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To cite this version:

Mohammed El Azhari. Positive linear functionals on BP*-algebras. Mathematica Slovaca, 1995, 45 (3), pp.281-285. hal-01297816

HAL Id: hal-01297816

https://hal.science/hal-01297816v1

Submitted on 4 Apr 2016

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POSITIVE LINEAR FUNCTIONALS ON BP*-ALGEBRAS

M. EL AZHARI

Abstract. Let A be a BP*-algebra with identity $e, P_1(A)$ be the set of all positive linear functionals f on A such that $f(e) = 1$, and let $M_s(A)$ be the set of all nonzero hermitian multiplicative linear functionals on A. We prove that $M_s(A)$ is the set of extreme points of $P_1(A)$. We also prove that, if $M_s(A)$ is equicontinuous, then every positive linear functional on A is continuous. Finally, we give an example of a BP*-algebra whose topological dual is not included in the vector space generated by $P_1(A)$, which gives a negative answer to a question posed by M. A. Hennings.

Mathematics Subject Classification 2010: 46K05, 46J05.

Keywords: positive linear functional, BP*-algebra, *-representation.

I. Preliminaries

Let A be a topological vector space, we denote by A^* the algebraic dual of A, and by A' the topological dual of A. Let A be an algebra and $x \in A$, we denote by $sp_A(x)$ the spectrum of x, by $\rho_A(x) = \sup\{|\lambda| : \lambda \in sp_A(x)\}$ the spectral radius of x, and by $R(A)$ the radical of A. A topological algebra is an algebra (over the complex field) which is also a Hausdorff topological vector space such that the multiplication is separately continuous. A locally convex algebra is a topological algebra whose topology is locally convex. Let A be a locally convex *-algebra. A linear functional f on A is called positive if $f(x^*x) \geq 0$ for all $x \in A$. A linear functional f on A is said to be hermitian if $f(x^*) = \overline{f(x)}$ for all $x \in A$. Let A be a commutative locally convex *-algebra with continuous involution and identity e. Let β be the family of all closed, bounded, absolutely convex, containing e, hermitian, idempotent subsets of A. For any $B \in \beta$, the linear span A_B of B, is a *-subalgebra of A which, when equipped with the Minkowski norm $\|.\|_B$, becomes a normed *-algebra with identity and isometric involution. If $A = \cup\{A_B, B \in \beta\}$ and $(A_B, \|.\|_B)$ is complete for all $B \in \beta$, we say that A is a BP*-algebra. We denote by $P(A)$ the set of all positive linear functionals on A, by $P_1(A)$ the subset of $P(A)$ consisting of those positive linear functionals f such that $f(e) = 1$, by $M(A)$ the set of all nonzero multiplicative linear functionals on A, and by $M_s(A)$ the set of all nonzero hermitian multiplicative linear functionals on A. A BP*-algebra A is called symmetric if $(e + x^*x)^{-1}$ exists for all $x \in A$.

II. Characterization of extreme points of $P_1(A)$

Let A be a BP*-algebra, $A = \cup\{A_B, B \in \beta\}$. In [5], T.Husain and S.A.Warsi tried to characterize the extreme points of $P_1(A)$, their method is essentially based on the fact that $P_1(A)$ is the the projective limit of $P_1(A_B), B \in \beta$, this method gave no result in the general case. Using the known method of Banach algebras, we characterize the extreme points of $P_1(A)$.

Lemma 1. ([6; Theorem 4.4.12]) Let H be a Hilbert space and L any *-subalgebra of $B(H)$. Then L is irreductible on H if and only if the centralizer of L in $B(H)$ contains only scalar multiples of the identity operator.

Proof. See [6].

Theorem 2. Let A be a BP*-algebra, then the extreme points of $P_1(A)$ are exactly the nonzero hermitian multiplicative linear functionals on A.

Proof. Let $\chi \in M_s(A)$ and $x \in A, \chi(x^*x) = \overline{\chi(x)}\chi(x) = |\chi(x)|^2 \geq 0$, therefore $M_s(A) \subset P_1(A)$. Let $f \in P_1(A), f$ is hermitian and admissible [4; Theorem 5.2], so we can associate to f a cyclic representation T_f on a Hilbert space H_f [6; Theorem 4.5.4]. Let f be an extreme point of $P_1(A), T_f$ is an irreductible representation [5; Theorem 5.1]. By Lemma 1, the centralizer Z of $\{T_f x, x \in A\}$ in $B(H_f)$ contains only scalar multiples of the identity operator. Since A is commutative, $\{T_f x, x \in A\} \subset Z$, so for each $x \in A$ there exists $\chi(x) \in C$ such that $T_f x = \chi(x)I$. It is easily shown that the map $x \to \chi(x)$ from A to C is a nonzero hermitian multiplicative linear functional on A. Let h be a cyclic vector of H_f such that $f(x) = \langle T_f x(h), h\rangle$ for all $x \in A$, then $f(x) = \chi(x)\|h\|^2$, thus $\|h\|^2 = 1$ since $f(e) = \chi(e) = 1$, hence $f = \chi \in M_s(A)$. Conversely, let $\chi \in M_s(A)$ and let (T_χ, H_χ) be the cyclic representation associated with χ. We have dim $H_\chi =$ dim $A/Ker\chi = 1$, therefore T_χ is irreductible, thus χ is an extreme point of $P_1(A)$ [5; Theorem 5.1].

III. On the symmetric spectral radius of a BP*-algebra

Let A be a BP*-algebra, we denote by ρ_s the symmetric spectral radius of A defined by $\rho_s(x) = \sup\{|\chi(x)| : \chi \in M_s(A)\}$ for all $x \in A$. We have $\rho_s(x) \leq \rho_A(x) < \infty$ for all $x \in A$.

Theorem 3. Let A be a BP*-algebra, then $\rho_s(x) = \sup\{f(x^*x) : f \in P_1(A)\}^{\frac{1}{2}}$ for all $x \in A$.

Proof. Put $|x| = \sup\{f(x^*x) : f \in P_1(A)\}^{\frac{1}{2}}$ for all $x \in A$. Let $\chi \in M_s(A) \subset P_1(A)$ and $x \in A, |\chi(x)|^2 = \chi(x^*x) \leq |x|^2$, thus $|\chi(x)| \leq |x|$, hence $\rho_s(x) \leq |x|$. Conversely, let $f \in P_1(A)$, by Theorem 2 and the Krein-Milman Theorem, there exists a net $(f_\alpha)_{\alpha\in\Lambda}$ in conv $M_s(A)$ such that $f_\alpha \to f$ in the weak topology $\sigma(A^*, A)$. For each $\alpha \in \Lambda$, there exist $\lambda_1, ..., \lambda_n \in R^+, \lambda_1 + ... + \lambda_n = 1$, and $\chi_1, ..., \chi_n \in M_s(A)$ such that $f_\alpha = \lambda_1\chi_1 + ... + \lambda_n\chi_n$. Let $x \in A, f_\alpha(x^*x) \leq$

$\lambda_1|\chi_1(x)|^2 + ... + \lambda_n|\chi_n(x)|^2 \leq \rho_s(x)^2$, since $f_\alpha(x^*x) \to f(x^*x)$, it follows that $f(x^*x) \leq \rho_s(x)^2$, therefore $\sup\{f(x^*x) : f \in P_1(A)\} \leq \rho_s(x)^2$, i.e. $|x| \leq \rho_s(x)$.

Corollary 4. Let A be a symmetric BP*-algebra, then $R(A) = \{x \in A : f(x^*x) = 0 \text{ for all } f \in P_1(A)\}$.

Proof. Each nonzero multiplicative linear functional on A is hermitian, therefore $R(A) = \{x \in A : \chi(x) = 0 \text{ for all } \chi \in M_s(A)\} = \{x \in A : \rho_s(x) = 0\}$.

Theorem 5. Let A be a BP*-algebra and $f \in P(A)$, then $|f(x)| \leq f(e)\rho_s(x)$ for all $x \in A$.

Proof. Let $f \in P(A)$. If $f(e) = 0$, by the Schwarz inequality, $|f(x)|^2 \leq f(e)f(x^*x)$ for all $x \in A$, then f vanishes on A. If $f(e) \neq 0$, put $f_1 = f(e)^{-1}f$, it is clear that $f_1 \in P_1(A)$. Let $x \in A, f_1(x^*x) \leq \rho_s(x)^2$ by Theorem 3, thus $|f_1(x)|^2 \leq f_1(x^*x) \leq \rho_s(x)^2$, therefore $|f_1(x)| \leq \rho_s(x)$, i.e. $|f(x)| \leq f(e)\rho_s(x)$.

Corollary 6. Let A be a BP*-algebra. If $M_s(A)$ is equicontinuous, then every positive linear functional on A is continuous.

Proof. If $M_s(A)$ is equicontinuous, then ρ_s is continuous.

Corollary 7. Let A be a barrelled BP*-algebra such that every nonzero hermitian multiplicative linear functional on A is continuous. Then every positive linear functional on A is continuous.

Proof $M_s(A)$ is bounded in the weak topology $\sigma(A', A)$. Since A is barrelled, $M_s(A)$ is equicontinuous by the Banach-Steinhaus Theorem.

Remarks.
1. Corollary 6 is an improvement of [4; Theorem 5.7].
2. Corollary 7 is a partial answer to the following problem [1]: is every positive linear functional on a barrelled BP*-algebra continuous ?

IV. On a question of M. A. Hennings

Let A be a BP*-algebra. In [3; question E], M. A. Hennings posed the following question: do the positive linear functionals on A spans the topological dual of A ?
M. A. Hennings [3, p.290 (lines 28-29)] stated that the general problem of whether this question is true or not is still open. Here we give a negative answer to this question, we first have the following proposition:

Proposition 8. Let A be a symmetric BP*-algebra. If $A' \subset span_{A^*}P(A)$, then A is semisimple.

Proof. We first remark that $span_{A^*}P(A) = span_{A^*}P_1(A)$. By Corollary 4, $R(A) = \{x \in A : f(x^*x) = 0 \text{ for all } f \in P_1(A)\}$. Let $x \in R(A)$, by the Schwarz inequality, $|f(x)|^2 \leq f(x^*x)$ for all $f \in P_1(A)$, thus $f(x) = 0$ for all $f \in P_1(A)$. Since $A' \subset span_{A^*}P_1(A)$, it follows that $g(x) = 0$ for all $g \in A'$, hence $x = 0$.

Remark. Proposition 8 is a generalization of the following Hennings result: If A is a symmetric unital MQ*-algebra such that $A' \subset span_{A^*}P(A)$, then A is semisimple.

Counter example Let E be the linear space of absolutely continuous functions on [0,1], with the norm $\|f\| = \int_0^1 |f(t)|dt$, the convolution-multiplication $(f \times g)(x) = \int_0^x f(x-t)g(t)dt$, and the natural involution $f^*(x) = \overline{f(x)}.(E, \|.\|)$ is a symmetric radical commutative Banach *-algebra [6; p.316]. Let $A = E \bigoplus Ce$ be the algebra obtained from E by adjunction of an identity e. A is a commutative Banach *-algebra with identity, A is symmetric [6; Theorem 4.7.9]. It is easily shown that $R(A) = E$, thus A is not semisimple, hence A' is not included in $span_{A^*}P(A)$ by Proposition 8.

V. On representations of BP*-algebras

Let A be a BP*-algebra and $f \in P(A)$. Let $L_f = \{x \in A : f(x^*x) = 0\}, L_f$ is a linear space and the quotient space A/L_f is a pre-Hilbert space with the inner product $\langle x+L_f, y+L_f\rangle = f(y^*x), x, y \in A$. Since f is hermitian and admissible, we can associate to f a cyclic representation on a Hilbert space H_f which is the completion of A/L_f. For each $x \in A, T_fx$ is defined by $T_fx(y + L_f) = xy + L_f$ for all $y \in A$.

Theorem 9. Let A be a BP*-algebra such that $M_s(A)$ is equicontinuous. Then for each $f \in P(A)$, the representation T_f is continuous.

Proof. Let $x, y \in A$, we have $f(y^*x^*xy) \leq f(y^*y)\rho_s(x^*x)$, thus $\langle xy + L_f, xy + L_f\rangle \leq \langle y + L_f, y + L_f\rangle\rho_s(x^*x)$, i.e. $\|T_fx(y + L_f)\| \leq \|y + L_f\|\rho_s(x^*x)^{\frac{1}{2}}$, hence $\|T_fx\| \leq \rho_s(x^*x)^{\frac{1}{2}} = \rho_s(x)$. Since $M_s(A)$ is equicontinuous, ρ_s is continuous, therefore T_f is continuous.

Remark. Theorem 9 is an improvement of [5; Theorem 4.2].

Acknowledgments. I would like to express my thanks to Professor T. Husain for help he has given me during the preparation of this work.

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