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Topological Resonances on Quantum Graphs

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Introduction

We will consider metric graphs G which consist of a finite graph Γ with some leads attached to some vertices. To this metric graph is associated a Laplacian using the Kirchhoff conditions. Resonances on such Quantum graphs are introduced in the book [BK13] and studied in several papers (see [EL10, DP11, LZ16]). In the paper [DP11], the following result is proved:

Theorem 0.1 *All resonances $k_j = \sigma_j + i\tau_j$, $j \in \mathbb{N}$, lie in a band $-M \leq \tau_j \leq 0$ and they have the large K asymptotic*

$$\#\{j \mid |\sigma_j| \leq K\} = \frac{2V}{\pi}K + O(1) ,$$

with $0 < V \leq |L|$ where $|L|$ is the total length of the finite graph Γ .

In this paper, we will be interested in resonances close to the real axis which in physics are the most important. They are linked to compactly supported eigenfunctions as anticipated in [EL10]. The goal of this paper is to describe some asymptotic properties of these resonances, called “topological resonances” in the paper [GSS13]. See also the paper [LZ16] for the explicit calculations of the related “Fermi golden rule”. We show that there is a dichotomy between graphs which can have eigenfunctions with compact support for some particular metrics and the other ones which are some specific trees, namely those with at most one vertex of degree one. In the first case, there are many resonances close to the real axis and we are able to say something on their asymptotics, while in the second one, there is a gap which is an invariant associated to the graph.

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We will follow the notations of the paper [CdV15] of the first author.

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1 Main results and conjectures

We consider a finite graph $\Gamma = (V, E)$ and fix a subset $V_0 \subset V$. To each vertex w of V_0 , we attach a non zero number n_w of infinite half-lines, called the “leads” in [GSS13]. The total number of leads is denoted by N with $N = \sum_{w \in V_0} n_w$.

Allowing loops and multiple edges, we can (and will) always assume that

$$\forall v \in V, \text{degree}_G(v) \neq 2 .$$

The metric graph denoted by G is the union of Γ with some lengths $l_e > 0$ for $e \in E$, and the attached infinite half-lines. Let us denote by \vec{l} the vector in \mathbb{R}^n with coordinates l_e , $e \in E$. To this set of data, we associate, using the usual Kirchhoff conditions at the vertices, a non negative self-adjoint Laplacian $\Delta_G^{\vec{l}}$ acting on $L^2(|G|, |dx|_{\vec{l}})$ where $|G|$ is the 1D singular manifold associated to G and $|dx|_{\vec{l}}$ is the Riemannian measure. Let us specify that, at any vertex of degree 1, we impose Neumann boundary conditions. This Laplacian has a discrete sequence of non negative eigenvalues (possibly empty) and a continuous spectrum $[0, +\infty[$ of multiplicity N . The Schwartz kernel of the resolvent $(k^2 - \Delta_G^{\vec{l}})^{-1}$ defined for $\Im k > 0$ extends to the lower half plane in a meromorphic way. The poles of this extension in $\Im k \leq 0$ are called the *resonances*. We denote by $\text{Res}_G^{\vec{l}}$ the set of resonances of $\Delta_G^{\vec{l}}$. Our goal is to study $\text{Res}_G^{\vec{l}}$, mainly in the case where the lengths $(l_e)_{e \in E}$ are independent over the integers (we will say that \vec{l} is “irrational”).

There are two mutually disjoint families of graphs G :

- *Type I*: the trees with at most one vertex of degree 1
- *Type II*: all other graphs.

The *Type I* graphs are the graphs G for which, for any choice of \vec{l} , the operator $\Delta_G^{\vec{l}}$ has no L^2 eigenfunctions or equivalently $\Delta_\Gamma^{\vec{l}}$ has no eigenfunctions vanishing on V_0 .

The main results of this note are

Theorem 1.1 *If the graph G is a tree with at most one vertex of degree 1 (G is of type I), there exists a minimal finite number $h(G) > 0$ so that, for any choice of the lengths $l_e > 0$ with $|L| := \sum l_e$, we have*

$$\text{Res}_G^{\vec{l}} \subset \{ \sigma + i\tau \mid \tau|L| \leq -h(G) \} .$$

The optimal constant $h(G)$ is an interesting graph parameter. It follows from the next results that $h(G) = 0$ for type II graphs. Moreover we have that

Theorem 1.2 *If G' is obtained from G by contracting some edges of Γ while keeping the leads in a natural way, then $h(G') \geq h(G)$.*

It would be interesting to say more on this graph parameter.

In order to state the second main result, we need the

Definition 1.1 *For a graph G as before, W_G is the sub-set of the torus $T^E := (e^{i\ell_e})_{e \in E}$ so that $\Delta_{\Gamma}^{\vec{l}}$ admits 1 as an eigenvalue with a non zero eigenfunction vanishing on V_0 ; we define also W_G^o as the subset of W_G where the eigenfunction of $\Delta_{\Gamma}^{\vec{l}}$ vanishing on V_0 is unique, up to scaling.*

Note that W_G depends only on the choice of V_0 . If G is of type II, the sets W_G and W_G^o are non-empty semi-algebraic sets (see [CdV15] and Appendices B and C). It follows from Theorem 4.1 that W_G^o is a smooth (non closed in general) submanifold of T^E which is an union of connected components (called the strata) of various dimensions. The maximal dimension of these strata is called the dimension of W_G^o . We will need the

Definition 1.2 *The number $d(G)$ is defined by*

$$d(G) := \#E - 1 - \dim W_G^o .$$

We will be interested in the following quantity:

Definition 1.3 *For $\varepsilon \geq 0$, let us define $N_{G,\vec{l}}(\varepsilon)$ as follows:*

$$N_{G,\vec{l}}(\varepsilon) := \liminf_{K \rightarrow +\infty} \frac{1}{K} \#\{\sigma_j + i\tau_j \in \text{Res}_{\vec{l}}^G \mid 0 \leq \sigma_j \leq K, \quad -\varepsilon \leq \tau_j \leq 0\} .$$

Finally let us define a combinatorial invariant $g(G)$ of the graph G :

Definition 1.4 *If G is of type I, $g(G) = +\infty$, otherwise $g(G)$ is the smallest number so that there exists either a simple cycle in G with $g(G)$ vertices or a path in G joining two vertices of degree 1 with $g(G) + 1$ vertices (and $g(G)$ edges).*

The number $g(G)$ can be smaller than the girth of G , for example if G is a tree of type II.

We have the

Theorem 1.3 *1. If the graph G is of type II and \vec{l} is irrational, there exist $\varepsilon_0 > 0$ and $C > 0$, depending on \vec{l} , so that, for $0 < \varepsilon \leq \varepsilon_0$, we have*

$$N_{G,\vec{l}}(\varepsilon) \geq C\varepsilon^{d(G)/2} .$$

2. Moreover, we have

- If there exists $a \in V_0$ so that there is no loops at the vertex a , we have $d(G) \geq 1$
- $d(G) \leq g(G) - 1$
- If γ is a simple cycle of Γ , then
 - either $d(G) \leq \#V_0$ for all $V_0 \subset V(\gamma)$
 - or there exists a vertex a of γ so that $d(G) \leq \#V_0$ if $V_0 \subset V(\gamma)$ and $a \notin V_0$.
- If Γ is 2-connected,
 - either $d(G) = 1$ for all sets V_0 with $\#V_0 = 1$
 - or there exists a unique vertex a of Γ so that $d(G) = 1$ for all sets V_0 with $\#V_0 = 1$ and $V_0 \neq \{a\}$.
- If $V_0 = V$, then $d(G) = g(G) - 1$.

We are not able to derive an upper bound in general, but we conjecture, following [GSS13], the following estimates:

Conjecture 1.1 *As $\varepsilon \rightarrow 0^+$, there exists $C > 0$ so that*

$$N_{G,\Gamma}(\varepsilon) \sim C\varepsilon^{d(G)/2} ,$$

and

$$d(G) = \min(g(G) - 1, \#V_0) .$$

Remark 1.1 **How is $N_{G,\Gamma}(\varepsilon)$ related to the Gnutzmann-Schanz-Smilansky paper [GSS13]?** *Let us choose the resonant state u_j associated to $k_j = \sigma_j + i\tau_j$ so that $\sum_{m=1}^N |t_m|^2 = 1$, then we look at its energy in Γ defined by*

$$\mathcal{E}(u) := \int_{|\Gamma|} |u|^2 dx|_{\Gamma} .$$

Proposition 2.1 gives that $\mathcal{E}(u) = 1/(2|\tau|)$. The previous authors look at the asymptotic behaviour, as $\alpha \rightarrow \infty$, of

$$P_{G,\Gamma}(\alpha) := \lim_{K \rightarrow \infty} \frac{1}{K} \#\{k_j = \sigma_j + i\tau_j | 0 \leq \sigma_j \leq K, \mathcal{E}(u_j) \geq \alpha\} ,$$

which is the same as $N_{G,\Gamma}(1/2\alpha)$ at which we are looking in the present paper.

2 Finding resonances

We will use the following notations: if z is a vector in \mathbb{C}^n , (z) will be the diagonal matrix with entries the coordinates of z and $(z)_2$ will denote the diagonal matrix of size $2n$

$$(z)_2 = \begin{pmatrix} (z) & 0 \\ 0 & (z) \end{pmatrix} .$$

We choose an orientation of each edge of Γ and parametrize the edge e by a real parameter x_e with $0 \leq x_e \leq l_e$ according to the orientation. The leads are parametrized by x_m , $m = 1, \dots, N$, with $0 \leq x_m < \infty$. We will denote by $|dx|_{\vec{l}} = \sum_{e \in E} |dx_e| + \sum_{m=1}^N |dx_m|$ the Riemannian measure on $|G|$ where $|G|$ is the 1D singular topological space associated to G .

Definition 2.1 *A complex number $k = \sigma + i\tau$ with $\tau \leq 0$ is a “resonance” of $\Delta_G^{\vec{l}}$ if and only if there exists a non zero “resonant state” u which satisfies “ $(\Delta_G^{\vec{l}} - k^2)u = 0$ ” and $\forall m = 1, \dots, N$, $\exists t_m \in \mathbb{C}$, $u(x_m) = t_m \exp(ikx_m)$.*

Equivalent definitions of resonances for Quantum graphs are given in [EL07].

Following [BK13] sec. 5.4., we can describe all solutions of $(\Delta_G^{\vec{l}} - k^2)u = 0$ in the following way: writing $u(x_e) = a_e \exp(ikx_e) + b_e \exp(-ikx_e)$ and $u(x_m) = t_m^{\text{out}} \exp(ikx_m) + t_m^{\text{in}} \exp(-ikx_m)$, and denoting by $e^{ik\vec{l}}$ the point in the \mathbb{C}^E of coordinates e^{ikl_e} , $e \in E$, the Kirchhoff conditions at the vertices express as

$$\begin{pmatrix} t^{\text{out}} \\ a \\ b \end{pmatrix} = \begin{pmatrix} R & T_o \\ T_i & U(e^{ik\vec{l}})_2 \end{pmatrix} \begin{pmatrix} t^{\text{in}} \\ a \\ b \end{pmatrix} . \quad (1)$$

The $2n + N$ square matrix

$$S = \begin{pmatrix} R & T_o \\ T_i & U(e^{ik\vec{l}})_2 \end{pmatrix}$$

is unitary for real k 's. The resonances are the value of k for which there exists a non trivial solution of Equation (1) with $t^{\text{in}} = 0$ and are hence given by the equation

$$\mathcal{R}_G(\exp(ikl_1), \dots, \exp(ikl_n)) = 0$$

where

$$\mathcal{R}_G(z) = \det(\text{Id} - U(z)_2) .$$

The associated resonant state is given by

$$U(e^{ik\vec{l}})_2 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} ,$$

$$t^{\text{in}} = 0, \quad t^{\text{out}} = T_o \begin{pmatrix} a \\ b \end{pmatrix} .$$

We will need the

Proposition 2.1 *If u is a resonant state for the resonance $k = \sigma + i\tau$ of $\Delta_G^{\vec{l}}$ with $\sigma \neq 0$, we have*

$$-2\tau \int_{|\Gamma|} |u|^2 |dx|_{\vec{l}} = \sum_{m=1}^N |t_m|^2 .$$

Proof. – Let us evaluate the integral

$$I = \int_{|\Gamma|} (u\Delta\bar{u} - \bar{u}\Delta u) |dx|_{\vec{l}}$$

in two ways: first, using the differential equation $\Delta u = k^2 u$, we have

$$I = -4i\sigma\tau \int_{|\Gamma|} |u|^2 |dx|_{\vec{l}} ,$$

whereas using the Green-Riemann formula, we get:

$$I = 2i\sigma \sum_{k=1}^N |t_m|^2 .$$

□

Corollary 2.1 *The resonant states with $\tau = 0$ are eigenstates of $\Delta_G^{\vec{l}}$ with support in $|\Gamma|$. And conversely, each such eigenfunction of $\Delta_G^{\vec{l}}$ is a resonant state of $\Delta_G^{\vec{l}}$.*

Proposition 2.1 implies that all t_m 's vanish, hence the conclusions.

3 Type I: trees with at most one vertex of degree one

Let us prove Theorem 1.1. We start with the following Lemma:

Lemma 3.1 *If G is a tree with at most one vertex of degree 1 and \vec{l} is given, then $\Delta_G^{\vec{l}}$ has no non vanishing L^2 eigenfunction.*

Proof. – Let us assume that the eigenvalue is $k^2 > 0$. It is clear that the corresponding eigenfunction u has to vanish on the leads because u is of the form $a_i \cos kx_i + b_i \sin kx_i$ on l_i and is square integrable. Let us assume that u does not vanish identically along an edge $e_1 = [v_1, v_2]$, then, either v_2 is of degree 1, or there exists an edge $e_2 = [v_2, v_3]$ with $v_3 \neq v_1$, so that u does not vanish identically on e_2 . Iterating, and possibly going in the other direction, we get that there exists either a cycle, or a path joining two vertices of degree 1 so that u has a finite number of zeroes on them. By assumption, G has no cycles, and all

paths starting from a vertex of degree 1 will end into a lead where it has to vanish identically. \square

Let us now give the proof of Theorem 1.1:

Proof.– By rescaling the lengths l_e , we can assume that $|L| = 1$. By contradiction, there exists a sequence of vectors \vec{l}_n so that u_n is a resonant state of $\Delta_n := \Delta_G^{\vec{l}_n}$ of resonance $k_n = \sigma_n + i\tau_n$, with $\int_{|\Gamma|} |u_n|^2 |dx|_n = 1$, and $\tau_n \rightarrow 0$. We have the equations

$$(U(z_n)_2 - \text{Id}) \begin{pmatrix} a_n \\ b_n \end{pmatrix} = 0, \quad t_n^{\text{out}} = T_o \begin{pmatrix} a_n \\ b_n \end{pmatrix},$$

with

$$(z_n)_e = e^{ik_n(l_n)_e}.$$

From Proposition 2.1 we already know that $t_n^{\text{out}} \rightarrow 0$. We can extract converging sub-sequences of the vectors $\exp(i\sigma_n l_e^n)$ going to $\exp(il_e^\infty)$ with $2\pi \leq l_e^\infty < 4\pi$. It is not possible that the vectors (a_n, b_n) tend to 0, because it would imply that the L^2 norms of u_n on $|\Gamma|$ tend to 0. On the other hand, if we denote by $M_n^2 = \max_e (|a_e^n|^2 + |b_e^n|^2)$, the sequence $(a_n, b_n)/M_n$ converges (up to extraction of a subsequence) to the coefficients of an eigenfunction with compact support of $\Delta_G^{\vec{l}_\infty}$ with eigenvalue 1 and the contradiction follows from Lemma 3.1. \square

Example 3.1 *We compute $h(G)$ defined in Theorem 1.1 in some simple examples:*

- G is a star graph where all edges are leads: there is no resonances.
- G is a star graph with only one edge of finite length l and $N > 1$ leads: the resonances are

$$k_j = \frac{1}{2l} \left((1 + 2j)\pi - i \log \frac{N+1}{N-1} \right), \quad j \in \mathbb{Z},$$

so that $h(G) = \frac{1}{2} \log \frac{N+1}{N-1}$.

- Γ is an interval $[v, v']$ of length $l > 0$ and $G_{N,N'}$, with $N, N' \geq 2$, is obtained from Γ by attaching N leads to v and N' leads to v' . The set of resonances is

$$\left\{ \frac{1}{2l} \left(2\pi j - i \log \frac{(N+1)(N'+1)}{(N-1)(N'-1)} \right) \mid j \in \mathbb{Z} \right\}$$

and

$$h(G) = \frac{1}{2} \log \frac{(N+1)(N'+1)}{(N-1)(N'-1)}.$$

4 Graphs of Type II

The goal of this section is to describe in a precise way the asymptotic behaviour of the resonances in the type II case.

4.1 Geometric preliminaries

Let us start introducing some algebraic sets:

- If \vec{l} is given,

$$R_G^{\vec{l}} := \{(z_e)_{e \in E} = (e^{i\alpha_e - \tau l_e})_{e \in E} \in \mathcal{R}_G^{-1}(0), \alpha_e, \tau \in \mathbb{R}\} \subset T^E \times \mathbb{R} .$$

We denote by $Y_G^{\vec{l}}$ the projection of $R_G^{\vec{l}}$ onto the torus T^E :

$$Y_G^{\vec{l}} := \{(e^{i\alpha_e})_{e \in E} \mid \exists \tau \in \mathbb{R} \text{ with } (e^{i\alpha_e - \tau l_e})_{e \in E} \in R_G^{\vec{l}}\} .$$

- Z_Γ is the determinant manifold of Γ defined by $(e^{il_e})_{e \in E} \in Z_\Gamma$ if and only if 1 is an eigenvalue of $\Delta_\Gamma^{\vec{l}}$.

It follows from Section 4 of [CdV15] that

Proposition 4.1 *The graph G is of type II if and only if W_G^o is non empty.*

For type I, W_G is empty. For type II, except for circles, the first author constructed, in Section 4 of [CdV15], non zero eigenstates associated to a non degenerate eigenvalue of some $\Delta_\Gamma^{\vec{l}}$ which vanish on all vertices of Γ , except may be two vertices of degree 1: in these cases, W_G^o is non empty. In the case of the circular graphs (i.e. $|\Gamma|$ is a circle), the eigenspaces of Δ_Γ are degenerate, but the resonant states are still non degenerate.

We have the

Theorem 4.1 *If $w_0 \in W_G^o$, then $R_G^{\vec{l}}$ is smooth near $(w_0, 0)$ and, if u_0 is a corresponding eigenfunction with coefficient $(a_e, b_e)_{e \in E}$, then*

$$T_{w_0, 0} R_G^{\vec{l}} = \left\{ \sum_e (|a_e|^2 + |b_e|^2) d\alpha_e = 0, d\tau = 0 \right\} .$$

Similarly, $Y_G^{\vec{l}}$ is smooth near w_0 and

$$T_{w_0} Y_G^{\vec{l}} = \left\{ \sum_e (|a_e|^2 + |b_e|^2) d\alpha_e = 0 \right\} .$$

If u_0 is also non degenerate as an eigenfunction of $\Delta_\Gamma^{\vec{l}}$, then

$$T_{w_0, 0} R_G^{\vec{l}} = T_{w_0} Z_\Gamma \oplus 0 ,$$

and

$$T_{w_0} Y_G^{\vec{l}} = T_{w_0} Z_\Gamma .$$

The set $R_G^{\vec{l}}$ is near w_0 a graph $\tau = \tau(e^{i\alpha_e})$ with $(e^{i\alpha_e}) \in Y_G^{\vec{l}}$.

The differential of τ vanishes at w_0 , so that the Hessian of τ is well defined on $T_{w_0} Y_G^{\vec{l}}$.

Proof.– Let $w_0 = (e^{i\alpha_e^0})$ be a point in W_G^o . The eigenvalue 1 of $U(w_0)_2$ is non degenerate and we choose an eigenvector $u_0 \in \mathbb{C}^{2E}$ of norm 1 so that $U(w_0)_2 u_0 = u_0$. If $w \in \mathbb{C}^E$ is close to w_0 , $U(w)_2$ admits an unique eigenvalue $\lambda(w)$ close to 1 which is non degenerate, and the associated eigenvector of norm 1, $u(w)$, is smooth w.r. to w . Let us assume that $w(t)$ depends smoothly of t with $w(0) = w_0$ and compute the derivative $\dot{\lambda}$ of $\lambda(w(t))$ at $t = 0$. Taking the derivative of the equation $U(w(t))_2 u(w(t)) = \lambda(t) u(w(t))$ w.r. to t at $t = 0$, we get with natural notations

$$U(\dot{w}_0)_2 u_0 + U(w_0)_2 \dot{u}_0 = \dot{\lambda}_0 u_0 + \dot{u}_0 . \quad (2)$$

Taking the scalar product of both sides of Equation (2) with $U(w_0)_2 u_0 = u_0$, we get, after simplifications and using the fact that we can choose $u(t)$ so that $\langle \dot{u}_0 | u_0 \rangle = 0$,

$$\dot{\lambda}_0 = \langle U((w_0)(w_0)^{-1} \dot{w}_0)_2 u_0 | U(w_0)_2 u_0 \rangle + \langle U(w_0)_2 \dot{u}_0 | U(w_0)_2 u_0 \rangle .$$

Now we can use the identity

$$\left\langle S \begin{pmatrix} 0 \\ v \end{pmatrix} \middle| S \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \right\rangle = \langle U(w_0)_2 v | U(w_0)_2 u_0 \rangle .$$

Because S is unitary at w_0 , we get

$$\langle U(w_0)_2 v | U(w_0)_2 u_0 \rangle = \langle v | u_0 \rangle .$$

Using this, we get

$$\dot{\lambda}_0 = \langle ((w_0)^{-1} \dot{w}_0)_2 u_0 | u_0 \rangle . \quad (3)$$

Now let us take, with \vec{l} fixed $w(\alpha, \tau) = (e^{i\alpha_e - \tau l_e})$, and compute the derivatives of λ w.r. to α and τ at w_0 ; from Equation (3), we get

$$\frac{\partial \lambda}{\partial \tau} = - \sum l_e m_e$$

and

$$\frac{\partial \lambda}{\partial \alpha_e} = i m_e$$

with $m_e = |a_e|^2 + |b_e|^2$. Now using the fact that the determinant is the product of all eigenvalues, we get that the set $\det(U(w)_2 - \text{Id}) = 0$ is also defined by $\lambda(w) = 1$ with the same non degeneracy properties. This gives the fact that $R_G^{\vec{l}}$ is near $(w_0, 0)$ a submanifold of codimension 2 of $T^E \oplus \mathbb{R}_\tau$ whose tangent space is what is given in the Theorem 4.1. \square

Theorem 4.2 *For almost all choices of \vec{l} , $\Delta_G^{\vec{l}}$ has no L^2 eigenfunctions; in particular $N_{G,\vec{l}}(\varepsilon = 0) = 0$.*

This is simply because W_G^o is of codimension at least 1 in Z_Γ ; hence the line $\sigma \rightarrow (\exp(i\sigma l_e))$ does not cross W_G^o for a generic choice of the lengths.

4.2 Asymptotics of $N_G^{\vec{l}}(\varepsilon)$

We now prove the first part of Theorem 1.3.

Proof. – The proof follows from the same kind of ergodicity argument than in the paper [CdV15]. More precisely, let $w_0 \in W_G^o$ so that W_G^o is of codimension $d(G)$ in Y_Γ^G near w_0 . Thanks to Theorem 4.1 there exists a compact neighborhood D of w_0 in Y_Γ^G with smooth boundary, so that $\tau(\alpha)$ is well defined and smooth on D . Then we have, following the argument in [CdV15],

$$N_G^{\vec{l}}(\varepsilon) \geq \text{vol}(\{\tau \geq -\varepsilon\} \cap D)$$

where the volume is computed w.r. to the Barra-Gaspard measure $\mu_{\vec{l}} = |\iota(\vec{l})d\alpha|$ which is smooth non negative on D . Because τ vanishes on W_G^o which is of codimension $d(G)$ in D , this volume is greater than $C\varepsilon^{d(G)/2}$ with $C > 0$ (equality holds if the Hessian of τ at w_0 is transversally non degenerate). \square

5 Bounds for $d(G)$

5.1 The bound $d(G) \geq 1$

Let us assume that there is no loops at the vertex $a \in V_0$. It follows from [BL16] Theorem 3.6 that for a generic choice of lengths the eigenvalue 1 is non degenerate and the corresponding eigenfunction does not vanish at a . Knowing that W_G^o is semi-algebraic, this implies that $d(G) \geq 1$.

5.2 Upper bounds for $d(G)$

In the following sections, we will prove the upper bounds for $d(G)$ given in the second part of Theorem 1.3. If γ is simple cycle of Γ , we will decompose the vector \vec{l} as $\vec{l} = (\vec{l}_\gamma, \vec{l}')$ where \vec{l}_γ is the set of lengths of the edges of the cycle γ . Let us recall from [CdV15], proof of Theorem 4.1. p. 356 (see also Appendix B) that, if $\vec{l}_\gamma = (2\pi, \dots, 2\pi)$, we can always choose \vec{l}' so that 1 is a non degenerate eigenvalue of $\Delta_\Gamma^{\vec{l}}$. The point $w_0 = (1, \dots, 1, e^{il'_e})$ belongs then to W_G^o and we will study the set W_G^o near such a point w_0 . A similar study can be done for paths joining two vertices of degree 1, but we will omit it.

5.3 The case of circular graphs

Let us assume that $|\Gamma|$ is a circle. We can always assume that $V_0 = V$. Then W_G^o is a finite set of points of the form (± 1) with the number of -1 even. Hence $d(G) = \#V_0 - 1 = g(G) - 1$.

From now, we will assume the $|\Gamma|$ is not a circle and $w_0 \in W_G^o$ is chosen as described before.

5.4 The bound $d(G) \leq g(G) - 1$

Let us now discuss the upper bounds for $d(G)$ in terms of the girth (part 2 of Theorem 1.3).

Let γ be a simple cycle in Γ with $g(G)$ vertices and consider the submanifold (not closed in general) W_G^γ of W_G^o defined by $\vec{l}_\gamma = (2\pi, \dots, 2\pi)$ and \vec{l}' so that 1 is a non degenerate eigenvalue of $\Delta_\Gamma^{\vec{l}'}$. Then $\Delta_\Gamma^{\vec{l}'}$ admits an eigenfunction with eigenvalue 1 supported on γ and vanishing on all vertices of G . This defines a stratum W_G^γ of W_G^o of dimension $\#E - g(G)$. The codimension of W_G^γ in $Y_G^{\vec{l}'}$ is $g(G) - 1$.

Similar estimates hold by using the smallest path ending at 2 vertices of degree 1.

5.5 The bound $d(G) \leq \#V_0$

Let us now discuss the upper bounds for $d(G)$ in terms of $\#V_0$ (part 2 of Theorem 1.3). Let γ be a simple cycle of Γ of length k . Let us choose $\vec{l}_0 = (2\pi, \dots, 2\pi, \vec{l}')$ (the lengths 2π occur k times as lengths of the edges of γ), so that 1 is a non degenerate eigenvalue of $\Delta_\Gamma^{\vec{l}_0}$ with an eigenfunction which restricts to each edge of γ to $\sin x_e$ and which vanishes outside γ . For \vec{l} in some neighborhood U_0 of \vec{l}_0 with $e^{i\vec{l}} \in Z_\Gamma$ we can choose some eigenfunction $u(\vec{l})$ smoothly dependent of \vec{l} . Let us denote by F the map from $\{w = e^{i\vec{l}} | \vec{l} \in U_0\} \cap Z_\Gamma$ to $\mathbb{R}^{V(\gamma)}$ which associates to w the restriction to $V(\gamma)$ of $u(\vec{l})$, and by \mathcal{E} the subspace of $T_{w_0} Z_\Gamma$ of variations $\delta \vec{l}$ of lengths supported by γ so that $\sum_{e \in \gamma} \delta l_e = 0$. The dimension of \mathcal{E} is clearly $k - 1$. Let us prove that the differential $L : \mathcal{E} \rightarrow \mathbb{R}^{V(\gamma)}$ of F is injective. Denoting by $\vec{l}(t)$ a variation of \vec{l}_0 and denoting derivatives at $t = 0$ by dots, we get

$$\dot{\Delta}u + (\Delta - 1)\dot{u} = 0 . \quad (4)$$

We have to prove that \dot{u} cannot vanish on all vertices of V_0 unless $\delta \vec{l}$ vanishes. Let us assume, by contradiction, that \dot{u} vanishes on V_0 . Using Lemma A.1, and ordering the vertices of γ using the orientation of γ as $V(\gamma) = \{v_1, \dots, v_k\}$, we get $\dot{u}(v_{i+1}) = \dot{u}(v_i) + \delta l_{e_i}$ with $e_i = (v_i, v_{i+1})$. We get that \dot{u} cannot vanish identically on V_0 unless $\delta \vec{l}$ does. The final result follows from linear algebra: If

$\mathcal{F} = F'(w_0)(\mathcal{E})$, $\dim \mathcal{F} = k - 1$ and hence there is at most one $a \in V(\gamma)$ so that $\mathcal{F} \subset \{v|v(a) = 0\}$. If $V_0 \subset V(\gamma)$ with $\#V_0 \leq k - 1$ and $a \notin V_0$, the map $\pi \circ F'(w_0)$ is surjective where π is the canonical projection from $\mathbb{R}^{V(\gamma)}$ onto \mathbb{R}^{V_0} .

5.6 The case $\#V_0 = 1$

Let us say that $a \in V(\Gamma)$ is irregular if, for all simple cycles γ with $a \in V(\gamma)$, the previous space \mathcal{F} is included in $\{v|v(a) = 0\}$. Using the 2-connectivity of γ , we see that there is at most one such vertex a . We conjecture that such a vertex never exists. In any case it does not exist if the graph is homogeneous, ie for any vertex a and b , there exists an automorphism of Γ sending a onto b .

5.7 The case $V_0 = V$

If $X \subset E$, we define

$$\mathcal{W}_X := \{[\vec{l}] \in W_G \mid \forall e \in X, \exists u \in \ker(\Delta_\Gamma^{\vec{l}} - 1), u|_e \neq 0\} .$$

Then the family of sets \mathcal{W}_X , $X \subset E$ is a partition of W_G . If $[\vec{l}] \in \mathcal{W}_X$, all lengths of $e \in X$ are multiple of π . The eigenvalue 1 is of multiplicity $b_1(\Gamma_X)$ where Γ_X is the subgraph of Γ whose edge set is X . Moreover, if $b_1(\Gamma_X) = 1$, Γ_X reduces to that simple cycle. Hence $\mathcal{W}_X \cap W_G^o \neq \emptyset$ if and only if X is a simple cycle. The dimension of the corresponding stratum is $\#E - \#X$ and the maximal dimension of such a stratum is obtained by taking the cycle of minimal length . Hence $d(G) = \min_{\{X \text{ simple cycle}\}}(\#X) - 1 = g(G) - 1$.

6 Examples

6.1 Homogeneous 2-connected graphs

If the graph Γ is homogeneous and 2-connected, there is no exceptional vertex, hence $d(G) = 1$ if $\#V_0 = 1$.

In the next examples, all subsets V_0 of $V(\Gamma)$ with $\#V_0 < g(\Gamma) - 1$ are contained in a simple cycle. It is known (see [Be73]) that this follows from the fact that Γ is $(g(\Gamma) - 1)$ -connected.

In fact, we can do better using the

Lemma 6.1 *If g is an automorphism of Γ leaving the simple cycle γ invariant and with no vertex of γ fixed by g , then there is no exceptional vertex on γ , ie $d(G) \leq \#V_0$ for all $V_0 \subset V(\gamma)$.*

Proof.– We can choose \vec{l}_0 invariant by g . It follows that the space $\mathcal{F} \subset \mathbb{R}^{V(\gamma)}$ defined in Section 5.5 is invariant by g and cannot be a coordinate hyperplane because g acts without fixed points on γ . \square

6.1.1 Complete graphs

If Γ is the complete graph

$$d(G) \leq \min(2, \#V_0) .$$

6.1.2 Cubes

All sets of 2 vertices are part vertices of a simple cycle which satisfies the assumptions of Lemma 6.1, hence

$$d(G) \leq \min(3, \#V_0) .$$

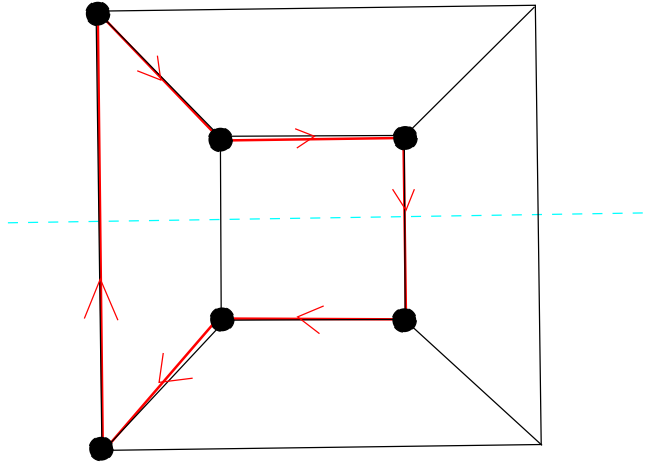


Figure 1: *cube with a simple cycle on Γ with a symmetry of order 2 without fixed vertices on γ . Any set of 2 vertices of Γ is part of such a cycle.*

6.1.3 Dodecahedron

All sets of 3 vertices are part vertices of a simple cycle which satisfies the assumptions of Lemma 6.1, hence

$$d(G) \leq \min(4, \#V_0) .$$

6.1.4 Petersen graph

All sets of 3 vertices are part vertices of a simple cycle which satisfies the assumptions of Lemma 6.1, hence

$$d(G) \leq \min(4, \#V_0) .$$

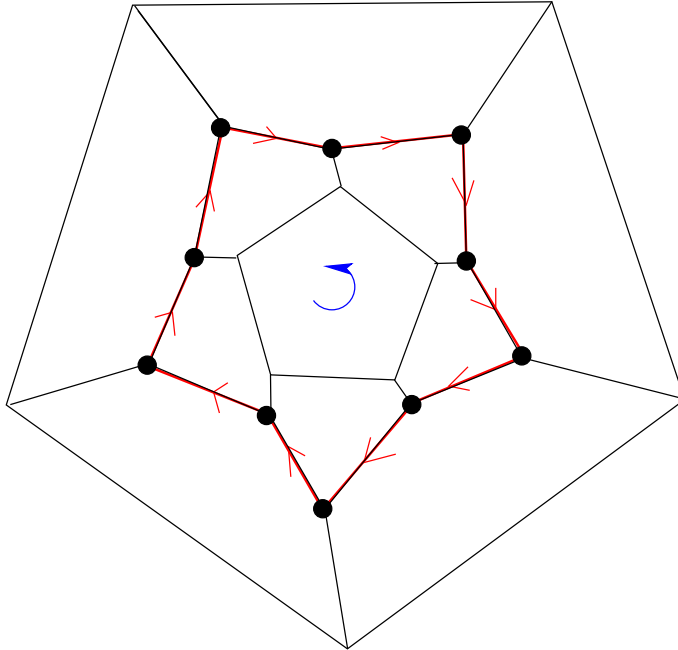


Figure 2: *dodecahedron with a simple cycle on Γ with a symmetry of order 5 without fixed vertices on γ .*

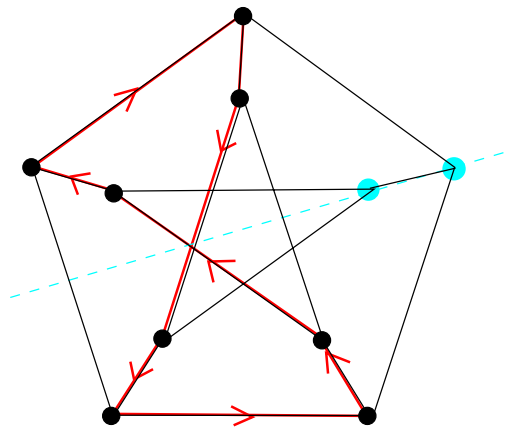


Figure 3: *Petersen graph with a simple cycle on Γ with a symmetry of order 2 without fixed vertices on γ . Any set of 3 vertices of Γ is part of such a cycle.*

6.1.5 Tetrahedron

The goal of this section is to compute the dimension $d(G)$ in the cases where Γ is a tetrahedron. Instead of making direct computations, we present geometrical arguments which could be extended to other graphs. The link with the minor relation between graphs is implicit in our construction of reduced graphs.

We will show the

Theorem 6.1 *If Γ is the tetrahedron, i.e. the clique with four vertices, we have $d(G) = \min(\#V_0, 2)$ where 2 is the girth of Γ minus 1.*

The result follows from Theorem 1.3 for $\#V_0 = 1$ and $\#V_0 = 4$. It remains to prove that $d(G) = 2$ if $\#V_0 = 2$ or 3. The bound $d(G) \leq 2$ comes from Theorem 1.3, because the girth of Γ is 3.

If $[\vec{l}]$ belongs to W_G^o and u is the unique associated eigenfunction vanishing on V_0 , let us denote by $n_{\vec{l}}$ the number of edges of Γ on which u vanishes identically. We decompose each stratum of W_G^o following the values of $n_{\vec{l}}$ into a finite number of sub-strata. The maximal dimension of these sub-strata is the same as the dimension of W_G^o . If $[\vec{l}] \in W_G^o$, we have to show that the dimension, denoted $\dim(\vec{l})$, of the sub-stratum of W_G^o containing $[\vec{l}]$ is smaller than 3.

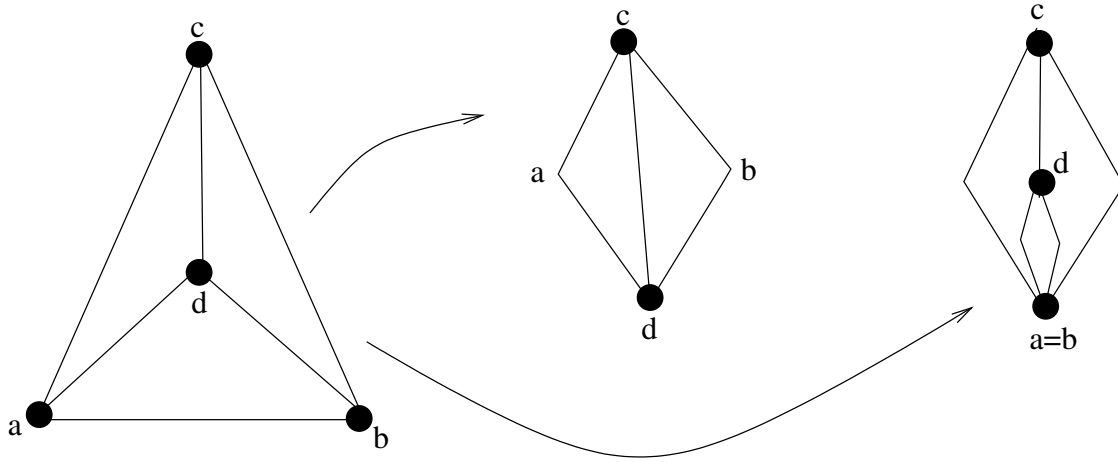


Figure 4: *the two reduced graphs.*

Let us discuss the different possibilities:

- $n_{\vec{l}} = 0$:
 - $\#V_0 = 2$: let us assume that $V_0 = \{a, b\}$. The length of the edge $e = \{a, b\}$ is equal to 0 modulo π . In these cases the function u on e is of the form $\mu \sin x_e$ with $\mu \neq 0$. Let us consider the graph Γ_c^{red}

obtained by identifying the vertices a and b (contracting the edge e). This reduced graph has 5 edges. Denote by A the vertex of degree four obtained by identifying a and b .

- * $l_e = \pi$ modulo 2π : the Kirchoff conditions at the vertex A are $u'_1 + u'_2 + u'_3 + u'_4 = 0$. We have to look at $W_{G_c^{\text{red}}}^o$ where $V_0 = \{A\}$. The dimension of this manifold is less than $(5 - 1) - 1 = 3$: we apply Theorem 1.3 to G_c^{red} with $V_0 = \{A\}$ and clearly there is no loop at the vertex A . Hence $\dim(\vec{l}) \leq 3$.
- * $l_e = 2\pi$ modulo 2π : the Kirchoff conditions at the vertex A are $u'_1 + u'_2 = u'_3 + u'_4$. Similar construction gives the same conclusion.
- $\#V_0 = 3$: in this case the lengths of the cycle whose vertices are V_0 are equal to 0 modulo π . This implies that the dimension $\dim(\vec{l})$ of the corresponding sub-stratum is at most 3.
- $n_{\vec{l}} = 1$: let us assume that u vanishes identically on the edge $e = \{a, b\}$. Let us consider the reduced graph Γ_r^{red} obtained by removing the edge e . Topologically, this graph has two vertices c and d of degree 3 and three edges joining them. The dimension of $Z_{\Gamma_r^{\text{red}}}$ is 2. If $[l'] \in Z_{\Gamma_r^{\text{red}}}$, then the position of the zeroes of the eigenfunction u (unique up to rescaling) on the two edges obtained by removing a and b determines all the remaining lengths except l_e . This implies that $\dim(\vec{l}) = 2 + 1$.
- If $n_{\vec{l}} \geq 2$, one has two cases
 - If the two edges where u vanishes identically have a common vertex, say a , it follows from the Kirchoff conditions at the vertex a that u vanishes also identically on the third edge with vertex a . Hence the support of u is the remaining cycle (otherwise $u \equiv 0$). This forces the lengths of the edges of this cycle to be congruent to 0 modulo π and hence $\dim(\vec{l}) = 3$.
 - If the two edges where u vanishes identically have no common vertex, then u vanishes at all vertices and the four remaining edges have lengths congruent to 0 modulo π . This implies that $\dim(\vec{l}) = 2$.

6.2 A simple example where we compute the asymptotics of $N_G^{\vec{l}}(\varepsilon)$: the Y-graph

Let us consider a Y graph with two edges of length l and L and an infinite edge. If $z = \exp(ikl)$, $w = \exp(ikL)$, we get that

$$\mathcal{R}_G(z, w) = z^2 w^2 - z^2 - w^2 - 3 ,$$

whereas Z_Γ is defined by $z = e^{i\alpha}, w = e^{i\beta}$ with $\alpha + \beta = 0 \pmod{\pi}$. Since $W_G = Z_\Gamma \cap R_G^{\vec{l}}$, with $\vec{l} = (l, L)$, it follows that $W_G = \{(\pm i, \pm i)\}$, and that the tangent space to $R_G^{\vec{l}}$ is $d\alpha + d\beta = 0$ (due to theorem 4.1). Let us consider a neighborhood D in $R_G^{\vec{l}}$ of $w_0 = (i, i)$, namely points $(z = \exp(i\pi/2 + i\alpha - \tau l), w = \exp(i\pi/2 + i\beta - \tau L))$ satisfying the resonance equation, with $\alpha = +u, \beta = -u$ and τ, u small. We get that

$$e^{2i(\alpha+\beta)}e^{-\tau(l+L)} + e^{2i\alpha}e^{-\tau l} + e^{2i\beta}e^{-\tau L} = 3,$$

which yields to the following asymptotics

$$\tau \sim -u^2/4(l+L),$$

as τ tends to 0. Then the Barra-Gaspard volume V_ε of the set $D_\varepsilon = D \cap \{\tau \geq -\varepsilon\}$ is given by

$$V_\varepsilon = \frac{1}{2\pi^2} \int_{D_\varepsilon} |Ld\beta - ld\alpha| = \frac{1}{2\pi^2} \int_{u^2/4(l+L) \leq \varepsilon} (l+L)|du|,$$

and thus we get the following asymptotics

$$N(\varepsilon) \sim \frac{4(l+L)^{3/2}}{\pi^2} \varepsilon^{1/2}.$$

6.3 Circular graphs

A graph G is called circular if $|\Gamma|$ is homeomorphic to a circle. A general circular graph is denoted C_{N_1, \dots, N_p} with $N_i > 0$: this is a subset of p points in circular order $\{v_1, \dots, v_p\}$ where N_i leads are attached to v_1, \dots

6.3.1 C_1

The resonances of C_1 are easily shown to be the spectrum of Γ :

$$\text{Res}_{C_1}^L = \{2\pi j/L \mid j \in \mathbb{Z}\}.$$

6.3.2 $C_{1,1}$

In this case, there are two lengths l, L and putting $z = e^{ikl}$ and $w = e^{ikL}$, we get

$$\mathcal{R}_{C_{1,1}}(z, w) = (zw - w - z - 3)(zw + z + w - 3).$$

Then $W_{C_{1,1}} = \{(-1, -1), (1, 1)\}$

6.3.3 $C_{1,1,1}$

It is still possible to compute

$$\mathcal{R}_{C_{1,1,1}}(z_1, z_2, z_3) = z_1^2 z_2^2 z_3^2 - (z_1^2 z_2^2 + z_2^2 z_3^2 + z_3^2 z_1^2) - 3(z_1^2 + z_2^2 + z_3^2) - 16z_1 z_2 z_3 + 27,$$

with $z_i = e^{ikl_i}$. In this case $W_{C_{1,1,1}}$ consists of four points: $w_1 = (1, -1, -1)$, $w_2 = (-1, 1, -1)$, $w_3 = (-1, -1, 1)$, $w_4 = (1, 1, 1)$.

6.3.4 $C_{1,\dots,1}$

For all values of N , $W_{C_{1,\dots,1}}$ is the finite set of vectors X in $\{-1, +1\}^N$ so that the number of negative components of X is even.

6.3.5 C_2

This example is studied in [DP11]. The resonances of C_2 are the same as for C_1 :

$$\text{Res}_{C_2}^L = \{2\pi j/L \mid j \in \mathbb{Z}\} .$$

7 Open questions

The main open question is clearly to get upper bounds of $N_G^{\vec{l}}(\varepsilon)$ as in the conjecture 1.1.

Two other questions seem interesting:

- Is it possible to get an estimate of the minimal imaginary part of resonances given by Theorem 0.1 of the form $M = M(G)/|L|$ with $M(G)$ depending only of the combinatorics of G ?
- Is it true that if \vec{l} is irrational there is no (or at most a finite number) of embedded eigenvalues? Compare this to Theorem 4.2.
- Is the constant $h(G)$ in Theorem 1.1 minor monotonic?

A Calculation of a derivative

The goal of this section is to prove the

Lemma A.1 *Let $\lambda = 1$ be a simple eigenvalue of $\Delta_{\Gamma}^{\vec{l}_0}$ with an eigenfunction u_0 . Let us assume that u_0 restricts to some edge $e = [a, b]$ of length 2π to $\sin x$ where $x \in [0, 2\pi]$ is an arc-length parametrization of e with origine a and end b . If $\vec{l}(t)$ is a smooth deformation of \vec{l}_0 so that the eigenvalue $\lambda(t)$ is constant equal to 1, we have*

$$\left(\frac{d}{dt}\right)_{t=0} (u(b) - u(a)) = \left(\frac{dl_e}{dt}\right)_{t=0} .$$

Proof. – We denote by dots the derivative at $t = 0$ and get

$$(\Delta - 1)\dot{u} + \dot{\Delta}u_0 = 0 .$$

The restriction of this equation to the edge e gives putting $v = \dot{u}$:

$$v''(x) + v(x) = \dot{\Delta} \sin x .$$

Taking the metric $g(t) = (1 + 2t)dx^2$, gives $\dot{l}_e = 2\pi$, while $\dot{\Delta} = 2\frac{d^2}{dx^2}$. The derivative v satisfies

$$v''(x) + v(x) = -2 \sin x .$$

Using the method of variation of constants, we get:

$$v(x) = \alpha \cos x + \beta \sin x + x \cos x ,$$

with some constants α and β , and hence

$$v(2\pi) - v(0) = 2\pi .$$

□

B Non degenerated eigenfunctions supported by simple cycles

Let γ be a simple cycle of Γ with ordered vertices $(x_0, x_1, x_2, \dots, x_n = x_0)$. Let us consider lengths $\vec{l} = (2\pi, \dots, 2\pi, \vec{l}')$ where the 2π 's are the lengths of the n edges of γ and \vec{l}' the other lengths. Let us assume in what follows that Γ is not homeomorphic to a circle, ie not reduced to γ . Let us reprove the following

Lemma B.1 *There exists an open dense subset Ω of $(\mathbb{R}^+)^{E(\Gamma) \setminus E(\gamma)}$ so that if \vec{l}' belongs to Ω , then 1 is a non degenerate eigenvalue of $\Delta_{\Gamma}^{\vec{l}}$.*

The fact that Ω is open is clear from general perturbation theory. Let us start now with $u_0 \in \mathcal{E} := \ker(\Delta_{\Gamma}^{\vec{l}_0} - 1)$ the function which restricts to $\sin t_e$ on each edge of γ and vanishes outside. Let us assume that $\dim \mathcal{E} > 1$ and look at the degenerate perturbation theory of the eigenvalue 1 while moving the vector \vec{l}' . The first order perturbation of the eigenvalues if \vec{l}' depends on a parameter t is given by the eigenvalues of the quadratic form

$$\mathcal{Q} := - \sum_{e \in E} m_e \frac{\partial l_e}{\partial t} \Big|_{t=0}$$

If the variation of lengths is given by

$$\vec{l}(t) = (2\pi, \dots, 2\pi, \vec{l}'_0 + t\mathbf{1}) ,$$

we get

$$\mathcal{Q} = - \sum_{e \in E} m_e$$

which is < 0 on the orthogonal of u_0 in \mathcal{E} . Hence, for u small, the eigenvalue 1 is non degenerate.

C Semi-algebraic sets

A semi-algebraic subset of \mathbb{R}^N is a set defined by a finite number of equations and inequations with polynomial entries. A fundamental result is the Tarski-Seidenberg Theorem which says

Theorem C.1 *The image of a semi-algebraic set by a linear map from \mathbb{R}^N onto \mathbb{R}^n is still semi-algebraic.*

One of the main properties of semi-algebraic sets is that they admit a stratification: such a set is a union of a finite number of sub-manifold of \mathbb{R}^N , called the strata, so that the closure of each of them is the union of a finite number of strata. The dimension of a semi-algebraic set is then the maximal dimension of these strata. More details can be found in the classical book [BCR98].

Let us show as a typical example that W_G is semi-algebraic as well as W_G^o . The equation for eigenfunctions of Δ_Γ^l with eigenvalue 1 is of the form $M(z)(\vec{a}, \vec{b}) = 0$ with $M(z)$ polynomial in $z = \exp(il)$. The corresponding eigenfunction vanishes at the vertex x_0 if $a_e = 0$ or $a_e \cos l_e + b_e \sin l_e = 0$, depending of the orientation of $e = \{x_0, x_1\}$. All these equations are polynomial in the coordinates of z and the vectors \vec{a}, \vec{b} . It is then enough to add $|z_e|^2 = 1$. All of this gives an algebraic sub-set of $\mathbb{R}^{4\#E}$. The set W_G is the image of this set by the projection on the $2\#E$ first factors. Concerning W_G^o , the uniqueness of the solution of a system of homogeneous linear equations up to scaling reduces to the non-vanishing of some minors. Algebraicity follows then by taking the sum of the squares of these minors.

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