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Abstract

Roughly speaking, let us say that a map between metric spaces is *large scale conformal* if it maps packings by large balls to large quasi-balls with limited overlaps. This quasi-isometry invariant notion makes sense for finitely generated groups. Inspired by work by Benjamini and Schramm, we show that under such maps, some kind of dimension increases: exponent of polynomial volume growth for nilpotent groups, conformal dimension of the ideal boundary for hyperbolic groups. A purely metric space notion of ℓ^p-cohomology plays a key role.

Résumé

Grosso modo, une application entre espaces métriques est *conforme à grande échelle* si elle envoie tout empilement de grandes boules sur une collection de grandes quasi-boules qui ne se chevauchent pas trop. Cette notion est un invariant de quasi-isométrie, elle s'étend aux groupes de type fini. En s'inspirant de travaux de Benjamini et Schramm, on montre qu'en présence d'une telle application, une sorte de dimension doit augmenter : il s'agit de l'exposant de croissance polynômiale du volume pour les groupes nilpotents, de la dimension conforme du bord pour les groupes hyperboliques. Une nouvelle définition, purement métrique, de la cohomologie ℓ^p joue un rôle important.

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Contents

1 Introduction

1.1 Microscopic conformality

Examples of conformal mappings arose pretty early in history: the stereographic projection, which is used in astrolabes, was known to ancient Greece. The metric distorsion of a conformal mapping can be pretty large. For instance, the Mercator planisphere (1569) is a

conformal mapping of the surface of a sphere with opposite poles removed onto an infinite cylinder. Its metric distorsion (Lipschitz constant) blows up near the poles, as everybody knows. Nevertheless, in many circumstances, it is possible to estimate metric distorsion, and this lead in the last century to metric space analogues of conformal mappings, known as quasi-symmetric or quasi-Möbius maps. Grosso modo, these are homeomorphisms which map balls to quasi-balls. Quasi means that the image $f(B)$ is jammed between two concentric balls, $B \subset f(B) \subset \ell B$, with a uniform ℓ, independent of location or radius of B.

1.2 Mesoscopic conformality

Some evidence that conformality may manifest itself in a discontinuous space shows up with Koebe's 1931 circle packing theorem. A circle packing of the 2-sphere is a collection of interior-disjoint disks. The incidence graph of the packing has one vertex for each circle and an edge between vertices whenever corresponding circles touch. Koebe's theorem states that every triangulation of the 2-sphere is the incidence graph of a disk packing, unique up to Möbius transformations. Thurston conjectured that triangulating a planar domain Ω with a portion of the incidence graph of the standard equilateral disk packing, and applying Koebe's theorem to it, one would get a numerical approximation to Riemann's conformal mapping of Ω to the round disk. This was proven by Rodin and Sullivan, [RS], in 1987. This leads us to interpret Koebe's circle packing theorem as a mesoscopic analogue of Riemann's conformal mapping theorem.

1.3 A new class of maps

In this paper, we propose to go one step further and define a class of large scale conformal maps. Roughly speaking, large scale means that our definitions are unaffected by local changes in metric or topology. Technically, it means that pre- or post-composition of large scale conformal maps with quasi-isometries are again large scale conformal. This allows to transfer some techniques and results of conformal geometry to discrete spaces like finitely generated groups, for instance.

1.4 Examples

In first approximation, a map between metric spaces is large scale conformal if it maps every packing by sufficiently large balls to a collection of large quasi-balls which can be split into the union of boundedly many packings. We postpone till next section the rather technical formal definition. Here are a few sources of examples.

- Quasi-isometric embeddings are large scale conformal.
- Snowflaking (i.e. replacing a metric by a power of it) is large scale conformal.
- Power maps $z \mapsto z|z|^{K-1}$ are large scale conformal for $K \geq 1$. They are not quasi-isometric, nor even coarse embeddings.
- Compositions of large scale conformal maps are large scale conformal.

For instance, every nilpotent Lie or finitely generated group can be large scale conformally embedded in Euclidean space of sufficiently high dimension, [A]. Every hyperbolic group can be large scale conformally embedded in hyperbolic space of sufficiently high dimension, [BoS].

1.5 Results

Our first main result is that a kind of dimension increases under large scale conformal maps. The relevant notion depends on classes of groups.

Theorem 1 *If G is a finitely generated or Lie nilpotent group, set $d_1(G) = d_2(G) =$ the exponent of volume growth of G. If G is a finitely generated or Lie hyperbolic group, let $d_1(G) := CohDim(G)$ be the infimal p such that the ℓ^p-cohomology of G does not vanish. Let $d_2(G) := ConfDim(\partial G)$ be the Ahlfors-regular conformal dimension of the ideal boundary of G.*

Let G and G' be nilpotent or hyperbolic groups. If there exists a large scale conformal map $G \to G'$, then $d_1(G) \leq d_2(G')$.

Theorem 1 is a large scale version of a result of Benjamini and Schramm, [BS], concerning packings in $\mathbb{R}^d$. The proof follows the same general lines but differs in details. The result is not quite sharp in the hyperbolic group case, since it may happen that $d_1(G) < d_2(G)$, [BP].

However equality $d_1(G) = d_2(G)$ holds for Lie groups, their lattices and also for a few other finitely generated examples.

Our second result is akin to the fact that maps between geodesic metric spaces which are uniform/coarse embeddings in both directions must be quasi-isometries.

Theorem 2 *Let X and X' be bounded geometry manifolds or polyhedra. Assume that X and X' have isoperimetric dimension > 1. Every homeomorphism $f : X \to X'$ such that f and f^{-1} are large scale conformal is a quasi-isometry.*

Isoperimetric dimension is defined in subsection 8.3. Examples of manifolds or polyhedra with isoperimetric dimension > 1 include universal coverings of compact manifolds or finite polyhedra whose fundamental group is not virtually cyclic.

This is a large scale version of the fact that every quasiconformal diffeomorphism of hyperbolic space is a quasi-isometry. This classical result generalizes to Riemannian n-manifolds whose isoperimetric dimension is $> n$. The new feature of the large scale version is that isoperimetric dimension > 1 suffices.

1.6 Proof of Theorem 1

Our main tool is a metric space avatar of energy of functions, and, more generally, norms on cocycles giving rise to L^p-cohomology. Whereas, on Riemannian n-manifolds, only n-energy $\int |\nabla u|^n$ is conformally invariant, all p-energies turn out to be large scale conformal invariants. Again, we postpone the rather technical definitions to Section 5 and merely give a rough sketch of the arguments.

Say a locally compact metric space is p-parabolic if for all (or some) point o, there exist compactly supported functions taking value 1 at o, of arbitrarily small p-energy. For instance, a nilpotent Lie or finitely generated group G is p-parabolic iff $p \geq d_1(G)$. Non-elementary hyperbolic groups are never p-parabolic. If X has a large scale conformal embedding into X' and X' is p-parabolic, so is X. This proves Theorem 1 for nilpotent targets.

If a metric space has vanishing L^p-cohomology, maps with finite p-energy have a limit at infinity. We show that a hyperbolic group G' admits plenty of functions with finite p-energy when $p > ConfDim(G')$. First, such functions separate points of the ideal boundary $\partial G'$ (this

result also appears in [BnK]). Second, for each ideal boundary point ξ, there exists a finite p-energy function with a pole at ξ. The first fact implies that, if $CohDim(G) > ConfDim(G')$, any large scale conformal map $G \to G'$ converges to some ideal boundary point ξ, the second leads to a contradiction. This proves Theorem 1 for hyperbolic domains and targets.

In [BS], Benjamini and Schramm observed that vanishing of reduced cohomology suffices for the previous argument to work, provided the domain is not p-parabolic. This proves Theorem 1 for nilpotent domains and hyperbolic targets.

In the hyperbolic to hyperbolic case, one expects $CohDim$ to be replaced by $ConfDim$. For this, one could try to reconstruct the ideal boundary of a hyperbolic group merely in terms of finite p-energy functions, in the spirit of [R] and [B].

1.7 Proof of Theorem 2

Following H. Grötzsch, [Gr], we define 1-capacities of compact sets K in a locally compact metric space X by minimizing 1-energies of compactly supported functions taking value 1 on K. Then we minimize capacities of compact connected sets joining a given pair of points to get a pseudo-distance δ on X. This is invariant under homeomorphisms which are large scale conformal in both directions. If X has bounded geometry, δ is finite. If X has isoperimetric dimension $d > 1$, then δ tends to infinity with distance. This shows that homeomorphisms which are large scale conformal maps in both directions are coarse embeddings in both directions, hence quasi-isometries. It turns out that all finitely generated groups have isoperimetric dimension > 1, but virtually cyclic ones.

1.8 Larger classes

Some of our results extend to wider classes of maps. If we merely require that balls of a given range of sizes are mapped to quasi-balls which are not too small, we get the class of *uniformly conformal maps*. It is stable under precomposition with arbitrary uniform (also known as coarse) embeddings. We can show that no such map can exist between nilpotent or hyperbolic groups unless the inequalities of Theorem 1 hold.

Corollary 1 *We keep the notation of Theorem 1. Let G and G' be hyperbolic groups. If there exists a uniform embedding $G \to G'$, then $d_1(G) \leq d_2(G')$.*

We note that special instances of this Corollary have been obtained by D. Hume, J. Mackay and R. Tessera by a different method, [HMT]. Their results apply in particular to M. Bourdon's rich class of (isometry groups of) Fuchsian buildings, see section 7.

If we give up the restriction on the size of the images of balls, we get the even wider class of *coarse conformal maps*. It is stable under post-composition with quasi-symmetric homeomorphisms. New examples arise, such as stereographic projections, or the Poincaré model of hyperbolic space and its generalizations to arbitrary hyperbolic groups. However, when targets are smooth, coarse conformal maps are automatically uniformly conformal, hence similar results hold.

Corollary 2 *No coarse conformal map can exist between a finitely generated group G and a nilpotent Lie group G' equipped with a Riemannian metric unless G is itself virtually nilpotent, and $d_1(G) \leq d_1(G')$. Also, no coarse conformal map can exist from a hyperbolic group G to a bounded geometry manifold quasiisometric to a hyperbolic group G' unless $d_1(G) \leq d_2(G')$.*

1.9 Organization of the paper

Section 2 contains definitions, basic properties and examples of coarse, uniform, rough and large scale conformal maps. In Section 3, the notion of a quasi-symmetry structure is introduced, as a tool to handle hyperbolic metric spaces: every such space has a rough conformal map onto a product of quasi-metric quasi-symmetry spaces, as shown in Section 4. The existence of this map has the effect of translating large scale problems into microscopic analytic issues. The definition of energy in Section 5 comes with moduli of curve families and parabolicity. It culminates with the proof that several families of quasi-symmetry spaces are parabolic. L^p-cohomology of metric spaces is defined in Section 6, where the main results relating parabolicity, L^p-cohomology and coarse conformal maps are proven. Section 7 draws consequences for nilpotent or hyperbolic groups, concluding the proof of Theorem 1. The material for the proof of Theorem 2 is collected in Section 8. As a byproduct, we find conditions on a pair of spaces X, X' in or-

der that coarse conformal maps $X \to X'$ be automatically uniformly conformal, this provides the generalizations selected in Corollary 2.

1.10 Acknowledgements

The present work originates from a discussion with James Lee and Itai Benjamini on conformal changes of metrics on graphs, cf. [ST], during the Institut Henri Poincaré trimester on Metric geometry, algorithms and groups, see [MAG]. It owes a lot to Benjamini and Schramm's paper [BS]. The focus on the category of metric spaces and large scale conformal maps was triggered by a remark by Jonas Kahn.

2 Coarse notions of conformality

A sphere packing in a metric space Y is a collection of interior-disjoint balls. The incidence graph X of the packing has one vertex for each ball and an edge between vertices whenever corresponding balls touch. A packing may be considered as a map from the vertex set X to Y, that maps the tautological packing of X (by balls of radius $1/2$) to the studied packing of Y.

We modify the notion of a sphere packing in order to make it more flexible. In the domain, we allow radii of balls to vary in some finite interval $[R, S]$. In the range, we replace collections of disjoint balls with collections of balls with bounded multiplicity (unions of boundedly many packings). We furthermore insist that ℓ-times larger concentric balls still form a bounded multiplicity packing.

The resulting notion is invariant under coarse embeddings between domains and quasi-symmetric maps between ranges. Therefore, it is a one-sided large scale concept (in terms of domain, not of range). It is reminiscent of conformality since it requires that spheres be (roughly) mapped to spheres. Whence the term "coarsely conformal". In order to get a class which is invariant under post-composition with quasi-isometries, we shall introduce a subclass of "large scale conformal" maps.

2.1 Coarse, uniform, rough and large scale conformality

Let X be a metric space. A ball B in X is the data of a point $x \in X$ and a radius $r \geq 0$. For brevity, we also denote $B(x, r)$ by B. If $\lambda \geq 0$, λB denotes $B(x, \lambda r)$. For $S \geq R \geq 0$, let $\mathcal{B}^X_{R,S}$ denote the set of balls whose radius r satisfies $R \leq r \leq S$.

Definition 3 *Let X be a metric space. An* (ℓ, R, S)-packing *is a collection of balls $\{B_j\}$, each with radius between R and S, such that the concentric balls ℓB_j are pairwise disjoint. An (N, ℓ, R, S)-packing is the union of at most N (ℓ, R, S)-packings.*

The balls of an (N, ℓ, R, S)-packing, $N \geq 2$, are not disjoint (I apologize for this distorted use of the word *packing*), but no more than N can contain a given point.

Definition 4 *Let X and X' be metric spaces. Let $f : X \to X'$ be a map. Say f is* (R, S, R', S')-coarsely conformal *if there exists a map*

$$B \mapsto B', \quad \mathcal{B}^X_{R,S} \to \mathcal{B}^{X'}_{R',S'},$$

and, for all $\ell' \geq 1$, an $\ell \geq 1$ and an N' such that

1. *For all $B \in \mathcal{B}^X_{R,S}$, $f(B) \subset B'$.*
2. *If $\{B_j\}$ is a (ℓ, R, S)-packing of X, then $\{B'_j\}$ is an (N', ℓ', R', S')-packing of X'.*

Definition 5 *Let $f : X \to X'$ be a map between metric spaces.*

1. *We say that f is* coarsely conformal *if there exists $R > 0$ such that for all finite $S \geq R$, f is $(R, S, 0, \infty)$-coarsely conformal.*
2. *We say that f is* uniformly conformal *if for every $R' > 0$, there exists $R > 0$ such that for all finite $S \geq R$, f is (R, S, R', ∞)-coarsely conformal.*
3. *We say that f is* roughly conformal *if there exists $R > 0$ such that f is $(R, \infty, 0, \infty)$-coarsely conformal.*
4. *We say that f is* large scale conformal *if for every $R' > 0$, there exists $R > 0$ such that f is (R, ∞, R', ∞)-coarsely conformal.*

Here is the motivation for these many notions. The main technical step in our theorems applies to the larger class of coarse conformal maps. To turn this into a large scale notion, one needs to forbid the occurrence of small balls, whence the slightly more restrictive uniform variant, to which all our results apply. Uniformly conformal maps do not form a category, it is the smaller class of large scale conformal maps which does. The Poincaré models of hyperbolic metric spaces are crucial tools, but these maps are not large scale conformal, since small balls do occur in the range, merely roughly conformal.

Proposition 6 *The four classes enjoy the following properties*

- *Large scale conformal* $\Longrightarrow$ *roughly conformal* $\Longrightarrow$ *coarsely conformal.*
- *Large scale conformal* $\Longrightarrow$ *uniformly conformal* $\Longrightarrow$ *coarsely conformal.*
- *Let X, X' and X'' be metric spaces. Let $f : X \to X'$ be (R, S, R', S')-coarsely conformal. Let $f' : X' \to X''$ be (R', S', R'', S'')-coarsely conformal. Then $f' \circ f : X \to X''$ is (R, S, R'', S'')-coarsely conformal.*
- *Large scale conformal maps can be composed. Large scale conformal maps up to* translations*, i.e. self-maps that move points a bounded distance away, constitute the morphisms of the* large scale conformal category*.*
- *Roughly conformal maps can be precomposed with large scale conformal maps. Precomposing a roughly conformal map with a uniformly conformal map yields a coarsely conformal map.*
- *Uniformly conformal maps between locally compact metric spaces are automatically proper.*

Proof Composing maps

$$\mathcal{B}^X_{R,S} \to \mathcal{B}^{X'}_{R',S'} \to \mathcal{B}^{X''}_{R'',S''},$$

we get, for every ball B in X, balls B' in X' and B'' in X'' such that $f(B) \subset B'$, $f'(B') \subset B''$, hence $f' \circ f(B) \subset B''$. Furthermore, we get, for every $\ell'' \geq 1$, a scaling factor ℓ' and a multiplicity N'', and then a scaling factor ℓ and a multiplicity N'. Given an (ℓ, R, S)-packing of X, the corresponding balls can be split into at most N' (ℓ', R', S')-packings of X'. For each sub-packing, the corresponding

balls in X'' can be split into at most N'' (ℓ'', R'', S'')-packings of X''. This yields a total of at most $N'N''$ (ℓ'', R'', S'')-packings of X'', i.e. a $(N'N'', \ell'', R'', S'')$-packing, as desired.

Properness of uniformly conformal maps is proven by contradiction. If $f : X \to X'$ is uniformly conformal but not proper, there exists a sequence $x_j \in X$ such that $f(x_j)$ has a limit $x' \in X'$. Fix $R' > 0$ and $\ell' \geq 1$, get $R > 0$, $\ell \geq 1$ and N'. One may assume that $d(x_j, x_{j'}) > 2\ell R$ for all $j' \neq j$. Then $\{B(x_j, R)\}$ is a (ℓ, R, R)-packing. There exist balls $B'_j \supset f(B(x_j, R))$ which form a (N', ℓ', R', ∞)-packing. For j large, $x' \in B'_j$, contradicting multiplicity $\leq N'$. One concludes that f is proper. ■

In the next subsections, we shall relate our large scale conformal definitions with classical notions and collect examples.

2.2 Quasi-symmetric maps

Notions of quasi-conformal maps on metric spaces have a long history, see [HK], [He], [Ty].

Example 7 *By definition, a homeomorphism $f : X \to X'$ is* quasi-symmetric *if there exists a homeomorphism $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every triple x, y, z of distinct points of X,*

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta(\frac{d(x, y)}{d(x, z)}).$$

Proposition 8 *Quasi-symmetric homeomorphisms are roughly (and thus coarsely) conformal.*

Proof When $B = B(x, r)$, we define B' to be the smallest ball centered at $f(x)$ which contains $f(B)$. Let ρ' be its radius. Let $y \in B$ be such that $d(f(x), f(y)) = \rho'$. If $z \in f^{-1}(\ell' B')$, $d(f(x), f(z)) \leq \ell' \rho'$, so

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \geq \frac{\rho'}{\ell' \rho'} = \frac{1}{\ell'}.$$

By quasi-symmetry, this implies that $\eta(\frac{d(x,y)}{d(x,z)}) \geq \frac{1}{\ell'}$, and thus $d(x, z) \leq \frac{1}{\eta^{-1}(\frac{1}{\ell'})} d(x, y)$. In other words, $z \in \ell B$ with $\ell = \frac{1}{\eta^{-1}(\frac{1}{\ell'})}$. We conclude that, for every $\ell' \geq 1$, there exist $\ell \geq 1$ and a correspondance $B \mapsto B'$ such that $f(B) \subset B'$ and $f^{-1}(\ell' B') \subset \ell B$.

If B_1 and B_2 are balls in X such that ℓB_1 and ℓB_2 are disjoint, then $f^{-1}(\ell' B_1') \subset \ell B_1$ and $f^{-1}(\ell' B_2') \subset \ell B_2$ are disjoint as well, hence $\ell' B_1'$ and $\ell' B_2'$ are disjoint, thus f is $(0, \infty, 0, \infty)$-coarsely conformal. A fortiori, f is $(R, \infty, 0, \infty)$-coarsely conformal for $R > 0$, so f is roughly conformal. ■

Note that R and S play no role when checking that quasi-symmetric maps are coarsely conformal, and no guarantee on radii of balls in the range is given (i.e. such maps are $(0, \infty, 0, \infty)$-coarsely conformal). In a sense, for the wider class of maps we are interested in, three of the quasi-symmetry assumptions are relaxed:

- Maps need not be homeomorphisms.
- The quasi-symmetry estimate applies only to balls in a certain range $[R, S]$ of radii.
- Centers need not be mapped to centers.

Proposition 9 *Globally defined quasiconformal mappings of Euclidean space $\mathbb{R}^n$, $n \geq 2$, are quasi-symmetric, hence roughly and coarsely conformal.*

Proof Although not explicitely stated, this follows from the proof of Theorem 22.3, page 78, in [V1]. In $\mathbb{R}^n$, there is a uniform lower bound $h(\frac{b}{a}) > 0$ for the conformal capacity of condensers (C_0, C_1) such that C_0 connects 0 to the a-sphere and C_1 connects the b-sphere to infinity. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be K-quasiconformal and map 0 to 0. Let $l = \min\{|f(x)|\,;\, |x| = a\}$ and $L = \max\{|f(x)|\,;\, |x| = b\}$, let $C_0 = f^{-1}(B(0, l))$ and $C_1 = f^{-1}(\mathbb{R}^n \setminus B(0, L))$. Then $cap_n(C_0, C_1) \geq h(\frac{b}{a})$. On the other hand,

$$cap_n(C_0, C_1) \leq K\, cap_n(f(C_0, C_1)) = K\omega_{n-1}\, \log(\frac{L}{l})^{1-n},$$

this yields an upper bound on $\frac{L}{l}$ in terms of $\frac{b}{a}$, proving that f is quasi-symmetric. ■

Example 10 *For all $K > 0$, the map $z \mapsto z|z|^{K-1}$ on $\mathbb{R}^n$ is roughly conformal. If $K \geq 1$, it sends large balls to large balls, hence it is large scale conformal.*

If $n \geq 2$, Proposition 9 applies. Its 1-dimensional analogue $f : x \mapsto x|x|^{K-1}$ is roughly conformal as well, and large scale conformal if $K \geq 1$. This can be checked directly.

2.3 Coarse quasi-symmetry

From the proof of Proposition 8, we see that the following is a sufficient condition for coarse conformality.

Definition 11 *Let X and X' be metric spaces. Let $f : X \to X'$ be a map. Say f is* (R, S, R', S')-coarsely quasi-symmetric *if it comes with a map*

$$B \mapsto B', \quad \mathcal{B}^X_{R,S} \to \mathcal{B}^{X'}_{R',S'},$$

and for all $\ell' \geq 1$, there exist $\ell \geq 1$ such that

1. *for all $B \in \mathcal{B}^X_{R,S}$, $f(B) \subset B'$.*
2. *If $\ell B_1 \cap \ell B_2 = \emptyset$, then $\ell' B'_1 \cap \ell' B'_2 = \emptyset$.*

Say that f is coarsely quasi-symmetric *if there exists $R > 0$ such that for all $S \geq R$, f is $(R, S, 0, \infty)$-coarsely quasi-symmetric.*

Lemma 12 *Let X and X' be metric spaces. Let $f : X \to X'$ be a (R, S, R', S')-coarsely quasi-symmetric map. Then f is (R, S, R', S')-coarsely conformal. In particular, coarsely quasi-symmetric maps are coarsely conformal.*

Proof Coarse quasi-symmetry amounts to requiring $N' = 1$ in the definition of coarse conformality. ■

Remark 13 *If a map $f : X \to X'$ is (R, S, R', S')-coarsely quasi-symmetric, then the ball correspondence $B \mapsto B'$ satisfies, for all $\ell' \geq 1$, $f^{-1}(\ell' B') \subset \ell B$ for a suitable ℓ.*

Proof If $x \notin 2\ell B$, the two element collection $\{B_1 = B, B_2 = B(x, R)\}$ is a (ℓ, R, S)-packing. Thus $\{B', B'_2\}$ is a (ℓ', R', S')-packing. Therefore $f(x) \notin \ell' B'$, $x \notin f^{-1}(\ell' B')$. This shows that $f^{-1}(\ell' B') \subset 2\ell B$. ■

2.4 Quasi-Möbius maps

Let X be a metric space. If $a, b, c, d \in X$ are distinct, their *cross-ratio* is

$$[a, b, c, d] = \frac{d(a, c)}{d(a, d)} \frac{d(b, d)}{d(b, c)}.$$

An embedding $f : X \to Y$ is *quasi-Möbius* if there exists a homeomorphism $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all quadruples of distinct points $a, b, c, d \in X$,

$$[f(a), f(b), f(c), f(d)] \leq \eta([a, b, c, d]).$$

Note that if f is a homeomorphism, f^{-1} is quasi-Möbius as well.

The main examples are

- inversions $x \mapsto \frac{x}{|x|^2}$ in Banach spaces,
- the stereographic projection $\mathbb{R}^n \to S^n$,
- its complex, quaternionic and octonionic versions, sometimes known as *Cayley transforms*, where $\mathbb{R}^n$ is replaced with a Heisenberg group equipped with a Carnot-Carathéodory metric.
- the action of a hyperbolic group on its ideal boundary is (uniformly) quasi-Möbius.

According to J. Väisälä, [V2], if X and Y are bounded, quasi-Möbius maps $X \to Y$ are quasi-symmetric. If X and Y are unbounded, a quasi-Möbius map $X \to Y$ that tends to infinity at infinity is quasi-symmetric. We note an other situation where a quasi-Möbius map is coarsely conformal.

Lemma 14 *Assume X is unbounded and Y is bounded. Let $f : X \to Y$ be a quasi-Möbius embedding. Assume that f has a limit at infinity. Then f is coarsely conformal.*

Proof Fix $S > 0$. Let $y = \lim_{x\to\infty} f(x)$. Fix some origin $o \in X$ et let $o' = f(o)$. Consider a triple $a, b, c \in X$ such that $d(c, o) \geq 4S$, $d(c, a) \leq S$ and $d(c, b) \leq S$. Then $d(a, o) \geq 3S$, $d(b, o) \geq 3S$ and $d(a, b) \leq 2S$, thus

$$\frac{1}{3} \leq 1 - \frac{2S}{d(b, o)} \leq \frac{d(a, o)}{d(b, o)} \leq 1 + \frac{2S}{d(b, o)} \leq 2,$$

Let $x \in X$ satisfy $d(c, x) \geq 4S$. Then $d(x, a) \geq 3S$ and $d(x, b) \geq 3S$ as well, so

$$\frac{1}{6} \leq [a, b, x, o] \leq 6.$$

It follows that $\frac{1}{\eta^{-1}(\frac{1}{6})} \leq [f(a), f(b), f(x), f(o)] \leq \eta(6)$. As x tends to ∞, this cross-ratio tends to

$$[f(a), f(b), y, o'] = \frac{d(f(a), y)}{d(f(a), o')} \frac{d(f(b), o')}{d(f(b), y)},$$

showing that $\frac{d(f(a),y)}{d(f(b),y)}$ stays bounded above and below.

Since

$$[a, b, c, \infty] = \frac{d(a, c)}{d(b, c)}, \quad [f(a), f(b), f(c), y] = \frac{d(f(a), f(c))}{d(f(b), f(c))} \frac{d(f(b), y)}{d(f(a), y)},$$

the ratio $\frac{d(f(a),f(c))}{d(f(b),f(c))}$ is bounded above and below in terms of $\frac{d(a,c)}{d(b,c)}$ only. The argument in the previous proof show that f is $(0, S, 0, \infty)$-large scale quasi-symmetric. One concludes that f is coarsely conformal. ■

2.5 The Cayley transform

The stereographic projection (or Cayley transform) extends to all metric spaces. It is specially well suited to the class of Ahlfors regular metric spaces.

Recall that a metric space X is *Q-Ahlfors regular at scale S* if it admits a measure μ and a constant $C(S)$ such that

$$\frac{1}{C} r^Q \leq \mu(B(x, r)) \leq C r^Q$$

for all $x \in X$ and $r \leq S$. We abbreviate it in Q-Ahlfors regular if $S =$ diameter(X). Locally Q-Ahlfors regular means Q-Ahlfors regular at all scales (with constants depending on scale).

Example 15 *$\mathbb{R}$, intervals of $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are 1-Ahlfors regular. Carnot groups are Ahlfors regular. Snowflaking, i.e. replacing the distance d by d^α for some $0 < \alpha < 1$, turns a Q-Ahlfors regular space into a $\frac{Q}{\alpha}$-Ahlfors regular space. The product of Q- and Q'-Ahlfors regular spaces is a $Q + Q'$-Ahlfors regular space. The ideal boundary of a hyperbolic group equipped with a visual quasi-metric is Ahlfors regular, [S], [C].*

Lemma 16 (Väisälä, [V2]) *Every metric space X has a quasi-Möbius embedding into a bounded metric space $\dot{X}$. If X is Q-Ahlfors regular, so is $\dot{X}$.*

Proof Use the Kuratowski embedding $X \to L$, where $L = L^\infty(X)$. Then embed L into $L \times \mathbb{R}$ and apply an inversion. This is a quasi-Möbius map onto its image, which is bounded and homeomorphic to the one-point completion $\dot{X}$ of X.

If X is Q-Ahlfors regular with Hausdorff measure μ, let ν be the measure with density $|\dot{x}|^{2Q}$ with respect to the pushed forward measure on $\dot{X}$. Since the inverse image of $B(\dot{x}, r)$ is roughly equal to $B(x, r|\dot{x}|^{-2})$, where $|\dot{x}| = |x|^{-1}$,

$$\nu(B(\dot{x}, r)) \sim |\dot{x}|^{2Q} \mu(B(x, r|\dot{x}|^{-2})) \sim \mu(B(x, r)) \sim r^Q.$$

This shows that $\dot{X}$ is Q-Ahlfors regular as well. ■

2.6 Uniform/coarse embeddings

Definition 17 *A map $f : X \to X'$ is a* uniform *or* coarse embedding *if for every $T > 0$ there exists $\tilde{T}$ such that for every x_1, $x_2 \in X$,*

$$\begin{aligned} d(x_1, x_2) \leq T &\implies d(f(x_1), f(x_2)) \leq \tilde{T}, \\ d(f(x_1), f(x_2)) \leq T &\implies d(x_1, x_2) \leq \tilde{T}. \end{aligned}$$

Lemma 18 *For every $0 < R \leq S < +\infty$, a uniforme (or coarse) embedding is (R, S, R', S')-large scale quasi-symmetric for some positive and finite R' and S', hence (R, S, R', S')-coarsely conformal. In particular, uniform (or coarse) embeddings are uniformly conformal.*

Proof Assume that $f : X \to X'$ is a coarse embedding controlled by function $T \mapsto \tilde{T}$. Pick an arbitrary $R > 0$ and an arbitrary $S \geq R$. Set $R' = S' = \tilde{S}$, $U = \ell' \tilde{S}$ and $\ell = \frac{\tilde{U}}{R}$.

When $B = B(x, r)$, $r \in [R, S]$, we define $B' = B(f(x), \tilde{S})$. Then $f(B) \subset B'$.

On the other hand, if $x' \in f^{-1}(\ell' B')$, then $d(f(x'), f(x)) \leq \ell' \tilde{S} = U$, thus $d(x', x) \leq \tilde{U}$. This shows that $f^{-1}(\ell' B') \subset \ell B$ with $\ell = \frac{\tilde{U}}{R}$. In other words, f is $(R, S, \tilde{S}, \tilde{S})$-large scale quasi-symmetric. ■

Remark 19 *Conversely, if the metric spaces are geodesic, every (R, S, R', S')-coarse conformal map with $0 < R \leq S < +\infty$ and $0 < R' \leq S' < +\infty$ is a coarse embedding.*

We see that (R, S, R', S')-coarse conformality does not bring anything new while $R, R' > 0$ and $S, S' < \infty$, at least in the geodesic world. Similarly, classical conformal mappings with Jacobian bounded

above and below are bi-Lipschitz. Conformal geometry begins when $S' = \infty$ *or* $R' = 0$.

Indeed, let $f : X \to X'$ be a (R, S, R', S')-coarse conformal map. Then R-balls are mapped into S'-balls. If X is geodesic, this implies that

$$d(f(x_1), f(x_2)) \leq \frac{S'}{R} d(x_1, x_2) + S'.$$

Conversely, set $\ell' = (T + 2S')/2R'$ and let ℓ be the corresponding scaling factor in the domain. Let $B_1 = B(x_1, S)$ and $B_2 = B(x_2, S)$. If $d(x_1, x_2) > 2\ell S$, then ℓB_1 and ℓB_2 are disjoint. Let $B_1' = B(x_1', r_1) \supset f(B_1)$ and $B_2' = B(x_2', r_2) \supset f(B_1)$ be the corresponding balls in X'. Since $\{B_1, B_2\}$ is a $(1, \ell, R, S)$-packing, $\{B_1', B_2'\}$ is a $(1, \ell', R', S'\}$ packing, so both $R' \leq r_i \leq S'$ and $\ell B_1' \cap \ell B_2' = \emptyset$, hence $d(x_1', x_2') > \ell' r_1 + \ell' r_2 \geq 2\ell' R'$. Since both $d(f(x_i), x_i') \leq S'$, $d(f(x_1), f(x_2)) > 2\ell' R' - 2S' \geq T$.

Lemma 20 *Quasi-isometric embeddings are large scale conformal.*

Proof The assumption means that

$$\frac{1}{L}\, d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq L\, d(x_1, x_2) + C.$$

To a ball $B = B(x, r)$ in X, we attach $B' = B(f(x), Lr + C)$. Given balls $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$, assume that concentric balls ℓB_1 and ℓB_2 are disjoint. Then, looking at an (almost) minimizing path joining x_1 to x_2, one sees that $d(x_1, x_2) \geq \ell(r_1 + r_2)$. This implies

$$d(f(x_1), f(x_2)) \geq \frac{1}{L}\ell(r_1 + r_2) - C \geq \ell'(r_1 + r_2)$$

provided $r_1 + r_2 \geq 2R$ and $\ell' \leq \frac{\ell}{L} - \frac{C}{2R}$. If so, the concentric balls $\ell' B_1'$ and $\ell' B_2'$ are disjoint. Therefore, for every $\ell' \geq 1$, there exists $\ell \geq 1$ such that f maps $(1, \ell, R, \infty)$-packings to $(1, \ell', \frac{R}{L} - C, \infty)$-packings, and thus (ℓ, R, ∞)-packings to $(\ell', \frac{R}{L} - C, \infty)$-packings for all N. ■

Corollary 21 *Quasi-isometries between geodesic metric spaces are large scale conformal.*

Indeed, quasi-isometries between geodesic metric spaces are controlled by affine functions in both directions.

Example 22 *Orbital maps of injective homomorphisms between isometry groups of locally compact metric spaces are coarse embeddings. They are rarely quasi-isometric.*

For instance, the control function $T \mapsto \tilde{T}$ of the horospherical embedding of $\mathbb{R}^n$ into H^{n+1} is logarithmic. Given a Euclidean ball B of radius R, the hyperbolic ball B' whose intersection with the horosphere is B and whose projection back to the horosphere is smallest is a horoball, of infinite radius. The radius of its projection is $\frac{1}{2}(R^2+1)$. In order to be mapped to disjoint horoballs, two Euclidean R-balls B_1 and B_2 must have centers at distance at least R^2+1. This makes the scaling factor ℓ depend on R. Thus this embedding is not large scale conformal.

Question. Are orbital maps of subgroups in nilpotent groups always large scale conformal ?

2.7 Assouad's embedding

Assouad's Embedding Theorem, [A] (see also [NN]), states that every snowflake $X^\alpha = (X, d_X^\alpha)$, $0 < \alpha < 1$, of a doubling metric space admits a bi-Lipschitz embedding into some Euclidean space.

Proposition 23 *Bi-Lipschitz embeddings of snowflakes are large scale conformal. In particular, the Assouad embedding of a doubling metric space into Euclidean space is large scale conformal.*

Proof Identity $X \to X^\alpha$ is large scale conformal. Indeed, the correspondence $B \to B'$ is the identity, concentric ball ℓB is mapped to $\ell^\alpha B'$. So given a scaling factor $\ell' > 1$ in the range, $\ell = \ell'^{1/\alpha}$ fits as a scaling factor in the domain.

Bi-Lipschitz maps are large scale conformal, so the composition is large scale conformal as well (Proposition 6). ■

2.8 Sphere packings

Following Benjamini and Schramm [BS], we view a sphere packing in $\mathbb{R}^d$ as a map from the vertex set of a graph, the incidence graph G of the packing, to $\mathbb{R}^d$. G carries a canonical packing by balls of radius $\frac{1}{2}$ (whose incidence graph is G itself). It is a $(1, \frac{1}{2}, \frac{1}{2})$-packing, which is mapped to the given $(1, 0, \infty)$-packing.

Coarse conformality is a very strong restriction on a sphere packing. It has something to do with uniformity in the sense of [BC]. Recall that a sphere packing is M-uniform if

1. the degree of the incidence graph is bounded by M,
2. the ratio of radii of adjacent spheres is $\leq M$.

But uniformity is not sufficient. For instance, let $\sigma(z) = az$ be a planar similarity. If $a = re^{i\theta}$ is suitably chosen (for intance, $\theta = \pi/10$ and $r < e^{-2\pi/100}$), there exists a circle C such that $\sigma(C)$ touches C but no iterate $\sigma^k(C)$, $k \neq 1, -1$, does. The collection of iterates $\sigma^k(C)$, $k \in \mathbb{Z}$, constitutes a uniform planar circle packing whose incidence graph is $\mathbb{Z}$ but which does not give rise to a roughly conformal map of $\mathbb{Z}$ to $\mathbb{R}^2$. Indeed, if R is large enough so that iterates $\sigma^k(C)$, $k \in \{-R, ..., R\}$ make a full turn around the origin, any Euclidean ball B' which contains $2R + 1$ consecutive circles of the packing contains the origin. So the image of any $(N, 1, R, R)$-packing of $\mathbb{Z}$ has infinite multiplicity at the origin.

However, the same construction with $a > 0$ gives rise to a coarse conformal map, composition of an inversion with the standard isometric embedding of $\mathbb{Z}$ in $\mathbb{R}^2$.

Every subgraph of the incidence graph of a sphere packing is again the incidence graph of a sphere packing. No such property holds for coarsely (resp. roughly, resp. large scale) conformal mappings.

3 Quasi-symmetry structures

This notion is needed in order to host an important example of rough conformal map, the Poincaré model of a hyperbolic metric space.

3.1 Definition

Definition 24 *A* quasi-symmetry *structure on a set X is the data of a set $\mathcal{B}$ with a family $(\Phi_\ell)_{\ell \in \mathbb{R}_+^*}$ of maps $\mathcal{B} \to$ Subsets(X) satisfying*

$$\ell \leq \ell' \quad \Longrightarrow \quad \Phi_\ell(B) \subset \Phi_{\ell'}(B).$$

(in the sequel, one will rarely distinguish an element of $\mathcal{B}$ from the corresponding subset $\Phi_1(B)$ of X; then $\Phi_\ell(B)$ will be denoted by ℓB). A set equipped with such a structure is called a q.s. space. *Elements of $\mathcal{B}$ (as well as the corresponding subsets of X) are called* balls.

Example 25 *Quasi-metric spaces come with a natural q.s. structure, where $\mathcal{B} = X \times (0, \infty)$, and a pair (x, r) is mapped to the ball $B(x, \ell r)$.*

Definition 26 *Let X be a q.s. space. An* (N, ℓ)-packing *is a collection of balls $\{B_j\}$ such that the collection of concentric balls ℓB_j has multiplicity $\leq N$, i.e. the collection $\{\ell B_j\}$ can be split into at most N sub-families, each consisting of pairwise disjoint balls.*

3.2 Coarsely and roughly conformal maps to q.s. spaces

Definition 27 *Let X be a metric space and X' a q.s. space. Let $f : X \to X'$ be a map. Say f is* (R, S)-coarsely conformal *if there exist a map*

$$B \mapsto B', \quad \mathcal{B}^X_{R,S} \to \mathcal{B}^{X'},$$

and for all $\ell' \geq 1$, an $\ell \geq 1$ and an N' such that

1. *for all $B \in \mathcal{B}^X_{R,S}$, $f(B) \subset B'$.*
2. *If $\{B_j\}$ is a (ℓ, R, S)-packing of X, then $\{B'_j\}$ is an (N', ℓ')-packing of X'.*

We say that f is coarsely conformal *if there exists $R > 0$ such that for all $S \geq R$, f is (R, S)-coarsely conformal.*

We say that f is roughly conformal *if there exists $R > 0$ such that f is (R, ∞)-coarsely conformal.*

It is harder to be roughly conformal than coarsely conformal.

3.3 Quasi-symmetric maps between q.s. spaces

Definition 28 *Let X and X' be q.s. spaces. A map $f : X \to X'$ is* quasi-symmetric *if for all $\ell' \geq 1$, there exist $\ell \geq 1$ and a correspondence between balls*

$$B \mapsto B', \quad \mathcal{B}^X \to \mathcal{B}^{X'},$$

such that for all $B \in \mathcal{B}^X$,

1. *$f(B) \subset B'$.*
2. *$f^{-1}(\ell' B') \subset \ell B$.*

Quasi-symmetric maps are morphisms in a category whose objects are q.s. spaces. For q.s. structures associated to metrics, isomorphisms in this category coincide with classical quasi-symmetric homeomorphisms.

Proposition 29 *Let X be a metric space, and X', X'' q.s. spaces. If $f : X \to X'$ is (R, S)-coarsely conformal and $f' : X' \to X''$ is quasi-symmetric, then the composition $f' \circ f$ is (R, S)-coarsely conformal.*

Proof Fix $\ell'' \geq 1$. Since f' is quasi-symmetric, there exists $\ell' \geq 1$ such that every ball B' in X' is mapped into a ball B'' of X'' such that $f'^{-1}(\ell'' B'') \subset \ell' B'$. If B'_1 and B'_2 are balls in X' such that $\ell' B'_1 \cap \ell' B'_2 = \emptyset$, then $f'^{-1}(\ell'' B''_1) \cap f'^{-1}(\ell'' B''_2) = \emptyset$, hence $\ell'' B''_1$ and $\ell'' B''_2$ are disjoint. Thus $f' \circ f$ is (R, S)-coarsely conformal. ■

4 The Poincaré model

4.1 The 1-dimensional Poincaré model

The following example of q.s. space may be called the half hyperbolic line.

Definition 30 *Let $\mathbb{D}$ denote the interval $[0, 1]$ equipped with the following q.s. structure. $\mathcal{B}$ is the set of closed intervals in $[0, 1]$. For an interval $I \subset (0, 1]$ there exists a unique pair (R, t) such that $B = [e^{-R-t}, e^{R-t}]$. Then $\ell B := [e^{-\ell R-t}, \min\{1, e^{\ell R-t}\}]$. For an interval of the form $I = [0, b]$, $\ell I = I$.*

In other words, the structure is the usual metric space structure of a half real line transported by the exponential function, and extended a bit arbitrarily to a closed interval.

Remark 31 *Here is a formula for ℓI when $I = [a, b]$, $a > 0$.*

$$\ell I = [a^{\frac{1+\ell}{2}} b^{\frac{1-\ell}{2}}, \min\{1, a^{\frac{1-\ell}{2}} b^{\frac{1+\ell}{2}}\}].$$

Indeed, if $[a, b] = [e^{-R-t}, e^{R-t}]$, then $R = \frac{1}{2}\log(b/a)$ and $t = -\frac{1}{2}\log(ab)$.

4.2 Warped products

There is no way to take products of q.s. spaces, so we use auxiliary quasi-metrics.

Definition 32 *A* q.m.q.s. space *is the data of a q.s. space and a quasi-metric which defines the same balls (but possibly a different $B \mapsto \ell B$ correspondence).*

Example 33 *We keep denoting by $\mathbb{D}$ the q.m.q.s. space $[0,1]$ equipped with its usual metric but the q.s. structure of $\mathbb{D}$.*

Definition 34 *The* product *of two q.m.q.s. spaces Z_1 and Z_2 is $Z_1 \times Z_2$ equipped with the product quasi-metric*

$$d((z_1, z_2), (z_1', z_2')) = \max\{d(z_1, z_1'), d(z_2, z_2')\}$$

and, if $B = I \times \beta$ is a ball in the product, $\ell B = \ell I \times \ell\beta$.

Example 35 *Our main example is $\mathbb{D} \times Z$, where Z is a metric space with its metric q.s. structure.*

4.3 The Poincaré model of a hyperbolic metric space

Proposition 36 *Let X be a geodesic hyperbolic metric space with ideal boundary ∂X. Fix an origin $o \in X$ and equip ∂X with the corresponding visual quasi-distance. Assume that there exists a constant D such that for any $x \in X$ there exists a geodesic ray γ from the base point $\gamma(0) = o$ and passing near x: $d(x, \gamma) < D$. Then there is a rough conformal map π, called the* Poincaré model*, of X to the q.s. space $\mathbb{D} \times \partial X$. Its image is contained in $(0,1] \times \partial X$. If $x \in X$ tends to $z \in \partial X$ in the natural topology of $X \cup \partial X$, then $\pi(x)$ tends to $(0, z)$.*

Proof For $x \in X$, pick a geodesic ray γ starting from o that passes at distance $< D$ from x. Set $\chi(x) = e^{-d(o,x)}$, $\phi(x) = \gamma(+\infty)$ and $\pi(x) = (\chi(x), \phi(x))$. π may be discontinuous, but we do not care.

Let $B = B(x, R)$ be a ball in X, and $t = d(o, x)$. Then $\phi(B)$ is contained in a ball of radius e^{R-t}, up to a multiplicative constant. Indeed, thanks to V. Shchur's Lemma, [Sh], there exists a constant C such that, if $y \in B$ and $t' = d(o, y)$, then $|t - t'| \leq R + C + 2D$ and $t + t' + 2\log d_o(\phi(x), \phi(y)) \leq R + C + 2D$. Thus $2t + 2\log d_o(\phi(x), \phi(y)) \leq 2R +$

$2C+4D$, whence $d_o(\phi(x),\phi(y)) \leq e^{R+C+2D-t}$. It turns out that $\chi(B)$ is also a ball of radius e^{R-t}, up to a multiplicative constant. Indeed, if $y \in B$ and $t' = d(o,y)$, then $|t-t'| \leq R$, thus $\chi(y)$ belongs to the interval $[1-e^{R-t}, 1-e^{-R-t}]$, centered at $1-e^{-t}\cosh(R)$, with radius $e^{-t}\sinh(R)$. From now on, we decide to multiply the quasi-metric on ∂X by the constant factor $e^{-C-2D-1}$, i.e. we use the quasi-metric $d' = e^{-C-2D-1}d_o$. In this way, $\pi(B)$ is contained in the ball B' of $[0,1]\times\partial X$ centered at $(1-e^{-t}\cosh(R), \phi(x))$, with radius $e^{-t}\sinh(R)$.

Conversely, if $(\tau,\eta) \in B'$, then $\eta \in \partial X$ belongs to $B(\phi(x), e^{-t}\sinh(R))$ (which means that $d'(\phi(x),\eta) = e^{-C-2D-1}d_o(\phi(x),\eta) \leq e^{-t}\sinh(R)$), and $\tau \in [1-e^{R-t}, 1-e^{-R-t}]$. Let t' be such that $1-e^{-t'} = \tau$. Then $t-R \leq t' \leq t+R$. Let y (resp. z) be the point of the geodesic from o to η such that $d(o,y) = t'$ (resp. $d(o,z) = t$). Then $d(x,z) \leq 2t + 2\log d_o(\phi(x),\eta) + C + 2D \leq 2R + 3C + 4D + 2$, and $d(z,y) = |t-t'| \leq R$. Thus $d(x,y) \leq 4R$ provided $R \geq 3C+4D+2$. We conclude that

$$\pi(B) \subset B', \quad \text{and} \quad B' \subset \pi(3B).$$

Let us show that

$$\pi(3\ell B) \supset \ell B'.$$

To avoid confusion, let us denote by m_ℓ the $\mathbb{R}^*$ action on intervals. If $B = B(x,R)$ and $\chi(x) = t$, then $\chi(B) = [e^{-R-t}, e^{R-t}]$ and $\chi(\ell B) = m_\ell(\chi(B))$ by definition of m_ℓ. On the other hand, $B' = \chi(B)\times\beta$ where $\phi(B) \subset \beta = B(\phi(x), e^{-t}\sinh(R))$,

$$\phi(\ell B) = B(\phi(x), e^{-t}\sinh(\ell R)) \supset B(\phi(x), e^{-t}e^{(\ell-1)R}\sinh(R)) \supset \ell B,$$

provided $R \geq 1$. Therefore

$$m_\ell(B') = m_\ell(\chi(\ell B)) \times \ell\beta \subset m_\ell(\chi(\ell B)) \times \phi(\ell B).$$

Since this is the ball $(\ell B)'$ corresponding to ℓB, it is contained in $\pi(3\ell B)$.

Since X is hyperbolic, there is a constant δ such that two geodesics with the same endpoints are contained in the δ-tubular neighborhood of each other. We can assume that $\delta \leq C$. It follows that if $\pi(x_1) = \pi(x_2)$, then $d(x_1,x_2) \leq \delta$. Let B_1 and B_2 be balls of radii $\geq C$. Let B_1', B_2' be the corresponding balls in $\mathbb{D}\times\partial X$. If $\ell' B_1' \cap \ell' B_2' \neq \emptyset$, then $\pi(3\ell' B_1) \cap \pi(3\ell' B_2) \neq \emptyset$. Thus $d(3\ell' B_1, 3\ell' B_2) \leq \delta \leq C$. If

$B_1 = B(x_1, r_1)$, $3\ell' B_1 = B(x_1, 3\ell' r_1)$, its C-neigborhood is contained in $B(x_1, 3\ell' r_1 + C) \subset B(x_1, 4\ell' r_1) = 4\ell' B_1$. Hence $4\ell' B_1 \cap 4\ell' B_2 \neq \emptyset$. We may set $\ell = 4\ell'$ and conclude that π is $(3C + 4D + 2, \infty, 0, \infty)$-coarsely quasi-symmetric, hence roughly conformal. ■

5 Energy

Classical analysis has made considerable use of the fact that, on n-space, the L^n norm of the gradient of functions is a conformal invariant. Benjamini and Schramm observe that only the dimension of the range matters. In fact, some coarse analogue of the L^p norm of the gradient of functions turns out to be natural under coarsely conformal mappings. This works for all p (this fact showed up in [P2]).

5.1 Energy and packings

Let X be a metric space. Recall that an (ℓ, R, S)-packing is a collection of balls $\{B_j\}$, each with radius between R and S, such that the concentric balls ℓB_j are pairwise disjoint. If X is merely a q.s. space, ℓ-packings make sense.

Here is one more avatar of the definition of Sobolev spaces on metric spaces. For earlier attempts, see the surveys [He], [GT].

Definition 37 *Let X, Y be metric spaces. Let $\ell \geq 1$. Let $u : X \to Y$ be a map. Define its p-energy at parameters ℓ, R and $S \geq R$ as follows.*

$$E^p_{\ell,R,S}(u) = \sup\{\sum_j \text{diameter}(u(B_j))^p \,;\, (\ell, R, S)\text{-packings } \{B_j\}\}.$$

Remark 38 *The definition of $E^p_{\ell,0,\infty}$ extends to q.s. spaces, we denote it simply by*

$$E^p_\ell(u) = \sup\{\sum_j \text{diameter}(u(B_j))^p \,;\, \ell\text{-packings } \{B_j\}\}.$$

Our main source of functions with finite energy are Ahlfors regular metric spaces.

Proposition 39 *Let X be a d-Ahlfors regular metric space and $u : X \to Y$ a C^α-Hölder continuous function with bounded support. Then $E^p_{\ell,0,\infty}(u)$ is finite for all $p \geq d/\alpha$ and $\ell > 1$. In particular, $E^{d/\alpha}_{\ell,R,S}(u)$ is finite for all R, S.*

Proof Let μ denote Hausdorff d-dimensional measure. Assume first that $\mu(X) < \infty$. For every ball B,

$$\begin{aligned}\operatorname{diam}(u(B))^p &\le \text{const.}\, \operatorname{diam}(\mathrm{B})^{p\alpha} \\ &\le \text{const.}\, \mu(B)^{p\alpha/d} \\ &\le \text{const.}\, \mu(X)^{-1+p\alpha/d}\mu(B),\end{aligned}$$

thus

$$\begin{aligned}\sum_j \operatorname{diam}(u(B_j))^p &\le \text{const.}\, \mu(X)^{-1+p\alpha/d} \sum_j \mu(B_j) \\ &\le \text{const.}\, \mu(X)^{p\alpha/d} < \infty.\end{aligned}$$

If X is unbounded, we assume that u has support in a ball B_0 of radius R_0. Given $\ell > 1$, if a ball B intersects B_0, and has radius $R > 2R_0/(\ell - 1)$, then ℓB contains B_0. If such a ball arises in an ℓ-packing, it is the only element of the packing, hence an upper bound on $\sum \operatorname{diameter}(u(B))^p \le (\max u - \min u)^p$. Otherwise, all balls of the ℓ-packing are contained in $(1 + (2R_0/(\ell - 1))B_0$, whence the above upper bound is valid with X replaced with that ball. ∎

Example 40 *Let $0 < \alpha \le 1$. Let Z be a compact Q-Ahlfors regular metric space. Let $Z^\alpha = (Z, d_Z^\alpha)$ be a snowflake of Z. Consider the q.m.q.s. space $X = \mathbb{D} \times Z$, with its projections*

$$u : X' \to Z,\ u(y,z) = z, \quad \textit{and} \quad v : X' \to Y,\ v(y,z) = y.$$

Then, for all ℓ and for all $p \ge \alpha + Q$,

$$E_\ell^p(u) < +\infty, \quad E_\ell^{p/\alpha}(v) < +\infty.$$

Proof Since the previous argument, in the compact case, never uses concentric balls ℓB, only the balls B themselves, the difference between $\mathbb{D} \times Z^\alpha$ and the metric space product $[0,1] \times Z^\alpha$ disappears. It is compact and $1 + \frac{Q}{\alpha}$-Ahlfors regular with Hausdorff measure $\nu = dt \otimes \mu$. For every ball $B \in \mathbb{R}_+ \times Z^\alpha$,

$$\operatorname{diameter}(u(B)) = \operatorname{diameter}(B)^{1/\alpha} \sim \nu(B)^{\frac{1}{\alpha}\frac{1}{1+\frac{Q}{\alpha}}} = \nu(B)^{\frac{1}{\alpha+Q}},$$

thus u has finite p-energy for $p \ge \alpha + Q$. ∎

5.2 Dependence on radii and scaling factor

The following lemma shows that for spaces with bounded geometry, taking the supremum over (ℓ, R, S)-packings does not play such a big role, provided $R > 0$ and $S < \infty$. However, $(\ell, 0, \infty)$-packings in the definition of $E^p_{\ell,0,\infty}$-energy are much harder to handle.

Definition 41 *A metric space X has the* tripling property *if*

1. *Balls are connected.*
2. *There is a function $N' = N'(N, \ell, R, S)$ (called the* tripling function *of X) with the following property. For every $N, \ell \geq 1, R > 0, S \geq R$, every (N, ℓ, R, S)-packing of X is an $(N', \frac{R+S}{R}, R, S)$-packing as well.*

A (N, ℓ, R, S)-packing of X is covering *if the interiors form an open covering of X.*

The tripling property is meaningful only for $\ell < \frac{R+S}{R}$. If a metric space X carries a measure μ such that the measure of R-balls is bounded above and below in terms of R only (this will be called a *metric space with controlled balls* below), then it has the tripling property.

Lemma 42 *Let X be a metric space which has the tripling property. Let $\{B_j\}$ be a covering (N, ℓ, R, S)-packing of X. Then for all maps $u : X \to Y$ to metric spaces,*

$$E^p_{\ell,R,S}(u) \leq N'(N, \ell, R, S)^p \sum_j \operatorname{diam}(u(B_j))^p,$$

where N' is the tripling function of X.

Proof Let $\{B_j\}$ be a covering (N, ℓ, R, S)-packings of X and $\{B'_k\}$ a (ℓ, R, S)-packing of X. Let B' be a ball from the second packing. Let $J(B')$ index the balls from $\{B_j\}$ whose interiors intersect B'. By assumption, $|J(B')| \leq N'(N, \ell, R, S)$. Indeed, if B intersects B', then the center of B' belongs to $\frac{R+S}{R}B$. Since B' is connected, given any two points x, $x' \in B'$, there exists a chain $B_{j_1}, \ldots, B_{j_k}$, $j_i \in J(B')$ with $x \in B_{j_1}$, $x' \in B_{j_k}$, and each B_{j_i} intersects $B_{j_{i+1}}$. This implies that $d(u(x), u(x')) \leq \sum_{i=1}^k \operatorname{diam}(u(B_{j_i}))$, and

$$\operatorname{diam}(u(B')) \leq \sum_{j \in J(B')} \operatorname{diam}(u(B_j)).$$

Hölder's inequality yields

$$\operatorname{diam}(u(B'))^p \leq N'^{p-1} \sum_{j \in J(B')} \operatorname{diam}(u(B_j))^p.$$

A given ball of $\{B_j\}$ appears in as many $J(B')$ as its interior intersects balls of $\{B'_k\}$. This happens at most $N'(N, \ell, R, S)$ times. Therefore

$$\sum_k \operatorname{diam}(u(B'_k))^p \leq N'^p \sum_j \operatorname{diam}(u(B_j))^p.$$

■

Corollary 43 *Let X be a metric space which has the tripling property. Up to multiplicative constants depending only on R, S and ℓ, energies $E^p_{\ell,R,S}(u)$ do not depend on the choices of $R > 0$, $S < \infty$ and $\ell \geq 1$.*

Proof Let us show that covering (N, ℓ, R, R)-packings exist for all $R > 0$ and $\ell \geq 1$, provided N is large enough, $N = N(\ell, R)$. Let $\{B_j = B(x_j, \frac{R}{2})\}$ be a maximal collection of disjoint $\frac{R}{2}$-balls in X. In particular, $\{B_j\}$ is a $(1, \frac{R}{2}, (2\ell - 1)\frac{R}{2})$-packing. By the tripling property, there exists $N = N'(1, 2\ell, \frac{R}{2}, (2\ell - 1)\frac{R}{2})$ such that $\{B_j\}$ is a $(N, 2\ell, \frac{R}{2}, \frac{R}{2})$-packing. Then $\{2B_j\}$ is a covering (N, ℓ, R, R)-packing.

Fix $0 < R \leq 1 \leq S < +\infty$. Let $\{B_j\}$ be a covering $(N(\ell, 1), \ell, 1, 1)$-packing. According to Lemma 42, for all maps u to metric spaces,

$$\begin{aligned} E^p_{\ell,1,1}(u) &\leq E^p_{\ell,R,S}(u) \\ &\leq N'(1, \ell, R, S)^p \sum_j \operatorname{diam}(u(B_j))^p \\ &\leq N'(1, \ell, R, S)^p N(\ell, 1) E^p_{\ell,1,1}(u), \end{aligned}$$

so changing radii is harmless. By definition of tripling, given $\ell' \geq \ell$, every $(\ell, 1, \ell')$-packing is simultaneously a $(N'', \ell' + 1, 1, \ell')$-packing, $N'' = N'(1, \ell, 1, \ell')$, hence

$$\begin{aligned} E^p_{\ell',1,1}(u) &\leq E^p_{\ell,1,1}(u) \\ &\leq E^p_{\ell,1,\ell'}(u) \\ &\leq N'' E^p_{\ell'+1,1,\ell'}(u) \\ &\leq N'' E^p_{\ell',1,\ell'}(u) \\ &\leq N'' N'(1, \ell', 1, \ell')^p N(\ell', 1) E^p_{\ell',1,1}(u), \end{aligned}$$

so changing scaling factor is also harmless. ■

If $p = 1$, even the upper bound on radii of balls does not play such a big role.

Lemma 44 *Let X be a geodesic metric space. For every function u on X,*

$$E^1_{\ell,R,R}(u) \leq E^1_{\ell,R,\infty}(u) \leq (2\ell+2)E^1_{\ell,R,R}(u).$$

Proof Let B be a large ball. Assume u achieves its maximum on B at x and its minimum at y. Along the geodesic from x to y, consider an array of touching R-balls B_j. Then

$$\text{diameter}(u(B)) \leq \sum_j \text{diameter}(u(B_j)).$$

Pick one ball every 2ℓ along the array, in order to get an (ℓ, R, R)-packing. The array is the union of 2ℓ such packings, whence

$$\sum_j \text{diameter}(u(B_j)) \leq E^1_{\ell,R,R}(u).$$

Summing up over balls of an arbitrary (ℓ, R, ∞)-packing yields the announced inequality. ■

5.3 Functoriality of energy

Lemma 45 *Let X, X' and Y be metric spaces. Let $f : X \to X'$ be (R, S, R', S')-coarsely conformal. Then for all $\ell' \geq 1$, there exists $\ell \geq 1$ and N' such that for all maps $u : X' \to Y$,*

$$E^p_{\ell,R,S}(u \circ f) \leq N' E^p_{\ell',R',S'}(u).$$

Proof Let $\{B_j\}$ be an (ℓ, R, S)-packing of X. Let $\{B'_j\}$ be the corresponding (N', ℓ', R', S')-packing of X'. Split it into N' sub-collections which are $(1, \ell', R', S')$-packings. By assumption, $f(B_j) \subset B'_j$, so $\text{diam}(u \circ f(B_j)) \leq \text{diam}(u(B'_j))$. This yields, for each sub-collection,

$$\sum_j diam(u \circ f(B_j))^p \leq \sum_j diam(u(B'_j))^p \leq E^p_{\ell',R',S'}(u).$$

Summing up and taking supremum, this shows that $E^p_{\ell,R,S}(u \circ f) \leq N' E^p_{\ell',R',S'}(u)$. ■

Example 46 *If Y is d-Ahlfors regular and compact, then the identity $Y \to Y$ has finite $E^p_{\ell',R',S'}$-energy for all ℓ', R', S', so coarsely conformal maps $X \to Y$ have finite $E^p_{\ell,R,S}$-energy themselves for suitable ℓ.*

5.4 $(1,1)$-curves

Definition 47 *A* $(1,1)$-curve *in a metric space* X*, is a map* $\gamma : \mathbb{N} \to X$ *which is* $(1,1,R,S)$*-coarsely conformal for all* $R > 0$ *and* $S \geq R$*. When* X *is locally compact and equipped with a base point* o*, a* based $(1,1)$-curve *is a proper* $(1,1)$*-curve such that* $\gamma(0) = o$*.*

An $(\ell,1,1)$-packing of $\mathbb{N}$ corresponds to a subset $A \subset \mathbb{N}$ such that every ℓ-ball centered at a point of A contains at most one points of A. The packing consists of unit balls centered at points of A. Let us call such a set an ℓ*-subset* of $\mathbb{N}$. A $(1,1)$-curve in X is a sequence $(x_i)_{i\in\mathbb{N}}$ such that for all R and all sufficiently large $S \gg R$, there exists a collection of balls B_i in X with radii between R and S such that

- B_i contains $\{x_{i-1}, x_i, x_{i+1}\}$.
- For all $\ell' \geq 1$, there exist ℓ ans N' such that for every ℓ-subset of $\mathbb{N}$, the collection of concentric balls $\{\ell' B'_i\,;\, i \in A\}$ has multiplicity $\leq N'$.

Thus a $(1,1)$-curve is a chain of slightly overlapping balls which do not overlap too much: if radii are enlarged ℓ' times, decimating (i.e. keeping only one ball every ℓ) keeps the collection disjoint or at least bounded multiplicity.

Example 48 *An isometric map* $\mathbb{N} \to X$ *is a* $(1,1)$*-curve. In particular, geodesic rays in Riemannian manifolds give rise to* $(1,1)$*-curves.*

This is a special case of Lemma 18.

Remark 49 *In the definition of based* $(1,1)$*-curves, the properness assumption is needed only if* $R = 0$*.*

Proof Let $\gamma : \mathbb{N} \to X$ be a based $(1,1)$-curve. Let $\ell' = 2$, let ℓ be the corresponding scaling factor in the domain. The covering of $\mathbb{N}$ by 1-balls $B(j,1)$ is mapped to balls B'_j of radii $\geq R > 0$, with $\gamma(j) \in B'_j$. Since the covering $\{B(j,1)\}$ is the union of exactly ℓ ℓ-packings, $\{B'_j\}$ is the union of ℓ $(N',2)$-packings, it is an $(\ell N',2)$-packing. In particular, no point of X is contained in more than $\ell N'$ balls $2B'_j$.

If γ is not proper, there exists a sequence i_j tending to infinity such that $\gamma(i_j)$ has a limit $x \in X$. Since B'_{i_j} has radius $\geq R$, for j large enough $2B'_{i_j}$ contains x, contradicting multiplicity $\leq \ell N'$. ∎

Definition 50 *Let X be a q.s. space, and $K \subset X$. A* coarse curve *in X is a coarse conformal map $\mathbb{N} \to X$. A coarse curve γ is* based at *K if $\gamma(0) \in K$.*

Example 51 *Given an isometric map $\gamma : [0,1] \to X$, set $\gamma'(j) = \gamma(\frac{1}{j+1})$. This is a coarse curve in $X \setminus \{\gamma(0)\}$. In particular, geodesic segments in punctured Riemannian manifolds give rise to $(0,\infty)$-curves.*

Since γ' is the composition of inversion $\mathbb{R} \to \mathbb{R}$ and an isometric embedding, this is a special case of Proposition 8.

Proposition 52 *Let X be a metric space and X' a q.s. space. If $\gamma : \mathbb{N} \to X$ is a $(1,1)$-curve and $f : X \to X'$ is a coarse conformal map, then $f \circ \gamma$ is a coarse curve.*

5.5 Modulus

We need to show that certain maps with finite energy have a limit along at least one based curve. To do this, we shall use the idea, that arouse in complex analysis, of a property satisfied by almost every curve.

Definition 53 *Let Y be a metric space. The* length *of a map $u : \mathbb{N} \to Y$ is*

$$\text{length}(u) = \sum_{i=0}^{\infty} d(u(i), u(i+1)).$$

Definition 54 *Let X be a metric space. Let Γ be a family of $(1,1)$-curves in X. The (p,ℓ,R,S)-*modulus $\text{mod}_{p,\ell,R,S}(\Gamma)$ *is the infimum of $E^p_{\ell,R,S}$-energies of maps $u : X \to Y$ to metric spaces such that for every curve $\gamma \in \Gamma$, $\text{length}(u \circ \gamma) \geq 1$.*

Remark 55 *The definition of $(p,\ell,0,\infty)$-modulus extends to q.s. spaces X,*

$$\text{mod}_{p,\ell}(\Gamma) = \inf\{E^p_\ell(u)\,;\, u : X \to Y,\, \text{length}(u \circ \gamma) \geq 1\, \forall \gamma \in \Gamma\}.$$

Lemma 56 *Let X be a metric space. The union of a countable collection of $(1,1)$-curve families which have vanishing (p,ℓ,R,S)-modulus also has vanishing (p,ℓ,R,S)-modulus.*

Proof Fix $\epsilon > 0$. Let $u_j : X \to \mathbb{R}$ be a function with $E^p_{\ell,R,S}(u_j) \leq 2^{-j}\epsilon$ such that for all curves γ in the j-th family Γ_j, length$(u_j \circ \gamma) \geq 1$. Consider the ℓ^p direct product $Y = \prod_j Y_j$, i.e.

$$d^Y((y_j),(y'_j)) = \left(\sum_j d(y_j, y'_j)^p\right)^{1/p},$$

and the product map $u = (u_j) : X \to Y$. Then

$$E^p_{\ell,R,S}(u) \leq \sum_j E^p_{\ell,R,S}(u_j) \leq const.\epsilon^p,$$

whereas for all curves γ in the union curve family,

$$\text{length}(u \circ \gamma) \geq \sup_j \text{length}(u_j \circ \gamma) \geq 1.$$

This shows that the union curve family has vanishing (p,ℓ,R,S)-modulus. ■

Lemma 57 *Let X be a metric space. Let Γ be a family of $(1,1)$-curves in X. Then Γ has vanishing (p,ℓ,R,S)-modulus if and only if there exists a function $u : X \to \mathbb{R}_+$ such that $E^p_{\ell,R,S}(u) < +\infty$ but* length$(u \circ \gamma) = +\infty$ *for every $\gamma \in \Gamma$.*

Proof One direction is obvious. In the opposite direction, first observe that by rescaling target metric spaces, one can assume that there exist maps $u_j : X \to Y_j$ such that for all $\gamma \in \Gamma$, length$(u_j \circ \gamma) \geq j$ and $E^p_{\ell,R,S}(u_j) < 2^{-j}$. Apply the ℓ^p-product construction again. Get $u = (u_j) : X \to Y$ such that $E^p_{\ell,R,S}(u) \leq 1$ and length$(u \circ \gamma) \geq \max_j \text{length}(u_j \circ \gamma) = +\infty$. ■

Lemma 58 *Let X be a metric space. Let Y be a complete metric space. Fix $R \leq S$ and $\ell \geq 1$. Let $u : X \to Y$ be a map of finite $E^p_{\ell,R,S}$ energy. The family of $(1,1)$-curves along which u does not have a limit has vanishing (p,ℓ,R,S)-modulus.*

Proof If length$(u \circ \gamma) < \infty$, then $u \circ \gamma$ has a limit in Y, since Y is complete. Let Γ_{nl} be the sub-family of curves along which the length of u is infinite. It contains all curves along which u does not have a limit. By assumption, $E^p_{\ell,R,S}(u) < \infty$, but length$(u \circ \gamma) = +\infty \geq 1$ for all curves $\gamma \in \Gamma_{nl}$. So $\text{mod}_{p,\ell,R,S}(\Gamma_{nl}) = 0$. ■

Remark 59 *The cases $R = 0, S = \infty$ of Lemmata 56 to 58 extend to q.s. spaces X.*

5.6 Parabolicity

Definition 60 *Let X be a locally compact metric space. Say X is* (p,ℓ,R,S)-parabolic *if the family of all $(1,1)$-curves based at some point has vanishing (p,ℓ,R,S)-modulus.*

If X is merely a locally compact q.s. space, (p,ℓ)-parabolicity means that the (p,ℓ)-modulus of the family of proper coarse curves based at any compact set with nonempty interior vanishes.

Remark 61 *If metric space X has the tripling property, (p,ℓ,R,S)-parabolicity does not depend on the choices of $R>0$, $S<\infty$ and $\ell\geq 1$, thanks to Corollary 43.*

On the other hand, (p,ℓ)-parabolicity may depend wether $\ell=1$ or $\ell>1$, as we shall see in the next subsections.

Proposition 62 *Let X be a metric space and X' a q.s. space. Let $f:X\to X'$ be a coarse conformal map. Let R be large enough, and $S\geq R$. Let Γ be a family of $(1,1)$-based curves in X. Then for all ℓ', there exist ℓ such that*

$$\mathrm{mod}_{p,\ell,R,S}(\Gamma)\leq \mathrm{mod}_{p,\ell}(f(\Gamma)).$$

Proof By the composition rule (Lemma 6), $f(\Gamma)$ is a family of $(0,\infty)$-based curves. Given $u:X'\to Y$ such that $\mathrm{length}(u\circ\gamma')\geq 1$ for all $\gamma'\in f(\Gamma)$, $\mathrm{length}(u\circ f\circ\gamma)\geq 1$ for all $\gamma\in\Gamma$. According to Lemma 45, for all $\ell'\geq 1$, there exists $\ell\geq 1$ such that

$$E^p_{\ell,R,S}(u\circ f)\leq E^p_{\ell'}(u).$$

Taking the infimum over such maps u yields the announced inequality. ∎

Corollary 63 *Let X be a metric space and X' a q.s. space. Let $f:X\to X'$ be a proper, coarse conformal map. Then there exists $R>0$ such that for all $S\geq R$, if X' is (p,ℓ')-parabolic for some ℓ', then X is (p,ℓ,R,S)-parabolic for a suitable ℓ.*

There is a similar statement for (R',S',R,S)-coarsely conformal maps. This shows that parabolicity does not depend on the choice of base point, provided one accepts to change parameters ℓ, R and S. Indeed, the map which is identity but for one point o which is mapped to o' is proper and (R',S',R,S)-coarsely conformal.

Corollary 64 *Let X and X' be a metric spaces. Let $f : X \to X'$ be a uniformly conformal map. For every $R' > 0$, there exists R such that for all $S \geq R$, if X' is (p, ℓ', R', ∞)-parabolic for some ℓ', then X is (p, ℓ, R, S)-parabolic for a suitable ℓ.*

Proof Uniformly conformal maps are proper. ∎

5.7 Parabolicity of Ahlfors-regular spaces

Proposition 65 *Let X be a non-compact Q-Ahlfors regular metric space with $Q > 1$. Let K be a ball. For all $\ell > 1$, there exists a finite (Q, ℓ)-energy function $w : X \to \mathbb{R}$ which has no limit along every coarse curve based on K. It follows that X is (p, ℓ)-parabolic for every $\ell > 1$ and every $p \geq Q$.*

Proof Let μ be a measure such that balls of radius ρ have measure ρ^Q up to multiplicative constants.

Let $m = \max\{\frac{\ell+1}{\ell-1}, e\}$. Fix an origin $o \in X$, set $r(x) = d(x, o)$, $v(r) = \log\log r$ and $w(x) = \sin(v(r(x)))$ if $r \geq m^2$, $= \log\log(m^2)$ otherwise. Let $\{B_j\}$ be a ℓ-packing of X. At most one ball B is such that $o \in \ell B$, it contributes to $\sum_j \text{diameter}(w(B_j))^p$ by at most 1. We shall ignore it henceforth. Other balls $B = B(x, \rho)$ are such that $o \notin \ell B$, hence $r(x) = d(o, x) > \ell\rho$. For all $x' \in B$, $r(x') = d(o, x')$ satisfies $r(x) - \rho \leq r(x') \leq r(x) + \rho$, hence

$$\frac{\sup_B r}{\inf_B r} \leq \frac{r(x) + \rho}{r(x) - \rho} = \frac{\frac{r(x)}{\rho} + 1}{\frac{r(x)}{\rho} - 1} \leq \frac{\ell + 1}{\ell - 1} \leq m.$$

For $i \in \mathbb{Z}$, let $r_i = m^i$ and define $Y_i = \{x \in X\,;\, r_{i-2} \leq r(x) \leq r_i\}$ and $L_i = Lip(v_{|Y_i})$. Note that $\mu(Y_i) \leq C\, r_i^p$. By construction, each ball B of the packing is contained in at least of the Y_i. If $i \leq 2$, w is constant on Y_i, such balls do not contribute. Let $i \geq 3$. For a ball $B \subset Y_i$,

$$\begin{aligned} \text{diameter}(w(B))^Q &\leq \text{diameter}(v(B))^Q \\ &\leq L_i^Q \text{diameter}(r(B))^Q \\ &\leq C\, L_i^Q \mu(B). \end{aligned}$$

Summing up over all balls of the packing contained in Y_i,

$$\sum_{B_j \subset Y_i} \text{diameter}(w(B_j))^Q \leq C\, L_i^Q \mu(Y_i) \leq C\, (L_i r_i)^Q.$$

Since $v'(t) = \frac{1}{t \log t}$, $L_i \leq \frac{1}{r_{i-2} \log(r_{i-2})}$, $L_i r_i \leq \frac{r_i}{r_{i-2} \log(r_{i-2})} = \frac{m^2}{(i-2) \log m}$. Summing up over $i \geq 3$ leads to

$$\sum_j \text{diameter}(B_j)^Q \leq C \sum_{i=3}^{\infty} (\frac{1}{i-2})^Q < \infty.$$

This shows that $E_\ell^p(w) < \infty$.

Let $\gamma : \mathbb{N} \to X$ be a proper coarse curve based at $K = B(o, \rho)$. Let us show that $w \circ \gamma$ has no limit. By definition, there exists $\tilde{\ell} \geq 1$ such that every $\tilde{\ell}$-packing of $\mathbb{N}$ is mapped to a ℓ-packing $\{B'_j\}$ of X. For each $i = 0, \ldots, \ell - 1$, this applies to the ℓ-packing of unit balls centered at $3\tilde{\ell}\mathbb{N} + i \subset \mathbb{N}$. We know that

$$\sum_{j \in \mathbb{N}} \text{diameter}(v(B'_{3\tilde{\ell}j+i}))^p < +\infty,$$

hence, summing over i,

$$\sum_{j \in \mathbb{N}} \text{diameter}(v(B'_j))^p < +\infty,$$

so these diameters tend to zero. Since B'_j contains $\gamma(j)$ and $\gamma(j+1)$, this implies that $|v \circ \gamma(j+1) - v \circ \gamma(j)|$ tends to 0. Therefore the ω-limit set of the sequence $w \circ \gamma$ is the whole interval $[-1, 1]$, $w \circ \gamma$ has no limit.

We conclude that the family of all proper coarse curves based at some ball has vanishing (p, ℓ)-modulus, i.e., X is (p, ℓ)-parabolic. A fortiori, the family of all proper coarse curves based at o has vanishing (p, ℓ, R, S)-modulus, i.e., X is (p, ℓ, R, S)-parabolic, for $R > 0$. ■

Remark 66 *If we are merely interested in (p, ℓ, R, S)-parabolicity for some $R > 0$, or (p, ℓ, R, ∞)-parabolicity, a weaker form of Ahlfors-regularity is required. It suffices that balls of radius $\rho \geq R$ satisfy $c\,\rho^Q \leq \mu(B) \leq C\,\rho^Q$. Let us call this* Q-Ahlfors regularity in the large.

Indeed, the argument uses only balls of radius $\geq R$.

Corollary 67 *Let X be a non-compact metric space. Let $Q > 1$. Assume that X is Q-Ahlfors regular in the large. Then X is (p, ℓ, R, ∞)-parabolic for every $\ell > 1$ and every $p \geq Q$.* A fortiori, *it is (p, ℓ, R, S)-parabolic for every $\ell > 1$, every $0 < R \leq S$ and every $p \geq Q$.*

Proposition 68 *Let X be a compact p-Ahlfors regular metric space, with $p > 1$. Let $x_0 \in X$. For every $\ell > 1$, there exists a function $w : X \setminus \{x_0\} \to \mathbb{R}$ such that $E^p_\ell(w) < +\infty$ and w has a limit along no coarse curve converging to x. It follows that $X \setminus \{x_0\}$ is (p, ℓ)-parabolic.*

Proof The same as for the non-compact case, replacing function r with $1/r$. ∎

5.8 Parabolicity of $\mathbb{D}$

The half real line is 1-Ahlfors regular, a case which is not covered by Proposition 65. It is not 1-parabolic. Indeed, any function of finite 1-energy on $\mathbb{R}_+$ has a limit at infinity. However, it is p-parabolic for every $p > 1$.

Lemma 69 *The half real line $\mathbb{R}_+$ equipped with its metric q.s. structure is p-parabolic for every $p > 1$.*

Proof Let $\ell > 1$. Denote by $m = \frac{\ell+1}{\ell-1}$. We can assume that $m \geq e$. Define

$$u(t) = \log\log|t| \text{ if } t \geq m, \quad u(t) = \log\log m \text{otherwise.}$$

Let us show that u has finite (p, ℓ)-energy for all $p > 1$ and $\ell > 1$. Let $\{B_j\}$ be an ℓ-packing of $\mathbb{R}_+$. By assumption, the collection of concentric balls $\{\ell B_j\}$ consists of disjoint intervals. For simplicity, assume that 0 belongs to one of the ℓB_j's, say ℓB_0 (otherwise, translate everything). Write $B_j = [a_j, b_j]$ and assume that $a_j \geq 0$. Since ℓB_0 and ℓB_j are disjoint,

$$\frac{a_j + b_j}{2} - \ell\frac{b_j - a_j}{2} \geq 0,$$

thus

$$b_j \leq \frac{\ell + 1}{\ell - 1} a_j = m a_j.$$

We split the sum $\sum_j |u(b_j) - u(a_j)|^p$ into sub-sums where $a_j \in [m^i, m^{i+1})$. Since intervals B_j are disjoint and u is nondecreasing,

$$\begin{aligned}\sum_{a_j \in [m^i, m^{i+1})} |u(b_j) - u(a_j)| &\leq u(m^{i+2}) - u(m^i) \\ &= \log((i+2)\log m) - \log(i \log m) \\ &= \log\frac{i+2}{i} \leq \frac{2}{i}.\end{aligned}$$

Next, we use the general inequality, for nonnegative numbers x_j,

$$\sum x_j^p \leq (\sum x_j)^p,$$

and get

$$\sum_{a_j \in [m^i, m^{i+1})} |u(b_j) - u(a_j)|^p \leq (\sum_{a_j \in [m^i, m^{i+1})} |u(b_j) - u(a_j)|)^p \leq (\frac{2}{i})^p.$$

This gives

$$\sum_{a_j \geq m} |u(b_j) - u(a_j)|^p \leq \sum_{i=1}^{\infty} (\frac{2}{i})^p < +\infty.$$

On the remaining intervals, u is constant, except possibly on one interval containing m. On this interval, $|u(b_j) - u(a_j)| \leq u(m^2)$, so its contribution is bounded independently of the packing. We conclude that the supremum over ℓ-packings of $\sum_j |u(b_j) - u(a_j)|^p$ is bounded, i.e. u has finite p-energy. ■

Since, as a q.s. space, $\mathbb{D}$ is isomorphic to the half real line, $\mathbb{D}$ is p-parabolic for all $p > 1$ as well. As a preparation for the next result, note that the isomorphism is the exponential map $t \mapsto \exp(-t) : \mathbb{R}_+ \to \mathbb{D}$. Therefore the function of finite energy on $\mathbb{D}$ is $w(y) = \sin \log |\log |\log y||$.

5.9 Parabolicity of warped products

Proposition 70 *Let Z be a compact Q-Ahlfors regular metric space. Let $0 < \alpha \leq 1$. Let $z_0 \in Z$. Then $\mathbb{D} \times Z^\alpha \setminus \{(0, z_0)\}$ is $(\alpha+Q)$-parabolic.*

Proof For $(y, z) \in [0, 1] \times Z$, let $r(y, z) = \min\{y, d^{Z^\alpha}(z, z_0)\}$ denote the distance to $(0, z_0)$. We use the bounded function $w = \sin v(r)$, where

$$v(t) = \log |\log |\log t|| \quad \text{if} \quad t \leq r_1, \quad v(t) = \log |\log |\log r_1|| \quad \text{otherwise.}$$

The constant $r_1 = r_1(\ell)$ is produced by the following Lemma.

Lemma 71 *Let $v(t) = \log |\log |\log t||$. Assume that $\ell > 1$, and that*

$$b \leq r_1 := \min\{\frac{\ell - 1}{\ell}, \ell^{-2}, e^{-e^2}\}.$$

Then, if $\dfrac{b}{a} \geq \dfrac{\ell}{\ell - 1}$,

$$v(a) - v(\ell b) \leq 16\,(v(a) - v(b)).$$

Proof We use the inequalities

$$0 \leq u \leq 1 \implies \log(1+u) \geq \frac{1}{2}u, \quad 0 \leq u \leq \frac{1}{2} \implies -\log(1-u) \leq 2u.$$

Set $t = \log \frac{1}{b}$ and $s' = \log \frac{1}{a} - \log \frac{1}{b} = \log \frac{b}{a} \geq s := \log \frac{\ell}{\ell - 1}$. Then

$$\begin{aligned} v(a) - v(b) &= \log\log\log \frac{1}{a} - \log\log\log \frac{1}{b} \\ &= \log\log(t+s) - \log\log(t) \\ &= \log\left(\frac{\log(t+s')}{\log(t)}\right) \\ &\geq \log\left(\frac{\log(t+s)}{\log(t)}\right) \\ &= \log\left(1 + \frac{\log(t+s) - \log(t)}{\log(t)}\right) \\ &= \log\left(1 + \frac{\log(1+(s/t))}{\log(t)}\right) \end{aligned}$$

If $t = \log \frac{1}{b} \geq \max\{s, e^2\}$, $s/t \leq 1$, $\log(t) \geq 2$, so $\frac{\log(1+(s/t))}{\log(t)} \leq 1$. Also $s/t \leq 1$, thus

$$\log(1+(s/t)) \leq \frac{1}{2}\frac{s}{t}, \quad \log\left(1 + \frac{\log(1+(s/t))}{\log(t)}\right) \leq \frac{1}{2}\frac{\log(1+(s/t))}{\log(t)} \leq \frac{1}{4}\frac{s}{t\log(t)},$$

and

$$v(a) - v(b) \geq \frac{s}{4t\log(t)}.$$

Conversely, setting $\sigma = \log\frac{1}{b} - \log\frac{1}{\ell b} = \log \ell$,

$$\begin{aligned} v(b) - v(\ell b) &= \log\log\log\frac{1}{b} - \log\log\log\frac{1}{\ell b} \\ &= \log\log(t) - \log\log(t-\sigma) \\ &= -\log\left(\frac{\log(t-\sigma)}{\log(t)}\right) \\ &= -\log\left(1 + \frac{\log(t-\sigma) - \log(t)}{\log(t)}\right) \\ &= -\log\left(1 + \frac{\log(1-(\sigma/t))}{\log(t)}\right) \\ &\leq -2\frac{\log(1-(\sigma/t))}{\log(t)} \\ &\leq 4\frac{\sigma}{t\log(t)}, \end{aligned}$$

provided $\sigma/t \leq \frac{1}{2}$ and $-\frac{\log(1-(\sigma/t))}{\log(t)} \leq \frac{1}{2}$, which holds if $t = \log\frac{1}{b} \geq \max\{2\sigma, e^2\}$. ∎

Back to the proof of Proposition 70.

Let $\{B_j\}$ be a ℓ-packing of $\mathbb{D} \times Z$. The packing splits into three sub-collections,

1. Balls that contain $(0, z_0)$.
2. Balls intersecting $\mathbb{R}_+ \times \{z_0\}$, but not containing $(0, z_0)$.
3. Balls which do not intersect $\mathbb{R}_+ \times \{z_0\}$.

We first assume that $\alpha = 1$, i.e. we ignore any special feature of snowflaked spaces.

1. The first sub-collection has at most one element. Since $|w| \leq 1$, it contributes at most 1 to energy.

2. The second sub-collection is nearly taken care of by Lemma 69. If two balls $B = I \times \beta$ and $B' = I' \times \beta'$ in $[0,1] \times Z$ are disjoint and both intersect $[0,1] \times \{z_0\}$, then the intervals I and I' are disjoint. In particular, if $\ell B \cap \ell B' = \emptyset$, then $m_\ell(I) \cap m_\ell(I') = \emptyset$. Furthermore, if $B = I \times \beta$ and $I = [a,b]$, the radius of B is $\frac{b-a}{2}$, thus for $z \in \beta$, $d(z, z_0) \leq 2\ell\frac{b-a}{2}$, so

$$\min_B r \geq a, \quad \max_B d(\cdot, z_0) \leq \ell(b-a) \leq \ell b,$$

whence

$$\max_B r \le \ell b.$$

The Z factor plays little role. We expect from Lemma 69 that the sum of p-th powers of diameters of images $w(B_j)$ should be bounded independently of the packing, as soon as $p > 1$.

Here comes the proof. First, the last estimate needs be sharpened. Either $\ell(b-a) \le b$, in which case $r(B) = [a,b]$ and $v((r(B))) = [v(b), v(a)]$, or $\frac{b}{a} \ge \frac{\ell}{\ell-1}$. Lemma 71 shows that, in both cases,

$$\text{diameter}(v(r(B))) \le C\,(v(a) - v(b)),$$

provided b is small enough.

If an interval $I = [a,b]$ of $(0,1]$ is such that $m_\ell(I)$ does not contain 1, then $a \ge b^m$, for $m = \frac{\ell+1}{\ell-1}$. Indeed, the upper bound of $m_\ell(I)$ is $a^{\frac{1-\ell}{2}} b^{\frac{1+\ell}{2}}$. Hence if $B = I \times \beta$ is a ball of $[0,1] \times Z$,

$$\min_B r = a \ge b^m, \quad \max_B r \le \ell b \le \ell (\min_B r)^{1/m}.$$

Define inductively a sequence r_i by $r_{i+1} = (\frac{r_i}{\ell})^m$ (this gives $r_i = \ell^{-m\frac{m^i-1}{m-1}}$). Also, define r_0 so that $r_1 = (\frac{r_0}{\ell})^m$. Let $\{B_j = I_j \times \beta_j\}$ be a ℓ-packing of $\mathbb{D} \times Z$. Assume that all B_j intersect $[0,1] \times \{z_0\}$. At most one ℓB_j contains $(1, z_0)$, let us put it aside (it contributes at most 1 to energy). All other intervals I_j are disjoint and each one is contained in at least one of the intervals $[r_{i+2}, r_i]$. Therefore the index set is contained in the union of subsets

$$J_i = \{j\,;\, I_j \subset [r_{i+2}, r_i]\},$$

and for all p,

$$\sum_j \text{diameter}(w(B_j))^p \le \sum_{i=0}^{\infty} \sum_{j \in J_i} \text{diameter}(v(r(B_j)))^p.$$

We split the sub-packing into two sub-families,

1. Balls $B = I \times \beta$ such that $v(r(B)) \ne v(I)$ and $\max I \ge r_1$.
2. Balls $B = I \times \beta$ such that $v(r(B)) = v(I)$ or $\max I \le r_1$.

The first sub-family satisfy $\max I \geq \frac{\ell}{\ell-1} \min I$ and $\max I \geq r_1$. Those which have $\max I \geq r_0$ have $\min I \geq r_1$, v is constant on such balls. The others are contained in Y_1. The number of such disjoint intervals is bounded in terms of ℓ only. The diameters $\operatorname{diameter}(v(r(B))) \leq v(r_2)$ are bounded in terms of ℓ. Therefore the sum of p-th powers $\operatorname{diameter}(v(r(B)))$ over this sub-family is apriori bounded in terms of ℓ and p only.

The second sub-family satisfy

$$\operatorname{diameter}(v(r(B))) \leq C \operatorname{diameter}(v(I)).$$

Since the intervals I_j are disjoint,

$$\begin{aligned}\sum_{j\in J_i} \operatorname{diameter}(w(B_j))^p &\leq \left(\sum_{j\in J_i} \operatorname{diameter}(v(r(B_j)))\right)^p \\ &\leq C\,|v(r_{i+2}) - v(r_i)|^p,\end{aligned}$$

$$\sum_j \operatorname{diameter}(w(B_j))^p \leq C \sum_{i=0}^{\infty} |v(r_{i+2}) - v(r_i)|^p.$$

With our choice of $v(t) = \log|\log|\log t||$,

$$v(r_{i+2}) - v(r_i) \sim |\log(i+2) - \log(i)| \sim \frac{2}{i},$$

so the sum of p-th powers converges to a bound that depends only on ℓ and p.

3. The third sub-collection consists of balls $B = [a,b] \times \beta$ such that $z_0 \notin \ell\beta$.

Let us study how r varies along B. Note that the radius of balls B and β equals $\frac{b-a}{2}$. Let $\Delta = \max_\beta d(\cdot, z_0)$ and $\delta = \min_\beta d(\cdot, z_0)$. Then

$$\max_B r = \max\{b, \Delta\}, \quad \min_B r = \max\{a, \delta\}.$$

Since $z_0 \notin \ell\beta$, $\delta \geq (\ell-1)\frac{b-a}{2}$. Then $\Delta \leq \delta + b - a \leq \delta + \frac{2}{\ell-1}\delta = m\delta$. On the other hand,

- either $a < \frac{b}{2}$, $\delta \geq (\ell-1)\frac{b-a}{2} \geq (\ell-1)\frac{b}{4}$,

- or $a \geq \frac{b}{2}$.

In the first case,

$$\max\{b, \Delta\} \leq \max\{\frac{4}{\ell - 1}\delta, m\delta\} \leq \max\{\frac{4}{\ell - 1}, m\} \max\{a, \delta\}.$$

In the second case,

$$\max\{b, \Delta\} \leq \max\{2a, m\delta\} \leq \max\{2, m\} \max\{a, \delta\}.$$

In either case,

$$\max_B r \leq M \min_B r,$$

where $M = \max\{m, 2, \frac{4}{\ell - 1}\}$.

Let $\nu = dt \otimes \mu$ denote $1 + Q$-dimensional Hausdorff measure. Set $r_i = M^{-i}$. Let

$$Y_i = \{(y, z) \in [0, 1] \times Z \,;\, r_{i+2} < r(y, z) \leq r_i\}, \quad L_i = Lip(v_{|[r_{i+2}, r_i]}).$$

Since r is 1-Lipschitz, $Lip(u_{|Y_i}) \leq L_i$. Each ball B_j of the sub-packing is entirely contained in one of the sets Y_i. Therefore the index set is contained in the union of subsets

$$J_i' = \{j \,;\, B_j \subset Y_i\},$$

and for all p,

$$\sum_j \text{diameter}(w(B_j))^p \leq \sum_{i=0}^{\infty} \sum_{j \in J_i'} \text{diameter}(u(B_j))^p.$$

If $j \in J_i'$,

$$\text{diameter}(w(B_j))^{1+Q} \leq L_i^{1+Q} \text{diameter}(B_j)^{1+Q} \leq C\, L_i^{1+Q} \nu(B_j).$$

Thus

$$\sum_{j \in J_i'} \text{diameter}(u(B_j))^{1+Q} \leq C\, L_i^{1+Q} \nu(Y_i) \leq C'\, (L_i r_i)^{1+Q},$$

since $Y_i \subset B((0, z_0), r_i)$.

In order to estimate the Lipschitz constant L_i, observe that r is 1-Lipschitz. With our choice of $v(t) = \log|\log\|\log t||$,

$$v'(t) = \frac{1}{t\log t\ \log|\log t|}$$

achieves its maximum on $[r_{i+2}, r_i]$ at $r_{i+2} = M^{-(i+2)}$, hence

$$L_i \leq \frac{1}{r_{i+2}(i+2)\log(i+2)}, \quad L_i r_i \leq \frac{r_i}{r_{i+2}(i+2)\log(i+2)} \leq \frac{M^2}{i+2}.$$

This bounds $\sum_j \text{diameter}(w(B_j))^{1+Q}$ by a quantity that depends only on ℓ and Q.

4. At last, we take into account the parameter α. When the second factor is the snowflake space Z^α, the previous general discussion provides the exponent $1+\frac{Q}{\alpha}$. This can be improved into $\alpha+Q$ for the following reason: on Z^α, the distance to z_0 is not merely 1-Lipschitz, it is $1/\alpha$-Hölder continuous (although with a constant that deteriorates when getting close to z_0). Indeed, if $z, z' \in Z$, with $d(z,z_0) \leq d(z',z_0)$,

$$\begin{aligned}
|d_{Z^\alpha}(z,z_0) - d_{Z^\alpha}(z',z_0)| &= |d_Z(z,z_0)^\alpha - d_Z(z',z_0)^\alpha| \\
&\leq \alpha d_Z(z,z_0)^{\alpha-1}|d_Z(z,z_0) - d_Z(z',z_0)| \\
&\leq d_Z(z,z_0)^{\alpha-1} d_Z(z,z') \\
&= d_{Z^\alpha}(z,z_0)^{\frac{\alpha-1}{\alpha}} d_{Z^\alpha}(z,z')^{1/\alpha}.
\end{aligned}$$

Nothing needs be changed for the first two sub-collections, since an upper bound on energy sums is obtained for any exponent $p > 1$. For the third one, consisting of balls $B = [a,b] \times \beta$ such that $z_0 \notin \ell\beta$, recall that $\delta \geq (\ell-1)\frac{b-a}{2}$, where

$$\Delta = \max_\beta d_{Z^\alpha}(\cdot, z_0) \quad \text{and} \quad \delta = \min_\beta d_{Z^\alpha}(\cdot, z_0).$$

This implies that

$$\delta^{\frac{\alpha-1}{\alpha}} \geq (\ell-1)^{\frac{\alpha-1}{\alpha}} (\frac{b-a}{2})^{\frac{\alpha-1}{\alpha}}$$

and

$$\frac{b-a}{2} \leq (\ell-1)^{\frac{1-\alpha}{\alpha}} \delta^{\frac{\alpha-1}{\alpha}} (\frac{b-a}{2})^{1/\alpha}.$$

On the other hand, the $1/\alpha$-Hölder character of d_{Z^α} leads to

$$\Delta - \delta \leq \delta^{\frac{\alpha-1}{\alpha}} (\frac{b-a}{2})^{1/\alpha}.$$

Since

$$\max_B r = \max\{b, \Delta\}, \quad \min_B r = \max\{a, \delta\},$$

$$\begin{aligned} \max_B r - \min_B r &\leq \max\{b-a, \Delta - \delta\} \\ &\leq 2(\ell-1)^{\frac{1-\alpha}{\alpha}} \delta^{\frac{\alpha-1}{\alpha}} (\frac{b-a}{2})^{1/\alpha} \\ &\leq M' (\max_B r)^{\frac{\alpha-1}{\alpha}} \text{diameter}(B)^{1/\alpha}, \end{aligned}$$

for $M' = (2/(\ell-1))^{(\alpha-1)/\alpha}$.

Keeping the notation

$$r_i = M^{-i}, \quad Y_i = \{(y,z) \in [0,1] \times Z \,;\, r_{i+2} < r(y,z) \leq r_i\}, \quad L'_i = Lip(v_{|[r_{i+2}, r_i]}),$$

we see that a ball B contained in Y_i satisfies

$$\begin{aligned} \text{diameter}(w(B)) &\leq \text{diameter}(v(B)) \\ &\leq L'_i (\max_B r - \min_B r) \\ &\leq L'_i M' \, r_i^{\frac{\alpha-1}{\alpha}} \text{diameter}(B)^{1/\alpha}. \end{aligned}$$

Thus

$$\begin{aligned} \text{diameter}(w(B))^{\alpha+Q} &\leq {L'_i}^{\alpha+Q} {M'}^{\alpha+Q} \, r_i^{\frac{(\alpha-1)(\alpha+Q)}{\alpha}} \text{diameter}(B)^{1+(Q/\alpha)} \\ &\leq C \, {L'_i}^{\alpha+Q} r_i^{\frac{(\alpha-1)(\alpha+Q)}{\alpha}} \nu(B). \end{aligned}$$

Since $\nu(Y_i) \leq r_i^{1+(Q/\alpha)}$, summing over all balls in the sub-packing contained in Y_i gives

$$\begin{aligned} \sum_{j \in J'_i} \text{diameter}(w(B))^{\alpha+Q} &\leq C \, {L'_i}^{\alpha+Q} r_i^{\frac{(\alpha-1)(\alpha+Q)}{\alpha}} r_i^{1+(Q/\alpha)} \\ &\leq C \, (L'_i r_i)^{\alpha+Q}. \end{aligned}$$

The choice of $v(t) = \log|\log|\log t||$ yields again $L_i r_i \leq M^2/(i+2)$, and the sum is bounded above in terms of ℓ, α and Q only.

5. The final argument, showing that $w \circ \gamma$ has no limit for every proper coarse curve γ is the same as in Proposition 65. ■

6 L^p-cohomology

6.1 Definition

Here is one more avatar of the definition of L^p cohomology for metric spaces. This one has the advantage that it does not require any measure. For earlier attempts, see [CT], [E].

Definition 72 *Let X be a metric space. A* k-simplex of size S *in X is a $k+1$-tuple of points belonging some ball of radius S. A* k-cochain of size S *on X is a real valued function κ defined on the set of k-simplices of size S. Its $L^p_{\ell,R,S}$-norm is*

$$\|\kappa\|_{L^p_{\ell,R,S}} = \sup\{\sum_j \sup_{(B_j)^{k+1}} |\kappa|^p \,;\, \textit{all } (\ell,R,S)\textit{-packings } \{B_j\}\}^{1/p}.$$

Let $L^p_{\ell,R,S}C^k(X)$ denote the space of k-cochains with finite $L^p_{\ell,R,S}$-norm.

The coboundary operator d maps k-cochains to $k+1$-cochains,

$$d\kappa(x_0,\dots,x_{k+1}) = \kappa(x_1,\dots,x_{k+1}) - \kappa(x_0,x_2,\dots,x_{k+1}) + \cdots + (-1)^{k+1}\kappa(x_0,\dots,x_k).$$

Denote by

$$\mathcal{L}^{q,p}_{\ell,R,S}C^k(X) = L^q_{\ell,R,S}C^k(X) \cap d^{-1}L^p_{\ell,R,S}C^{k+1}(X),$$

in order to turn d into a bounded operator $\mathcal{L}^{q,p}_{\ell,R,S}(X) \to L^p_{\ell,R,S}(X)$. The $L^{q,p}$-cohomology *of X is*

$$L^{q,p}_{\ell,R,S}H^k(X) = \left(\ker(d) \cap L^p_{\ell,R,S}C^k(X)\right) / d\left(\mathcal{L}^{q,p}_{\ell,R,S}C^{k-1}(X)\right).$$

The exact $L^{q,p}$-cohomology *of X is the kernel of the forgetful map $L^{q,p}_{\ell,R,S}H^k(X) \to H^k(X)$.*

When $p=q$, $\mathcal{L}^{p,p} = L^p$ and $L^{p,p}$-cohomology is simply called L^p-cohomology.

Remark 73 *The definition of $L^{q,p}_{\ell,0,\infty}H^k(X)$ extends to q.s. spaces X, and is simply denoted by $L^{q,p}_{\ell}H^k(X)$.*

For instance, a function $u : X \to \mathbb{R}$ can be viewed as a 0-cochain of infinite size, $du(x_1,x_2) = u(x_2) - u(x_1)$ is a 1-cochain of infinite size, and

$$E^p_{\ell,R,S}(u) = \|du\|^p_{L^p_{\ell,R,S}}.$$

Example 74 *If X is a compact infinite d-Ahlfors regular metric space, then, for all $S > 0$, $L^p_{\ell,0,S}H^1(X) \neq 0$ for $p \geq d$.*

Indeed, in an infinite metric space, one can ℓ-pack infinitely many small balls. Therefore a function which is ≥ 1 has infinite $L^p_{\ell,0,S}$ norm. Since non constant Lipschitz functions on X have finite energy, and do not belong to any $L^p_{\ell,0,S}C^0(X)$, their $L^p_{\ell,0,S}$ cohomology classes do not vanish.

6.2 Link to usual L^p-cohomology

L^p-cohomology calculations on manifolds (resp. on simplicial complexes) require the classical de Rham (resp. simplicial) definition of cohomology. There is a de Rham style theorem relating Definition 72 to smooth differential forms (resp. simplicial cochains). It shows up in [Ge]. We shall need the more general case of $L^{q,p}$-cohomology, which appears in [D].

Proposition 75 ([P6]) *Let $1 \leq p \leq q < +\infty$. Let X be a bounded geometry Riemannian manifold with boundary or simplicial complex. In the n-manifold case, assume that*

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}.$$

and $p > 1$ if degree $k = n$ (these limitations are absent in the simplicial complex case). Assume that cohomology of X vanishes uniformly up to degree k, i.e. for all $T > 0$, there exists $\tilde{T}$ such that for all $x \in X$, the inclusion $B(x,T) \to B(x,\tilde{T})$ induces the 0 map in cohomology up to degree k.

Then for every $R > 0$ and $S < +\infty$ and for large enough $\ell \geq 1$, there is a natural isomorphism of $L^{q,p}$-cohomologies $L^{q,p}_{\ell,R,S}H^k(X) \simeq L^{q,p}H^k(X)$. In degree $k+1$, the isomorphism persists provided the space $L^{q,p}H^k(X)$ is replaced with exact cohomology*, i.e. the kernel $EL^{q,p}H^k(X)$ of the forgetful map $L^{q,p}H^k(X) \to H^k(X)$. This isomorphism is compatible with multiplicative structures.*

For degree 1 cohomology, the size limit plays no role. Indeed, a 1-cocycle of size S on a simply connected space is the differential of a function, and thus uniquely extends into a 1-cocycle without size limit. The considerations of subsection 5.2 show that, *a priori*, all $L^p_{\ell,R,S}$-norms are equivalent on 1-cocycles. For higher degree cohomology,

this holds only at the cohomology level, under suitable assumptions, thanks to Proposition 75.

Remark 76 *[P6] contains the following useful remark. On an n-manifold, one can define de Rham L^p versus L^q+L^s-cohomology. One could do so with cochains, but this would be useless since $\ell^s \subset \ell^q$ when $s \leq q$. Fortunately, under the assumptions of Proposition 75, and if $s \leq q$, this de Rham L^p versus $L^q + L^s$-cohomology is isomorphic to $L^{q,p}_{\ell,R,S}H^\cdot$, and so to $L^{q,p}H^\cdot$.*

6.3 Functoriality of L^p-cohomology

For every $\ell' \geq 1$, (R, S, R', S')-coarse conformal map $f : X \to X'$ induces a bounded linear map

$$f^* : L^{q,p}_{\ell',R',S'}H^k(X') \to L^{q,p}_{\ell,R,S}H^k(X)$$

for suitable $\ell \geq 1$. So, under the assumptions of Proposition 75, a coarse conformal map of a bounded geometry Riemannian manifold with boundary or simplicial complex to a q.s. space X' induces maps

$$f^* : L^{q,p}_{\ell'}H^k(X') \to L^{q,p}H^k(X),$$

for all large enough ℓ'. Exact cohomology is natural as well.

Under the assumptions of Proposition 75, $L^{q,p}$-cohomology is natural under coarse embeddings and a quasi-isometry invariant. On the other hand, it is not clear wether it is natural under large scale conformal maps.

6.4 Vanishing of 1-cohomology and limits

Definition 77 *Let X be a metric space, Y a topological space, and $y \in Y$. Assume X is unbounded. Say a map $f : X \to Y$ tends to y at infinity if for every neighborhood V of y, there exists a bounded set K such that $f(x) \in V$ when $x \notin K$.*

Lemma 78 *Let X be an unbounded metric space. Let $q < \infty$. Then every function $u \in L^q_{\ell,R,S}C^0(X)$ tends to 0 at infinity.*

Proof Fix $\epsilon > 0$. Let $\{B_j\}$ be an (ℓ, R, S)-packing such that

$$\sum_j (\sup_{B_j} |u|)^q > \|u\|^q_{L^q_{\ell,R,S}} - \epsilon.$$

Pick a finite subfamily which achieves the sum minus ϵ. The union of this finite subfamily is contained in a ball K. If $d(x, \ell K) > \ell R$, add $B(x, R)$ to the finite subfamily to get a larger (ℓ, R, S)-packing. By definition of energy, $\sup_{B(x,R)} |u|^q < 2\epsilon$. In particular, $|u(x)| < (2\epsilon)^{1/q}$ outside a bounded set. This shows that u tends to 0 at infinity. ■

Corollary 79 *Let X be an unbounded metric space. Let $q < \infty$. Assume that the $L^{q,p}$-cohomology of X vanishes, i.e. $L^{q,p}_{\ell,R,S} H^1(X) = 0$. Let Y be a complete metric space. Then every map from X to Y with finite $E^p_{\ell,R,S}$ energy has a limit at infinity.*

Proof Let $u : X \to Y$ have finite energy. For $y \in Y$, set $v_y(x) = d(u(x), y)$. Then v_y has finite $E^p_{\ell,R,S}$ energy. By assumption, there exists a 0-cochain $w \in L^q_{\ell,R,S} C^0(X)$ such that $dw = dv_y$. This implies that v_y has a finite limit $\alpha(y)$ at infinity. α belongs to the closure of the Kuratowski embedding of Y in $L^\infty(Y)$. Since Y is complete, the embedding has a closed image, so there exists a point $z \in Y$ such that $\alpha(y) = d(z, y)$ for all $y \in Y$. In particular, $d(u(x), z) = v_z(x)$ tends to $\alpha(z) = 0$. ■

6.5 Vanishing of reduced 1-cohomology and limits

Definition 80 *Let X be a metric space. The* reduced $L^{q,p}$-cohomology *of X is obtained by modding out by the L^p-closure of the image of the coboundary operator d,*

$$L^{q,p}_{\ell,R,S} \bar{H}^k(X) = \left(\ker(d) \cap L^p_{\ell,R,S} C^k(X)\right) / \overline{d\left(\mathcal{L}^{q,p}_{\ell,R,S} C^{k-1}(X)\right)}.$$

Lemma 81 *Let X be an unbounded metric space with a base point. Then for every finite energy function u such that the reduced $L^{q,p}_{\ell,R,S}$-cohomology class of du vanishes, there exists $c \in \mathbb{R}$ such that u converges to c along p-almost every based curve.*

Proof Assume that $u_j \in L^q_{\ell,R,S} C^0(X)$ and

$$\|du_j - du\|_{L^p_{\ell,R,S}} \text{ tends to } 0.$$

For $t \in \mathbb{R}$, let $\Gamma_{t,+}$ (resp. $\Gamma_{t,-}$) be the family of based curves γ along which u has a finite limit and $\lim u \circ \gamma \geq t$ (resp. $\leq t$). Fix $s < t$. Let $v_j = \frac{2}{t-s}(u - u_j)$.

Assume that there exists a based curve $\gamma \in \Gamma_{t,+}$ such that, for infinitely many j, $\text{length}(v_j \circ \gamma) \leq 1$. For those j's, for every $\gamma' \in \Gamma_{s,-}$, $\text{length}(v_j \circ \gamma) \leq 1$. Indeed, along the bi-infinite curve obtained by concatenating γ and γ', the total variation of v_j is ≥ 2. Therefore, for infinitely many j's,

$$\text{mod}_{p,\ell,R,S}(\Gamma_{s,-}) \leq E(v_j) = (\frac{2}{t-s}\|du_j - du\|_{L^p_{\ell,R,S}})^p,$$

and $\text{mod}_{p,\ell,R,S}(\Gamma_{s,-}) = 0$.

Otherwise, for each $\gamma \in \Gamma_{t,+}$, for all but finitely many j's, $\text{length}(v_j \circ \gamma) \geq 1$. $\Gamma_{t,+}$ is the union of subfamilies

$$\Gamma_{t,+,J} = \{\gamma \in \Gamma_{t,+} \, ; \, \forall j \geq J, \, \text{length}(v_j \circ \gamma) \geq 1\},$$

each of which has vanishing modulus. By stability under countable unions, $\text{mod}_{p,\ell,R,S}(\Gamma_{t,+}) = 0$.

Let c be the supremum of all $t \in \mathbb{R}$ such that $\text{mod}_{p,\ell,R,S}(\Gamma_{t,+}) > 0$. By stability under countable unions, the family of based curves along which u has a finite limit $> c$ has vanishing modulus. If $c = -\infty$, for every $n \in \mathbb{Z}$, the family of based curves along which u has a finite limit $\leq n$ has vanishing modulus. Thus the family of all based curves has vanishing modulus, and the Lemma is proved. Otherwise, $\text{mod}_{p,\ell,R,S}(\Gamma_{s,-}) = 0$ for all $s < c$. By stability under countable unions, the family of based curves along which u has a finite limit $< c$ has vanishing modulus. This shows that u tends to c along almost every based curve. ■

Corollary 82 *Let X be an unbounded metric space. Let Y be a complete metric space. Let $u : X \to Y$ have finite $E^p_{\ell,R,S}$ energy. Assume that the reduced $L^{q,p}$-cohomology of X vanishes, i.e. $L^{q,p}_{\ell,R,S}\bar{H}^1(X) = 0$. Then u has a common limit along p-almost every based curve.*

Proof For $y \in Y$, set $v_y(x) = d(u(x), y)$. Then v_y has finite $E^p_{\ell,R,S}$ energy. By assumption, dv_y belongs to the $L^p_{\ell,R,S}$-closure of $d\mathcal{L}^{q,p}_{\ell,R,S}C^0(X)$. This implies that v_y has a finite limit $\alpha(y)$ along almost every based curve. α belongs to the closure of the Kuratowski embedding of Y in $L^\infty(Y)$. Since Y is complete, the embedding has a closed image, so there exists a point $z \in Y$ such that $\alpha(y) = d(z, y)$ for all $y \in Y$. In particular, $d(u(x), z) = v_z(x)$ tends to $\alpha(z) = 0$ along almost every based curve. ■

6.6 p-separability

Definition 83 *Let X be a q.s. space. Let $\mathcal{E}_{p,\ell}$ denote the space of continuous functions on X with finite (p,ℓ)-energy.*

Definition 84 *Say a q.s. space X is p-separated if for every large enough ℓ, $\mathcal{E}_{p,\ell}$ separates points and complements of points are (p,ℓ)-parabolic. If X is non-compact, one requires further that X itself be (p,ℓ)-parabolic.*

Example 85 *Q-Ahlfors-regular metric spaces are p-separated for all $p \geq Q$.*

Proof Proposition 39 shows that Lipschitz functions with bounded support have finite energy. They separate points. Propositions 65 and 68 establish parabolicity of X and of point complements. ■

Proposition 86 *Let $1 \leq p, q < +\infty$. Let X be a locally compact metric space. Let X' be a locally compact p-separated q.s. space. Let $f : X \to X'$ be a coarse conformal map. Then there exists $R > 0$ such that for all $S \geq R$ and $\ell' > 1$, for all large enough ℓ,*

- *either the induced map*

$$f^* : EL^{q,p}_{\ell'}\bar{H}^1(X') \to EL^{q,p}_{\ell,R,S}\bar{H}^1(X)$$

  in reduced exact $L^{q,p}$-cohomology does not vanish,

- *or X is (p,ℓ,R,S)-parabolic.*

Proof Let us treat first the simpler case when unreduced L^p-cohomology vanishes. Let $f : X \to X'$ be a coarse conformal map. Assume that f has distinct accumulation points x'_1 and x'_2 at infinity. By assumption, there exists a continuous function v on X' with finite (p,ℓ')-energy such that $v(x'_1) \neq v(x'_2)$. Then, for all $R \leq S$, $v \circ f$ has finite $E^p_{\ell,R,S}$ energy for suitable ℓ. If $L^{q,p}_{\ell,R,S}H^1(X)$ vanishes, according to Corollary 79, $v \circ f$ has a limit at infinity. This should be at the same time $v(x'_1)$ and $v(x'_2)$, contradiction. We conclude that f has at most one accumulation point at infinity. Hence either it has a limit x', or it tends to infinity.

In either case, there exists a finite energy function $w : X' \to \mathbb{R}$ that has no limit along p-almost every coarse curve converging to x' (resp. to infinity). Then $w \circ f$ has finite $E^p_{\ell,R,S}$ energy as well, it must have

a finite limit in $\mathbb{R}$. This contradicts the fact that the family of based $(1,1)$-curves in X has positive (p,ℓ,R,S)-modulus. We conclude that f^* does not vanish on $L^{q,p}_\ell H^1(X')$.

Assume that f induces a trivial map in reduced cohomology and that X is non-(p,ℓ,R,S)-parabolic. Equip $\mathcal{E}_{p,\ell}$ with the topology of uniform convergence on compact sets. Let $D \subset \mathcal{E}_{p,\ell}$ be a countable dense subset. Then D still separates points. We know that for all $v \in D$, $v \circ f \circ \gamma$ has a common limit y_v for almost every based $(1,1)$-curve γ in X. For $v \in D$, let Γ_v be the family of (R,S)-based curves $\gamma \subset X$ such that $v \circ f \circ \gamma$ does not have a limit or has a limit which differs from y_v. Then $\Gamma = \bigcup_{v \in D} \Gamma_v$ has vanishing (p,ℓ,R,S)-modulus. Let Γ' be the complementary family. Since X is non-(p,ℓ,R,S)-parabolic, Γ' is non-empty. Fix two based curves $\gamma, \gamma' \in \Gamma'$. Assume that $f \circ \gamma$ and $f \circ \gamma'$ have distinct accumulation points x'_1 and x'_2 in X'. Let $v \in D$ be such that $v(x'_1) \neq v(x'_2)$. By construction, $v \circ \gamma$ and $v \circ \gamma'$ converge to y_v. Since v is continuous, $v \circ \gamma$ subconverges to $v(x'_1)$ and $v \circ \gamma'$ to $v(x'_2)$, contradiction. We conclude that f has a common limit x' along all $\gamma \in \Gamma'$. The argument ends in the same manner. ■

6.7 Relative p-separability

Here comes a relative version of Proposition 86, motivated by the case of warped products.

Definition 87 *Let X be a q.s. space. Let $u : X \to Y$ be a continuous map to a topological space. Say X is* p-separated relatively to u *if for every large enough ℓ,*

1. *$\mathcal{E}_{p,\ell} \cup \{u\}$ separates points.*
2. *Complements of points in X are (p,ℓ)-parabolic.*

Example 88 *Let Z be a compact Q-Ahlfors-regular metric space. Let $0 < \alpha \leq 1$. Let $Z^\alpha = (Z, d_Z^\alpha)$ be a snowflaked copy of Z. Let $X = \mathbb{D} \times Z^\alpha$. Let u be the projection to the first factor $Y = [0,1]$. Then X is p-separated relative to u for all $p \geq \alpha + Q$.*

Indeed, the projection to the second factor equipped with d_Z has finite (p,ℓ)-energy for all $p \geq \alpha + Q$ and all $\ell > 1$, see Example 40. Together with u, it separates points. Proposition 70 states that complements of points in X are (p,ℓ)-parabolic.

Proposition 89 *Let $1 \leq p, q < +\infty$. Let X be a locally compact metric space containing at least one based $(1,1)$-curve. Let X' be a compact q.s. space which is p-separated relatively to a map $u : X' \to Y$. Let $f : X \to X'$ be a coarse conformal map. Assume that the map $u \circ f$ tends to some point $y \in Y$ at infinity. Then there exists $R > 0$ such that for all $S \geq R$ and $\ell' > 1$, for all large enough ℓ,*

- *either the induced map*

$$f^* : EL^{q,p}_{\ell'} \bar{H}^1(X') \to EL^{q,p}_{\ell,R,S} \bar{H}^1(X)$$

  in reduced exact $L^{q,p}$-cohomology does not vanish,

- *or X is (p, ℓ, R, S)-parabolic.*

Proof By contradiction. Assume that f induces a trivial map in reduced cohomology and that X is non-(p, ℓ, R, S)-parabolic. Equip $\mathcal{E}_{p,\ell}$ with the topology of uniform convergence. Let $D \subset \mathcal{E}_{p,\ell}$ be a countable dense subset. Then $D \cup \{u\}$ still separates points. For all $v \in D$, $v \circ f \circ \gamma$ has a common limit t_v for almost every based $(1,1)$-curve γ in X. For $v \in D$, let Γ_v be the family of (R,S)-based curves $\gamma \subset X$ such that $v \circ f \circ \gamma$ does not have a limit or has a limit which differs from t_v. Then $\Gamma = \bigcup_{v \in D} \Gamma_v$ has vanishing (p, ℓ, R, S)-modulus. Let Γ' be the complementary family. Since X is non-(p, ℓ, R, S)-parabolic, Γ' is non-empty. Fix two based curves $\gamma, \gamma' \in \Gamma'$. Assume that $f \circ \gamma$ and $f \circ \gamma'$ have distinct accumulation points x'_1 and x'_2 in X'. Since $u(x'_1) = u(x'_2) = y$, there exists $v \in D$ such that $v(x'_1) \neq v(x'_2)$. By construction, $v \circ \gamma$ and $v \circ \gamma'$ converge to t_v. Since v is continuous, $v \circ \gamma$ subconverges to $v(x'_1)$ and $v \circ \gamma'$ to $v(x'_2)$, contradiction. We conclude that f has a common limit x' along all $\gamma \in \Gamma'$.

Let $w : X' \to \mathbb{R}$ be a finite energy function that has no limit along (p, ℓ)-almost every coarse curve converging to x'. Then $w \circ f$ has finite $E^p_{\ell,R,S}$ energy as well, it must have a finite limit in $\mathbb{R}$. This contradicts the fact that the family of based $(1,1)$-curves in X has positive (p, ℓ, R, S)-modulus. We conclude that either f^* does not vanish on $L^{q,p}_\ell H^1(X')$ or X is p-parabolic. ∎

7 Lack of coarse conformal maps

Theorem 3 *Let $1 < p, q < +\infty$. Let X be a Riemannian manifold or a simplicial complex with bounded geometry. In the n-manifold case,*

assume further that $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$. *Assume that* X *is non-*p*-parabolic and that* $EL^{q,p}\bar{H}^1(X) = 0$.

1. *Let* X' *be a* p*-Ahlfors-regular metric space. Then there can be no coarse conformal maps* $X \to X'$.
2. *Let* $0 < \alpha \leq 1$. *Let* X' *be a warped product* $\mathbb{D} \times Z^\alpha$ *where* Z *is a compact* $p - \alpha$*-Ahlfors-regular metric space. For every coarse conformal map* $X \to X'$, *the projected map to the first factor* $X \to [0, 1]$ *cannot tend to 0.*

Proof It follows from Examples 85 and 88, Propositions 86, 89 and 75. ∎

7.1 Examples

For nilpotent groups, reduced L^p-cohomology vanishes. Indeed, such groups admit unbounded central subgroups. A central element in G acts by a translation of G, i.e. moves points a bounded distance away. The Corollary on page 221 of [G2] applies: reduced L^p-cohomology vanishes in all degrees, in particular in degree 1.

A nilpotent group of homogeneous dimension Q is p-parabolic if and only if $p \geq Q$. (p, ℓ, R, ∞)-parabolicity for $p \geq Q$, $R > 0$ and $\ell > 1$ follows from the asymptotics of volume of balls, [P1], and Remark 66. Non-(p, ℓ, R, S)-parabolicity for $p < Q$ will be proved below, in Proposition 107 and Corollary 106. Carnot groups in their Carnot-Carathéodory metrics are even (p, ℓ)-parabolic for $p \geq Q$ and $\ell > 1$, according to Proposition 65, since they are Q-Ahlfors regular.

Non-elementary hyperbolic groups, [GH], have infinite isoperimetric dimension, hence they are never p-parabolic (again, this follows from Proposition 107 and Corollary 106). Their L^p-cohomology vanishes for p in an interval starting from 1, whose upper bound is denoted by CohDim ([BP]).

Conformal dimension ConfDim arises as the infimal Hausdorff dimension of Ahlfors-regular metrics in the quasi-symmetric gauge of the ideal boundary. By definition, the quasi-symmetric gauge is the set of metrics which are quasi-symmetric to a visual quasi-metric (all such quasi-metrics are mutually quasi-symmetric), [BP]. Quite a number of results on CohDim and ConfDim can be found in recent works by Marc Bourdon, [BnK], John Mackay [McK] and their co-authors.

7.2 Maps to nilpotent groups

Corollary 90 *Let G and G' be nilpotent Lie group of homogeneous dimensions Q and Q'. Assume that G' is Carnot and equipped with a homogeneous Carnot-Carathéodory metric. If there exists a coarse conformal map $G \to G'$, then $Q \leq Q'$.*

Proof G', a Carnot group in its Carnot-Carathéodory metric, is Q'-Ahlfors-regular. Since reduced L^p-cohomology vanishes, Theorem 3 forbids existence of a coarse conformal map $G \to G'$ unless G is Q'-parabolic. This implies that $Q \leq Q'$. ■

Note that no properness assumption was made. Also, a stronger result will be obtained by a different method in Corollary 120.

Corollary 91 *Let G be a finitely generated group. Let G' be a nilpotent Lie or finitely generated group. If there exists a uniformly conformal map $G \to G'$, then G is itself virtually nilpotent, and $d(G) \leq d(G')$.*

Proof It is Q'-Ahlfors regularity of G' in the large which is used here, and Corollaries 64 and 67. Indeed, G' is p-parabolic for $p = d(G')$, therefore so is G. Proposition 113 implies that the isoperimetric dimension of G' is at most p. Proposition 107 tells us that G must be virtually nilpotent and $d(G) \leq p = d(G')$. ■

Corollary 92 *Let G be a non-elementary hyperbolic group. Let G' be a Carnot group of homogeneous dimension Q' equipped with its Carnot-Carathéodory metric. If there exists a coarse conformal map $G \to G'$, then* $\mathrm{CohDim}(G) \leq Q'$.

Proof Since G' is Q'-Ahlfors regular and non-elementary hyperbolic groups are never p-parabolic, Theorem 3 provides this upper bound on $\mathrm{CohDim}(G)$. ■

Corollary 93 *Let G be a non-elementary hyperbolic group. Let G' be a nilpotent group of homogeneous dimension Q'. If there exists a uniformly conformal map $G \to G'$, then* $\mathrm{CohDim}(G) \leq Q'$.

Proof Corollaries 64 and 67 apply, since non-elementary hyperbolic groups are never p-parabolic. ■

7.3 Maps to hyperbolic groups

Corollary 94 *Let G, G' be non-elementary hyperbolic groups. If there exists a uniformly conformal map $G \to G'$, then*

$$\mathrm{CohDim}(G) \leq \mathrm{ConfDim}(G').$$

Proof Let d' be an Ahlfors-regular metric in the gauge of $\partial G'$, of Hausdorff dimension Q'. According to [Ca], there exist a bounded geometry hyperbolic graph X, a visual quasi-metric d_o on ∂X and a bi-Lipschitz homeomorphism $q : (\partial G', d') \to (\partial X, d_o)$, arising from a quasi-isometry $q : G' \to X$. Set $Z = (\partial X, (q^{-1})^* d')$. Pick $0 < \alpha \leq 1$. Let $X' = \mathbb{D} \times Z^\alpha$. According to Proposition 36, the Poincaré model of X is a roughly conformal map $\pi : X \to X'$. If $f : G \to G'$ is uniformly conformal, then $f' = \pi \circ q \circ f : G \to X'$ is coarsely conformal (Proposition 6). Furthermore, $q \circ f$ is proper, so the projection of f' to the first factor tends to 0.

Since G is never p-parabolic, Theorem 3 asserts that $\mathrm{CohDim}(G) \leq \alpha + Q'$. Taking the infimum over $\alpha \in (0,1)$ and $Q' \geq \mathrm{ConfDim}(G)$, we get $\mathrm{CohDim}(G) \leq \mathrm{ConfDim}(G')$. ∎

Example 95 *Fuchsian buildings.*

Right-angled Fuchsian buildings (also known as Bourdon buildings) $X_{p,q}$ are universal covers of orbihedra having one p-sided polygon, p even, with trivial face group, cyclic $\mathbb{Z}/q\mathbb{Z}$ edge groups and direct product $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ vertex groups. The conformal dimension and the cohomological dimension of $X_{p,q}$ are both equal to $1 + \frac{\log q}{\arg\cosh(\frac{p-2}{2})}$, [Bo], [BP]. As p and q vary, these numbers fill a dense subset of $[1, +\infty)$.

There are obvious isometric embeddings $X_{p,q} \to X_{p,q'>q}$ and a jungle of bi-Lipschitz embeddings $X_{2p-4,q} \to X_{p,q}$, $X_{3p-8,q} \to X_{p,q}$,.... The only known restriction on the existence of uniform/coarse embeddings $X_{p,q} \to X_{p',q'}$ is provided by Corollary 94 or, alternatively, by D. Hume, J. Mackay and R. Tessera's p-separation estimates, [HMT].

Corollary 96 *Let G be a nilpotent Lie group of homogeneous dimension Q. Let G' be a hyperbolic group. If there exists a uniformly conformal map $G \to G'$, then $Q \leq \mathrm{ConfDim}(G')$.*

Proof Only the last paragraph of the proof of Corollary 94 needs be changed. In this case, reduced L^p-cohomology vanishes always,

Theorem 3 asserts that G must be p-parabolic for $p = \alpha + Q'$, hence $Q \leq \alpha + Q'$. Taking the infimum over $\alpha \in (0,1)$ and Q', we get $Q \leq \mathrm{ConfDim}(G')$. ■

Example 97 *This is sharp. For instance, the uniform embeddings $\mathbb{R}^{n-1} \to H^n_{\mathbb{R}}$ and $Heis^{2m-1} \to H^m_{\mathbb{R}}$ are uniformly conformal.*

More generally, every Carnot group G is a subgroup of the hyperbolic Lie group $G' = \mathbb{R} \ltimes G$, where $\mathbb{R}$ acts on G through Carnot dilations. The homogeneous dimension of G is equal to the conformal dimension of G', [P2]. This provides a uniformly conformal map $G \to G'$, according to Lemma 18.

7.4 Proof of Theorem 1 and Corollary 1

Theorem 1 is a combination of Corollaries 91, 93, 94 and 96 applied to the subclass of large scale conformal maps. Corollary 1 is the special case of uniform/coarse embeddings.

8 Large scale conformal isomorphisms

8.1 Capacities

Definition 98 *Let X be a metric space, let $K \subset X$ be a bounded set. The (p, ℓ, R, S)-*capacity *of K, $\mathrm{cap}_{p,\ell,R,S}(K)$, is the infimum of $E^p_{\ell,R,S}$-energies of functions $u : X \to [0,1]$ which take value 1 on K and have bounded support.*

Remark 99 *If $\mathrm{cap}_{p,\ell,R,S}(\{o\}) = 0$, then X is (p, ℓ, R, S)-parabolic.*

Proof The capacity of the one point set $\{o\}$ bounds from above the (p, ℓ, R, S)-modulus of the family of all $(1,1)$-curves based at o. ■

Note that $\mathrm{cap}_{p,\ell,R,S}(\{o\}) = 0$ implies that for every bounded set K, $\mathrm{cap}_{p,\ell',R',S'}(K) = 0$ for suitable constants.

Proposition 100 *Let X and X' be locally compact, noncompact metric spaces. Let $f : X \to X'$ be a large scale conformal map. Then, for every $R' > 0$, there exists $R > 0$ and for every $\ell' \geq 1$, there exist $\ell \geq 1$ and N' such that, for all compact sets $K \subset X$,*

$$\mathrm{cap}_{p,\ell,R,\infty}(K) \leq N' \, \mathrm{cap}_{p,\ell',R',\infty}(f(K)).$$

Proof If $u : X' \to [0,1]$ has compact support and $u(f(K)) = 1$, then $u \circ f$ has compact support and $(u \circ f)(K) \geq 1$, thus $E^p_{\ell,R,\infty}(u \circ f) \geq \mathrm{cap}_{p,\ell,R,\infty}(K)$. We know from Lemma 45 that

$$E^p_{\ell,R,\infty}(u \circ f) \leq N' \, E^p_{\ell',R',\infty}(u).$$

Taking the infimum over all such functions u,

$$\mathrm{cap}_{p,\ell,R,\infty}(K) \leq N' \, \mathrm{cap}_{p,\ell',R',\infty}(f(K)).$$

■

8.2 Non-parabolicity and $L^{q,p}$ cohomology

Definition 101 *A metric space X is* uniformly perfect in the large *if there exists a constant $c > 0$ such that, for all $x \in X$ and large enough T, $B(x,T) \setminus B(x,cT) \neq \emptyset$.*

Unbounded geodesic spaces (e.g. graphs, Riemannian manifolds), and spaces roughly isometric to such (e.g. locally compact groups) are uniformly perfect in the large. The point of this property is to ensure that R-volumes (meaning the number of disjoint R-balls that one can pack inside) of large balls are large.

Lemma 102 *Let X be a metric space which is uniformly perfect in the large. Fix a radius $R > 0$. Let $\mathrm{vol}_R(B)$ denote the maximal number of disjoint R-balls that can be packed in B, and $v_R(T) = \inf_{x \in X} \mathrm{vol}_R(B(x,T))$. Then $v_R(T)$ tends to infinity with T.*

Proof Given a ball $B = B(x_0, T)$, uniform perfectness, applied in $B(x_0, \frac{2T}{2+c})$, provides a point $x_1 \in B(x_0, \frac{2T}{2+c}) \setminus B(x, \frac{2cT}{2+c})$. Then $B(x_1, \frac{c}{2+c}T) \subset B(x,T) \setminus B(x_0, \frac{c}{2+c}T)$. Iterating the construction produces a sequence of disjoint balls $B(x_j, (\frac{c}{2+c})^{j+1}T)$ in $B(x,T)$. If $n = \lfloor \log_{c/2+c}(T/R) \rfloor$, we get n disjoint R-balls in $B(x,T)$. ■

Lemma 103 *Let X be a metric space which is uniformly perfect in the large. Fix $\ell \geq 1$ and $S \geq R > 0$. If $L^{q,p}_{\ell,R,S} H^1(X) = L^{q,p}_{\ell,R,S} \bar{H}^1(X)$ for some finite q, then the capacity of balls tends to infinity uniformly with their radius: there exists a function $\kappa_{\ell,R}$ such that $\kappa_{\ell,R}(T)$ tends to infinity as $T \to \infty$, and such that for every ball $B(x,T)$ of radius T,*

$$\mathrm{cap}^p_{\ell,R,S}(B(x,T)) \geq \kappa_{\ell,R}(T).$$

In particular, X is non-(p,ℓ,R,S)-parabolic.

Proof By assumption, the coboundary $d : \mathcal{L}^{q,p}_{\ell,R,S}C^0(X) \to L^p_{\ell,R,S}C^1(X)$ has a closed image. Its kernel consists of constant functions, which can be modded out. d becomes a continuous isomorphism between Banach spaces. According to the isomorphism theorem, d has a bounded inverse. Thus there exists a constant C such that, for every function $u \in L^q_{\ell,R,S}C^0(X)$, there exists a constant c_u such that

$$\|u - c_u\|_{L^p_{\ell,R,S}C^0(X)} \leq C \, \|du\|_{L^p_{\ell,R,S}C^1(X)}.$$

Since $vol_R(X)$ is infinite, constants do not belong to $L^q_{\ell,R,S}C^0(X)$, so $c_u = 0$. For functions $u : X \to [0,1]$ with bounded support, this translates into

$$\|u\|_{L^q_{\ell,R,S}C^0(X)} \leq C \, E^p_{\ell,R,S}(u)^{1/p}.$$

Let $B(x,T)$ be a large ball. Uniform perfectness ensures that a logarithmic number of disjoint balls of radius ℓR can be packed into $B(x,T)$. If $u = 1$ on $B(x,T)$, using this packing, we get a lower bound on $\|u\|^q_{L^q_{\ell,R,S}C^0}$ of the order of $\log(T/\ell R)$ which depends only on R, ℓ and T. This shows that the p-capacity of $B(x,T)$ tends to infinity with T. ∎

8.3 $L^{q,p}$-cohomology and isoperimetric dimension

Definition 104 *Let X be a Riemannian manifold. Say that X has isoperimetric dimension $\geq d$ if compact subsets $D \subset X$ with smooth boundary and sufficiently large volume satisfy*

$$\text{volume}(D) \leq C \, \text{volume}(\partial D)^{\frac{d}{d-1}}.$$

Lemma 105 *Let X be a Riemannian n-manifold with bounded geometry. If X has isoperimetric dimension $\geq d > 1$, then, for $p < d$, $EL^{q,p}H^1(X) = EL^{q,p}\bar{H}^1(X)$ for all $1 \leq p < d$ and $q < \infty$ such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{\max\{n,d\}}$.*

Proof The isoperimetric profile of a Riemannian manifold X is

$$I_X(v) = \inf\{\text{volume}(\partial D) \, ; \, \text{volume}(D) = v\}.$$

For instance, if $X = \mathbb{R}^n$ is Euclidean n-space,

$$I_{\mathbb{R}^n}(v) = c_n v^{\frac{n-1}{n}}.$$

According to [MN], the isoperimetric profile of a bounded geometry Riemannian manifold is continuous. According to [BM], for a bounded geometry Riemannian n-manifold, for every $C < c_n$, there exists v_0 such that

$$I_X(v) \geq C\, v^{\frac{n-1}{n}} \quad \text{if} \quad v \leq v_0.$$

By assumption, there exists a $c > 0$ such that $I_X(v) \geq c\, v^{\frac{d-1}{d}}$ for large v. Up to reducing the constant c, we can assume that this estimate holds as soon as $v \geq v_0$.

Following a classical argument, let us check that an L^1 Sobolev inequality holds. Let u be a smooth compactly supported function on X. Let u_t be the indicator function of the superlevel set $\{|u| > t\}$, i.e. $u_t(x) = 1$ if $|u(x)| > t$, $u_t(x) = 0$ otherwise. Let $t_0 = \sup\{t \geq 0\,;\, \text{volume}(\{|u| > t\}) \geq v_0$. Let u' be the function that is equal to $|u|$ on $\{|u| \leq t_0\}$, and to t_0 elsewhere. Let $u'' = |u| - u'$. Then

$$\|d|u|\|_1 = \|du'\|_1 + \|du''\|_1.$$

and

$$u' = \int_0^{t_0} u_t\, dt, \quad u'' = \int_{t_0}^{+\infty} u_t\, dt.$$

Using the isoperimetric inequality for volumes $\geq v_0$, we estimate the $L^{d'}$-norm of u', where $d' = \frac{d}{d-1}$,

$$\begin{aligned}
\|u'\|_{d'} &\leq \int_0^{t_0} \|u_t\|_{d'}\, dt \\
&= \int_0^{t_0} (\text{volume}(\{|u| > t\})^{1/d'}\, dt \\
&\leq \frac{1}{c} \int_0^{t_0} (\text{volume}(\{|u| = t\})\, dt \\
&= \frac{1}{c} \|du'\|_1.
\end{aligned}$$

A similar estimate, using the isoperimetric inequality for volumes $\leq v_0$, gives

$$\|u''\|_{n'} \leq \qquad\qquad \frac{1}{C}\|du''\|_1.$$

If $d \geq n$, since u'' vanishes outside a set of volume $\leq v_0$, Hölder's inequality gives

$$\|u''\|_{d'} \leq v_0^{\frac{1}{n}-\frac{1}{d}} \|u''\|_{n'},$$

hence

$$\|u\|_{d'} \leq C \, \|du\|_1. \tag{1}$$

It follows that $\|u\|_q$ is controlled by $\|du\|_p$ provided $p < d$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{d}$. Indeed, replacing $|u|$ with $|u|^r$, $r \geq 1$ in inequality (1), and applying Hölder's inequality, one finds that

$$\left(\int |u|^{rd'}\right)^{1/d'} \leq C \int |u|^{r-1}|du'| \leq C \left(\int |u|^{rd'}\right)^{\frac{r-1}{rd'}} \left(\int |du|^{\frac{rd'}{rd'-r+1}}\right)^{\frac{rd'-r+1}{rd'}}.$$

If $p < d$, on can pick q such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{d}$ and $r = q/d' \geq 1$. Then

$$\|u\|_q \leq C \, \|du\|_p.$$

If $d < n$, one more step is needed. Note that, for any $r \geq 1$, the decomposition of $v = |u|^r$ is $v' = |u'|^r$ and $v'' = |u''|^r$. Therefore one can apply the above method to $|u'|^r$ and $|u''|^r$ respectively. For u', one can use q such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{d}$ and get

$$\|u'\|_q \leq C \, \|du'\|_p.$$

For u'', since $p < d < n$, one can pick s such that $\frac{1}{p} - \frac{1}{s} = \frac{1}{n}$ and get

$$\|u''\|_s \leq C \, \|du''\|_p.$$

Summing up, since $\|du\|_p^p = \|du'\|_p^p + \|du''\|_p^p$, one finds that

$$\||u|\|_{L^s+L^q} \leq \|du\|_p.$$

This says that reduced and unreduced de Rham L^p versus $L^q + L^s$-cohomology coincide. According to Remark 76, this is equivalent to

$$EL^{q,p}_{\ell,R,S} H^1(X) = EL^{q,p}_{\ell,R,S} \bar{H}^1(X),$$

for all p, q such that $1 \leq p < d$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{\max\{n,d\}}$. ∎

Corollary 106 *A bounded geometry Riemannian manifold which has isoperimetric dimension $\geq d > 1$ is non-(p, ℓ, R, S)-parabolic for all $1 \leq p < d$ and all large enough ℓ, R, $S \geq R$.*

Proof This follows from Lemmata 103 and 105. ■

It turns out that the isoperimetric dimensions of finitely generated groups are known.

Proposition 107 (Compare M. Troyanov, [T]) *Let G be a finitely generated group. Then the isoperimetric dimension of G is*

- *either equal to 1 if G is virtually cyclic,*
- *or equal to its homogeneous dimension, an integer larger than 1 if G is virtually nilpotent but not virtually cyclic.*
- *Otherwise, it is infinite.*

If follows that a finitely generated group is p-parabolic if and only if it is virtually nilpotent of homogeneous dimension $\leq p$.

Proof According to T. Coulhon-L. Saloff Coste, [CSC], for finitely generated (or Lie) groups, volume growth provides an estimate on isoperimetric dimension. In particular, it implies that isoperimetric dimension is infinite unless volume growth is polynomial, in which case isoperimetric dimension is equal to the polynomial degree of volume growth. The only finitely generated groups of linear growth are virtually cyclic ones. That groups of polynomial growth are virtually nilpotent is M. Gromov's theorem of [G1]. The isoperimetry of nilpotent groups was originally due to N. Varopoulos, [VSC]. ■

8.4 Grötzsch invariant

Following [Gr], we use capacities to define a kind of large scale conformally invariant distance on a metric space.

Definition 108 *Let X be a metric space. Fix parameters p, ℓ, R, S. For x_1, $x_2 \in X$, let*

$$\delta_{p,\ell,R,S,\phi}(x_1, x_2) = \inf\{\mathrm{cap}_{p,\ell,R,S}(\mathrm{im}(\gamma)) \,; \\ \gamma \text{ continuous arc in } X \text{ from } x_1 \text{ to } x_2\}.$$

Lemma 109 *Let $f : X \to X'$ be a large scale conformal map. Assume that f is a bijection and that $f^{-1} : X' \to X$ is continuous. For all $R' > 0$, there exists $R > 0$ such that for all $\ell' \geq 1$, there exists $\ell \geq 1$ and N' such that, for all x_1, $x_2 \in X$ and all $S \geq R$,*

$$\delta_{p,\ell,R,S}(x_1, x_2) \leq N' \, \delta_{p,\ell',R',\infty}(f(x_1), f(x_2)).$$

Proof If γ' is a continuous arc joining $f(x_1)$ to $f(x_2)$, $f^{-1}\circ\gamma'$ is a continuous arc joining x_1 to x_2, thus $\delta_{p,\ell,R,S}(x_1,x_2) \leq \mathrm{cap}_{p,\ell,R,\infty}(f^{-1}\circ\gamma')$. Therefore $\delta_{p,\ell,R,\infty}(x_1,x_2) \leq N'\mathrm{cap}_{p,\ell,R,\infty}(\gamma')$, and taking an infimum,

$$\delta_{p,\ell,R,S}(x_1,x_2) \leq \delta_{p,\ell,R,\infty}(x_1,x_2) \leq N'\,\delta_{p,\ell',R',\infty}(f(x_1),f(x_2)).$$

where the first inequality exploits the fact that adding constraints on packings decreases energies and capacities. ■

8.5 Upper bounds on capacities

Definition 110 *Say a metric space X has* controlled balls *if there exist $R>0$ and a measure μ and continuous functions $v>0$ and $V<\infty$ on $[R,+\infty)$ such that for every $x\in X$ and every $r\geq R$,*

$$v(r) \leq \mu(B(x,r)) \leq V(r).$$

If such an estimate holds also for $r\in(0,R]$ and furthermore

$$\forall r\in(0,R], \quad v(r)\geq C\,r^Q,$$

one says that X is locally Q-Ahlfors regular*.*

In a Riemannian manifold or a simplicial complex with bounded geometry, balls are automatically controlled.

Lemma 111 *Let X be a geodesic metric space which has controlled balls. Let $p\geq 1$. Then $\delta_{p,\ell,R,\infty}$ is bounded above uniformly in terms of distance d. I.e. there exists a function $\Pi_{p,\ell,R}$ such that, if $d(x_1,x_2)\geq R$,*

$$\delta_{p,\ell,R,\infty}(x_1,x_2) \leq \Pi_{p,\ell,R}(d(x_1,x_2)).$$

If furthermore X is locally Q-Ahlfors regular for some $Q\leq p$, then such an upper bound still holds with $R=0$, i.e. there exists a function $\Pi_{p,\ell}$ such that

$$\delta_{p,\ell,0,\infty}(x_1,x_2) \leq \Pi_{p,\ell}(d(x_1,x_2)).$$

Proof Fix $r\geq R$, set $T(r)=2+\frac{8r}{\ell-1}$ and

$$C(r) = \sup_{\rho\in[R,T]} \frac{(2\rho)^p}{v(\rho)}.$$

For each $x \in X$, define a function $u_{x,r}$ as follows: $u_{x,r} = 1$ on $B = B(x,r)$, vanishes outside $2B$ and is linear in the distance to x in between. Let us estimate its p-energy. Let $\{B_j\}$ be a (ℓ, R, ∞)-packing of X. If a ball B_i intersects $2B$, and has radius $> 4r/(\ell-1)$, then ℓB_i contains $2B$. No other ball of the packing can intersect $2B$, hence an upper bound on $\sum_j \text{diameter}(u_{x,r}(B_j))^p \leq 1$. Otherwise, all balls of the ℓ-packing contributing to energy are contained in TB, for $T = 2 + (8r/(\ell-1)$. For each ball B_j of radius ρ_j, $\rho_j \in [R,T]$ and

$$\begin{aligned}\text{diameter}(u_{x,r}(B_j))^p &\leq \text{diameter}(B_j)^p \\ &\leq (2\rho_j)^p \\ &\leq C(r)\, v(\rho_j) \\ &\leq C(r)\, \mu(B_j).\end{aligned}$$

Summing up,

$$\begin{aligned}\sum_j \text{diameter}(B_j)^p &\leq C(r) \sum_j \mu(B_j) \\ &\leq C(r)\, \mu(B(x,T)) \\ &\leq C(r) V(T(r)).\end{aligned}$$

This gives an upper bound on $\text{cap}_{p,\ell,R,\infty}(B(x,r))$ which depends on its radius r, on p, on R and on ℓ only.

If X is locally Q-Ahlfors regular and $Q \leq p$, then

$$C'(r) = \sup_{\rho \in (0,T]} \frac{(2\rho)^p}{v(\rho)} < \infty,$$

so the argument generalizes to arbitrary $(\ell, 0, \infty)$-packings.

Balls in locally compact geodesic metric spaces contain geodesics which are continuous arcs. Thus the lower bound $\delta_{p,\ell,R,\infty}$ is bounded above by the capacity of a geodesic segment, which in turn is bounded above by the capacity of a ball, which is estimated in terms of its radius, on p and on ℓ only. ■

8.6 Strong non-parabolicity

Here, we are concerned with lower bounds on Grötzsch' invariant δ.

Definition 112 *Let X be a metric space. Say that X is* strongly non-(p, ℓ, R, S)-parabolic *if $\delta_{p,\ell,R,S}(x_1, x_2)$ tends to infinity uniformly*

with $d(x_1, x_2)$. In other words, for every $R \leq S$ and $\ell > 1$, there exists a function $\pi_{p,\ell,R,S}$ such that $\pi_{p,\ell,R,S}(T)$ tends to infinity when $T \to \infty$, and such that

$$\delta_{p,\ell,R,S}(x_1, x_2) \geq \pi_{p,\ell,R,S}(d(x_1, x_2)).$$

Proposition 113 *Let X be a metric space. Fix $\ell \geq 1$ and $S \geq R > 0$. If $L^{q,p}_{\ell,R,S} H^1(X) = L^{q,p}_{\ell,R,S} \bar{H}^1(X)$ for some finite q, then X is strongly non-(p, ℓ, R, S)-parabolic.*

Proof Let $x_1, x_2 \in X$. Let γ be a continuous arc joining x_1 to x_2. Fix $R > 0$ and $\ell \geq 1$. Assume that $d(x_1, x_2) \geq 2R$. For each $j = 0, \ldots, k := \lfloor d(x_1, x_2)/2\ell R \rfloor$, pick a point y_j on γ such that $d(y_j, x_1) = 2\ell R j$. Let $B_j = B(y_j, R)$. By construction, $\{B_j\}$ is a (ℓ, R, R)-packing of X. Let $u : X \to [0, 1]$ be a function of bounded support such that $u = 1$ on γ. Then $\sup_{B_j} u = 1$, thus

$$\|u\|_{L^q_{\ell,R,R}} \geq k^{1/q}.$$

By assumption, there exists C such that, for every function u of bounded support,

$$\|u\|_{L^q_{\ell,R,R}} \leq C\, E^p_{\ell,R,R}(u)^{1/p}.$$

This shows that

$$\mathrm{cap}_p(\gamma) \geq C\, k^{p/q} = C\, \lfloor \frac{d(x_1, x_2)}{2\ell R} \rfloor^{p/q},$$

this is a lower bound on $\delta_{p,\ell,R,R}(x_1, x_2)$. This yields a lower bound on $\delta_{p,\ell,R,S}(x_1, x_2)$ for any S. ∎

8.7 Consequences

Corollary 114 *Let X and X' be geodesic metric spaces which are strongly non-(p, ℓ, R, S)-parabolic for some $p \geq 1$. Assume that both have controlled balls. Let $f : X \to X'$ be a homeomorphism such that both f and f^{-1} are large scale conformal maps. Then f is a quasi-isometry.*

Proof By strong non-(p, ℓ, R, S)-parabolicity, δ invariants in X are bounded below,

$$\delta_{p,\ell,R,S}(x_1, x_2) \geq \pi_{p,\ell,R,S}(d(x_1, x_2)).$$

According to Lemma 111, they are bounded above in X',

$$\delta_{p,\ell,R,\infty}(x'_1, x'_2) \leq \Pi_{p,\ell,R}(d(x'_1, x'_2)).$$

If $f : X \to X'$ is a large scale conformal homeomorphism, $N'\delta \circ f \geq \delta$ up to changes in parameters (Lemma 109),

$$N'\delta_{p,\ell',R',\infty}(f(x_1), f(x_2)) \geq \delta_{p,\ell,R,S}(x_1, x_2).$$

Combining these inequalities, we get for f,

$$N'\Pi_{p,\ell',R'}(d(f(x_1), f(x_2))) \geq \pi_{p,\ell,R,S}(d(x_1, x_2)),$$

and for f^{-1},

$$N'\Pi_{p,\ell',R'}(d(f^{-1} \circ f(x_1), f^{-1} \circ f(x_2))) \geq \pi_{p,\ell,R,S}(d(f(x_1), f(x_2))),$$

hence

$$\pi_{p,\ell,R,S}(d(f(x_1), f(x_2))) \leq N'\Pi_{p,\ell',R'}(d(x_1, x_2)).$$

These inequalities show that f is a quasi-isometry. ■

Corollary 115 *Let M, M' be bounded geometry Riemannian manifolds quasi-isometric to non virtually cyclic groups. Then homeomorphisms $M \to M'$ which are large scale conformal in both directions must be quasi-isometries.*

Remark 116 *This applies to Euclidean spaces of dimension ≥ 2. Note that Examples 10 are roughly conformal in both directions, but large scale conformal in only one direction.*

Remark 117 *A natural question (Sylvain Maillot) is wether two spaces X and X' can have a large scale conformal map $X \to X'$ and a large scale conformal map $X' \to X$ without being quasi-isometric.*

8.8 From coarse to uniformly conformal maps

Proposition 118 *Let X be a metric space which is strongly non-(p, ℓ, R, S)-parabolic for some $p \geq 1$ and all $\ell \geq 1$. Let X' be a metric space which is locally Q-Ahlfors regular for some $Q \leq p$. Every coarsely conformal map $X \to X'$ is uniformly conformal. Every roughly conformal map $X \to X'$ is large scale conformal.*

Proof Fix $S \geq R > 0$. Assume that $f : X \to X'$ is coarsely (resp. roughly) conformal. It suffices to show that for all $T' > 0$, there exists $T_0 > 0$ such that f maps no T-ball B, $T \geq T_0$, into a T'-ball B'. Given $\ell' \geq 1$, there exist $\ell \geq 1$ and N' such that if $f(B) \subset B'$, $\mathrm{cap}_{p,\ell,R,S}(B) \leq N' \mathrm{cap}_{p,\ell',0,\infty}(B')$. This upper bound fails if T is sufficiently large, $T \geq T_0$. This shows that T-balls are never mapped into T'-balls. So if f is coarsely conformal, it is in fact (T_0, S_0, T', ∞)-coarsely conformal, for large enough T_0 and for every $S_0 \geq T_0$, hence it is uniformly conformal. If f is roughly conformal, it is in fact $(T_0, \infty, T', \infty)$-coarsely conformal, for large enough T_0 thus f is large scale conformal. ■

Remark 119 *The assumptions of Proposition 118 are satisfied for $X = \mathbb{R}^n$ provided $p < n$ and for $X' = \mathbb{R}^{n'}$ for $p \geq n'$. So Proposition 118 applies if $n' < n$, i.e. exactly when there are no coarse conformal maps $X \to X'$. In fact, the conclusion fails if $n = n'$, as Examples 10 show.*

Proposition 118 allows to modify the assumptions in the corollaries of subsections 7.2 and 7.3. For instance,

Corollary 120 *Let G be a finitely generated group. Let G' be a nilpotent Lie group equipped with a left-invariant Riemannian metric. If there exists a coarse conformal map $G \to G'$, then G is virtually nilpotent and $d(G) \leq d(G')$.*

Proof If G is virtually cyclic, then it is virtually nilpotent and $d(G) = 1 \leq d(G')$. Otherwise, G has isoperimetric dimension $Q > 1$, thus it is strongly non-(p, ℓ, R, S)-parabolic for all $1 < p < Q$. G' is locally n'-Ahlfors regular for $n' = \mathrm{dimension}(G')$. Assume that $Q > Q'$. Since $n' \leq Q'$, one can pick p such that $\max\{1, n'\} < p < Q$. Proposition 118 asserts that a coarse conformal map $G \to G'$ is automatically uniformly conformal. Corollary 91 shows that such a map cannot exist. ■

Corollary 121 *Let G, G' be non-elementary hyperbolic groups. Let M' be a Riemannian manifold of bounded geometry, which is quasi-isometric to G'. If there exists a coarse conformal map $G \to M'$, then*

$$\mathrm{CohDim}(G) \leq \mathrm{ConfDim}(G').$$

Proof Non-elementary hyperbolic groups have infinite isoperimetric dimensions. Thus G is strongly non-(p, ℓ, R, S)-parabolic for all p. By assumption, M' is locally n'-Ahlfors regular for $n' = \text{dimension}(M')$. Choose some $p \geq n'$. Proposition 118 asserts that a coarse conformal map $G \to M'$ is automatically uniformly conformal. Composing with a quasi-isometry, we get a uniformly conformal map $G \to G'$, so Corollary 94 applies. ■

Corollary 2 is a combination of Corollaries 120 and 121.

8.9 Large scale conformality in one dimension

We have been unable to extend Theorem 2 to the virtually cyclic case. Here is a partial result.

Lemma 122 *Let f and g be continuous maps $\mathbb{R} \to \mathbb{R}$ such that*

- *f and g are large scale conformal;*
- *$g \circ f$ and $f \circ g$ are coarse embeddings.*

Then f is a quasi-isometry.

Proof Fix $R' \leq R$, ℓ', ℓ, N' as given by the definition of large scale conformality. Let $\tilde{R}$ be given by the definition of coarse embeddings: $g \circ f$ maps R-balls to $\tilde{R}$-balls. To save notation, assume that the same constants serve for g. Since we are on the real line, balls of radius R are intervals of length $2R$. In the correspondence between balls $B \mapsto B'$, one can assume that B' is a minimal interval containing $f(B)$ and of length $\geq 2R'$, i.e. $B' = f(B)$ itself if $\text{length}(f(B)) \geq 2R'$.

The balls $B_j = B(2\ell R j, R)$ are mapped into balls B'_j forming an (N', ℓ', R', ∞)-packing. Assume that $f(B_0)$ has length $2R'_0 \geq 2R'$, in order that $B'_0 = f(B_0)$. Let $\{B''_j\}$ be an (ℓ, R, ∞)-packing of B'_0. The number of balls in this packing is at least $\frac{R'_0}{2\ell R}$. In turn, g maps B''_j into B'''_j, which form an (N', ℓ', R', ∞)-packing, which is the union of N' (ℓ', R', ∞)-packings. One of them has at least $\frac{R'_0}{2\ell R N'}$ elements. Every ball B'''_j contains

$$g(B''_j) \subset g(B'_0) = g \circ f(B_0) \subset \tilde{B} := B(g \circ f(0), \tilde{R}),$$

so B'''_j intersects $\tilde{B}$. At most two of these balls contains boundary points, so all others are contained in $\tilde{B}$. At least one of these balls has

radius ρ no larger than

$$\frac{\tilde{R}}{\frac{R'_0}{2\ell RN'} - 2} = \frac{2\ell RN'\tilde{R}}{R'_0 - 4\ell RN'}.$$

Since $\rho \geq R'$, we obtain an upper bound on R'_0. This shows that all R-balls are mapped to R'_0-balls, i.e. f is a coarse embedding. ■

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