e} = V\mathfrak{A}$ so that $S(g^e)^{g^e}$ is a finite extension of \mathfrak{A} . Since \mathfrak{A} is integrally closed in the fraction field of $S(g^e)^{g^e}$, $\mathfrak{A} = S(g^e)^{g^e}$. \square

Proof of Theorem 1.4. The “if” part results from [CM16, Theorem 1.5] (see here Theorem 1.3).

Suppose that e is good. By Definition 1.1 and Theorem 1.2, $S(g^e)^{g^e}$ is a polynomial algebra and the nullvariety of $S(g^e)_+^{g^e}$ in g^f is equidimensional of dimension $r - \ell$. On the other hand, by Lemma 3.1, (iv), \mathfrak{A} is the integral closure of A_0 in the fraction field of $S(g^e)$. Hence the nullvarieties of \mathfrak{A}_+ and $A_{0,+}$ in g^f are the same. But by Corollary 3.2, $\mathfrak{A} = S(g^e)^{g^e}$, so \mathcal{N}_0 has dimension $r - \ell$ since e is good. On the other hand, A_0 is contained in $\varepsilon(A)$ by construction of $\varepsilon(A)$, and $\varepsilon(A)$ is contained in $S(g^e)^{g^e}$ by [PPY07, Proposition 0.1], whence $\mathcal{N} = \mathcal{N}_0$.

As a result, \mathcal{N} has dimension $r - \ell$ and so by Theorem 1.5, for some homogeneous generating sequence q_1, \dots, q_ℓ of $S(g)^g$, the element ${}^eq_1, \dots, {}^eq_\ell$ are algebraically independent over \mathbb{k} . \square

APPENDIX A. GENERAL FACTS ON COMMUTATIVE ALGEBRA

In this section t is an indeterminate and the base ring R is \mathbb{k} , $\mathbb{k}[t]$ or $\mathbb{k}[[t]]$. For M a graded space over \mathbb{N} and for j in \mathbb{N} , denote by $M^{[j]}$ the space of degree j and

by M_+ the sum of $M^{[j]}$, $j > 0$. Let A be a finitely generated graded R -algebra over \mathbb{N} such that $A^{[0]} = R$ and such that $A^{[j]}$ is a free R -module of finite rank for any $j \in \mathbb{N}$. Moreover, A is an integral domain. Denoting by $\dim A$ the Krull dimension of A , we set:

$$\ell := \begin{cases} \dim A & \text{if } R = \mathbb{k} \\ \dim A - 1 & \text{if } t \in R \end{cases}$$

As a rule, for B an integral algebra, we denote by $K(B)$ its fraction field.

A.1. Let B be a graded subalgebra of A .

- Lemma A.1.** (i) *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be pairwise different graded prime ideals contained in A_+ . If they are the minimal prime ideals containing their intersection, then for some homogeneous element p of A_+ , the element p is not in the union of $\mathfrak{p}_1, \dots, \mathfrak{p}_m$.*
- (ii) *For some homogeneous sequence p_1, \dots, p_ℓ in A_+ , A_+ is the radical of the ideal generated by p_1, \dots, p_ℓ .*
- (iii) *Suppose that A_+ is the radical of AB_+ . Then for some homogeneous sequence p_1, \dots, p_ℓ in B_+ , A_+ is the radical of the ideal generated by p_1, \dots, p_ℓ .*

Proof. (i) Prove by induction on j that for some homogeneous element p_j of A_+ , p_j is not in the union of $\mathfrak{p}_1, \dots, \mathfrak{p}_j$. Since \mathfrak{p}_1 is a graded ideal strictly contained in A_+ , it is true for $j = 1$. Suppose that it is true for $j - 1$. If p_{j-1} is not in \mathfrak{p}_j , there is nothing to prove. Suppose that p_{j-1} is in \mathfrak{p}_j . According to the hypothesis, \mathfrak{p}_j is strictly contained in A_+ and it does not contain the intersection of $\mathfrak{p}_1, \dots, \mathfrak{p}_{j-1}$. So, since $\mathfrak{p}_1, \dots, \mathfrak{p}_j$ are graded ideals, for some homogeneous sequence r, q in A_+ ,

$$r \in \bigcap_{k=1}^{j-1} \mathfrak{p}_k \setminus \mathfrak{p}_j, \quad \text{and} \quad q \in A_+ \setminus \mathfrak{p}_j.$$

Denoting by m and n the respective degrees of p_{j-1} and rq , $p_{j-1}^n + (rq)^m$ is homogeneous of degree mn and it is not in $\mathfrak{p}_1, \dots, \mathfrak{p}_j$ since these ideals are prime.

(ii) Prove by induction on i that for some homogeneous sequence p_1, \dots, p_i in A_+ , the minimal prime ideals of A containing p_1, \dots, p_i have height i . Let p_1 be in $A_+ \setminus \{0\}$. By [Ma86, Ch. 5, Theorem 13.5], all minimal prime ideal containing p_1 has height 1. Suppose that it is true for $i - 1$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals containing p_1, \dots, p_{i-1} . Since A_+ has height $\ell > i - 1$, A_+ strictly contains $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. By (i), there exists a homogeneous element p_i in A_+ not in the union of $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. Then, by [Ma86, Ch. 5, Theorem 13.5], the minimal prime ideals containing p_1, \dots, p_i have height i . For $i = \ell$, the minimal prime ideals containing p_1, \dots, p_ℓ have height ℓ . Hence they are equal to A_+ since A_+ is a prime ideal of height ℓ containing p_1, \dots, p_ℓ , whence the assertion.

(iii) The ideal AB_+ is generated by a homogeneous sequence a_1, \dots, a_m in B_+ . Denote by B' the subalgebra of A generated by a_1, \dots, a_m . Then B' is a finitely

generated graded subalgebra of A such that A_+ is the radical of AB'_+ . If $R = \mathbb{k}$, denote by d its dimension and if $t \in R$, denote by $d + 1$ its dimension. By (ii), for some homogeneous sequence p_1, \dots, p_d in B'_+ , B'_+ is the radical of the ideal generated by p_1, \dots, p_d . Then A_+ is the radical of the ideal of A generated by p_1, \dots, p_d . Since A_+ has height ℓ , $\ell \leq d$ by [Ma86, Ch. 5, Theorem 3.5]. Since B' is a subalgebra of A , its dimension is at most $\dim A$. Hence $d = \ell$. \square

Proposition A.2. *Suppose that A_+ is the radical of AB_+ . Then B is finitely generated and A is a finite extension of B .*

Proof. Since A is a noetherian ring, for some homogeneous sequence a_1, \dots, a_m in B_+ , AB_+ is the ideal generated by this sequence. Denote by C the R -subalgebra of A generated by a_1, \dots, a_m . Then C is a graded subalgebra of A . Denote by π the morphism

$$\mathrm{Specm}(A) \xrightarrow{\pi} \mathrm{Specm}(C)$$

whose comorphism is the canonical injection $C \rightarrow A$. Let \bar{A} and \bar{C} be the integral closures of A and C in $K(A)$. Since C is contained in A , \bar{C} is contained in \bar{A} . Let α and β be the morphisms

$$\mathrm{Specm}(\bar{A}) \xrightarrow{\alpha} \mathrm{Specm}(A) \quad \text{and} \quad \mathrm{Specm}(\bar{C}) \xrightarrow{\beta} \mathrm{Specm}(C)$$

whose comorphisms are the canonical injections $A \rightarrow \bar{A}$ and $C \rightarrow \bar{C}$ respectively. Then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Specm}(\bar{A}) & \xrightarrow{\bar{\pi}} & \mathrm{Specm}(\bar{C}) \\ \alpha \downarrow & & \downarrow \beta \\ \mathrm{Specm}(A) & \xrightarrow{\pi} & \mathrm{Specm}(C) \end{array}$$

with $\bar{\pi}$ the morphism whose comorphism is the canonical injection $\bar{C} \rightarrow \bar{A}$.

The action of $G_m(\mathbb{k})$ in A extends to an action of $K(A)$, and \bar{A} is invariant under this action. Denoting by \bar{R} the integral closure of R in $K(A)$, \bar{R} is the set of fixed points under the action of $G_m(\mathbb{k})$ in \bar{A} . Since C is invariant under $G_m(\mathbb{k})$ so is \bar{C} . For \mathfrak{m} a maximal ideal of \bar{R} , the ideal $\mathfrak{m} + \bar{C}_+$ is the maximal ideal of \bar{C} containing \mathfrak{m} and invariant under $G_m(\mathbb{k})$. Then, for \mathfrak{p} a maximal ideal of \bar{C} , $\mathfrak{p} \cap \bar{R} + \bar{C}_+$ is in the closure of the orbit of \mathfrak{p} under $G_m(\mathbb{k})$. Moreover,

$$\{\mathfrak{m} + \bar{A}_+\} = \bar{\pi}^{-1}\{\mathfrak{m} + \bar{C}_+\}$$

for all maximal ideal \mathfrak{m} of \bar{R} . Hence $\bar{\pi}$ is quasi finite. Moreover $\bar{\pi}$ is birational. Then, by Zariski's main theorem [Mu88], $\bar{\pi}$ is an open immersion. As a result, $\bar{\pi}$ is surjective. Hence $\bar{\pi}$ is an isomorphism and $\bar{A} = \bar{C}$. As a result, \bar{A} is a finite extension of C since β is a finite morphism. As submodules of the finite module \bar{A} over the noetherian ring C , A and B are finite C -modules. Hence A is a finite

extension of B . Denoting by $\omega_1, \dots, \omega_d$ a generating family of the C -module B , B is the subalgebra of A generated by $a_1, \dots, a_m, \omega_1, \dots, \omega_d$. \square

Denote by $\mathbb{k}[t]_*$ the localization of $\mathbb{k}[t]$ at the prime ideal $t\mathbb{k}[t]$ and set:

$$R_* := \begin{cases} \mathbb{k} & \text{if } R = \mathbb{k} \\ \mathbb{k}[t]_* & \text{if } R = \mathbb{k}[t] \\ R & \text{if } R = \mathbb{k}[[t]] \end{cases}, \quad \widehat{R} := \begin{cases} \mathbb{k} & \text{if } R = \mathbb{k} \\ \mathbb{k}[[t]] & \text{if } R = \mathbb{k}[t] \\ R & \text{if } R = \mathbb{k}[[t]] \end{cases}.$$

For M a R -module, set $\widehat{M} := \widehat{R} \otimes_R M$.

Lemma A.3. *Suppose $R = \mathbb{k}[t]$. Let M be a torsion free R -module and let N be a submodule of M . Then for a in $\widehat{N} \cap M$, ra is in N for some r in R such that $r(0) \neq 0$.*

Proof. Since M is torsion free, the canonical map $M \rightarrow \widehat{M}$ is an embedding. Moreover, the canonical map $\widehat{N} \rightarrow \widehat{M}$ is an embedding since \widehat{R} is flat over R . Let a be in $\widehat{N} \cap M$ and let \bar{a} be its image in M/N by the quotient map. Denote by J_a the annihilator of \bar{a} in R , whence a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{d} & M & \xrightarrow{d} & M/N \longrightarrow 0 \\ & & & & \uparrow \delta & & \uparrow \delta \\ 0 & \longrightarrow & J_a & \xrightarrow{d} & R & \xrightarrow{d} & R\bar{a} \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

with exact lines and columns. Since \widehat{R} is a flat extension of R , tensoring this diagram by \widehat{R} gives the following diagram with exact lines and columns:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{N} & \xrightarrow{d} & \widehat{M} & \xrightarrow{d} & \widehat{R} \otimes_R M/N \longrightarrow 0 \\ & & & & \uparrow \delta & & \uparrow \delta \\ 0 & \longrightarrow & \widehat{R}J_a & \xrightarrow{d} & \widehat{R} & \xrightarrow{d} & \widehat{R}\bar{a} \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

For b in \widehat{R} , $(\delta \circ d)b = (d \circ \delta)b = 0$ since a is in \widehat{N} by (i), whence $db = 0$. As a result, $\widehat{R}J_a = \widehat{R}$. Hence J_a contains an element r , invertible in \widehat{R} , that is $r(0) \neq 0$, whence the lemma. \square

For $p = (p_1, \dots, p_\ell)$ a homogeneous sequence in A_+ , we set:

$$\underline{p} := \begin{cases} p & \text{if } R = \mathbb{k} \\ (t, p_1, \dots, p_\ell) & \text{if } R = \mathbb{k}[[t]] \end{cases}, \quad A_* := R_* \otimes_R A, \quad \widehat{A} := \widehat{R} \otimes_R A,$$

and we denote by $J_{\underline{p}}$ the ideal of A generated by the sequence \underline{p} . Since $A^{[0]} = R$, the gradation on A extends to a gradation on A_* and \widehat{A} such that $A_*^{[0]} = R_*$ and $\widehat{A}^{[0]} = \widehat{R}$. When $R = \mathbb{k}$ or $R = \mathbb{k}[[t]]$, then $A_* = A$ and $\widehat{A} = A$.

Lemma A.4. *Suppose that A is Cohen-Macaulay. Let $p = p_1, \dots, p_\ell$ be a homogeneous sequence in A_+ such that A_+ is the radical of the ideal of A generated by p_1, \dots, p_ℓ and let V be a graded complement in A to the \mathbb{k} -subspace $J_{\underline{p}}$.*

- (i) *The space V has finite dimension.*
- (ii) *The space A_* is equal to $VR_*[p_1, \dots, p_\ell]$.*
- (iii) *The algebra A is a flat extension of $R[p_1, \dots, p_\ell]$.*
- (iv) *For all homogeneous sequence a_1, \dots, a_n in A , linearly free over \mathbb{k} modulo $J_{\underline{p}}$, a_1, \dots, a_n is linearly free over $R[p_1, \dots, p_\ell]$.*
- (v) *The linear map*

$$V \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow A_*, \quad v \otimes a \longmapsto va$$

is an isomorphism.

Proof. According to Lemma A.1, (ii), the sequence p does exist.

(i) Let J_p be the ideal of A generated by p_1, \dots, p_ℓ . Since A_+ is the radical of J_p , $A^{[d]} = J_p^{[d]}$ for d sufficiently big. When $t \in R$, for all d , then $tA^{[d]}$ has finite codimension in $A^{[d]}$ since $A^{[d]}$ is a finite free R -module. Hence $J_{\underline{p}}$ has finite codimension in A so that V has finite dimension.

(ii) Suppose that t is in R . First of all, we prove by induction on d the inclusion

$$A^{[d]} \subset (VR[p_1, \dots, p_\ell])^{[d]} + tA^{[d]}$$

when $t \in R$. Since $A^{[0]}$ is the direct sum of $V^{[0]}$ and $J_{\underline{p}}^{[0]}$, $V^{[0]}$ is contained in $\mathbb{k} + tR$, whence the inclusion for $d = 0$. Suppose that it is true for all j smaller than d . Since p_1, \dots, p_ℓ have positive degree, by induction hypothesis,

$$J_{\underline{p}}^{[d]} \subset (VR[p_1, \dots, p_\ell])^{[d]} + tA^{[d]},$$

whence the inclusion for d . Then, by induction on m ,

$$A^{[d]} \subset (VR[p_1, \dots, p_\ell])^{[d]} + t^m A^{[d]}.$$

As a result, since $A^{[d]}$ is a finite R -module,

$$A^{[d]} \subset (V\widehat{R}[p_1, \dots, p_\ell])^{[d]},$$

whence $\widehat{A} = V\widehat{R}[p_1, \dots, p_\ell]$. This equality remains true when $R = \mathbb{k}$ by an analogous and simpler argument.

When $R = \mathbb{k}[[t]]$, according to Lemma A.3, for a in A , ra is in $VR[p_1, \dots, p_\ell]$ for some r in R such that $r(0) \neq 0$ by Lemma A.3. As a result, $A_* = VR_*[p_1, \dots, p_\ell]$.

(iii) By Proposition A.2, A is a finite extension of $R[p_1, \dots, p_\ell]$. In particular, $R[p_1, \dots, p_\ell]$ has dimension $\ell + \dim R$ so that p_1, \dots, p_ℓ are algebraically independent over R . Hence $R[p_1, \dots, p_\ell]$ is a regular algebra, whence the assertion by [Ma86, Ch. 8, Theorem 23.1].

(iv) Prove the assertion by induction on n . Since A is an integral domain, the assertion is true for $n = 1$. Suppose the assertion true for $n - 1$. Let (b_1, \dots, b_n) be a homogeneous sequence in $R[p_1, \dots, p_\ell]$ such that

$$b_1 a_1 + \dots + b_n a_n = 0.$$

Let K and I be the kernel and the image of the linear map

$$R[p_1, \dots, p_\ell]^n \longrightarrow R[p_1, \dots, p_\ell], \quad (c_1, \dots, c_n) \longmapsto c_1 b_1 + \dots + c_n b_n,$$

whence the short exact sequence of $R[p_1, \dots, p_\ell]$ modules

$$0 \longrightarrow K \longrightarrow R[p_1, \dots, p_\ell]^n \longrightarrow I \longrightarrow 0.$$

The gradation of $R[p_1, \dots, p_\ell]$ induces a gradation of $R[p_1, \dots, p_\ell]^n$ and K is a graded submodule of $R[p_1, \dots, p_\ell]^n$ since b_1, \dots, b_n is a homogeneous sequence in $R[p_1, \dots, p_\ell]$. Denote by y_1, \dots, y_m a generating homogeneous sequence of the $R[p_1, \dots, p_\ell]$ -module K . By (iii), the short sequence of A -modules

$$0 \longrightarrow A \otimes_{R[p_1, \dots, p_\ell]} K \longrightarrow A^n \longrightarrow A \otimes_{R[p_1, \dots, p_\ell]} I \longrightarrow 0.$$

is exact. So, for some homogeneous sequence x_1, \dots, x_m in A ,

$$a_i = \sum_{j=1}^m x_j y_{j,i}$$

for $i = 1, \dots, n$. Since a_n is not in $J_{\underline{p}}$, for some j_* , the element $y_{j_*,i}$ is an invertible element of R_* , whence

$$b_n y_{j_*,n} = - \sum_{i=1}^{n-1} b_i y_{j_*,i} \quad \text{and} \quad \sum_{i=1}^{n-1} b_i (y_{j_*,n} a_i - a_n y_{j_*,i}) = 0.$$

So, by induction hypothesis,

$$b_1 = \dots = b_{n-1} = 0$$

since the sequence

$$y_{j_*,n} a_1 - a_n y_{j_*,1}, \dots, y_{j_*,n} a_{n-1} - a_n y_{j_*,n-1}$$

is linearly free over \mathbb{k} modulo $J_{\underline{p}}$. Then $b_n = 0$ since $a_n \neq 0$.

(v) Let (v_1, \dots, v_n) be a homogeneous basis of V . Since the space of relations of linear dependence over $R[p_1, \dots, p_\ell]$ of v_1, \dots, v_n is graded, it is equal to $\{0\}$ by (iv), whence the assertion by (ii). \square

- Corollary A.5.** (i) *The algebra A_* is Cohen-Macaulay if and only if for some homogeneous sequence p_1, \dots, p_ℓ in A_+ , the algebra A_* is a finite free extension of $R_*[p_1, \dots, p_\ell]$.*
- (ii) *Suppose that A_* is Cohen-Macaulay. For a homogeneous sequence q_1, \dots, q_ℓ in A_+ , A_* is a finite free extension of $R_*[q_1, \dots, q_\ell]$ if and only if R_*A_+ is the radical of the ideal of A_* generated by q_1, \dots, q_ℓ .*

Proof. (i) The “only if” part results from Lemma A.4.(v). Suppose that for some homogeneous sequence p_1, \dots, p_ℓ in A_+ , the algebra A_* is a finite free extension of $R_*[p_1, \dots, p_\ell]$. In particular, $R_*[p_1, \dots, p_\ell]$ is a polynomial algebra over R_* since A_* has dimension $\dim A$. Let \mathfrak{p} be a prime ideal of A_* and let \mathfrak{q} be its intersection with $R_*[p_1, \dots, p_\ell]$. Denote by $A_{\mathfrak{p}}$ and $R[p_1, \dots, p_\ell]_{\mathfrak{q}}$ the localizations of A_* and $R_*[p_1, \dots, p_\ell]$ at \mathfrak{p} and \mathfrak{q} respectively. Since A_* is a finite extension of $R_*[p_1, \dots, p_\ell]$, these local rings have the same dimension. Denote by d this dimension. By flatness, all regular sequence a_1, \dots, a_d in $R[p_1, \dots, p_\ell]_{\mathfrak{q}}$ is regular in $A_{\mathfrak{p}}$ so that $A_{\mathfrak{p}}$ is Cohen-Macaulay. Hence A_* is Cohen-Macaulay.

(ii) The “only if” part results from (i) and Proposition A.2. Suppose that A_* is a finite free extension of $R_*[q_1, \dots, q_\ell]$. Let \mathfrak{p} be a minimal prime ideal of A_* containing q_1, \dots, q_ℓ and let \mathfrak{q} be its intersection with $R_*[q_1, \dots, q_\ell]$. Then \mathfrak{q} is generated by q_1, \dots, q_ℓ . In particular it has height ℓ . So \mathfrak{p} has height ℓ since A_* is a finite extension of $R_*[q_1, \dots, q_\ell]$. As a result, $\mathfrak{p} = R_*A_+$ since R_*A_+ is a prime ideal of height ℓ , containing q_1, \dots, q_ℓ , whence the assertion. \square

Set $B_* := R_* \otimes_R B$ and for \mathfrak{p} a prime ideal of B denote by $B_{\mathfrak{p}}$ its localization at \mathfrak{p} .

Proposition A.6. *Suppose that the following conditions are verified:*

- (1) *B is normal,*
- (2) *A_+ is the radical of AB_+ ,*
- (3) *A is Cohen-Macaulay.*
- (i) *Let p_1, \dots, p_ℓ be a homogeneous sequence in B_+ such that B_+ is the radical of the ideal of B generated by this sequence. Then for some graded subspace V of A , having finite dimension, the linear morphisms*

$$V \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow A_*, \quad v \otimes a \longmapsto va,$$

$$(V \cap B) \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow B_*, \quad v \otimes a \longmapsto va$$

are isomorphisms.

- (ii) *If $R = \mathbb{k}$ or $R = \mathbb{k}[[t]]$, the algebra B_* is Cohen-Macaulay.*
- (iii) *For \mathfrak{p} prime ideal of B , containing t , the local ring $B_{\mathfrak{p}}$ is Cohen-Macaulay.*

Proof. (i) By Proposition A.2 and by Condition (2), B is finitely generated and A is a finite extension of B . By Condition (2) and by Lemma A.1.(iii), for some

homogeneous sequence (p_1, \dots, p_ℓ) in B_+ , A_+ is the radical of the ideal generated by p_1, \dots, p_ℓ . Set:

$$\underline{p} := \begin{cases} t, p_1, \dots, p_\ell & \text{if } \dim A = \ell + 1 \\ p_1, \dots, p_\ell & \text{if } \dim A = \ell \end{cases}$$

Denote by m the degree of the extension $K(A)$ of $K(B)$. For a in A_* , set:

$$a^\# := \frac{1}{m} \text{tr } a$$

with tr the trace map. By Condition (1), B_* is normal and the map $a \mapsto a^\#$ is a projection from A_* onto B_* whose restriction to A is a projection onto B . Moreover, it is a graded morphism of B -modules. Let M be its kernel. Let J_0 and J be the ideals of B and A generated by \underline{p} respectively. Since t, p_1, \dots, p_ℓ are in B , J is the direct sum of J_0 and MJ_0 . Let \bar{V}_0 be a graded complement in B to the \mathbb{k} -space J_0 and let V_1 be a graded complement in M to the \mathbb{k} -space MJ_0 . Setting $V := \bar{V}_0 + V_1$, V is a graded complement in A to the \mathbb{k} -space J . By Condition (3) and Lemma A.4, V has finite dimension and the linear map

$$V \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow A_*, \quad v \otimes a \mapsto va$$

is an isomorphism. So, since $V_0 = V^\#$, the linear map

$$V_0 \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow B_*, \quad v \otimes a \mapsto va$$

is an isomorphism, whence the assertion.

(ii) results from (i) and Corollary A.5.

(iii) By (i) and Corollary A.5, A_* is Cohen-Macaulay. For \mathfrak{p} a prime ideal of B , containing t , $B_{\mathfrak{p}}$ is the localization of B_* at the prime ideal $B_*\mathfrak{p}$, whence the assertion by (ii). \square

A.2. In this subsection $R = \mathbb{k}[t]$ and set $\widehat{R} = \mathbb{k}[[t]]$. For M a graded module over R such that $M^{[j]}$ is a free submodule of finite rank for all j , we denote by $P_{M,R}(T)$ its Hilbert series:

$$P_{M,R}(T) := \sum_{j \in \mathbb{N}} \text{rk } M^{[j]} T^j.$$

For V a graded space over \mathbb{k} such that $V^{[j]}$ has finite dimension, we denote by $P_{V,\mathbb{k}}(T)$ its Hilbert series:

$$P_{V,\mathbb{k}}(T) := \sum_{j \in \mathbb{N}} \dim V^{[j]} T^j.$$

Let S be a graded polynomial algebra over \mathbb{k} such that $S^{[0]} = \mathbb{k}$ and $S^{[j]}$ has finite dimension for all j . Consider on $S[t]$ and $S[[t]]$ the gradations extending that of S and such that t has degree 0. Consider the following conditions on A :

- (1) A is graded subalgebra of $S[t]$,

- (2) for some homogeneous sequence a_1, \dots, a_ℓ in A_+ , $A = \mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell] \cap S[t]$,
- (3) A is Cohen-Macaulay.

If Condition (2) holds, then $A[t^{-1}] = R[a_1, \dots, a_\ell][t^{-1}]$. Moreover, if so, since A has dimension $\ell + 1$, then the elements t, a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} . Set $\widehat{A} := \widehat{R} \otimes_R A$.

Lemma A.7. *Assume that Conditions (1) and (2) hold.*

- (i) *The element t is a prime element of A .*
- (ii) *The algebra A is a factorial ring.*
- (iii) *The Hilbert series of the R -module A is equal to*

$$P_{A,R}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}},$$

with d_1, \dots, d_ℓ the degrees of a_1, \dots, a_ℓ respectively.

Proof. (i) Let a and b be in A such that ab is in tA . Since $tS[t]$ is a prime ideal of $S[t]$, a or b is in $tS[t]$. Suppose $a = ta'$ for some a' in $S[t]$. Then a' is in $A[t^{-1}]$. By Condition (2), $A[t^{-1}] = R[a_1, \dots, a_\ell][t^{-1}]$. Hence a' is in A by Condition (2) again. As a result, tA is a prime ideal of A .

(ii) Since A is finitely generated, it suffices to prove that all prime ideal of height 1 is principal by [Ma86, Ch. 7, Theorem 20.1]. Let \mathfrak{p} be a prime ideal of height 1. If t is in \mathfrak{p} , then $\mathfrak{p} = tA$ by (i). Suppose that t is not in \mathfrak{p} and set $\overline{\mathfrak{p}} = A[t^{-1}]\mathfrak{p}$. Then $\overline{\mathfrak{p}}$ is a prime ideal of height 1 of $R[a_1, \dots, a_\ell][t^{-1}]$ by Condition (2). For a in $\overline{\mathfrak{p}}$, $t^m a$ is in \mathfrak{p} for some nonnegative integer m . Hence

$$\mathfrak{p} = \overline{\mathfrak{p}} \cap A$$

since \mathfrak{p} is prime. As a polynomial ring over the principal ring $\mathbb{k}[t, t^{-1}]$, the ring $R[a_1, \dots, a_\ell][t^{-1}]$ is a factorial ring. Then $\overline{\mathfrak{p}}$ is generated by an element a in \mathfrak{p} . Since S is a polynomial ring, $S[t]$ is a factorial ring. So, for some nonnegative integer m and for some a' in $S[t]$, prime to t , $a = t^m a'$. By Condition (2), a' is in A . Then a' is an element of \mathfrak{p} , generating $\overline{\mathfrak{p}}$ and not divisible by t in A . Let b and c be in A such that bc is in Aa' . Then b or c is in $A[t^{-1}]a'$. Suppose b in $A[t^{-1}]a'$. So, for some l in \mathbb{N} , $t^l b = b'a'$ for some b' in A . We choose l minimal verifying this condition. By (i), since a' is not divisible by t in A , b' is divisible by t in A if $l > 0$. By minimality of l , $l = 0$ and b is in Aa' . As a result, Aa' is a prime ideal and $\mathfrak{p} = Aa'$ since \mathfrak{p} has height 1.

(iii) By Condition (2),

$$A[t^{-1}] = \mathbb{k}[t, t^{-1}] \otimes_{\mathbb{k}} \mathbb{k}[a_1, \dots, a_\ell] \quad \text{whence} \quad \text{rk } A^{[d]} = \dim \mathbb{k}[a_1, \dots, a_\ell]^{[d]}$$

for all nonnegative integer d . Since a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} ,

$$P_{\mathbb{k}[a_1, \dots, a_\ell], \mathbb{k}}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}},$$

whence the assertion. \square

Let p_1, \dots, p_ℓ be a homogeneous sequence in A such that A_+ is the radical of the ideal of A generated by this sequence. By Lemma A.1, (ii), such a sequence does exist. Denote by C the integral closure of $\mathbb{k}[p_1, \dots, p_\ell]$ in $\mathbb{k}(t, a_1, \dots, a_\ell)$.

Lemma A.8. *Assume that Conditions (1), (2) and (3) hold.*

- (i) *The algebra C is a graded subalgebra of A and t is not algebraic over C .*
- (ii) *The algebra C is Cohen-Macaulay. Moreover, C is a finite free extension of $\mathbb{k}[p_1, \dots, p_\ell]$.*
- (iii) *The algebra $C + tA$ is normal.*

Proof. (i) By Lemma A.7, (ii), A is a normal ring such that $K(A) = \mathbb{k}(t, a_1, \dots, a_\ell)$ by Condition (2). Then C is contained in A since $\mathbb{k}[p_1, \dots, p_\ell]$ is contained in A . Moreover, C is a graded algebra since so is $\mathbb{k}[p_1, \dots, p_\ell]$. By Proposition A.2, A is a finite extension of $R[p_1, \dots, p_\ell]$. So, since A has dimension $\ell + 1$, the elements t, p_1, \dots, p_ℓ are algebraically independent over \mathbb{k} . As a result, t is not algebraic over C .

(ii) By (i), $C[t]$ is a polynomial ring over C . So, by [Ma86, Ch. 8, Corollary of Theorem 23.9], $C[[t]]$ is a normal ring. By definition, A_+ is the radical of the ideal of A generated by p_1, \dots, p_ℓ . So \widehat{A}_+ is the radical of $AC[[t]]_+$ since p_1, \dots, p_ℓ are in $C[[t]]_+$. Since \widehat{R} is a flat extension of R , the algebra \widehat{A} is Cohen-Macaulay by Condition (3). Then, by Proposition A.6, (ii), $C[[t]]$ is Cohen-Macaulay. Let V be a graded complement in C to the ideal of C generated by p_1, \dots, p_ℓ . Since t is not algebraic over C , the space V is a complement in $C[t]$ to the ideal of $C[t]$ generated by t, p_1, \dots, p_ℓ . Then, by Lemma A.4, V has finite dimension and the linear morphism

$$V \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow R_*C, \quad v \otimes a \longmapsto va$$

is an isomorphism. As a result, the linear morphism

$$V \otimes_{\mathbb{k}} \mathbb{k}[p_1, \dots, p_\ell] \longrightarrow C, \quad v \otimes a \longmapsto va$$

is an isomorphism, whence the assertion by Corollary A.5, (ii).

(iii) Set $\tilde{A} := C + tA$. At first, \tilde{A} is a graded subalgebra of A since C is a graded algebra and tA is a graded ideal of A . According to Proposition A.6, (i), for some graded subspace V of A , having finite dimension, the linear morphisms

$$\begin{aligned} V \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] &\longrightarrow A_*, & v \otimes a &\longmapsto va, \\ (V \cap C[t]) \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] &\longrightarrow R_*C, & v \otimes a &\longmapsto va \end{aligned}$$

are isomorphisms. Let v_1, \dots, v_n be a basis of V such that v_1, \dots, v_m is a basis of $V \cap C[t]$. For a in A_* , the element a has unique expansion

$$a = v_1 a_1 + \dots + v_n a_n$$

with a_1, \dots, a_n in $R_*[p_1, \dots, p_\ell]$. If a is in tA_* , a_1, \dots, a_n are in $tR_*[p_1, \dots, p_\ell]$ and if a is in R_*C , a_1, \dots, a_m are in $\mathbb{k}[p_1, \dots, p_\ell]$ and a_{m+1}, \dots, a_n are equal to 0, whence $R_*C \cap tA_* = tR_*C$ and $C \cap tA = \{0\}$. In particular, C is the quotient of \tilde{A} by $t\tilde{A}$.

For \mathfrak{p} a prime ideal of \tilde{A} , denote by $\tilde{A}_{\mathfrak{p}}$ the localization of \tilde{A} at \mathfrak{p} . If t is not in \mathfrak{p} , then $A[t^{-1}]$ is contained in $\tilde{A}_{\mathfrak{p}}$ so that $\tilde{A}_{\mathfrak{p}}$ is a localization of the regular algebra $R[a_1, \dots, a_\ell][t^{-1}]$ by Condition (2). Hence $\tilde{A}_{\mathfrak{p}}$ is a regular local algebra. Suppose that t is in \mathfrak{p} . Denote by $\bar{\mathfrak{p}}$ the image of \mathfrak{p} in C by the quotient map. Then $\tilde{A}_{\mathfrak{p}}/t\tilde{A}_{\mathfrak{p}}$ is the localization $C_{\bar{\mathfrak{p}}}$ of C at the prime ideal $\bar{\mathfrak{p}}$. Since C is Cohen-Macaulay, so are $C_{\bar{\mathfrak{p}}}$ and $\tilde{A}_{\mathfrak{p}}$. As a result, \tilde{A} is Cohen-Macaulay.

Let \mathfrak{p} be a prime ideal of height 1 of \tilde{A} . If t is not in \mathfrak{p} , $\tilde{A}_{\mathfrak{p}}$ is a regular local algebra as it is already mentioned. Suppose that t is in \mathfrak{p} . By Lemma A.7(i), $t\tilde{A} = \mathfrak{p}$ so that all element of $C \setminus \{0\}$ is invertible in $\tilde{A}_{\mathfrak{p}}$, whence

$$\tilde{A}_{\mathfrak{p}} = K(C) + t\tilde{A}_{\mathfrak{p}} \quad \text{and} \quad t\tilde{A}_{\mathfrak{p}} = tK(C) + t^2\tilde{A}_{\mathfrak{p}}.$$

Hence $\tilde{A}_{\mathfrak{p}}$ is a regular local ring of dimension 1. As a result, \tilde{A} is regular in codimension 1. Then, by Serre's normality criterion [B98, §1, n°10, Théorème 4], \tilde{A} is normal since \tilde{A} is Cohen-Macaulay. \square

Corollary A.9. *Assume that Conditions (1), (2) and (3) hold.*

- (i) *The algebra \widehat{A} is equal to $C[[t]]$.*
- (ii) *For a in A , the element ra is in $C[t]$ for some r in $\mathbb{k}[t]$ such that $r(0) \neq 0$.*

Proof. (i) Since tA is contained in A , we have $K(A) = K(\tilde{A})$. Since C_+ is contained in \tilde{A}_+ , A_+ is the radical of $A\tilde{A}_+$. Then, by Proposition A.2, A is a finite extension of \tilde{A} . So, by Lemma A.8(iii), $A = \tilde{A}$ and by induction on m ,

$$A \subset C[t] + t^m A$$

for all positive integer m . Since A and $C[t]$ are graded and since the R -module $A^{[d]}$ is finitely generated for all d , $\widehat{A} = C[[t]]$.

- (ii) The assertion results from (i) and Lemma A.3. \square

Proposition A.10. *Assume that Conditions (1), (2) and (3) hold. Then the algebra A_* is polynomial over R_* . Moreover, for some homogeneous sequence q_1, \dots, q_ℓ in A_+ such that q_1, \dots, q_ℓ have degree d_1, \dots, d_ℓ respectively, $A_* = R_*[q_1, \dots, q_\ell]$.*

Proof. According to Corollary A.9 and Lemma A.8(i), it suffices to prove that C is a polynomial algebra over \mathbb{k} generated by a homogeneous sequence q_1, \dots, q_ℓ such

that q_1, \dots, q_ℓ have degree d_1, \dots, d_ℓ respectively. According to Corollary A.9,(i), Lemma A.8,(i) and Lemma A.7,(iii),

$$P_{C,\mathbb{k}}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}}.$$

By Corollary A.9,(ii), for $i = 1, \dots, \ell$, for some r_i in R such that $r_i(0) \neq 0$, $r_i a_i$ has an expansion

$$r_i a_i = \sum_{m \in \mathbb{N}} c_{i,m} t^m$$

with $c_{i,m}, m \in \mathbb{N}$ in $C^{[d_i]}$, with finite support. For z in \mathbb{k} and $i = 1, \dots, \ell$, set:

$$b_i(z) = \sum_{m \in \mathbb{N}} c_{i,m} z^m$$

so that $b_i(z)$ is in $C^{[d_i]}$ for all z . As already mentioned, t, a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} by Condition (2) since A has dimension $\ell + 1$. Then, so are $r_1 a_1, \dots, r_\ell a_\ell$ and for some z in \mathbb{k} , $b_1(z), \dots, b_\ell(z)$ are algebraically independent over \mathbb{k} . Denoting by C' the subalgebra of C generated by this sequence,

$$P_{C',\mathbb{k}}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}},$$

whence $C = C'$ so that C is a polynomial algebra. □

REFERENCES

- [B98] N. Bourbaki, *Algèbre commutative, Chapitre 10, Éléments de mathématiques*, Masson (1998), Paris.
- [CM10] J.-Y. Charbonnel and A. Moreau, *The index of centralizers of elements of reductive Lie algebras*, Documenta Mathematica, **15** (2010), 387–421.
- [CM16] J.-Y. Charbonnel and A. Moreau, *The symmetric invariants of centralizers and Slodowy grading*, Math. Zeitschrift **282** (2016), n° 1-2, 273–339.
- [JS10] A. Joseph and D. Shafrir, *Polynomiality of invariants, unimodularity and adapted pairs*, Transformation Groups, **15**, (2010), 851–882.
- [Ma86] H. Matsumura, *Commutative ring theory* Cambridge studies in advanced mathematics (1986), n° 8, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney.
- [Mu88] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics (1988), n° 1358, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.
- [PPY07] D.I. Panyushev, A. Premet and O. Yakimova, *On symmetric invariants of centralizers in reductive Lie algebras*, Journal of Algebra **313** (2007), 343–391.
- [Y07] O. Yakimova, *A counterexample to Premet's and Joseph's conjecture*, Bulletin of the London Mathematical Society **39** (2007), 749–754.

JEAN-YVES CHARBONNEL, UNIVERSITÉ PARIS DIDEROT - CNRS, INSTITUT DE MATHÉMATIQUES DE JUSSIEU
- PARIS RIVE GAUCHE, UMR 7586, GROUPES, REPRÉSENTATIONS ET GÉOMÉTRIE, BÂTIMENT SOPHIE GERMAIN,
CASE 7012, 75205 PARIS CEDEX 13, FRANCE

E-mail address: `jean-yves.charbonnel@imj-prg.fr`

ANNE MOREAU, LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS, TÉLÉPORT 2 - BP 30179, BOULE-
VARD MARIE ET PIERRE CURIE, 86962 FUTUROSCOPE CHASSENEUIL CEDEX, FRANCE

E-mail address: `anne.moreau@math.univ-poitiers.fr`