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THE SYMMETRIC INVARIANTS OF CENTRALIZERS AND SLODOWY GRADING II

JEAN-YVES CHARBONNEL AND ANNE MOREAU

ABSTRACT. Let \mathfrak{g} be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field \mathbb{k} of characteristic zero, and let (e, h, f) be an \mathfrak{sl}_2 -triple of \mathfrak{g} . Denote by \mathfrak{g}^e the centralizer of e in \mathfrak{g} and by $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ the algebra of symmetric invariants of \mathfrak{g}^e . We say that e is good if the nullvariety of some ℓ homogenous elements of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ in $(\mathfrak{g}^e)^*$ has codimension ℓ . If e is good then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. In this paper, we prove that the converse of the main result of [CM16] is true. Namely, we prove that e is good if and only if for some homogenous generating sequence q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the initial homogenous components of their restrictions to $e + \mathfrak{g}^f$ are algebraically independent over \mathbb{k} .

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1. INTRODUCTION

1.1. Let \mathfrak{g} be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field \mathbb{k} of characteristic zero, let $\langle \cdot, \cdot \rangle$ be the Killing form of \mathfrak{g} and let G be the adjoint group of \mathfrak{g} . If \mathfrak{a} is a subalgebra of \mathfrak{g} , we denote by $S(\mathfrak{a})$ the symmetric algebra of \mathfrak{a} . For $x \in \mathfrak{g}$, we denote by \mathfrak{g}^x the centralizer of x in \mathfrak{g} and by G^x the stabilizer of x in G . Then $\text{Lie}(G^x) = \text{Lie}(G_0^x) = \mathfrak{g}^x$ where G_0^x is the identity component of G^x . Moreover, $S(\mathfrak{g}^x)$ is a \mathfrak{g}^x -module and $S(\mathfrak{g}^x)^{\mathfrak{g}^x} = S(\mathfrak{g}^x)^{G_0^x}$.

In [CM16], we continued the investigations of [PPY07] and we studied the question on whether the algebra $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial in ℓ variables; see [Y07, CM10, JS10] for other references related to the topic.

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1.2. Let us first summarize the main results of [CM16].

Definition 1.1 ([CM16, Definition 1.3]). An element $x \in \mathfrak{g}$ is called a *good element* of \mathfrak{g} if for some homogenous sequence (p_1, \dots, p_ℓ) in $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$, the nullvariety of p_1, \dots, p_ℓ in $(\mathfrak{g}^x)^*$ has codimension ℓ in $(\mathfrak{g}^x)^*$.

Thus an element $x \in \mathfrak{g}$ is good if the nullcone of $S(\mathfrak{g}^x)$, that is, the nullvariety in $(\mathfrak{g}^x)^*$ of the augmentation ideal $S(\mathfrak{g}^x)_+^{\mathfrak{g}^x}$ of $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$, is a complete intersection in $(\mathfrak{g}^x)^*$ since the transcendence degree over \mathbb{k} of the fraction field of $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is ℓ by the main result of [CM10].

For example, regular nilpotent elements are good; see the introduction of [CM16] for more details and other examples.

Theorem 1.2 ([CM16, Theorem 3.3]). *Let x be a good element of \mathfrak{g} . Then $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $S(\mathfrak{g}^x)$ is a free extension of $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$.*

Furthermore, x is good if and only if so is its nilpotent component in the Jordan decomposition [CM16, Proposition 3.5]. As a consequence, we can restrict the study to the case of nilpotent elements.

Let e be a nilpotent element of \mathfrak{g} . By the Jacobson-Morosov Theorem, e is embedded into a \mathfrak{sl}_2 -triple (e, h, f) of \mathfrak{g} . Denote by $S_e := e + \mathfrak{g}^f$ the *Slodowy slice associated with e* . Identify \mathfrak{g}^* with \mathfrak{g} , and $(\mathfrak{g}^e)^*$ with \mathfrak{g}^f , through the Killing form $\langle \cdot, \cdot \rangle$ of \mathfrak{g} . For p in $S(\mathfrak{g}) \simeq \mathbb{k}[\mathfrak{g}^*] \simeq \mathbb{k}[\mathfrak{g}]$, denote by ${}^e p$ the initial homogenous component of its restriction to S_e . According to [PPY07, Proposition 0.1], if p is in $S(\mathfrak{g})^{\mathfrak{g}}$, then ${}^e p$ is in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

Theorem 1.3 ([CM16, Theorem 1.5]). *Suppose that for some homogenous generators q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} . Then e is a good element of \mathfrak{g} . In particular, $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $S(\mathfrak{g}^e)$ is a free extension of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Moreover, ${}^e q_1, \dots, {}^e q_\ell$ is a regular sequence in $S(\mathfrak{g}^e)$.*

In other words, Theorem 1.3 provides a sufficient condition for that $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial. By [PPY07], one knows that for homogenous elements q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent if and only if

$$(1) \quad \sum_{i=1}^{\ell} \deg {}^e q_i = \frac{\dim \mathfrak{g}^e + \ell}{2}.$$

So we have a practical criterion to verify the sufficient condition of Theorem 1.3. However, note that even if the condition of Theorem 1.3 holds, that is, if (1) holds, $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not necessarily generated by the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$. As a matter of fact, there are nilpotent elements e satisfying this condition and for which $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not generated by some ${}^e q_1, \dots, {}^e q_\ell$, for any choice of homogenous generators q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$ (cf. [CM16, Remark 2.25]).

Theorem 1.3 can be applied to a great number of nilpotent orbits in the simple classical Lie algebras, and for some nilpotent orbits in the exceptional Lie algebras, see [CM16, Sections 5 and 6]. We also provided in [CM16, Example 7.8] an example of a nilpotent element e for which $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not polynomial.

1.3. In this note, we prove that the converse of Theorem 1.3 also holds. Namely, we prove the following result.

Theorem 1.4. *The nilpotent element e of \mathfrak{g} is good if and only if for some homogeneous generating sequence q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} .*

Theorem 1.4 was conjectured in [CM16, Conjecture 7.11]. Notice that it may happen that for some r_1, \dots, r_ℓ in $S(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent over \mathbb{k} , and that however e is not good. This is the case for instance for the nilpotent elements in $\mathfrak{so}(\mathbb{k}^{12})$ associated with the partition $(5, 3, 2, 2)$, cf. [CM16, Example 7.6]. In fact, according to [PPY07, Corollary 2.3], for any nilpotent element e of \mathfrak{g} , there exist r_1, \dots, r_ℓ in $S(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent over \mathbb{k} .

1.4. We introduce in this subsection the main notations of the paper and explain our approach to prove Theorem 1.4.

Let x_1, \dots, x_r be a basis of \mathfrak{g}^e such that for $i = 1, \dots, r$, $[h, x_i] = n_i x_i$ with n_i a nonnegative integer. For $\mathbf{j} = (j_1, \dots, j_r)$ in \mathbb{N}^r , set:

$$|\mathbf{j}| := j_1 + \dots + j_r, \quad |\mathbf{j}|_e := j_1(n_1 + 2) + \dots + j_r(n_r + 2), \quad x^{\mathbf{j}} := x_1^{j_1} \dots x_r^{j_r}.$$

There are two gradations on $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, the standard one and the Slodowy gradation. For all \mathbf{j} in \mathbb{N}^r , $x^{\mathbf{j}}$ has standard degree $|\mathbf{j}|$ and Slodowy degree $|\mathbf{j}|_e$. Denoting by $t \mapsto h(t)$ the one parameter subgroup of G generated by $\text{ad} h$, the Slodowy slice $e + \mathfrak{g}^f$ is invariant under the one parameter subgroup $t \mapsto t^{-2}h(t)$ and for all \mathbf{j} in \mathbb{N}^r and for all y in \mathfrak{g}^f ,

$$x^{\mathbf{j}}(t^{-2}h(t)(e + y)) = t^{-|\mathbf{j}|_e} x^{\mathbf{j}}(e + y).$$

Let t be an indeterminate and let R be the polynomial algebra $\mathbb{k}[t]$. The polynomial algebra

$$S(\mathfrak{g}^e)[t] := \mathbb{k}[t] \otimes_{\mathbb{k}} S(\mathfrak{g}^e)$$

identifies with the algebra of polynomial functions on $\mathfrak{g}^f \times \mathbb{k}$. The gradation of $S(\mathfrak{g}^e)$ induces a gradation of $S(\mathfrak{g}^e)[t]$ such that t has degree 0.

Denote by ε the evaluation map at $t = 0$ so that ε is a graded morphism from $S(\mathfrak{g}^e)[t]$ onto $S(\mathfrak{g}^e)$. For p in $S(\mathfrak{g})$, denote by $\kappa(p)$ the polynomial function $x \mapsto p(e + x)$ on \mathfrak{g}^f so that $\kappa(p) \in S(\mathfrak{g}^e)$. Then ${}^e p$ is the homogeneous component of $\kappa(p)$ of minimal degree with respect to the standard gradation on $S(\mathfrak{g}^e)$. Let τ be the embedding of $S(\mathfrak{g}^e)$ into $S(\mathfrak{g}^e)[t]$ such that $\tau(x_i) := tx_i$ for $i = 1, \dots, r$.

Denote by A the intersection of $S(\mathfrak{g}^e)[t]$ with the sub- $\mathbb{k}[t, t^{-1}]$ -module of

$$S(\mathfrak{g}^e)[t, t^{-1}] := \mathbb{k}[t, t^{-1}] \otimes_{\mathbb{k}} S(\mathfrak{g}^e)$$

generated by $\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})$. Let \mathcal{V} be the nullvariety of A_+ in $\mathfrak{g}^f \times \mathbb{k}$ where A_+ is the augmentation ideal of A . Let \mathcal{V}_* be the union of the irreducible components of \mathcal{V} which are not contained in $\mathfrak{g}^f \times \{0\}$ and let \mathcal{N} be the nullvariety of $\varepsilon(A)_+$ in \mathfrak{g}^f , with $\varepsilon(A)_+$ the augmentation ideal of $\varepsilon(A)$. Then \mathcal{V} is the union of \mathcal{V}_* and $\mathcal{N} \times \{0\}$. Using properties of the varieties \mathcal{V} and \mathcal{V}_* , we prove the following result (see Section 2).

Theorem 1.5. *Suppose that \mathcal{N} has dimension $r - \ell$. Then for some homogeneous generating sequence q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} .*

The key point is to show that, under the hypothesis of Theorem 1.5, $\varepsilon(A)$ is the subalgebra of $S(\mathfrak{g}^e)$ generated by the family ${}^e p$, $p \in S(\mathfrak{g})^{\mathfrak{g}}$, and hence that \mathcal{N} coincides with the nullvariety in \mathfrak{g}^f of ${}^e q_1, \dots, {}^e q_\ell$. So, if \mathcal{N} has dimension $r - \ell$, then the elements ${}^e q_1, \dots, {}^e q_\ell$ must be algebraically independent over \mathbb{k} .

The proof of Theorem 1.5 is done in Section 2. Then Theorem 1.4 will be a consequence of Theorem 1.5 and it will be proven in Section 3. In Appendix A, we state general facts on commutative algebra that are used throughout the paper.

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2. PROOF OF THEOREM 1.5

In this section, unless otherwise specified, the gradation on $S(\mathfrak{g}^e)$ is the Slodowy gradation.

For m a nonnegative integer, $S(\mathfrak{g}^e)^{[m]}$ denotes the space of degree m of $S(\mathfrak{g}^e)$. We retain the notations of the introduction, in particular of Subsection 1.4.

2.1. Let R be the ring $\mathbb{k}[t]$. As in Appendix A, for M a graded subspace of $S(\mathfrak{g}^e)[t] = R \otimes_{\mathbb{k}} S(\mathfrak{g}^e)$, its subspace of degree m is denoted by $M^{[m]}$. In particular, $S(\mathfrak{g}^e)[t]^{[m]}$ is equal to $S(\mathfrak{g}^e)^{[m]}[t]$ and it is a free R -module of finite rank. As a result, for all graded R -submodule M of $S(\mathfrak{g}^e)[t]$, its Hilbert series is well defined.

For m a nonnegative integer, denote by F_m the space of elements of $\kappa(S(\mathfrak{g})^{\mathfrak{g}})$ whose component of minimal standard degree is at least m . Then F_0, F_1, \dots is a decreasing filtration of the algebra $\kappa(S(\mathfrak{g})^{\mathfrak{g}})$. Let d_1, \dots, d_ℓ be the standard degrees of a homogeneous generating sequence of $S(\mathfrak{g})^{\mathfrak{g}}$. We assume that the sequence d_1, \dots, d_ℓ is increasing.

Recall that A is the intersection of $S(\mathfrak{g}^e)[t]$ with the sub- $\mathbb{k}[t, t^{-1}]$ -module of $S(\mathfrak{g}^e)[t, t^{-1}]$ generated by $\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})$, and that A_+ is the augmentation ideal of A .

Lemma 2.1. (i) *For p a homogeneous element of standard degree d in $S(\mathfrak{g})^{\mathfrak{g}}$, the element $\kappa(p)$ and ${}^e p$ have degree $2d$.*

- (ii) For some homogeneous sequence a_1, \dots, a_ℓ in A_+ , the elements t, a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} , and A is the intersection of $S(\mathfrak{g}^e)[t]$ with $\mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell]$.
- (iii) The Hilbert series of the R -algebra A is equal to

$$P_{A,R}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}.$$

- (iv) The Hilbert series of the \mathbb{k} -algebra $\varepsilon(A)$ is equal to

$$P_{\varepsilon(A),\mathbb{k}}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}.$$

- (v) The subalgebra $\varepsilon(A)$ is the graded algebra associated with the filtration F_0, F_1, \dots

Proof. (i) For y in \mathfrak{g}^f and s in \mathbb{k}^* ,

$$p(s^{-2}h(s)(e + y)) = s^{-2d}p(h(s)(e + y)) = s^{-2d}p(e + y)$$

since p is invariant under the one parameter subgroup $s \mapsto h(s)$. Hence $\kappa(p)$ is homogeneous of degree $2d$. Since the monomials x^j are homogeneous, ${}^e p$ has degree $2d$.

(ii) Let q_1, \dots, q_ℓ be a homogeneous generating sequence of $S(\mathfrak{g})^{\mathfrak{g}}$. By a well known fact [CM16, Lemma 4.4,(i)], the morphism

$$G \times (e + \mathfrak{g}^f) \longrightarrow \mathfrak{g}, \quad (g, x) \longmapsto g(x)$$

is dominant. Then $\kappa(S(\mathfrak{g})^{\mathfrak{g}})$ is a polynomial algebra generated by $\kappa(q_1), \dots, \kappa(q_\ell)$. So, setting $a_i := \tau \circ \kappa(q_i)$ for $i = 1, \dots, \ell$, the sequence a_1, \dots, a_ℓ is a homogeneous sequence in A_+ such that

$$\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}] = \mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell].$$

Let $\bar{\tau}$ be the automorphism of $S(\mathfrak{g}^e)[t, t^{-1}]$ extending τ and such that $\bar{\tau}(t) = t$. Then

$$\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}] = \bar{\tau}(\kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]).$$

Since $\kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]$ has dimension $\ell + 1$, $\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]$ has dimension $\ell + 1$ too, and t, a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} . By definition, $A = S(\mathfrak{g}^e)[t] \cap \tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]$. Hence

$$A[t^{-1}] = \mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell] \quad \text{and} \quad A = S(\mathfrak{g}^e)[t] \cap \mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell].$$

(iii) Since t has degree 0, the gradation of $S(\mathfrak{g}^e)[t]$ extends to a gradation of $S(\mathfrak{g}^e)[t, t^{-1}]$ such that for all m , its space of degree m is equal to $S(\mathfrak{g}^e)^{[m]}[t, t^{-1}]$. Then for all $\mathbb{k}[t, t^{-1}]$ -submodule M of $S(\mathfrak{g}^e)[t, t^{-1}]$, M has a Hilbert series:

$$P_{M,\mathbb{k}[t,t^{-1}]}(T) := \sum_{m \in \mathbb{N}} \text{rk } M^{[m]} T^m$$

with $M^{[m]}$ the subspace of degree m of M . From the equality $A[t^{-1}] = \mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell]$, we deduce

$$P_{A[t^{-1}], \mathbb{k}[t, t^{-1}]}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}$$

since for $i = 1, \dots, \ell$, the element a_i has degree $2d_i$ by (i). For all m , the rank of the R -module $A^{[m]}$ is equal to the rank of the $\mathbb{k}[t, t^{-1}]$ -module $A[t^{-1}]^{[m]}$, whence

$$P_{A,R}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}.$$

(iv) Let m be a nonnegative integer. The R -module $A^{[m]}$ is free of finite rank and for (v_1, \dots, v_n) a basis of this module, (tv_1, \dots, tv_n) is a basis of the R -module $tA^{[m]}$. Since $\varepsilon(A)^{[m]}$ is the quotient of $A^{[m]}$ by $tA^{[m]}$,

$$\dim \varepsilon(A)^{[m]} = n = \text{rk} A^{[m]},$$

whence the assertion by (iii).

(v) Let $\text{gr}_F A$ be the graded algebra associated with the filtration F_0, F_1, \dots of $\kappa(\mathbb{S}(\mathfrak{g})^{\mathfrak{g}})$. Denote by $a \mapsto a(1)$ the evaluation map at $t = 1$ from $\mathbb{S}(\mathfrak{g}^e)[t]$ to $\mathbb{S}(\mathfrak{g}^e)$. For a in A such that $\varepsilon(a) \neq 0$, $a(1)$ is in $\kappa(\mathbb{S}(\mathfrak{g})^{\mathfrak{g}})$ and $\varepsilon(a)$ is the component of minimal degree of $a(1)$ with respect to the standard gradation, whence $\varepsilon(A) \subset \text{gr}_F A$. Conversely, let \bar{a} be a homogeneous element of degree m of $\text{gr}_F A$ and let a be a representative of \bar{a} in F_m . Then $\tau(a) = t^m b$ with b in A such that $\varepsilon(b) = \bar{a}$, whence $\text{gr}_F A \subset \varepsilon(A)$ and the assertion. \square

Let R_* be the localization of R at the prime ideal tR and set

$$\widehat{R} := \mathbb{k}[[t]], \quad A_* := R_* \otimes_R A, \quad \widehat{A} := \widehat{R} \otimes_R A.$$

The gradation of A extends to gradations on A_* and \widehat{A} such that $A_*^{[0]} = R_*$ and $\widehat{A}^{[0]} = \widehat{R}$.

Proposition 2.2. (i) *The algebra $\varepsilon(A)$ is polynomial if and only if for some standard homogeneous generating sequence q_1, \dots, q_ℓ of $\mathbb{S}(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} . Moreover, in this case, A is a polynomial algebra.*

(ii) *If A_* is a polynomial algebra, then for some homogeneous sequence p_1, \dots, p_ℓ in A_+ , we have $A_* = R_*[p_1, \dots, p_\ell]$, the elements t, p_1, \dots, p_ℓ are algebraically independent over \mathbb{k} and p_1, \dots, p_ℓ have degree $2d_1, \dots, 2d_\ell$ respectively.*

Proof. (i) Let q_1, \dots, q_ℓ be a homogeneous generating sequence of $\mathbb{S}(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} . We can assume that for $i = 1, \dots, \ell$, q_i has standard degree d_i . For $i = 1, \dots, \ell$, ${}^e q_i$ has degree $2d_i$ by Lemma 2.1(i), and we set

$$Q_i := t^{-2d_i} \tau \circ \kappa(q_i).$$

For $\mathbf{i} = (i_1, \dots, i_\ell)$ in \mathbb{N}^ℓ , set:

$$q^{\mathbf{i}} := q_1^{i_1} \cdots q_\ell^{i_\ell}, \quad Q^{\mathbf{i}} := Q_1^{i_1} \cdots Q_\ell^{i_\ell}, \quad {}^e q^{\mathbf{i}} := {}^e q_1^{i_1} \cdots {}^e q_\ell^{i_\ell},$$

$$|\mathbf{i}|_{\min} := 2i_1 d_1 + \cdots + 2i_\ell d_\ell.$$

Then, for all \mathbf{i} in \mathbb{N}^ℓ ,

$$\tau \circ \kappa(q^{\mathbf{i}}) = t^{|\mathbf{i}|_{\min}} Q^{\mathbf{i}}.$$

Moreover,

$$\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}] = \mathbb{k}[t, t^{-1}, Q_1, \dots, Q_\ell].$$

Let a be in A . For some l in \mathbb{N} and for some sequence $c_{\mathbf{i}, m}$, $(\mathbf{i}, m) \in \mathbb{N}^\ell \times \mathbb{N}$ in \mathbb{k} , of finite support,

$$t^l a = \sum_{(\mathbf{i}, m) \in \mathbb{N}^\ell \times \mathbb{N}} c_{\mathbf{i}, m} t^m Q^{\mathbf{i}} \quad \text{whence} \quad \sum_{\mathbf{i} \in \mathbb{N}^\ell} c_{\mathbf{i}, m} {}^e q^{\mathbf{i}} = 0$$

for $m < l$. Hence a is in $R[Q_1, \dots, Q_\ell]$ since the sequence ${}^e q^{\mathbf{i}}$, $\mathbf{i} \in \mathbb{N}^\ell$, is linearly free over \mathbb{k} . As a result,

$$A = R[Q_1, \dots, Q_\ell] \quad \text{and} \quad \varepsilon(A) = \mathbb{k}[{}^e q_1, \dots, {}^e q_\ell]$$

so that A and $\varepsilon(A)$ are polynomial algebras over \mathbb{k} since ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} .

Conversely, suppose that $\varepsilon(A)$ is a polynomial algebra. By Lemma 2.1, (i) and (iv), the algebra $\varepsilon(A)$ is Slodowy graded and standard graded. Let d be the dimension of $\varepsilon(A)$. As $\varepsilon(A)$ is a polynomial algebra, it is regular so that the \mathbb{k} -space $\varepsilon(A)_+ / \varepsilon(A)_+^2$ has dimension d . Moreover, the two gradations on $\varepsilon(A)$ induce gradations on $\varepsilon(A)_+ / \varepsilon(A)_+^2$. Hence $\varepsilon(A)_+ / \varepsilon(A)_+^2$ has a bihomogeneous basis. Then some bihomogeneous sequence u_1, \dots, u_d in $\varepsilon(A)_+$ represents a basis of $\varepsilon(A)_+ / \varepsilon(A)_+^2$. As a result, the \mathbb{k} -algebra $\varepsilon(A)$ is generated by the bihomogeneous sequence u_1, \dots, u_d . For $i = 1, \dots, d$, denote by δ_i the Slodowy degree of u_i . As ε is homogeneous with respect to the Slodowy gradation, $u_i = \varepsilon(r_i)$ for some homogeneous element r_i of degree δ_i of A . Let m_i be the smallest nonnegative integer such that $t^{m_i} r_i$ is in $\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})$. According to Lemma 2.1, (i), δ_i is even and for some standard homogeneous element p_i of standard degree $\delta_i/2$ of $S(\mathfrak{g})^{\mathfrak{g}}$, $t^{m_i} r_i = \tau \circ \kappa(p_i)$. Then $u_i = {}^e p_i$ since p_i is standard homogeneous.

Let \mathfrak{B} be the subalgebra of $S(\mathfrak{g})$ generated by p_1, \dots, p_d . Suppose that \mathfrak{B} is strictly contained in $S(\mathfrak{g})^{\mathfrak{g}}$. A contradiction is expected. For some positive integer m , the space $S(\mathfrak{g})_m^{\mathfrak{g}}$ of standard degree m of $S(\mathfrak{g})^{\mathfrak{g}}$ is not contained in \mathfrak{B} . Let q be in $(S(\mathfrak{g})^{\mathfrak{g}})_m \setminus \mathfrak{B}$ such that ${}^e q$ has maximal standard degree. By Lemma 2.1, (i), ${}^e q$ is a polynomial in u_1, \dots, u_d , of degree $2m$. So, for some polynomial q' of degree m in \mathfrak{B} , ${}^e(q - q')$ has standard degree bigger than the standard degree of ${}^e q$. So, by maximality of the standard degree of ${}^e q$, the elements $q - q'$ and q are in \mathfrak{B} , whence the contradiction. As a result, $\mathfrak{B} = S(\mathfrak{g})^{\mathfrak{g}}$ and $d = \ell$.

(ii) Suppose that A_* is a polynomial algebra. Denoting by J the ideal of A_* generated by t and A_+ , the \mathbb{k} -space J/J^2 is a graded space of dimension ℓ since A_* is a regular algebra of dimension $\ell + 1$. Then for some homogeneous sequence p_1, \dots, p_ℓ in A_+ , (t, p_1, \dots, p_ℓ) is a basis of J modulo J^2 . Since p_1, \dots, p_ℓ have positive degree, we prove by induction on d that

$$A_*^{[d]} \subset R_*[p_1, \dots, p_\ell]^{[d]} + tA_*^{[d]}.$$

Then by induction on m , we get

$$A_*^{[d]} \subset R_*[p_1, \dots, p_\ell] + t^m A_*^{[d]}.$$

So, since the R_* -module $A_*^{[d]}$ is finitely generated,

$$A_*^{[d]} \subset \widehat{R}[p_1, \dots, p_\ell]^{[d]}$$

whence $\widehat{A} = \widehat{R}[p_1, \dots, p_\ell]$ and $A_* = R_*[p_1, \dots, p_\ell]$ by Lemma A.3.

Denote by $\delta_1, \dots, \delta_\ell$ the respective degrees of p_1, \dots, p_ℓ . We can suppose that p_1, \dots, p_ℓ is ordered so that $\delta_1 \leq \dots \leq \delta_\ell$. Prove by induction on i that $\delta_j = 2d_j$ for $j = 1, \dots, i$. By Lemma 2.1,(iii), $2d_1$ is the smallest positive degree of the elements of A . Moreover, δ_1 is the smallest positive degree of the elements of $R[p_1, \dots, p_\ell]$, whence $\delta_1 = 2d_1$. Suppose $\delta_j = 2d_j$ for $j = 1, \dots, i - 1$. Set $A_i := R[p_i, \dots, p_\ell]$. Then, by induction hypothesis and Lemma 2.1,(iii),

$$P_{A_i, R}(T) = \prod_{j=i}^{\ell} \frac{1}{1 - T^{\delta_j}} = \prod_{j=i}^{\ell} \frac{1}{1 - T^{2d_j}}.$$

By the first equality, δ_i is the smallest positive degree of the elements of A_i and by the second equality, $2d_i$ is the smallest positive degree of the elements of A_i too, whence $\delta_i = 2d_i$. Then with $i = \ell$, we get that $\delta_j = 2d_j$ for $j = 1, \dots, \ell$. \square

Recall that $\widehat{R} = \mathbb{k}[[t]]$.

Corollary 2.3. *Suppose that A_* is a polynomial algebra. Then for some standard homogeneous generating sequence q_1, \dots, q_ℓ in $S(\mathfrak{g})^{\mathfrak{g}}$,*

$$A_* = R_*[t^{-2d_1} \tau \circ \kappa(q_1), \dots, t^{-2d_\ell} \tau \circ \kappa(q_\ell)].$$

Proof. For m nonnegative integer, denote by $S(\mathfrak{g})_m^{\mathfrak{g}}$ the space of standard degree m of $S(\mathfrak{g})^{\mathfrak{g}}$. By Proposition 2.2,(ii), for some homogeneous sequence p_1, \dots, p_ℓ in A_+ such that p_1, \dots, p_ℓ have degree $2d_1, \dots, 2d_\ell$ respectively,

$$A_* = R_*[p_1, \dots, p_\ell].$$

For $i = 1, \dots, \ell$, let m_i be the smallest integer such that $t^{m_i} p_i$ is in $\tau \circ \kappa(S(\mathfrak{g})^{\mathfrak{g}})$. By Lemma 2.1,(i), $t^{m_i} p_i$ has an expansion

$$t^{m_i} p_i = \sum_{j \in \mathbb{N}} t^j \tau \circ \kappa(q_{i,j})$$

with $q_{i,j}$, $j \in \mathbb{N}$, in $S(\mathfrak{g})_{d_i}^{\mathfrak{g}}$ of finite support. Denoting by $\delta_{i,j}$ the standard degree of ${}^e q_{i,j}$, set:

$$J'_i := \{j \in \mathbb{N} ; m_i = j + \delta_{i,j}\}, \quad \delta_i := \inf\{\delta_{i,j} ; j \in J'_i\},$$

$$j_i := m_i - 2d_i, \quad Q_i := t^{-2d_i} \tau \circ \kappa(q_{i,j_i}).$$

For $i = 1, \dots, \ell$, since p_i is not divisible by t in A ,

$$p_i - Q_i \in tA,$$

whence

$$A_* \subset R[Q_1, \dots, Q_\ell] + tA_*.$$

Then, by induction m ,

$$A_* \subset R_*[Q_1, \dots, Q_m] + t^m A_*$$

for all m . As a result,

$$\widehat{A} = \widehat{R}[Q_1, \dots, Q_\ell],$$

since for all d , the R_* -module $A_*^{[d]}$ is finitely generated. Then, by Lemma A.3,

$$A_* = R_*[Q_1, \dots, Q_\ell].$$

As a result, since A has dimension $\ell + 1$, the elements t, Q_1, \dots, Q_ℓ are algebraically independent over \mathbb{k} and so are $q_{1,j_1}, \dots, q_{\ell,j_\ell}$. Moreover the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ is generated by $q_{1,j_1}, \dots, q_{\ell,j_\ell}$ since they have degree d_1, \dots, d_ℓ respectively. \square

2.2. Denote by \mathcal{V} the nullvariety of A_+ in $\mathfrak{g}^f \times \mathbb{k}$. Let \mathcal{V}_* be the union of the irreducible components of \mathcal{V} which are not contained in $\mathfrak{g}^f \times \{0\}$. The following result is proven in [CM16, Corollary 4.4,(i)]. Indeed, the proof of this result does not use the assumption of [CM16, Section 4] that for some homogeneous generators q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent.

Lemma 2.4 ([CM16, Corollary 4.4,(i)]). (i) *The variety \mathcal{V}_* is equidimensional of dimension $r + 1 - \ell$.*

(ii) *For all irreducible component X of \mathcal{V}_* and for all z in \mathbb{k} , X is not contained in $\mathfrak{g}^f \times \{z\}$.*

Let \mathcal{N} be the nullvariety of $\varepsilon(A)_+$ in \mathfrak{g}^f . Then \mathcal{V} is the union of \mathcal{V}_* and $\mathcal{N} \times \{0\}$.

Lemma 2.5. (i) *All irreducible component of \mathcal{N} has dimension at least $r - \ell$ and all irreducible component of \mathcal{V} has dimension at least $r + 1 - \ell$.*

(ii) *Assume that \mathcal{N} has dimension $r - \ell$. Then for some homogeneous sequence $p_1, \dots, p_{r-\ell}$ in $S(\mathfrak{g}^e)_+$, the nullvariety of $t, p_1, \dots, p_{r-\ell}$ in \mathcal{V} is equal to $\{0\}$.*

Proof. (i) By Lemma 2.1,(ii), for some homogeneous sequence a_1, \dots, a_ℓ in A_+ , the elements t, a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} . Let b_1, \dots, b_m be a homogeneous sequence in A_+ , generating the ideal $S(\mathfrak{g}^\ell)[t]A_+$ of $S(\mathfrak{g}^\ell)[t]$. Set:

$$B := \mathbb{k}[a_1, \dots, a_\ell, b_1, \dots, b_m], \quad B_+ := Ba_1 + \dots + Ba_\ell + Bb_1 + \dots + Bb_m,$$

$$C := B[t], \quad C_{++} := B_+[t] + Ct.$$

Then B and C are graded subalgebras of A and B_+ and C_{++} are maximal ideals of B and C respectively. Moreover, C has dimension $\ell + 1$. We have a commutative diagram

$$\begin{array}{ccc} & \mathfrak{g}^f \times \mathbb{k} & \\ \alpha \swarrow & & \searrow \beta \\ \text{Specm}(C) & \xrightarrow{\pi} & \text{Specm}(B) \end{array}$$

with α, β, π the morphisms whose comorphisms are the canonical injections

$$C \longrightarrow S(\mathfrak{g}^\ell)[t], \quad B \longrightarrow S(\mathfrak{g}^\ell)[t], \quad B \longrightarrow C$$

respectively. Since C has dimension $\ell + 1$, the irreducible components of the fibers of α have dimension at least $r - \ell$, whence the result for \mathcal{N} since $\mathcal{N} \times \{0\} = \alpha^{-1}(C_{++})$. Moreover, $\mathcal{V} = \beta^{-1}(B_+)$ and $\pi^{-1}(B_+)$ is a subvariety of dimension 1 of $\text{Specm}(C)$. Hence all irreducible component of \mathcal{V} has dimension at least $r + 1 - \ell$.

(ii) Prove by induction on i that there exists a homogeneous sequence p_1, \dots, p_i in $S(\mathfrak{g}^\ell)_+$ such that the minimal prime ideals of $S(\mathfrak{g}^\ell)$ containing $\varepsilon(A)_+$ and p_1, \dots, p_i have height $\ell + i$. First of all, $S(\mathfrak{g}^\ell)\varepsilon(A)_+$ is graded. Then the minimal prime ideals of $S(\mathfrak{g}^\ell)$ containing $\varepsilon(A)_+$ are graded too. By, (i), $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ have height ℓ since \mathcal{N} has dimension $r - \ell$ by hypothesis. In particular, they are strictly contained in $S(\mathfrak{g}^\ell)_+$. Hence, by Lemma A.1,(ii), for some homogeneous element p_1 in $S(\mathfrak{g}^\ell)$, p_1 is not in the union of these ideals so that the statement is true for $i = 1$ by [Ma86, Ch. 5, Theorem 13.5]. Suppose that it is true for $i - 1$. Then the minimal prime ideals containing $\varepsilon(A)_+$ and p_1, \dots, p_{i-1} are graded and strictly contained in $S(\mathfrak{g}^\ell)_+$ by the induction hypothesis. So, by Lemma A.1,(ii), for some homogeneous element p_i in $S(\mathfrak{g}^\ell)$, p_i is not in the union of these ideals and the sequence p_1, \dots, p_i verify the condition of the statement by [Ma86, Ch. 5, Theorem 13.5]. For $i = r - \ell$, the nullvariety of $p_1, \dots, p_{r-\ell}$ in \mathcal{N} has dimension 0. Then it is equal to $\{0\}$ as the nullvariety of a graded ideal, whence the assertion since $\mathcal{N} \times \{0\}$ is the nullvariety of t in \mathcal{V} . \square

2.3. We assume in this subsection that \mathcal{N} has dimension $r - \ell$. Let $p_1, \dots, p_{r-\ell}$ be as in Lemma 2.5,(ii), and set

$$C := A[p_1, \dots, p_{r-\ell}].$$

Then $p_1, \dots, p_{r-\ell}$ are algebraically independent over A since \mathcal{N} has dimension $r - \ell$.

Lemma 2.6. *The ideal $S(\mathfrak{g}^e)[t]_+$ of $S(\mathfrak{g}^e)[t]$ is the radical of $S(\mathfrak{g}^e)[t]C_+$.*

Proof. Let Y be an irreducible component of the nullvariety of C_+ in $\mathfrak{g}^f \times \mathbb{k}$. Then Y has dimension at least 1. By definition the nullvariety of t in Y is equal to $\{0\}$. Hence Y has dimension 1. The gradation on $S(\mathfrak{g}^e)[t]$ induces an action of the one-dimensional multiplicative group G_m on $\mathfrak{g}^f \times \mathbb{k}$ such that for all (x, z) in $\mathfrak{g}^f \times \mathbb{k}$, $(0, z)$ is in the closure of the orbit of (x, z) under G_m . Since C_+ is graded, Y is invariant under G_m . As a result, $Y = \{0\} \times \mathbb{k}$ or for some x in $\mathfrak{g}^f \times \mathbb{k}$, Y is the closure of the orbit of $(x, 0)$ under G_m since 0 is the nullvariety of t in Y . In the last case, x is a zero of $p_1, \dots, p_{r-\ell}$ in \mathcal{N} , that is $x = 0$. Hence $Y = \{0\} \times \mathbb{k}$. As a result, the nullvariety of C_+ in $\mathfrak{g}^f \times \mathbb{k}$ is equal to $\{0\} \times \mathbb{k}$ that is the nullvariety of $S(\mathfrak{g}^e)[t]_+$, whence the assertion since $S(\mathfrak{g}^e)[t]_+$ is a prime ideal of $S(\mathfrak{g}^e)[t]$. \square

For \mathfrak{p} a prime ideal of A , denote by $A_{\mathfrak{p}}$ the localization of A at \mathfrak{p} and by $\bar{\mathfrak{p}}$ the ideal of C generated by \mathfrak{p} . Since C is a polynomial algebra over A , $\bar{\mathfrak{p}}$ is a prime ideal of C and the localization $C_{\bar{\mathfrak{p}}}$ of C at $\bar{\mathfrak{p}}$ is a localization of the polynomial algebra $A_{\mathfrak{p}}[p_1, \dots, p_{r-\ell}]$. According to [Ma86, Ch. 6, Theorem 17.4], if $C_{\bar{\mathfrak{p}}}$ is Cohen-Macaulay so is $A_{\mathfrak{p}}$ as a quotient of $C_{\bar{\mathfrak{p}}}$ by an ideal generated by a regular sequence.

Proof of Theorem 1.5. By Lemma 2.6 and Proposition A.2, the algebra C is finitely generated. Then A is finitely generated as a quotient of C . So, by Lemma A.7,(ii), A is a factorial ring and so is C as a polynomial ring over A . As a result, C is normal so that $S(\mathfrak{g}^e)[t]$ and C verify Conditions (1), (2), (3) of Proposition A.6. Hence by Proposition A.6, for all prime ideal \mathfrak{p} of A , containing t , $C_{\bar{\mathfrak{p}}}$ is Cohen-Macaulay, whence $A_{\mathfrak{p}}$ is Cohen-Macaulay. By Lemma 2.1,(ii), for \mathfrak{p} a prime ideal of A , not containing t , $A_{\mathfrak{p}}$ is the localization of $\mathbb{k}[t, t^{-1}, a_1, \dots, a_{\ell}]$ at the prime ideal generated by \mathfrak{p} . Hence $A_{\mathfrak{p}}$ is Cohen-Macaulay since the algebra $\mathbb{k}[t, t^{-1}, a_1, \dots, a_{\ell}]$ is regular. As a result A is Cohen-Macaulay. In particular, A verifies Conditions (1), (2), (3) of Subsection A.2. So, by Proposition A.10, A_* is a polynomial algebra. Then by Corollary 2.3, for some homogeneous generating sequence q_1, \dots, q_{ℓ} in $S(\mathfrak{g})^{\mathfrak{g}}$,

$$A_* = R_*[t^{-2d_1} \tau \circ \kappa(q_1), \dots, t^{-2d_{\ell}} \tau \circ \kappa(q_{\ell})].$$

Form the above equality, we deduce that any element of A is the product of an element of the algebra $R[t^{-2d_1} \tau \circ \kappa(q_1), \dots, t^{-2d_{\ell}} \tau \circ \kappa(q_{\ell})]$ by a polynomial in t with nonzero constant term, whence

$$A = R[t^{-2d_1} \tau \circ \kappa(q_1), \dots, t^{-2d_{\ell}} \tau \circ \kappa(q_{\ell})] \quad \text{and so} \quad \varepsilon(A) = \mathbb{k}[{}^e q_1, \dots, {}^e q_{\ell}]$$

since for $i = 1, \dots, \ell$,

$${}^e q_i := \varepsilon(t^{-2d_i} \tau \circ \kappa(q_i)).$$

Since $\mathcal{N} \times \{0\}$ is the nullvariety of t and A_+ in $\mathfrak{g}^f \times \mathbb{k}$, \mathcal{N} is the nullvariety in \mathfrak{g}^f of ${}^e q_1, \dots, {}^e q_\ell$. Hence ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} since \mathcal{N} has dimension $r - \ell$. \square

3. PROOF OF THEOREM 1.4

Let (e, h, f) be an \mathfrak{sl}_2 -triple in \mathfrak{g} . We use the notations κ and ${}^e p, p \in S(\mathfrak{g})^{\mathfrak{g}}$, as in Introduction. In this section, we use the standard gradations on $S(\mathfrak{g})$ and $S(\mathfrak{g}^e)$. Let A_0 be the subalgebra of $S(\mathfrak{g}^e)$ generated by the family ${}^e p, p \in S(\mathfrak{g})^{\mathfrak{g}}$, and let \mathcal{N}_0 be the nullvariety of $A_{0,+}$ in \mathfrak{g}^f where $A_{0,+}$ denotes the augmentation ideal of A_0 .

Let a_1, \dots, a_m be a homogeneous sequence in $A_{0,+}$ generating the ideal of $S(\mathfrak{g}^e)$ generated by $A_{0,+}$. According to [PPY07, Corollary 2.3], A_0 contains a homogeneous sequence b_1, \dots, b_ℓ algebraically free over \mathbb{k} .

Lemma 3.1. *Let \mathfrak{A} be the integral closure of $\mathbb{k}[a_1, \dots, a_m, b_1, \dots, b_\ell]$ in the fraction field of $S(\mathfrak{g}^e)$.*

- (i) *The algebra \mathfrak{A} is contained in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ and its fraction field is the fraction field of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$.*
- (ii) *Let a in $S(\mathfrak{g}^e)_+^{\mathfrak{g}^e}$. If a is equal to 0 on \mathcal{N}_0 , then a is in \mathfrak{A}_+ .*
- (iii) *The ideal $S(\mathfrak{g}^e)^{\mathfrak{g}^e} \mathfrak{A}_+$ of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is equal to \mathfrak{A}_+ .*
- (iv) *The algebra \mathfrak{A} is the integral closure of A_0 in the fraction field of $S(\mathfrak{g}^e)$.*

Proof. (i) Let K_0 be the field of invariant elements under the adjoint action of \mathfrak{g}^e in the fraction field of $S(\mathfrak{g}^e)$. According to [CM16, Lemma 3.1], K_0 is the fraction field of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Since $a_1, \dots, a_m, b_1, \dots, b_\ell$ are in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, \mathfrak{A} is contained in K_0 . Moreover, \mathfrak{A} is contained in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ since $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is integrally closed in K_0 . Since K_0 has transcendence degree ℓ over \mathbb{k} and since b_1, \dots, b_ℓ are algebraically independent over \mathbb{k} , K_0 is the fraction field of \mathfrak{A} .

(ii) Since \mathcal{N}_0 is the nullvariety of $a_1, \dots, a_m, b_1, \dots, b_\ell$ in \mathfrak{g}^f , \mathcal{N}_0 is the nullvariety of \mathfrak{A}_+ in \mathfrak{g}^f . Let a be in $S(\mathfrak{g}^e)_+^{\mathfrak{g}^e}$ such that a is equal to 0 on \mathcal{N}_0 . Since \mathcal{N}_0 is a cone, all homogeneous components of a is equal to 0 on \mathcal{N}_0 . So it suffices to prove the assertion for a homogeneous. We have a commutative diagram

$$\begin{array}{ccc}
 \mathfrak{g}^f & \xrightarrow{\pi} & \text{Specm}(\mathfrak{A}[a]) \\
 & \searrow \alpha & \swarrow \beta \\
 & & \text{Specm}(\mathfrak{A})
 \end{array}$$

with π, α, β the comorphisms of the canonical injections

$$\mathfrak{A}[a] \longrightarrow S(\mathfrak{g}^e), \quad \mathfrak{A} \longrightarrow S(\mathfrak{g}^e), \quad \mathfrak{A} \longrightarrow \mathfrak{A}[a].$$

Since \mathcal{N}_0 is the nullvariety of $\mathfrak{A}[a]_+$ and \mathfrak{A}_+ in \mathfrak{g}^f , $\beta^{-1}(\mathfrak{A}_+) = \mathfrak{A}[a]_+$. The gradations of \mathfrak{A} and $\mathfrak{A}[a]$ induce actions of G_m on $\text{Specm}(\mathfrak{A})$ and $\text{Specm}(\mathfrak{A}[a])$ such that β is equivariant. Moreover, \mathfrak{A}_+ is in the closure of all orbit under G_m in $\text{Specm}(\mathfrak{A})$.

Hence β is a quasi finite morphism. Moreover, β is a birational since \mathfrak{A} and $\mathfrak{A}[a]$ have the same fraction field by (i). Hence, by Zariski's main theorem [Mu88], β is an open immersion from $\text{Specm}(\mathfrak{A}[a])$ into $\text{Specm}(\mathfrak{A})$. So, β is surjective since \mathfrak{A}_+ is in the image of β and since it is in the closure of all G_m -orbit in $\text{Specm}(\mathfrak{A})$. As a result, β is an isomorphism and a is in \mathfrak{A} , whence the assertion.

(iii) For a in \mathfrak{A}_+ and b in $S(\mathfrak{g}^e)$, ab is equal to 0 on \mathcal{N}_0 . Hence ab is in \mathfrak{A}_+ by (ii).

(iv) By (ii), A_0 is contained in \mathfrak{A} . Moreover, since $a_1, \dots, a_m, b_1, \dots, b_\ell$ are in A_0 , \mathfrak{A} is contained in the integral closure of A_0 in the fraction field of $S(\mathfrak{g}^e)$, whence the assertion. \square

Corollary 3.2. *Suppose that the algebra $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is finitely generated. Then \mathfrak{A} is equal to $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$.*

Proof. Let C be the quotient of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ by the ideal $S(\mathfrak{g}^e)^{\mathfrak{g}^e}\mathfrak{A}_+$. By hypothesis, C is finitely generated. Then it has finitely many minimal prime ideals. Denote them by $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. For a in the radical of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}\mathfrak{A}_+$, a is equal to 0 on \mathcal{N}_0 . Moreover, it is in $S(\mathfrak{g}^e)_+^{\mathfrak{g}^e}$. Then, by Lemma 3.1,(ii), a is in \mathfrak{A}_+ . As a result, C is a reduced algebra and the canonical map

$$A_0 \longrightarrow A_0/\mathfrak{p}_1 \times \cdots \times A_0/\mathfrak{p}_m$$

is injective. Since \mathfrak{A} and $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ have the same fraction field, they have the same Krull dimension. Then, by Lemma 3.1,(iii), $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ are maximal ideals of C . As a result, the \mathbb{k} -algebra C is finite dimensional. Let V be a graded complement to $S(\mathfrak{g}^e)^{\mathfrak{g}^e}\mathfrak{A}_+$ in $S(\mathfrak{g}^e)$. According to Lemma 3.1,(iii), $S(\mathfrak{g}^e)^{\mathfrak{g}^e} = V\mathfrak{A}$ so that $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a finite extension of \mathfrak{A} . Since \mathfrak{A} is integrally closed in the fraction field of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, $\mathfrak{A} = S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. \square

Proof of Theorem 1.4. The “if” part results from [CM16, Theorem 1.5] (see here Theorem 1.3).

Suppose that e is good. By Definition 1.1 and Theorem 1.2, $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and the nullvariety of $S(\mathfrak{g}^e)_+^{\mathfrak{g}^e}$ in \mathfrak{g}^f is equidimensional of dimension $r - \ell$. On the other hand, by Lemma 3.1,(iv), \mathfrak{A} is the integral closure of A_0 in the fraction field of $S(\mathfrak{g}^e)$. Hence the nullvarieties of \mathfrak{A}_+ and $A_{0,+}$ in \mathfrak{g}^f are the same. But by Corollary 3.2, $\mathfrak{A} = S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, so \mathcal{N}_0 has dimension $r - \ell$ since e is good. On the other hand, A_0 is contained in $\varepsilon(A)$ by construction of $\varepsilon(A)$, and $\varepsilon(A)$ is contained in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ by [PPY07, Proposition 0.1], whence $\mathcal{N} = \mathcal{N}_0$.

As a result, \mathcal{N} has dimension $r - \ell$ and so by Theorem 1.5, for some homogeneous generating sequence q_1, \dots, q_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, the element ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over \mathbb{k} . \square

APPENDIX A. GENERAL FACTS ON COMMUTATIVE ALGEBRA

In this section t is an indeterminate and the base ring R is \mathbb{k} , $\mathbb{k}[t]$ or $\mathbb{k}[[t]]$. For M a graded space over \mathbb{N} and for j in \mathbb{N} , denote by $M^{[j]}$ the space of degree j and

by M_+ the sum of $M^{[j]}$, $j > 0$. Let A be a finitely generated graded R -algebra over \mathbb{N} such that $A^{[0]} = R$ and such that $A^{[j]}$ is a free R -module of finite rank for any $j \in \mathbb{N}$. Moreover, A is an integral domain. Denoting by $\dim A$ the Krull dimension of A , we set:

$$\ell := \begin{cases} \dim A & \text{if } R = \mathbb{k} \\ \dim A - 1 & \text{if } t \in R \end{cases}$$

As a rule, for B an integral algebra, we denote by $K(B)$ its fraction field.

A.1. Let B be a graded subalgebra of A .

- Lemma A.1.** (i) *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be pairwise different graded prime ideals contained in A_+ . If they are the minimal prime ideals containing their intersection, then for some homogeneous element p of A_+ , the element p is not in the union of $\mathfrak{p}_1, \dots, \mathfrak{p}_m$.*
- (ii) *For some homogeneous sequence p_1, \dots, p_ℓ in A_+ , A_+ is the radical of the ideal generated by p_1, \dots, p_ℓ .*
- (iii) *Suppose that A_+ is the radical of AB_+ . Then for some homogeneous sequence p_1, \dots, p_ℓ in B_+ , A_+ is the radical of the ideal generated by p_1, \dots, p_ℓ .*

Proof. (i) Prove by induction on j that for some homogeneous element p_j of A_+ , p_j is not in the union of $\mathfrak{p}_1, \dots, \mathfrak{p}_j$. Since \mathfrak{p}_1 is a graded ideal strictly contained in A_+ , it is true for $j = 1$. Suppose that it is true for $j - 1$. If p_{j-1} is not in \mathfrak{p}_j , there is nothing to prove. Suppose that p_{j-1} is in \mathfrak{p}_j . According to the hypothesis, \mathfrak{p}_j is strictly contained in A_+ and it does not contain the intersection of $\mathfrak{p}_1, \dots, \mathfrak{p}_{j-1}$. So, since $\mathfrak{p}_1, \dots, \mathfrak{p}_j$ are graded ideals, for some homogeneous sequence r, q in A_+ ,

$$r \in \bigcap_{k=1}^{j-1} \mathfrak{p}_k \setminus \mathfrak{p}_j, \quad \text{and} \quad q \in A_+ \setminus \mathfrak{p}_j.$$

Denoting by m and n the respective degrees of p_{j-1} and rq , $p_{j-1}^n + (rq)^m$ is homogeneous of degree mn and it is not in $\mathfrak{p}_1, \dots, \mathfrak{p}_j$ since these ideals are prime.

(ii) Prove by induction on i that for some homogeneous sequence p_1, \dots, p_i in A_+ , the minimal prime ideals of A containing p_1, \dots, p_i have height i . Let p_1 be in $A_+ \setminus \{0\}$. By [Ma86, Ch. 5, Theorem 13.5], all minimal prime ideal containing p_1 has height 1. Suppose that it is true for $i - 1$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals containing p_1, \dots, p_{i-1} . Since A_+ has height $\ell > i - 1$, A_+ strictly contains $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. By (i), there exists a homogeneous element p_i in A_+ not in the union of $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. Then, by [Ma86, Ch. 5, Theorem 13.5], the minimal prime ideals containing p_1, \dots, p_i have height i . For $i = \ell$, the minimal prime ideals containing p_1, \dots, p_ℓ have height ℓ . Hence they are equal to A_+ since A_+ is a prime ideal of height ℓ containing p_1, \dots, p_ℓ , whence the assertion.

(iii) The ideal AB_+ is generated by a homogeneous sequence a_1, \dots, a_m in B_+ . Denote by B' the subalgebra of A generated by a_1, \dots, a_m . Then B' is a finitely

generated graded subalgebra of A such that A_+ is the radical of AB'_+ . If $R = \mathbb{k}$, denote by d its dimension and if $t \in R$, denote by $d + 1$ its dimension. By (ii), for some homogeneous sequence p_1, \dots, p_d in B'_+ , B'_+ is the radical of the ideal generated by p_1, \dots, p_d . Then A_+ is the radical of the ideal of A generated by p_1, \dots, p_d . Since A_+ has height ℓ , $\ell \leq d$ by [Ma86, Ch. 5, Theorem 3.5]. Since B' is a subalgebra of A , its dimension is at most $\dim A$. Hence $d = \ell$. \square

Proposition A.2. *Suppose that A_+ is the radical of AB_+ . Then B is finitely generated and A is a finite extension of B .*

Proof. Since A is a noetherian ring, for some homogeneous sequence a_1, \dots, a_m in B_+ , AB_+ is the ideal generated by this sequence. Denote by C the R -subalgebra of A generated by a_1, \dots, a_m . Then C is a graded subalgebra of A . Denote by π the morphism

$$\mathrm{Specm}(A) \xrightarrow{\pi} \mathrm{Specm}(C)$$

whose comorphism is the canonical injection $C \rightarrow A$. Let \bar{A} and \bar{C} be the integral closures of A and C in $K(A)$. Since C is contained in A , \bar{C} is contained in \bar{A} . Let α and β be the morphisms

$$\mathrm{Specm}(\bar{A}) \xrightarrow{\alpha} \mathrm{Specm}(A) \quad \text{and} \quad \mathrm{Specm}(\bar{C}) \xrightarrow{\beta} \mathrm{Specm}(C)$$

whose comorphisms are the canonical injections $A \rightarrow \bar{A}$ and $C \rightarrow \bar{C}$ respectively. Then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Specm}(\bar{A}) & \xrightarrow{\bar{\pi}} & \mathrm{Specm}(\bar{C}) \\ \alpha \downarrow & & \downarrow \beta \\ \mathrm{Specm}(A) & \xrightarrow{\pi} & \mathrm{Specm}(C) \end{array}$$

with $\bar{\pi}$ the morphism whose comorphism is the canonical injection $\bar{C} \rightarrow \bar{A}$.

The action of $G_m(\mathbb{k})$ in A extends to an action of $K(A)$, and \bar{A} is invariant under this action. Denoting by \bar{R} the integral closure of R in $K(A)$, \bar{R} is the set of fixed points under the action of $G_m(\mathbb{k})$ in \bar{A} . Since C is invariant under $G_m(\mathbb{k})$ so is \bar{C} . For \mathfrak{m} a maximal ideal of \bar{R} , the ideal $\mathfrak{m} + \bar{C}_+$ is the maximal ideal of \bar{C} containing \mathfrak{m} and invariant under $G_m(\mathbb{k})$. Then, for \mathfrak{p} a maximal ideal of \bar{C} , $\mathfrak{p} \cap \bar{R} + \bar{C}_+$ is in the closure of the orbit of \mathfrak{p} under $G_m(\mathbb{k})$. Moreover,

$$\{\mathfrak{m} + \bar{A}_+\} = \bar{\pi}^{-1}\{\mathfrak{m} + \bar{C}_+\}$$

for all maximal ideal \mathfrak{m} of \bar{R} . Hence $\bar{\pi}$ is quasi finite. Moreover $\bar{\pi}$ is birational. Then, by Zariski's main theorem [Mu88], $\bar{\pi}$ is an open immersion. As a result, $\bar{\pi}$ is surjective. Hence $\bar{\pi}$ is an isomorphism and $\bar{A} = \bar{C}$. As a result, \bar{A} is a finite extension of C since β is a finite morphism. As submodules of the finite module \bar{A} over the noetherian ring C , A and B are finite C -modules. Hence A is a finite

extension of B . Denoting by $\omega_1, \dots, \omega_d$ a generating family of the C -module B , B is the subalgebra of A generated by $a_1, \dots, a_m, \omega_1, \dots, \omega_d$. \square

Denote by $\mathbb{k}[t]_*$ the localization of $\mathbb{k}[t]$ at the prime ideal $t\mathbb{k}[t]$ and set:

$$R_* := \begin{cases} \mathbb{k} & \text{if } R = \mathbb{k} \\ \mathbb{k}[t]_* & \text{if } R = \mathbb{k}[t] \\ R & \text{if } R = \mathbb{k}[[t]] \end{cases}, \quad \widehat{R} := \begin{cases} \mathbb{k} & \text{if } R = \mathbb{k} \\ \mathbb{k}[[t]] & \text{if } R = \mathbb{k}[t] \\ R & \text{if } R = \mathbb{k}[[t]] \end{cases}.$$

For M a R -module, set $\widehat{M} := \widehat{R} \otimes_R M$.

Lemma A.3. *Suppose $R = \mathbb{k}[t]$. Let M be a torsion free R -module and let N be a submodule of M . Then for a in $\widehat{N} \cap M$, ra is in N for some r in R such that $r(0) \neq 0$.*

Proof. Since M is torsion free, the canonical map $M \rightarrow \widehat{M}$ is an embedding. Moreover, the canonical map $\widehat{N} \rightarrow \widehat{M}$ is an embedding since \widehat{R} is flat over R . Let a be in $\widehat{N} \cap M$ and let \bar{a} be its image in M/N by the quotient map. Denote by J_a the annihilator of \bar{a} in R , whence a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{d} & M & \xrightarrow{d} & M/N \longrightarrow 0 \\ & & & & \uparrow \delta & & \uparrow \delta \\ 0 & \longrightarrow & J_a & \xrightarrow{d} & R & \xrightarrow{d} & R\bar{a} \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

with exact lines and columns. Since \widehat{R} is a flat extension of R , tensoring this diagram by \widehat{R} gives the following diagram with exact lines and columns:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{N} & \xrightarrow{d} & \widehat{M} & \xrightarrow{d} & \widehat{R} \otimes_R M/N \longrightarrow 0 \\ & & & & \uparrow \delta & & \uparrow \delta \\ 0 & \longrightarrow & \widehat{R}J_a & \xrightarrow{d} & \widehat{R} & \xrightarrow{d} & \widehat{R}\bar{a} \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

For b in \widehat{R} , $(\delta \circ d)b = (d \circ \delta)b = 0$ since a is in \widehat{N} by (i), whence $db = 0$. As a result, $\widehat{R}J_a = \widehat{R}$. Hence J_a contains an element r , invertible in \widehat{R} , that is $r(0) \neq 0$, whence the lemma. \square

For $p = (p_1, \dots, p_\ell)$ a homogeneous sequence in A_+ , we set:

$$\underline{p} := \begin{cases} p & \text{if } R = \mathbb{k} \\ (t, p_1, \dots, p_\ell) & \text{if } R = \mathbb{k}[[t]] \end{cases}, \quad A_* := R_* \otimes_R A, \quad \widehat{A} := \widehat{R} \otimes_R A,$$

and we denote by $J_{\underline{p}}$ the ideal of A generated by the sequence \underline{p} . Since $A^{[0]} = R$, the gradation on A extends to a gradation on A_* and \widehat{A} such that $A_*^{[0]} = R_*$ and $\widehat{A}^{[0]} = \widehat{R}$. When $R = \mathbb{k}$ or $R = \mathbb{k}[[t]]$, then $A_* = A$ and $\widehat{A} = A$.

Lemma A.4. *Suppose that A is Cohen-Macaulay. Let $p = p_1, \dots, p_\ell$ be a homogeneous sequence in A_+ such that A_+ is the radical of the ideal of A generated by p_1, \dots, p_ℓ and let V be a graded complement in A to the \mathbb{k} -subspace $J_{\underline{p}}$.*

- (i) *The space V has finite dimension.*
- (ii) *The space A_* is equal to $VR_*[p_1, \dots, p_\ell]$.*
- (iii) *The algebra A is a flat extension of $R[p_1, \dots, p_\ell]$.*
- (iv) *For all homogeneous sequence a_1, \dots, a_n in A , linearly free over \mathbb{k} modulo $J_{\underline{p}}$, a_1, \dots, a_n is linearly free over $R[p_1, \dots, p_\ell]$.*
- (v) *The linear map*

$$V \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow A_*, \quad v \otimes a \longmapsto va$$

is an isomorphism.

Proof. According to Lemma A.1(ii), the sequence p does exist.

(i) Let J_p be the ideal of A generated by p_1, \dots, p_ℓ . Since A_+ is the radical of J_p , $A^{[d]} = J_p^{[d]}$ for d sufficiently big. When $t \in R$, for all d , then $tA^{[d]}$ has finite codimension in $A^{[d]}$ since $A^{[d]}$ is a finite free R -module. Hence $J_{\underline{p}}$ has finite codimension in A so that V has finite dimension.

(ii) Suppose that t is in R . First of all, we prove by induction on d the inclusion

$$A^{[d]} \subset (VR[p_1, \dots, p_\ell])^{[d]} + tA^{[d]}$$

when $t \in R$. Since $A^{[0]}$ is the direct sum of $V^{[0]}$ and $J_{\underline{p}}^{[0]}$, $V^{[0]}$ is contained in $\mathbb{k} + tR$, whence the inclusion for $d = 0$. Suppose that it is true for all j smaller than d . Since p_1, \dots, p_ℓ have positive degree, by induction hypothesis,

$$J_{\underline{p}}^{[d]} \subset (VR[p_1, \dots, p_\ell])^{[d]} + tA^{[d]},$$

whence the inclusion for d . Then, by induction on m ,

$$A^{[d]} \subset (VR[p_1, \dots, p_\ell])^{[d]} + t^m A^{[d]}.$$

As a result, since $A^{[d]}$ is a finite R -module,

$$A^{[d]} \subset (V\widehat{R}[p_1, \dots, p_\ell])^{[d]},$$

whence $\widehat{A} = V\widehat{R}[p_1, \dots, p_\ell]$. This equality remains true when $R = \mathbb{k}$ by an analogous and simpler argument.

When $R = \mathbb{k}[[t]]$, according to Lemma A.3, for a in A , ra is in $VR[p_1, \dots, p_\ell]$ for some r in R such that $r(0) \neq 0$ by Lemma A.3. As a result, $A_* = VR_*[p_1, \dots, p_\ell]$.

(iii) By Proposition A.2, A is a finite extension of $R[p_1, \dots, p_\ell]$. In particular, $R[p_1, \dots, p_\ell]$ has dimension $\ell + \dim R$ so that p_1, \dots, p_ℓ are algebraically independent over R . Hence $R[p_1, \dots, p_\ell]$ is a regular algebra, whence the assertion by [Ma86, Ch. 8, Theorem 23.1].

(iv) Prove the assertion by induction on n . Since A is an integral domain, the assertion is true for $n = 1$. Suppose the assertion true for $n - 1$. Let (b_1, \dots, b_n) be a homogeneous sequence in $R[p_1, \dots, p_\ell]$ such that

$$b_1 a_1 + \dots + b_n a_n = 0.$$

Let K and I be the kernel and the image of the linear map

$$R[p_1, \dots, p_\ell]^n \longrightarrow R[p_1, \dots, p_\ell], \quad (c_1, \dots, c_n) \longmapsto c_1 b_1 + \dots + c_n b_n,$$

whence the short exact sequence of $R[p_1, \dots, p_\ell]$ modules

$$0 \longrightarrow K \longrightarrow R[p_1, \dots, p_\ell]^n \longrightarrow I \longrightarrow 0.$$

The gradation of $R[p_1, \dots, p_\ell]$ induces a gradation of $R[p_1, \dots, p_\ell]^n$ and K is a graded submodule of $R[p_1, \dots, p_\ell]^n$ since b_1, \dots, b_n is a homogeneous sequence in $R[p_1, \dots, p_\ell]$. Denote by y_1, \dots, y_m a generating homogeneous sequence of the $R[p_1, \dots, p_\ell]$ -module K . By (iii), the short sequence of A -modules

$$0 \longrightarrow A \otimes_{R[p_1, \dots, p_\ell]} K \longrightarrow A^n \longrightarrow A \otimes_{R[p_1, \dots, p_\ell]} I \longrightarrow 0.$$

is exact. So, for some homogeneous sequence x_1, \dots, x_m in A ,

$$a_i = \sum_{j=1}^m x_j y_{j,i}$$

for $i = 1, \dots, n$. Since a_n is not in $J_{\underline{p}}$, for some j_* , the element $y_{j_*,i}$ is an invertible element of R_* , whence

$$b_n y_{j_*,n} = - \sum_{i=1}^{n-1} b_i y_{j_*,i} \quad \text{and} \quad \sum_{i=1}^{n-1} b_i (y_{j_*,n} a_i - a_n y_{j_*,i}) = 0.$$

So, by induction hypothesis,

$$b_1 = \dots = b_{n-1} = 0$$

since the sequence

$$y_{j_*,n} a_1 - a_n y_{j_*,1}, \dots, y_{j_*,n} a_{n-1} - a_n y_{j_*,n-1}$$

is linearly free over \mathbb{k} modulo $J_{\underline{p}}$. Then $b_n = 0$ since $a_n \neq 0$.

(v) Let (v_1, \dots, v_n) be a homogeneous basis of V . Since the space of relations of linear dependence over $R[p_1, \dots, p_\ell]$ of v_1, \dots, v_n is graded, it is equal to $\{0\}$ by (iv), whence the assertion by (ii). \square

- Corollary A.5.** (i) *The algebra A_* is Cohen-Macaulay if and only if for some homogeneous sequence p_1, \dots, p_ℓ in A_+ , the algebra A_* is a finite free extension of $R_*[p_1, \dots, p_\ell]$.*
- (ii) *Suppose that A_* is Cohen-Macaulay. For a homogeneous sequence q_1, \dots, q_ℓ in A_+ , A_* is a finite free extension of $R_*[q_1, \dots, q_\ell]$ if and only if R_*A_+ is the radical of the ideal of A_* generated by q_1, \dots, q_ℓ .*

Proof. (i) The “only if” part results from Lemma A.4,(v). Suppose that for some homogeneous sequence p_1, \dots, p_ℓ in A_+ , the algebra A_* is a finite free extension of $R_*[p_1, \dots, p_\ell]$. In particular, $R_*[p_1, \dots, p_\ell]$ is a polynomial algebra over R_* since A_* has dimension $\dim A$. Let \mathfrak{p} be a prime ideal of A_* and let \mathfrak{q} be its intersection with $R_*[p_1, \dots, p_\ell]$. Denote by $A_{\mathfrak{p}}$ and $R[p_1, \dots, p_\ell]_{\mathfrak{q}}$ the localizations of A_* and $R_*[p_1, \dots, p_\ell]$ at \mathfrak{p} and \mathfrak{q} respectively. Since A_* is a finite extension of $R_*[p_1, \dots, p_\ell]$, these local rings have the same dimension. Denote by d this dimension. By flatness, all regular sequence a_1, \dots, a_d in $R[p_1, \dots, p_\ell]_{\mathfrak{q}}$ is regular in $A_{\mathfrak{p}}$ so that $A_{\mathfrak{p}}$ is Cohen-Macaulay. Hence A_* is Cohen-Macaulay.

(ii) The “only if” part results from (i) and Proposition A.2. Suppose that A_* is a finite free extension of $R_*[q_1, \dots, q_\ell]$. Let \mathfrak{p} be a minimal prime ideal of A_* containing q_1, \dots, q_ℓ and let \mathfrak{q} be its intersection with $R_*[q_1, \dots, q_\ell]$. Then \mathfrak{q} is generated by q_1, \dots, q_ℓ . In particular it has height ℓ . So \mathfrak{p} has height ℓ since A_* is a finite extension of $R_*[q_1, \dots, q_\ell]$. As a result, $\mathfrak{p} = R_*A_+$ since R_*A_+ is a prime ideal of height ℓ , containing q_1, \dots, q_ℓ , whence the assertion. \square

Set $B_* := R_* \otimes_R B$ and for \mathfrak{p} a prime ideal of B denote by $B_{\mathfrak{p}}$ its localization at \mathfrak{p} .

Proposition A.6. *Suppose that the following conditions are verified:*

- (1) *B is normal,*
 - (2) *A_+ is the radical of AB_+ ,*
 - (3) *A is Cohen-Macaulay.*
- (i) *Let p_1, \dots, p_ℓ be a homogeneous sequence in B_+ such that B_+ is the radical of the ideal of B generated by this sequence. Then for some graded subspace V of A , having finite dimension, the linear morphisms*

$$V \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow A_*, \quad v \otimes a \longmapsto va,$$

$$(V \cap B) \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow B_*, \quad v \otimes a \longmapsto va$$

are isomorphisms.

- (ii) *If $R = \mathbb{k}$ or $R = \mathbb{k}[[t]]$, the algebra B_* is Cohen-Macaulay.*
- (iii) *For \mathfrak{p} prime ideal of B , containing t , the local ring $B_{\mathfrak{p}}$ is Cohen-Macaulay.*

Proof. (i) By Proposition A.2 and by Condition (2), B is finitely generated and A is a finite extension of B . By Condition (2) and by Lemma A.1,(iii), for some

homogeneous sequence (p_1, \dots, p_ℓ) in B_+ , A_+ is the radical of the ideal generated by p_1, \dots, p_ℓ . Set:

$$\underline{p} := \begin{cases} t, p_1, \dots, p_\ell & \text{if } \dim A = \ell + 1 \\ p_1, \dots, p_\ell & \text{if } \dim A = \ell \end{cases}$$

Denote by m the degree of the extension $K(A)$ of $K(B)$. For a in A_* , set:

$$a^\# := \frac{1}{m} \text{tr } a$$

with tr the trace map. By Condition (1), B_* is normal and the map $a \mapsto a^\#$ is a projection from A_* onto B_* whose restriction to A is a projection onto B . Moreover, it is a graded morphism of B -modules. Let M be its kernel. Let J_0 and J be the ideals of B and A generated by \underline{p} respectively. Since t, p_1, \dots, p_ℓ are in B , J is the direct sum of J_0 and MJ_0 . Let \bar{V}_0 be a graded complement in B to the \mathbb{k} -space J_0 and let V_1 be a graded complement in M to the \mathbb{k} -space MJ_0 . Setting $V := \bar{V}_0 + V_1$, V is a graded complement in A to the \mathbb{k} -space J . By Condition (3) and Lemma A.4, V has finite dimension and the linear map

$$V \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow A_*, \quad v \otimes a \mapsto va$$

is an isomorphism. So, since $V_0 = V^\#$, the linear map

$$V_0 \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow B_*, \quad v \otimes a \mapsto va$$

is an isomorphism, whence the assertion.

(ii) results from (i) and Corollary A.5.

(iii) By (i) and Corollary A.5, A_* is Cohen-Macaulay. For \mathfrak{p} a prime ideal of B , containing t , $B_{\mathfrak{p}}$ is the localization of B_* at the prime ideal $B_*\mathfrak{p}$, whence the assertion by (ii). \square

A.2. In this subsection $R = \mathbb{k}[t]$ and set $\widehat{R} = \mathbb{k}[[t]]$. For M a graded module over R such that $M^{[j]}$ is a free submodule of finite rank for all j , we denote by $P_{M,R}(T)$ its Hilbert series:

$$P_{M,R}(T) := \sum_{j \in \mathbb{N}} \text{rk } M^{[j]} T^j.$$

For V a graded space over \mathbb{k} such that $V^{[j]}$ has finite dimension, we denote by $P_{V,\mathbb{k}}(T)$ its Hilbert series:

$$P_{V,\mathbb{k}}(T) := \sum_{j \in \mathbb{N}} \dim V^{[j]} T^j.$$

Let S be a graded polynomial algebra over \mathbb{k} such that $S^{[0]} = \mathbb{k}$ and $S^{[j]}$ has finite dimension for all j . Consider on $S[t]$ and $S[[t]]$ the gradations extending that of S and such that t has degree 0. Consider the following conditions on A :

- (1) A is graded subalgebra of $S[t]$,

- (2) for some homogeneous sequence a_1, \dots, a_ℓ in A_+ , $A = \mathbb{k}[t, t^{-1}, a_1, \dots, a_\ell] \cap S[t]$,
- (3) A is Cohen-Macaulay.

If Condition (2) holds, then $A[t^{-1}] = R[a_1, \dots, a_\ell][t^{-1}]$. Moreover, if so, since A has dimension $\ell + 1$, then the elements t, a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} . Set $\widehat{A} := \widehat{R} \otimes_R A$.

Lemma A.7. *Assume that Conditions (1) and (2) hold.*

- (i) *The element t is a prime element of A .*
- (ii) *The algebra A is a factorial ring.*
- (iii) *The Hilbert series of the R -module A is equal to*

$$P_{A,R}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}},$$

with d_1, \dots, d_ℓ the degrees of a_1, \dots, a_ℓ respectively.

Proof. (i) Let a and b be in A such that ab is in tA . Since $tS[t]$ is a prime ideal of $S[t]$, a or b is in $tS[t]$. Suppose $a = ta'$ for some a' in $S[t]$. Then a' is in $A[t^{-1}]$. By Condition (2), $A[t^{-1}] = R[a_1, \dots, a_\ell][t^{-1}]$. Hence a' is in A by Condition (2) again. As a result, tA is a prime ideal of A .

(ii) Since A is finitely generated, it suffices to prove that all prime ideal of height 1 is principal by [Ma86, Ch. 7, Theorem 20.1]. Let \mathfrak{p} be a prime ideal of height 1. If t is in \mathfrak{p} , then $\mathfrak{p} = tA$ by (i). Suppose that t is not in \mathfrak{p} and set $\overline{\mathfrak{p}} = A[t^{-1}]\mathfrak{p}$. Then $\overline{\mathfrak{p}}$ is a prime ideal of height 1 of $R[a_1, \dots, a_\ell][t^{-1}]$ by Condition (2). For a in $\overline{\mathfrak{p}}$, $t^m a$ is in \mathfrak{p} for some nonnegative integer m . Hence

$$\mathfrak{p} = \overline{\mathfrak{p}} \cap A$$

since \mathfrak{p} is prime. As a polynomial ring over the principal ring $\mathbb{k}[t, t^{-1}]$, the ring $R[a_1, \dots, a_\ell][t^{-1}]$ is a factorial ring. Then $\overline{\mathfrak{p}}$ is generated by an element a in \mathfrak{p} . Since S is a polynomial ring, $S[t]$ is a factorial ring. So, for some nonnegative integer m and for some a' in $S[t]$, prime to t , $a = t^m a'$. By Condition (2), a' is in A . Then a' is an element of \mathfrak{p} , generating $\overline{\mathfrak{p}}$ and not divisible by t in A . Let b and c be in A such that bc is in Aa' . Then b or c is in $A[t^{-1}]a'$. Suppose b in $A[t^{-1}]a'$. So, for some l in \mathbb{N} , $t^l b = b'a'$ for some b' in A . We choose l minimal verifying this condition. By (i), since a' is not divisible by t in A , b' is divisible by t in A if $l > 0$. By minimality of l , $l = 0$ and b is in Aa' . As a result, Aa' is a prime ideal and $\mathfrak{p} = Aa'$ since \mathfrak{p} has height 1.

(iii) By Condition (2),

$$A[t^{-1}] = \mathbb{k}[t, t^{-1}] \otimes_{\mathbb{k}} \mathbb{k}[a_1, \dots, a_\ell] \quad \text{whence} \quad \text{rk} A^{[d]} = \dim \mathbb{k}[a_1, \dots, a_\ell]^{[d]}$$

for all nonnegative integer d . Since a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} ,

$$P_{\mathbb{k}[a_1, \dots, a_\ell], \mathbb{k}}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}},$$

whence the assertion. \square

Let p_1, \dots, p_ℓ be a homogeneous sequence in A such that A_+ is the radical of the ideal of A generated by this sequence. By Lemma A.1,(ii), such a sequence does exist. Denote by C the integral closure of $\mathbb{k}[p_1, \dots, p_\ell]$ in $\mathbb{k}(t, a_1, \dots, a_\ell)$.

Lemma A.8. *Assume that Conditions (1), (2) and (3) hold.*

- (i) *The algebra C is a graded subalgebra of A and t is not algebraic over C .*
- (ii) *The algebra C is Cohen-Macaulay. Moreover, C is a finite free extension of $\mathbb{k}[p_1, \dots, p_\ell]$.*
- (iii) *The algebra $C + tA$ is normal.*

Proof. (i) By Lemma A.7,(ii), A is a normal ring such that $K(A) = \mathbb{k}(t, a_1, \dots, a_\ell)$ by Condition (2). Then C is contained in A since $\mathbb{k}[p_1, \dots, p_\ell]$ is contained in A . Moreover, C is a graded algebra since so is $\mathbb{k}[p_1, \dots, p_\ell]$. By Proposition A.2, A is a finite extension of $R[p_1, \dots, p_\ell]$. So, since A has dimension $\ell + 1$, the elements t, p_1, \dots, p_ℓ are algebraically independent over \mathbb{k} . As a result, t is not algebraic over C .

(ii) By (i), $C[t]$ is a polynomial ring over C . So, by [Ma86, Ch. 8, Corollary of Theorem 23.9], $C[[t]]$ is a normal ring. By definition, A_+ is the radical of the ideal of A generated by p_1, \dots, p_ℓ . So \widehat{A}_+ is the radical of $AC[[t]]_+$ since p_1, \dots, p_ℓ are in $C[[t]]_+$. Since \widehat{R} is a flat extension of R , the algebra \widehat{A} is Cohen-Macaulay by Condition (3). Then, by Proposition A.6,(ii), $C[[t]]$ is Cohen-Macaulay. Let V be a graded complement in C to the ideal of C generated by p_1, \dots, p_ℓ . Since t is not algebraic over C , the space V is a complement in $C[t]$ to the ideal of $C[t]$ generated by t, p_1, \dots, p_ℓ . Then, by Lemma A.4, V has finite dimension and the linear morphism

$$V \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] \longrightarrow R_*C, \quad v \otimes a \longmapsto va$$

is an isomorphism. As a result, the linear morphism

$$V \otimes_{\mathbb{k}} \mathbb{k}[p_1, \dots, p_\ell] \longrightarrow C, \quad v \otimes a \longmapsto va$$

is an isomorphism, whence the assertion by Corollary A.5,(ii).

(iii) Set $\tilde{A} := C + tA$. At first, \tilde{A} is a graded subalgebra of A since C is a graded algebra and tA is a graded ideal of A . According to Proposition A.6,(i), for some graded subspace V of A , having finite dimension, the linear morphisms

$$\begin{aligned} V \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] &\longrightarrow A_*, & v \otimes a &\longmapsto va, \\ (V \cap C[t]) \otimes_{\mathbb{k}} R_*[p_1, \dots, p_\ell] &\longrightarrow R_*C, & v \otimes a &\longmapsto va \end{aligned}$$

are isomorphisms. Let v_1, \dots, v_n be a basis of V such that v_1, \dots, v_m is a basis of $V \cap C[t]$. For a in A_* , the element a has unique expansion

$$a = v_1 a_1 + \dots + v_n a_n$$

with a_1, \dots, a_n in $R_*[p_1, \dots, p_\ell]$. If a is in tA_* , a_1, \dots, a_n are in $tR_*[p_1, \dots, p_\ell]$ and if a is in R_*C , a_1, \dots, a_m are in $\mathbb{k}[p_1, \dots, p_\ell]$ and a_{m+1}, \dots, a_n are equal to 0, whence $R_*C \cap tA_* = tR_*C$ and $C \cap tA = \{0\}$. In particular, C is the quotient of \tilde{A} by $t\tilde{A}$.

For \mathfrak{p} a prime ideal of \tilde{A} , denote by $\tilde{A}_{\mathfrak{p}}$ the localization of \tilde{A} at \mathfrak{p} . If t is not in \mathfrak{p} , then $A[t^{-1}]$ is contained in $\tilde{A}_{\mathfrak{p}}$ so that $\tilde{A}_{\mathfrak{p}}$ is a localization of the regular algebra $R[a_1, \dots, a_\ell][t^{-1}]$ by Condition (2). Hence $\tilde{A}_{\mathfrak{p}}$ is a regular local algebra. Suppose that t is in \mathfrak{p} . Denote by $\bar{\mathfrak{p}}$ the image of \mathfrak{p} in C by the quotient map. Then $\tilde{A}_{\mathfrak{p}}/t\tilde{A}_{\mathfrak{p}}$ is the localization $C_{\bar{\mathfrak{p}}}$ of C at the prime ideal $\bar{\mathfrak{p}}$. Since C is Cohen-Macaulay, so are $C_{\bar{\mathfrak{p}}}$ and $\tilde{A}_{\mathfrak{p}}$. As a result, \tilde{A} is Cohen-Macaulay.

Let \mathfrak{p} be a prime ideal of height 1 of \tilde{A} . If t is not in \mathfrak{p} , $\tilde{A}_{\mathfrak{p}}$ is a regular local algebra as it is already mentioned. Suppose that t is in \mathfrak{p} . By Lemma A.7,(i), $t\tilde{A} = \mathfrak{p}$ so that all element of $C \setminus \{0\}$ is invertible in $\tilde{A}_{\mathfrak{p}}$, whence

$$\tilde{A}_{\mathfrak{p}} = K(C) + t\tilde{A}_{\mathfrak{p}} \quad \text{and} \quad t\tilde{A}_{\mathfrak{p}} = tK(C) + t^2\tilde{A}_{\mathfrak{p}}.$$

Hence $\tilde{A}_{\mathfrak{p}}$ is a regular local ring of dimension 1. As a result, \tilde{A} is regular in codimension 1. Then, by Serre's normality criterion [B98, §1, n°10, Théorème 4], \tilde{A} is normal since \tilde{A} is Cohen-Macaulay. \square

Corollary A.9. *Assume that Conditions (1), (2) and (3) hold.*

- (i) *The algebra \widehat{A} is equal to $C[[t]]$.*
- (ii) *For a in A , the element ra is in $C[t]$ for some r in $\mathbb{k}[t]$ such that $r(0) \neq 0$.*

Proof. (i) Since tA is contained in A , we have $K(A) = K(\tilde{A})$. Since C_+ is contained in \tilde{A}_+ , A_+ is the radical of $A\tilde{A}_+$. Then, by Proposition A.2, A is a finite extension of \tilde{A} . So, by Lemma A.8,(iii), $A = \tilde{A}$ and by induction on m ,

$$A \subset C[t] + t^m A$$

for all positive integer m . Since A and $C[t]$ are graded and since the R -module $A^{[d]}$ is finitely generated for all d , $\widehat{A} = C[[t]]$.

- (ii) The assertion results from (i) and Lemma A.3. \square

Proposition A.10. *Assume that Conditions (1), (2) and (3) hold. Then the algebra A_* is polynomial over R_* . Moreover, for some homogeneous sequence q_1, \dots, q_ℓ in A_+ such that q_1, \dots, q_ℓ have degree d_1, \dots, d_ℓ respectively, $A_* = R_*[q_1, \dots, q_\ell]$.*

Proof. According to Corollary A.9 and Lemma A.8,(i), it suffices to prove that C is a polynomial algebra over \mathbb{k} generated by a homogeneous sequence q_1, \dots, q_ℓ such

that q_1, \dots, q_ℓ have degree d_1, \dots, d_ℓ respectively. According to Corollary A.9,(i), Lemma A.8,(i) and Lemma A.7,(iii),

$$P_{C,\mathbb{k}}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}}.$$

By Corollary A.9,(ii), for $i = 1, \dots, \ell$, for some r_i in R such that $r_i(0) \neq 0$, $r_i a_i$ has an expansion

$$r_i a_i = \sum_{m \in \mathbb{N}} c_{i,m} t^m$$

with $c_{i,m}, m \in \mathbb{N}$ in $C^{[d_i]}$, with finite support. For z in \mathbb{k} and $i = 1, \dots, \ell$, set:

$$b_i(z) = \sum_{m \in \mathbb{N}} c_{i,m} z^m$$

so that $b_i(z)$ is in $C^{[d_i]}$ for all z . As already mentioned, t, a_1, \dots, a_ℓ are algebraically independent over \mathbb{k} by Condition (2) since A has dimension $\ell + 1$. Then, so are $r_1 a_1, \dots, r_\ell a_\ell$ and for some z in \mathbb{k} , $b_1(z), \dots, b_\ell(z)$ are algebraically independent over \mathbb{k} . Denoting by C' the subalgebra of C generated by this sequence,

$$P_{C',\mathbb{k}}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}},$$

whence $C = C'$ so that C is a polynomial algebra. □

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