Local rule distributions, language complexity and non-uniform cellular automata

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Alberto Dennunzio[a,*], Enrico Formenti[b,*], Julien Provillard[b]

[a] *Università degli Studi di Milano–Bicocca, Dipartimento di Informatica, Sistemistica e Comunicazione, Viale Sarca 336, 20126 Milano (Italy)*
[b] *Université Nice Sophia Antipolis, Laboratoire I3S, 2000 Route des Colles, 06903 Sophia Antipolis (France)*

Abstract

This paper investigates a variant of cellular automata, namely ν-CA. Indeed, ν-CA are cellular automata which can have different local rules at each site of their lattice. The assignment of local rules to sites of the lattice completely characterizes ν-CA. In this paper, sets of assignments sharing some interesting properties are associated with languages of bi-infinite words. The complexity classes of these languages are investigated providing an initial rough classification of ν-CA.

Keywords: non-uniform cellular automata, bi-infinite words, ζ-rational languages

1. Introduction

Cellular automata (CA) are discrete dynamical systems consisting in an infinite number of finite automata arranged on a regular lattice. All automata of the lattice are identical and work synchronously. The new state of each automaton is computed by a local rule on the basis of the current state and the one of a fixed set of neighboring automata. This simple definition contrasts the huge number of different dynamical behaviors that made the model widely used in many scientific disciplines for simulating phenomena characterized by the emergency of complex behaviors from simple local interactions (chemical reactions, disease diffusion, particle reaction-diffusion, pseudo-random number generation, cryptography, *etc.*). For recent results on CA dynamics and an up-to-date bibliography see for instance [18, 12, 10, 1, 6, 9, 8].

In many cases, the uniformity of the local rule is more a constraint than a helping feature. Indeed, the uniformity constraint has been relaxed, for example,

*Corresponding author
Email addresses: `dennunzio@disco.unimib.it` (Alberto Dennunzio), `enrico.formenti@unice.fr` (Enrico Formenti), `julien.provillard@unice.fr` (Julien Provillard)

for modeling cell colonies growth, fast pseudo-random number generation, and VLSI circuit design and testing. This gave rise to new models, called non-uniform cellular automata (ν-CA) or hybrid cellular automata (HCA), in which the local rule of the finite automaton at a given site depends on its position. If the study of dynamical behavior has just started up [5, 7], applications and analysis of structural properties have already produced a wide literature (see, for instance, [13, 14]).

In this paper, we adopt a formal languages complexity point of view. Consider a finite set $\mathcal{R}$ of local rules defined over the same finite state set A. A (one-dimensional) ν-CA is essentially defined by the *distribution* or assignment of local rules in $\mathcal{R}$ to sites of the lattice. Whenever $\mathcal{R}$ contains a single rule, the standard cellular automata model is obtained. Therefore, each ν-CA can be associated with a unique bi-infinite word over $\mathcal{R}$. Consider now the class C of ν-CA defined over $\mathcal{R}$ and sharing a certain property P (for example surjectivity, injectivity, *etc.*). Clearly, C can be identified as a set of bi-infinite words contained in ${}^{\omega}\mathcal{R}^{\omega}$. In this paper, we analyze the language complexity of C *w.r.t.* several well-known properties, namely number-conservation, surjectivity, injectivity, sensitivity to initial conditions and equicontinuity. We have proved that C is a subshift of finite type and sofic, respectively, for the first two properties, while it is ζ-rational for the last three properties in the list. Remark that for sensitivity to initial conditions and equicontinuity, the results are proved when $\mathcal{R}$ contains only linear local rules (*i.e.* local rules satisfying a certain additivity property) with radius 1. The general case seems very complicated and it is still open.

In order to prove the main theorems, some auxiliary results, notions and constructions have been introduced (variants of De Bruijn graphs and their products, *etc.*). We believe that they can be interesting in their own to prove further properties.

2. Notations and definitions

For all $i, j \in \mathbb{Z}$ with $i \leq j$ (resp., $i < j$), let $[i, j] = \{i, i+1, \ldots, j\}$ (resp., $[i, j) = \{i, \ldots, j-1\}$).

Configurations and non uniform automata. Let A be a finite *alphabet*. A *configuration* or *bi-infinite word* is a function from $\mathbb{Z}$ to A. For any configuration x and any integer i, x_i denotes the element of x at index i. The *configuration set* $A^{\mathbb{Z}}$ is usually equipped with the metric d defined as follows

$$\forall x, y \in A^{\mathbb{Z}},\ d(x, y) = 2^{-n},\ \text{where } n = \min\{i \geq 0 \,:\, x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\} \ .$$

For any pair $i, j \in \mathbb{Z}$, with $i \leq j$, and any configuration $x \in A^{\mathbb{Z}}$ we denote by $x_{[i,j]}$ the word $w = x_i \ldots x_j \in A^{j-i+1}$, *i.e.*, the portion of x inside $[i, j]$, and we say that the word w *occurs* in x. Similarly, $u_{[i,j]} = u_i \ldots u_j$ is the portion of a word $u \in A^l$ inside $[i, j]$ (here, $i, j \in [0, l)$). In both the previous notations, $[i, j]$ can be replaced by $[i, j)$ with the obvious meaning. For any word $u \in A^*$,

$|u|$ denotes its length. A *cylinder* of block $u \in A^k$ and position $i \in \mathbb{Z}$ is the set $[u]_i = \{x \in A^{\mathbb{Z}} : x_{[i,i+k)} = u\}$. Cylinders are clopen sets *w.r.t.* the metric d and they form a basis for the topology induced by d. For $0 \in A$, a configuration x is said to be *finite* if the number of positions i at which $x_i \neq 0$ is finite.

A *local rule* of radius $r \in \mathbb{N}$ on the alphabet A is a map from A^{2r+1} to A. Local rules are crucial in both the definitions of cellular automata and non-uniform cellular automata. A function $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a *cellular automaton* (CA) if there exist $r \in \mathbb{N}$ and a local rule f of radius r such that

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad F(x)_i = f(x_{[i-r,i+r]}) \ .$$

The *shift* map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined as $\sigma(x)_i = x_{i+1}, \forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}$ is one among the simplest examples of CA.

Let $\mathcal{R}$ be a set of local rules on A. A *distribution* on $\mathcal{R}$ is an application θ from $\mathbb{Z}$ to $\mathcal{R}$, *i.e.*, a bi-infinite word on $\mathcal{R}$. Denote by Θ the set of all distributions on $\mathcal{R}$. A *non-uniform cellular automaton* (ν-CA) is a triple $(A, \theta, (r_i)_{i\in\mathbb{N}})$ where A is an alphabet, θ a distribution on the set of all possible local rules on A and r_i is the radius of θ_i. A ν-CA defines a global transition function $H_\theta : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad H_\theta(x)_i = \theta_i(x_{[i-r_i,i+r_i]}) \ .$$

In the sequel, when no misunderstanding is possible, we will identify a ν-CA with its global transition function. From [5], recall that a function $H : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is the global transition function of a ν-CA if and only if it is continuous. For all integer k and $H : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, let H^k denote the composition of H with itself k times, *i.e.* for all configuration $x \in A^{\mathbb{Z}}$, $H^0(x) = x$ and for $k > 0$, $H^k(x) = H(H^{k-1}(x))$. In this paper, we will consider distributions on a finite set of local rules. In that case, one can assume without loss of generality that there exists an integer r such that all the rules in $\mathcal{R}$ have the same radius r. All ν-CA constructed on such finite sets of local rules are called rν-CA (of radius r).

A *finite distribution* is a word $\psi \in \mathcal{R}^n$, *i.e.*, a sequence of n rules of $\mathcal{R}$. Each finite distribution ψ defines a function $h_\psi : A^{n+2r} \to A^n$ by

$$\forall u \in A^{n+2r}, \forall i \in [0, n), \quad h_\psi(u)_i = \psi_i(u_{[i,i+2r]}) \ .$$

These functions are called partial transition functions since they express the behavior of a ν-CA on a finite set of sites: if θ is a distribution and $i \leq j$ are integers, then

$$\forall x \in A^{\mathbb{Z}}, \quad H_\theta(x)_{[i,j]} = h_{\theta_{[i,j]}}(x_{[i-r,j+r]}) \ .$$

Languages. Recall that a *language* is any set $\mathcal{L} \subseteq A^*$ and a *finite state automaton* is a tuple $\mathcal{A} = (Q, A, T, I, F)$, where Q is a finite set of states, A is the alphabet, $T \subseteq Q \times A \times Q$ is the set of *transitions*, and $I, F \subseteq Q$ are the sets of *initial* and *final* states, respectively. A *path* p in $\mathcal{A}$ is a finite sequence $q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} q_2 \ldots q_{n-1} \xrightarrow{a_{n-1}} q_n$ visiting the states $q_0, \ldots, q_n$ and with label $a_1 \ldots a_{n-1}$ such that $(q_i, a_i, q_{i+1}) \in T$ for each $i \in [0, n)$. A path is *successful* if $q_0 \in I$ and $q_n \in F$. The language $\mathcal{L}(\mathcal{A})$ recognized by $\mathcal{A}$ is the set of labels of all successful paths in $\mathcal{A}$. A language $\mathcal{L}$ is *rational* if there exists a finite automaton $\mathcal{A}$ such that $\mathcal{L} = \mathcal{L}(\mathcal{A})$.

A bi-infinite language is any subset of $A^{\mathbb{Z}}$. Let $\mathcal{A} = (Q, A, T, I, F)$ be a finite automaton. A bi-infinite path p in $\mathcal{A}$ is a bi-infinite sequence $\ldots \xrightarrow{a_{-2}} q_{-1} \xrightarrow{a_{-1}} q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} \ldots$ such that $(q_i, a_i, q_{i+1}) \in T$ for each $i \in \mathbb{Z}$. The bi-infinite word $\ldots a_{-1}a_0a_1 \ldots$ is the label of the bi-infinite path p. A bi-infinite path is *successful* if the sets $\{i \in \mathbb{N} : q_{-i} \in I\}$ and $\{i \in \mathbb{N} : q_i \in F\}$ are infinite. This condition is known as the *Büchi acceptance condition*. The bi-infinite language $\mathcal{L}^{\zeta}(\mathcal{A})$ recognized by $\mathcal{A}$ is the set of labels of all successful bi-infinite paths in $\mathcal{A}$. A bi-infinite language $\mathcal{L}$ is *ζ–rational* if there exists a finite automaton $\mathcal{A}$ such that $\mathcal{L} = \mathcal{L}^{\zeta}(\mathcal{A})$.

A bi-infinite language X is a *subshift* if X is (topologically) closed and σ–invariant, *i.e.*, $\sigma(X) = X$. Let $\mathcal{F} \subseteq A^*$ and $X_{\mathcal{F}}$ be the bi-infinite language of all bi-infinite words x such that no word $u \in \mathcal{F}$ occurs in x. It is known that a bi-infinite language X is a subshift iff $X = X_{\mathcal{F}}$ for some $\mathcal{F} \subseteq A^*$ [20]. The set $\mathcal{F}$ is a set of *forbidden words* for X. A subshift X is said to be a *subshift of finite type* (resp. *sofic*) iff $X = X_{\mathcal{F}}$ for some finite (resp. rational) $\mathcal{F}$.

For a more in deep introduction to the theory of formal languages, the reader can refer to [16] for rational languages, [3, 20] for subshifts and [22] for ζ-rational languages.

Properties of non-uniform CA. A ν-CA is *sujective* (resp., *injective*) iff its global transition function H is surjective (resp., injective). A ν-CA H is *equicontinuous* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A^{\mathbb{Z}}$, $d(x, y) < \delta$ implies that $\forall n \in \mathbb{N}$, $d(H^n(x), H^n(y)) < \varepsilon$. A ν–CA H is *sensitive to the initial conditions* (or simply *sensitive*) if there exists a constant $\varepsilon > 0$ such that for all element $x \in A^{\mathbb{Z}}$, for all $\delta > 0$ there is a point $y \in A^{\mathbb{Z}}$ such that $d(x, y) < \delta$ and $d(H^n(x), H^n(y)) > \varepsilon$ for some $n \in \mathbb{N}$.

3. Number conservation

In physics, a lot of transformations are conservative: a certain quantity remains invariant along time (conservation laws of mass and energy for example). Both CA and ν-CA are used to represent phenomena from physics and it is therefore interesting to decide when they represent a conservative transformation. The case of uniform CA has been treated in a number of papers, see for instance [4, 11]. Here, we generalize those results to ν-CA. Indeed, we prove that the language of the set of distributions representing number conserving rν-CA is a subshift of finite type (SFT).

In this section, without loss of generality, A is $\{0, 1, \ldots, s-1\}$. Denote by $\underline{0}$ the configuration in which every element is 0. For all configuration $x \in A^{\mathbb{Z}}$, define the *partial charge* of x between the index $-n$ and n as $\mu_n(x) = \sum_{i=-n}^{n} x_i$ and the *global charge* of x as $\mu(x) = \lim_{n \to \infty} \mu_n(x)$. Clearly $\mu(x) = \infty$, if x is not a finite configuration.

Definition 1 (FNC). *A ν-CA H is* number-conserving on finite configurations *(FNC) if for all finite configuration x, $\mu(x) = \mu(H(x))$.*

Remark that if H is FNC then $H(\underline{0}) = \underline{0}$ and, for all finite configuration x, $H(x)$ is a finite configuration.

Definition 2 (NC). *A ν-CA H is said to be* number-conserving *(NC) if both the following conditions hold*

(1) $H(\underline{0}) = \underline{0}$

(2) $\forall x \in A^{\mathbb{Z}} \setminus \{\underline{0}\}, \quad \lim_{n\to\infty} \frac{\mu_n(H(x))}{\mu_n(x)} = 1.$

Remark 1. *Condition* (1) *in Definition 2 is implied by* (2) *for all rν-CA while it is not redundant in the more general case of ν-CA.*

Indeed, for a rν-CA H of radius r, assume that (2) *holds but $H(\underline{0}) \neq \underline{0}$ and let $k \in \mathbb{Z}$ be such that $H(\underline{0})_k \neq 0$. For all integer i, denote by $\delta_i \in A^{\mathbb{Z}}$ the configuration defined as $\forall j \in \mathbb{Z}, (\delta_i)_j = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker function (*i.e., *$\delta_{i,j} = 1$ if $i = j$, 0, otherwise). Clearly, $H(\delta_{k-r-1})_k = H(\delta_{k+r+1})_k = H(\underline{0})_k \neq 0$, and, by condition* (2)*, $1 = \mu(H(\delta_j)) \geq H(\underline{0})_k > 0$ for both $j = k-r-1$ and $j = k+r+1$. Hence, it holds that $H(\delta_{k-r-1}) = H(\delta_{k+r+1}) = \delta_k$ and for the configuration $x = \delta_{k-r-1} + \delta_{k+r+1}$ we get $H(x) = \delta_k$ and so $2 = \mu(x) \neq \mu(H(x)) = 1$, which contradicts* (2)*. Therefore,* (2) $\Rightarrow$ (1)*.*

Consider now the ν-CA H defined on $A = \{0,1\}$ as $\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}$,

$$H(x)_i = \begin{cases} 1 & \text{if } (i=0) \vee (i>0 \wedge x_{[-i+1,i]} = 1^{2i-1}0) \vee (i<0 \wedge x_{[i,-i]} = 01^{-2i}) \\ x_i & \text{otherwise} \end{cases}$$

For the ν-CA H, condition (2) *holds but $H(\underline{0}) \neq \underline{0}$. Therefore, condition* (1) *is not redundant for ν-CA.*

Proposition 1. *Let H be a rν-CA of radius r. Then, H is NC if and only if it is FNC.*

Proof. Assume that H is NC. Since $H(\underline{0}) = \underline{0}$, the images of finite configurations are finite configurations. Then, for all finite configuration $x \neq \underline{0}$, $\frac{\mu(H(x))}{\mu(x)} = \lim_{n\to\infty} \frac{\mu_n(H(x))}{\mu_n(x)} = 1$. Therefore, $\mu(x) = \mu(H(x))$ and H is FNC.

Conversely, suppose that H is not NC. By Remark 1, we can assume that condition (2) does not hold. So, there exists a configuration $x \in A^{\mathbb{Z}} \setminus \{\underline{0}\}$ such that either $M = \limsup_{n\to\infty} \frac{\mu_n(H(x))}{\mu_n(x)} > 1$ or $m = \liminf_{n\to\infty} \frac{\mu_n(H(x))}{\mu_n(x)} < 1$. If x is a finite configuration then $\mu(x) \neq \mu(H(x))$ and, hence, H is not FNC. We now deal with the case in which x is not finite. Assume that $M > 1$ (the proof for $m < 1$ is similar).

$M = \limsup_{n\to\infty} \frac{\mu_n(H(x))}{\mu_n(x)}$ then there exists an increasing sequence $(n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $\lim_{i\to\infty} \frac{\mu_{n_i}(H(x))}{\mu_{n_i}(x)} = M$ and, as $\lim_{i\to\infty} \mu_{n_i}(x) = \infty$, there exists some $j \in \mathbb{N}$ such that $\mu_{n_j}(H(x)) > \mu_{n_j}(x) + 2r(s-1)$. Let $n = n_j$ and y be the finite configuration such that $y_{[-n,n]} = x_{[-n,n]}$ and $\forall i \notin [-n,n], y_i = 0$. We have

$$\mu(H(y)) = \mu_{n+r}(H(y)) \geq \mu_{n-r}(H(y)) = \mu_{n-r}(H(x)) \geq \mu_n(H(x)) - 2r(s-1) \\ > \mu_n(x) = \mu(y).$$

Hence, H is not FNC. $\square$

Remark 2. *The Proposition 1 does not hold in the general case. For example the ν-CA H on $A = \{0,1\}$ defined by $\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, H(x)_{2i} = x_i$ and $H(x)_{2i+1} = 0$ is FNC but not NC. For the configuration x such that $\forall i \in \mathbb{Z}, x_i = 1$ we have $\lim_{n\to\infty} \frac{\mu_n(H(x))}{\mu_n(x)} = \frac{1}{2}$.*

Theorem 2. *Given a finite set of local rules $\mathcal{R}$, let $\mathcal{L} = \{\theta \in \Theta : H_\theta \text{ is NC}\}$. Then, $\mathcal{L}$ is a subshift of finite type.*

Proof. We are going to prove that $\mathcal{L} = X_{\mathcal{F}}$ where

$$\mathcal{F} = \left\{ \psi \in \mathcal{R}^{2r+1} : \exists u \in A^{2r+1}, \psi_{2r}(u) \neq u_0 + \sum_{i=0}^{2r-1} \psi_{i+1}(0^{2r-i}u_{[1,i+1]}) - \psi_i(0^{2r-i}u_{[0,i]}) \right\} .$$

Assume that $\theta \in \mathcal{L}$ and let $j \in \mathbb{Z}$. For all $u \in A^{2r+1}$, let x, y be two finite configurations such that $x_{[j-r,j+r]} = u$ and $y_{[j-r,j+r]} = 0u_{[1,2r]}$. Since H_θ is NC, by Proposition 1, $\mu(H(x)) = \mu(x)$ and $\mu(H(y)) = \mu(y)$, and hence

$$\sum_{i=0}^{2r} \theta_{j+i-2r}(0^{2r-i}u_{[0,i]}) + \sum_{i=1}^{2r} \theta_{j+i}(u_{[i,2r]}0^i) = \sum_{i=0}^{2r} u_i \ , \tag{1}$$

$$\sum_{i=1}^{2r} \theta_{j+i-2r}(0^{2r-i+1}u_{[1,i]}) + \sum_{i=1}^{2r} \theta_{j+i}(u_{[i,2r]}0^i) = \sum_{i=1}^{2r} u_i \ . \tag{2}$$

Subtracting (2) to (1), we obtain

$$\theta_j(u) = u_0 + \sum_{i=1}^{2r} \theta_{j+i-2r}(0^{2r-i+1}u_{[1,i]}) - \sum_{i=0}^{2r-1} \theta_{j+i-2r}(0^{2r-i}u_{[0,i]})$$

which can be rewritten as

$$\theta_j(u) = u_0 + \sum_{i=0}^{2r-1} \theta_{j+i+1-2r}(0^{2r-i}u_{[1,i+1]}) - \theta_{j+i-2r}(0^{2r-i}u_{[0,i]}) \ .$$

Thus, for all $j \in \mathbb{Z}$, $\theta_{[j-2r,j]} \notin \mathcal{F}$, meaning that $\theta \in X_{\mathcal{F}}$. So, $\mathcal{L} \subseteq X_{\mathcal{F}}$

Suppose now that $\theta \in X_{\mathcal{F}}$, *i.e.*, for all integer j, $\theta_{[j-2r,j]} \notin \mathcal{F}$. Taking $u = 0^{2r+1}$, for all j we have

$$\theta_{j+2r}(0^{2r+1}) = 0 + \sum_{i=0}^{2r-1} \theta_{j+i+1}(0^{2r+1}) - \theta_{j+i}(0^{2r+1})$$

which leads to $\theta_j(0^{2r+1}) = 0$. For all finite configuration x, it holds that

$$
\begin{aligned}
\mu(H_\theta(x)) &= \sum_{j\in\mathbb{Z}} H_\theta(x)_j = \sum_{j\in\mathbb{Z}} \theta_j(x_{[j-r,j+r]}) \\
&= \sum_{j\in\mathbb{Z}} \Bigg(x_j + \sum_{i=0}^{2r-1} \theta_{j+i+1-2r}(0^{2r-i}x_{[j-r+1,j-r+i+1]}) - \\
&\qquad - \theta_{j+i-2r}(0^{2r-i}x_{[j-r,j-r+i]}) \Bigg) \\
&= \sum_{j\in\mathbb{Z}} x_j + \sum_{i=0}^{2r-1} \Bigg(\sum_{j\in\mathbb{Z}} \theta_{j+i+1-2r}(0^{2r-i}x_{[j-r+1,j-r+i+1]}) - \\
&\qquad - \sum_{j\in\mathbb{Z}} \theta_{j+i-2r}(0^{2r-i}x_{[j-r,j-r+i]}) \Bigg)
\end{aligned}
$$

Since

$$
\sum_{j\in\mathbb{Z}} \theta_{j+i+1-2r}(0^{2r-i}x_{[j-r+1,j-r+i+1]}) = \sum_{j\in\mathbb{Z}} \theta_{j+i-2r}(0^{2r-i}x_{[j-r,j-r+i]}) \ ,
$$

we obtain

$$
\mu(H_\theta(x)) = \sum_{j\in\mathbb{Z}} H_\theta(x)_j = \sum_{j\in\mathbb{Z}} x_j = \mu(x).
$$

Thus, H_θ is FNC and, by Proposition 1, NC. Hence, $\theta \in \mathcal{L}$. So, $X_{\mathcal{F}} \subseteq \mathcal{L}$. $\square$

The following example shows that number conservation property can sometimes be the result of some kind of "cooperation" between rules that when considered as local rules of a CA might not be number-conserving.

Example 1. *Let $\mathcal{R} = \{f, g, h\}$ where f, g, h are the rules of radius 1 on the alphabet $A = \{0, 1\}$ defined as follows: $\forall x, y, z \in A$*

$$
f(x,y,z) = \begin{cases} 1 & \text{if } y = 1 \text{ and } z = 1 \\ 0 & \text{otherwise} \end{cases}
$$
$$
g(x,y,z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 0, \text{ or } y = 1 \text{ and } z = 0 \\ 0 & \text{otherwise} \end{cases}
$$
$$
h(x,y,z) = \begin{cases} 0 & \text{if } x = 0 \text{ and } y = 0 \\ 1 & \text{otherwise} \ . \end{cases}
$$

Note that f, g, h are the so-called elementary rules 136, 184 and 252, respectively. If we are interested in the language $\mathcal{L}$ of distributions θ on $\mathcal{R}$ such that H_θ is number-conserving, according to the proof of Theorem 2, $\mathcal{L}$ is the subshift of finite type $X_{\mathcal{F}}$ where the set of forbidden words is

$$
\mathcal{F} = \{fff, fgf, fhg, fhh, gff, ggf, ghg, ghh, hff, hgf, hhg, hhh\} ,
$$

Remark that $X_{\mathcal{F}} = X_{\{ff,gf,hh,hg\}}$. Moreover, since ff and hh are forbidden patterns, the rules f and h define CA which are not number-conserving but there exist suitable distributions $\theta \in \Theta$ giving number-conserving ν-CA, namely, all those which do not contain patterns from $\{ff, gf, hh, hg\}$.

4. Surjectivity and injectivity

In standard CA setting, injectivity is a fundamental property which is equivalent to reversibility [15]. It is well-known that it is decidable for one-dimensional CA and undecidable in higher dimensions [2, 17]. Surjectivity is also a dimension sensitive property (*i.e.* decidable in dimension one and undecidable for higher dimensions) and it is a necessary condition for many types of chaotic behaviors.

In this paper, we prove that the language associated with distributions inducing surjective (resp. injective) ν-CA is sofic (resp. ζ-rational). Remark that constructions for surjectivity and injectivity are sensibly different, contrary to what happens for the classical CA when dealing with the decidability of those properties.

Before proceeding to the main results of the section we need some technical lemma and new constructions. We believe that these constructions, inspired by [23], might be of interest in their own and could be of help for proving new results.

Lemma 3. *For any fixed $\theta \in \Theta$, the ν–CA H_θ is surjective if and only if $h_{\theta_{[i,j]}}$ is surjective for all integers i, j with $i \leq j$.*

Proof. Fix $\theta \in \Theta$ and assume now that H_θ is surjective. Let i, j be two integers such that $i \leq j$ and a word $w \in A^{j-i+1}$. Let x be any configuration such that $x_{[i,j]} = w$. By hypothesis, there exists y such that $H_\theta(y) = x$. Then, $h_{\theta_{[i,j]}}(y_{[i-r,j+r]}) = w$ and, hence, $h_{\theta_{[i,j]}}$ is surjective.

As to the converse, suppose that $h_{\theta_{[i,j]}}$ is surjective for all integers i, j with $i \leq j$. Let $x \in A^{\mathbb{Z}}$ and, for all $n \in \mathbb{N}$, define $Y_n = \{y \in A^{\mathbb{Z}} : H_\theta(y)_{[-n,n]} = x_{[-n,n]}\}$. Every Y_n is non-empty by hypothesis and compact as the pre-image of a cylinder by a continuous function. Moreover, $Y_{n+1} \subseteq Y_n$. Thus, $Y = \bigcap_{n \in \mathbb{N}} Y_n \neq \emptyset$ and $H(Y) = \{x\}$. Therefore, H_θ is surjective. □

Definition 3. *Let $\mathcal{R}$ be a finite set of rules of radius r. The* De Bruijn graph *of $\mathcal{R}$ is the labeled multi-edge graph $\mathcal{G} = (V, E)$, where $V = A^{2r}$ and edges in E are all the pairs (aw, wb) with label $(f, f(awb))$, obtained varying $a, b \in A$, $w \in A^{2r-1}$, and $f \in \mathcal{R}$.*

Example 2. *Let $A = \{0, 1\}$ and consider the set $\mathcal{R} = \{\oplus, id\}$ where $\oplus$ and id are the rules of radius 1 defined as $\forall x, y, z \in A$, $\oplus(x, y, z) = (x + z) \bmod 2$, and $id(x, y, z) = y$. The De Bruijn graph of $\mathcal{R}$ is the graph $\mathcal{G}$ in Figure 1.*

Lemma 4. *Let $\mathcal{G}$ be the De Bruijn graph of a finite set of rules $\mathcal{R}$. Consider $\mathcal{G}$ as an automaton where all states are both initial and final. Then, $\mathcal{L}(\mathcal{G}) = \{(\psi, u) \in (\mathcal{R} \times A)^* : h_\psi^{-1}(u) \neq \emptyset\}$.*

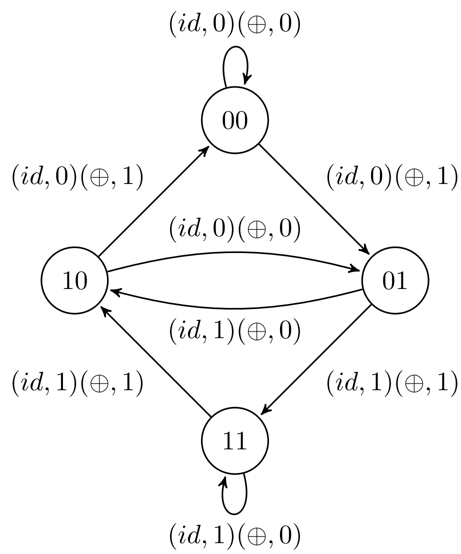

Figure 1: De Bruijn graph of $\mathcal{R} = \{\oplus, id\}$ (every printed edge represents two edges of the graph, labels are concatenated).

Proof. Any $(\psi, u) \in (\mathcal{R} \times A)^n$ is such that $h_\psi^{-1}(u) \neq \emptyset$ iff there exists $w \in A^{n+2r}$ such that $h_\psi(w) = u$. By Definition 3, this happens iff there is a path in $\mathcal{G}$ visiting the states $w_{[0,2r)}, \ldots, w_{[n,n+2r)}$ with labels $(\psi_0, u_0), \ldots, (\psi_{n-1}, u_{n-1})$, *i.e.*, iff $(\psi, u) \in \mathcal{L}(\mathcal{G})$. $\square$

Theorem 5. *Given a finite set of local rules $\mathcal{R}$, let $\mathcal{L} = \{\theta \in \Theta : H_\theta$ is surjective$\}$. Then, $\mathcal{L}$ is a sofic subshift.*

Proof. Let $\mathcal{F} = \{\psi \in \mathcal{R}^* : h_\psi$ is not surjective$\}$. By Lemma 3, $\mathcal{L}$ is just the subshift $X_\mathcal{F}$. Consider the De Bruijn graph $\mathcal{G}$ of $\mathcal{R}$ as an automaton $\mathcal{A}$ where all states are both initial and final. By Lemma 4, $\mathcal{L}(\mathcal{A}) = \{(\psi, u) \in (\mathcal{R} \times A)^* : h_\psi^{-1}(u) \neq \emptyset\}$. Build now the automaton $\mathcal{A}^c$ that recognizes $\mathcal{L}^c = \{(\psi, u) \in (\mathcal{R} \times A)^* : h_\psi^{-1}(u) = \emptyset\}$. Remove from $\mathcal{A}^c$ all second components of edge labels and let $\tilde{\mathcal{A}}$ be the obtained automaton. A word $\psi \in \mathcal{R}^*$ is recognized by $\tilde{\mathcal{A}}$ if and only if there exists $u \in A^*$ such that $(\psi, u) \in \mathcal{L}^c$, *i.e.*, iff h_ψ is not surjective. Thus, $\mathcal{L}(\tilde{\mathcal{A}}) = \mathcal{F}$ and $\mathcal{L} = X_\mathcal{F}$ is a sofic subshift. $\square$

The proof of the Theorem 5 provides an algorithm to build an automaton that recognizes the language $\mathcal{F}$ of the forbidden words for the sofic subshift $\mathcal{L}$. It consists of the following steps

1. Build the De Bruijn graph $\mathcal{G}$ of $\mathcal{R}$.
2. Consider $\mathcal{G}$ as an automaton in which all states are both initial and final and determinize it to obtain the automaton $\mathcal{A}$.

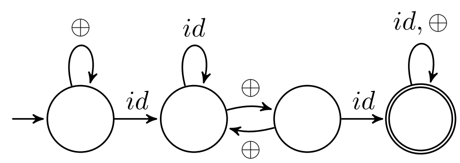

Figure 2: The automaton $\tilde{\mathcal{A}}$ obtained from the set $\mathcal{R} = \{id, \oplus\}$.

3. Complete $\mathcal{A}$ if necessary and make all final states non-final and vice versa to obtain $\mathcal{A}^c$.
4. Delete all second components of edge labels of $\mathcal{A}^c$ to obtain $\tilde{\mathcal{A}}$.

Example 3. *With the set $\mathcal{R}$ from the Example 2 as input, this algorithm gives the automaton in Figure 2. Thus, we deduce that $\mathcal{F} = \mathcal{R}^* id \oplus (\oplus\oplus)^* id \mathcal{R}^*$ and $\mathcal{L}_P$ is the well-known even subshift.*

Definition 4. *Let $\mathcal{R}$ be a finite set of rules of radius r and $\mathcal{G} = (V, E)$ the De Bruijn graph of $\mathcal{R}$. The* product graph $\mathcal{P}$ *of $\mathcal{R}$ is the labeled graph $(V \times V, W)$ where $((u, u'), (v, v')) \in W$ with label $(f, a) \in \mathcal{R} \times A$ if and only if (u, v) and (u', v') belong to E both with the same label (f, a).*

Theorem 6. *Given a finite set of local rules $\mathcal{R}$, let $\mathcal{L} = \{\theta \in \Theta : H_\theta$ is injective$\}$. Then, $\mathcal{L}$ is ζ-rational.*

Proof. Let $\mathcal{P}$ be the product graph of $\mathcal{R}$. Consider now $\mathcal{P}$ as a finite automaton where all the states are initial and the final states are the pairs (u, u') with $u \neq u'$. Remove from $\mathcal{P}$ all second components of edge labels and let $\tilde{\mathcal{P}}$ be the obtained automaton. We said that a bi-infinite path is successful in $\tilde{\mathcal{P}}$ if and only if it visits an accepting state. It is well known that the set of labels of successful paths defines a ζ-rational language [19, 22].

Any bi-infinite path in $\mathcal{P}$ with label $(\theta_i, z_i)_{i \in \mathbb{Z}} \in \Theta \times A^{\mathbb{Z}}$ corresponds to two bi-infinite paths in $\mathcal{G}$ in which the visited vertexes define two configurations x and y such that $H_\theta(x) = H_\theta(y) = z$. Then, a path p in $\tilde{\mathcal{P}}$ labeled by θ defines two configurations x and y such that $H_\theta(x) = H_\theta(y)$. Conversely, a distribution $\theta \in \Theta$ and two configurations x and y such that $H_\theta(x) = H_\theta(y)$ define an unique path p in $\tilde{\mathcal{P}}$. Moreover p visits an accepting state if and only if $x \neq y$. Then the language recognized by $\tilde{\mathcal{P}}$ is the set $\{\theta \in \Theta : \exists x, y \in A^{\mathbb{Z}}, x \neq y$ and $H_\theta(x) = H_\theta(y)\} = \mathcal{L}^c$.

Since the complementary of a ζ-rational language is ζ-rational, $\mathcal{L}$ is ζ-rational. $\square$

Example 4. *Let $\mathcal{R}$ be the set of rules from the Example 2. The graph $\tilde{\mathcal{P}}$ obtained by the product graph $\mathcal{P}$ of $\mathcal{R}$ is shown in Figure 3. According to the proof of Theorem 6, $\tilde{\mathcal{P}}$ is obtained by removing from $\mathcal{P}$ all second components of edge labels.*

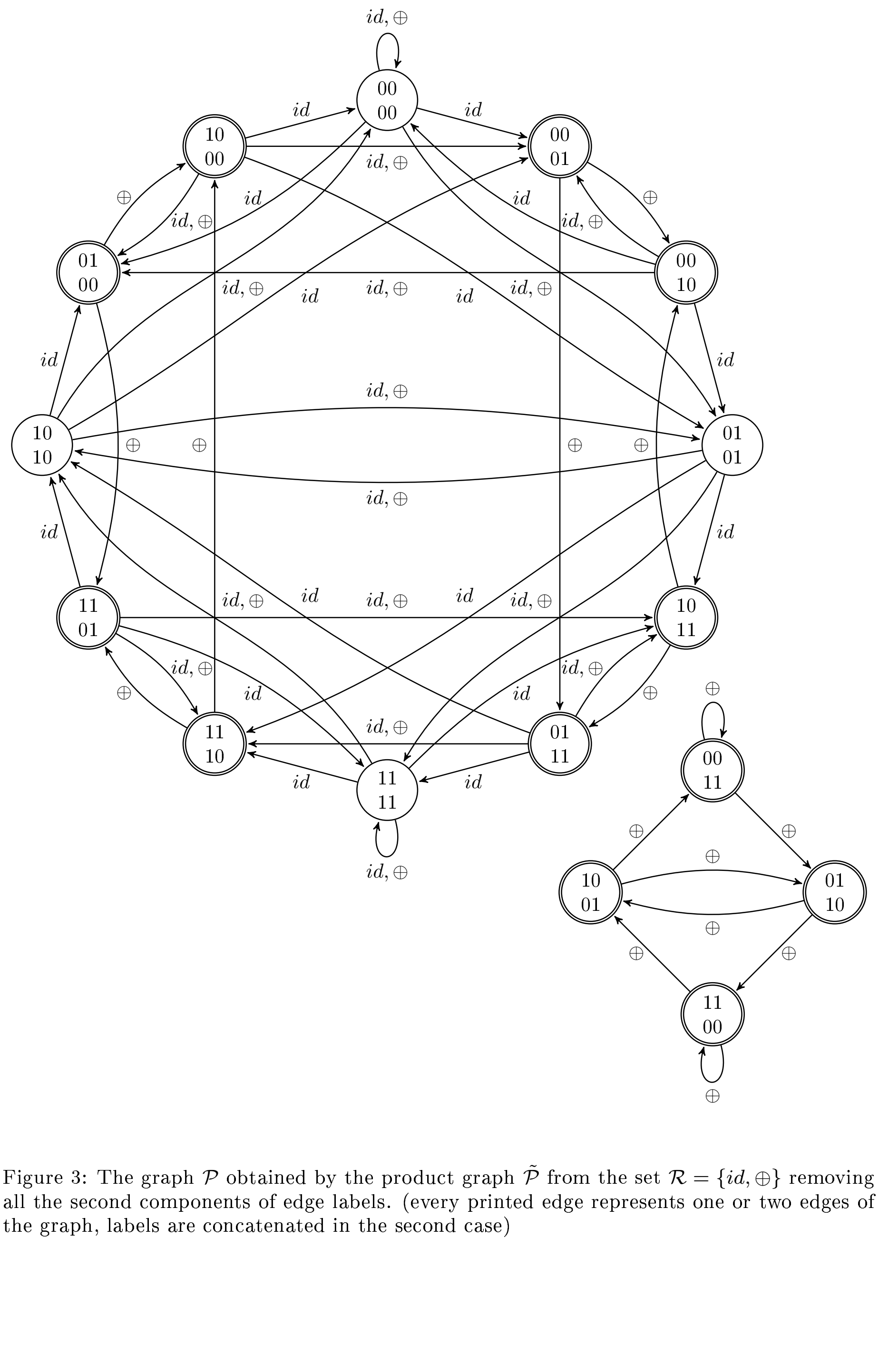

Figure 3: The graph $\mathcal{P}$ obtained by the product graph $\tilde{\mathcal{P}}$ from the set $\mathcal{R} = \{id, \oplus\}$ removing all the second components of edge labels. (every printed edge represents one or two edges of the graph, labels are concatenated in the second case)

5. Equicontinuity and sensitivity for linear rν-CA

Sensitivity to initial conditions is a widely known property indicating a possible chaotic behavior of a dynamical system and it is often popularized under the metaphor of the *butterfly effect*. At the opposite, equicontinuity is an element of stability of a system. In this section, we are going to study these properties in the context of linear ν-CA.

In order to consider linear ν-CA, the alphabet A is endowed with a sum $(+)$ and a product $(\cdot)$ operations that make it a commutative ring and we denote by 0 and 1 the neutral elements of $+$ and $\cdot$, respectively. Of course, A^n and $A^{\mathbb{Z}}$ are also commutative rings where sum and product are defined component-wise and, with some abuse of notation, they will be denoted by the the same symbols.

Remark that in the sequel uv still denote the concatenation of words u and v and u^n the concatenation of u with itself n times. The multiplication will always be denoted by a $\cdot$ or the usual symbol Π.

Definition 5. *A local rule f of radius r is* linear *if and only if there exists a word $\lambda \in A^{2r+1}$ such that $\forall u \in A^{2r+1}, f(u) = \sum_{i=0}^{2r} \lambda_i \cdot u_i$. A ν-CA H is* linear *if it is defined by a distribution of linear local rules.*

Remark 3. *The notion of linearity defined here matches with the usual notion of linearity in linear algebra,* i.e. *a ν-CA H is linear (in our sens) iff for all configurations x and y and for all $a \in A$, $H(a \cdot x + y) = a \cdot H(x) + H(y)$. This is also true for partial transition functions.*

Proposition 7. *Any linear ν-CA H is either sensitive or equicontinuous.*

Proof. For all integers $k \in \mathbb{N}$ and $i \in \mathbb{Z}$, let $\lambda_{i,k}$ be the word expressing the i–th linear local rule of radius $r_{i,k}$ in a family defining the (linear) ν–CA H^k. Without loss of generality, we can assume that either $(\lambda_{i,k})_0 \neq 0$ or $(\lambda_{i,k})_{2r_{i,k}} \neq 0$. Consider the following statement: "for all integer $i \in \mathbb{Z}$ the sequence $(r_{i,k})_{k\in\mathbb{N}}$ is bounded (by some integer $M_i > 0$)". We are going to show that if this statement is true, resp., false, then H is equicontinuous, resp., sensitive.

Assume that the statement is true. Let $n \in \mathbb{N}$ and $m = n + M$ where $M = \max\{M_i : -n \leq i \leq n\}$. Let x and y be two configurations such that $x_{[-m,m]} = y_{[-m,m]}$. We have that $H^k(x)_{[-n,n]} = H^k(y)_{[-n,n]}$, for all integer $k \in \mathbb{N}$. We have shown that for all $\varepsilon = 2^{-n}$, there exists $\delta = 2^{-m}$ such that for all $x, y \in A^{\mathbb{Z}}$, $d(y, x) < \delta$ implies that $\forall k \in \mathbb{N}$, $d(H^k(y), H^k(x)) < \varepsilon$ and, hence, H is equicontinuous.

If the statement is false, there exists $i \in \mathbb{Z}$ such that the sequence $(r_{i,k})_{k\in\mathbb{N}}$ is not bounded. Let $x \in A^{\mathbb{Z}}$, $m \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $r_{i,k} > 2|i| + 1 + m$. Define $y^-, y^+ \in A^{\mathbb{Z}}$ as follows

$$\forall j \in \mathbb{Z}, \quad y^-_j = \begin{cases} 1 & \text{if } j = i - r_{i,k} \\ 0 & \text{otherwise} \end{cases} \quad \text{and } y^+_j = \begin{cases} 1 & \text{if } j = i + r_{i,k} \\ 0 & \text{otherwise} \end{cases} \quad .$$

Then, $x_{[-m,m]} = (x + y^-)_{[-m,m]} = (x + y^+)_{[-m,m]}$ and either $H^k(x)_{[-i,i]} \neq H^k(x+y^-)_{[-i,i]}$ (if $(\lambda_{i,k})_0 \neq 0$) or $H^k(x)_{[-i,i]} \neq H^k(x+y^+)_{[-i,i]}$ (if $(\lambda_{i,k})_{2r_{i,k}} \neq$

Fixed	Application of h_ψ	Fixed
0^r	$0^n = u_\psi(v)_0$	v
0^r	$u_\psi(v)_1$	0^r
0^r	$u_\psi(v)_2$	0^r
$\vdots$	$\vdots$	$\vdots$
0^r	$u_\psi(v)_k$	0^r
$\vdots$	$\vdots$	$\vdots$

Figure 4: The sequence $u_\psi(v)$.

0). We have shown there exists $\varepsilon = 2^{-i}$ such that for all element $x \in A^{\mathbb{Z}}$, for all $\delta = 2^{-m}$ there exists $y \in A^{\mathbb{Z}}$ such that $d(x, y) < \delta$ and $d(H^k(x), H^k(y)) > \varepsilon$ for some $k \in \mathbb{N}$. Thus, H is sensitive with sensitivity constant 2^{-i}. $\square$

Remark 4. *In the non-linear case, there exists ν-CA which are neither sensitive nor equicontinuous [5].*

The previous definition and proposition allow linear ν-CA defined on a possibly infinite set of local rules. However, from now on we consider finite sets $\mathcal{R}$ in which all rules are linear and have radius r.

Definition 6 (Wall). *A* right-wall *is any element $\psi \in \mathcal{R}^*$ of length $n \geq r$ such that, for all word $v \in A^r$, the sequence $u_\psi(v) : \mathbb{N} \to A^n$ recursively defined by*

$$\begin{array}{rcl} u_\psi(v)_0 & = & 0^n \\ u_\psi(v)_1 & = & h_\psi(0^r u_\psi(v)_0 v) \\ u_\psi(v)_{k+1} & = & h_\psi(0^r u_\psi(v)_k 0^r) \text{ for } k > 1 \end{array}$$

verifies $\forall k \in \mathbb{N}, (u_\psi(v)_k)_{[0,r-1]} = 0^r$. Left-walls *are defined similarly.*

Roughly speaking, the sequence $u_\psi(v)$ gives the dynamical evolution of the function h_ψ when the leftmost and rightmost inputs are fixed (see Figure 4). The idea we develop here, in view of Propositon 12, is that a right (resp., left) wall completely filters out the information coming from its right (resp., left) while it may allow information coming from the opposite direction pass through.

Lemma 8. *For all right-wall $\psi \in \mathcal{R}^n$ and any $f \in \mathcal{R}$, $f\psi$ is a right-wall.*

Proof. Let $v \in A^r$. We are going to prove by induction that for all $k \in \mathbb{N}$ $u_{f\psi}(v)_k = 0u_\psi(v)_k$. This is enough to conclude that $f\psi$ is a right-wall.

Clearly, for $k = 0$, it holds that $u_{f\psi}(v)_0 = 0u_\psi(v)_0$. For $k = 1$, we obtain

$$u_{f\psi}(v)_1 = h_{f\psi}(0^{n+r+1}v) = f(0^{2r+1})h_\psi(0^{n+r}v) = 0u_\psi(v)_1 \ .$$

Assume now that $u_{f\psi}(v)_k = 0u_\psi(v)_k$ for $k > 0$. Then,

$$\begin{aligned} u_{f\psi}(v)_{k+1} &= h_{f\psi}(0^r u_{f\psi}(v)_k 0^r) \\ &= h_{f\psi}(0^{r+1} u_\psi(v)_k 0^r) \\ &= f(0^{2r+1}) h_\psi(0^r u_\psi(v)_k 0^r) \\ &= 0u_\psi(v)_{k+1} \ . \end{aligned}$$

□

Lemma 9. *For all right-wall $\psi \in \mathcal{R}^n$ and any $f \in \mathcal{R}$, ψf is a right-wall.*

Proof. Let $v \in A^r$. Denote by α_k the last letter of $u_{\psi f}(v)_k$ and let $\beta_k = \alpha_k 0^{r-1}$ and $\gamma = 0v_{[0,r-2]}$. We are going to prove by induction that for all $k \in \mathbb{N}$

$$u_{\psi f}(v)_k = \left(u_\psi(\gamma)_k + \sum_{i=1}^{k-1} u_\psi(\beta_{k-i})_i \right) \alpha_k \ . \tag{3}$$

This would permit to conclude that, using the fact that ψ is a right-wall,

$$(u_{\psi f}(v)_k)_{[0,r-1]} = (u_\psi(\gamma)_k)_{[0,r-1]} + \sum_{i=1}^{k-1} (u_\psi(\beta_{k-i})_i)_{[0,r-1]} = 0^r \ ,$$

i.e., ψf is a right-wall.

Clearly, for $k = 0$, it holds that $u_{\psi f}(v)_0 = 0^{n+1} = u_\psi(\gamma)_0 \alpha_0$. For $k = 1$, we have

$$u_{\psi f}(v)_1 = h_{\psi f}(0^r u_{\psi f}(v)_0 v) = h_{\psi f}(0^{n+r+1}v) = h_\psi(0^{n+r}\gamma)\alpha_1 = u_\psi(\gamma)_1 \alpha_1 \ .$$

Assume now that (3) holds for $k > 0$. Then,

$$u_{\psi f}(v)_{k+1} = h_{\psi f}(0^r u_{\psi f}(v)_k 0^r) \ . \tag{4}$$

Using the induction hypothesis on $u_{\psi f}(v)_k$, Equation 4 turns into

$$u_{\psi f}(v)_{k+1} = h_{\psi f}\left(0^r \left(u_\psi(\gamma)_k + \sum_{i=1}^{k-1} u_\psi(\beta_{k-i})_i \right) \alpha_k 0^r \right) \ . \tag{5}$$

Now, rewriting the previous equation using the definitions of β_k and α_{k+1}, one finds

$$\begin{aligned} u_{\psi f}(v)_{k+1} &= h_\psi \left(0^r \left(u_\psi(\gamma)_k + \sum_{i=1}^{k-1} u_\psi(\beta_{k-i})_i \right) \beta_k \right) \alpha_{k+1} && (6) \\ &= h_\psi \left(0^r u_\psi(\gamma)_k 0^r + \sum_{i=1}^{k-1} 0^r u_\psi(\beta_{k-i})_i 0^r + 0^{n+r}\beta_k \right) \alpha_{k+1} && (7) \end{aligned}$$

Finally, using the linearity of h_ψ in Equation 7

$$\begin{aligned}
u_{\psi f}(v)_{k+1} &= \Bigg(h_\psi(0^r u_\psi(\gamma)_k 0^r) + \\
&\qquad \sum_{i=1}^{k-1} h_\psi(0^r u_\psi(\beta_{k-i})_i 0^r) + h_\psi(0^{n+r}\beta_k)\Bigg)\alpha_{k+1} \\
&= \left(u_\psi(\gamma)_{k+1} + \sum_{i=1}^{k-1} u_\psi(\beta_{k-i})_{i+1} + u_\psi(\beta_k)_1\right)\alpha_{k+1} \\
&= \left(u_\psi(\gamma)_{k+1} + \sum_{i=1}^{k} u_\psi(\beta_{k+1-i})_i\right)\alpha_{k+1} \quad .
\end{aligned}$$

□

Proposition 10. *If $\psi \in \mathcal{R}^*$ is a right-wall, then $\psi'\psi\psi''$ is a right-wall for all $\psi', \psi'' \in \mathcal{R}^*$.*

Proof. This is a direct consequence of Lemmata 8 and 9. □

Similar results hold for left-walls.

Lemma 11. *Let $\theta \in \Theta$, $n \in \mathbb{Z}$, $m \geq n + r$ and $x \in A^{\mathbb{Z}}$ such that for all $l \leq m$, $x_l = 0$. Denote $\psi = \theta_{[n+1,m]}$ and, for any $i \in \mathbb{N}$, $\alpha_i = H_\theta^i(x)_{[m+1,m+r]}$. Then, the statement*

$$Q(k) = \left[\left(\forall i \in [0,k), H_\theta^i(x)_{[n-r+1,n]} = 0^r\right) \Rightarrow H_\theta^k(x)_{[n+1,m]} = \sum_{j=0}^{k} u_\psi(\alpha_{k-j})_j\right]$$

is true for all integer $k \geq 0$.

Proof. $Q(0)$ is clearly true. Assume that $Q(k)$ is true for an integer $k \in \mathbb{N}$ and suppose that $\forall i \in [0,k]$, $H_\theta^i(x)_{[n-r+1,n]} = 0^r$. Since $H_\theta^k(x)_{[n+1,m]} = \sum_{i=0}^{k} u_\psi(\alpha_{k-i})_i$, we obtain

$$\begin{aligned}
H_\theta^{k+1}(x)_{[n+1,m]} &= h_\psi(H_\theta^k(x)_{[n-r+1,m+r]}) \\
&= h_\psi\left(0^r\left(\sum_{i=0}^{k} u_\psi(\alpha_{k-i})_i\right)\alpha_k\right) \\
&= h_\psi\left(0^{n+2r} + 0^r u_\psi(\alpha_k)_0\alpha_k + \sum_{i=1}^{k} 0^r u_\psi(\alpha_{k-i})_i)0^r\right) \\
&= 0^n + h_\psi\left(0^r u_\psi(\alpha_k)_0\alpha_k\right) + \sum_{i=1}^{k} h_\psi\left(0^r u_\psi(\alpha_{k-i})_i)0^r\right) \quad .
\end{aligned}$$

By linearity of h_ψ, the previous equation becomes

$$\begin{aligned} H_\theta^{k+1}(x)_{[n+1,m]} &= u_\psi(\alpha_{k+1})_0 + u_\psi(\alpha_k)_1 + \sum_{i=1}^{k} u_\psi(\alpha_{k-i})_{i+1} \\ &= \sum_{i=0}^{k+1} u_\psi(\alpha_{k+1-i})_i \quad . \end{aligned}$$

Hence, $Q(k+1)$ is true. □

Proposition 12. *Let $\theta \in \Theta$, H_θ is sensitive if and only if one of the two following conditions holds.*

1. *There exists $n \in \mathbb{N}$ such that for all integer $m \geq n + r$, $\theta_{[n+1,m]}$ is not a right-wall.*
2. *There exists $n \in \mathbb{N}$ such that for all integer $m \leq -n - r$, $\theta_{[m,-n-1]}$ is not a left-wall.*

Proof. Suppose that condition 1. holds (the proof with 2. as assumption is similar). Let $m \geq n + r$. Since $\psi := \theta_{[n+1,m]}$ is not a right-wall there exists $v \in A^r$ and $k > 0$ such that $(u_\psi(v)_k)_{[0,r-1]} \neq 0^r$. Let v be such that k is minimal. Let x be the configuration such that $x_{[m+1,m+r]} = v$ and $x_i = 0$ for $i \notin [m+1, m+r]$. Let $\alpha_i = H_\theta^i(x)_{[m+1,m+r]}$. We are going to prove that for all $i \in [0, k]$, the statement $S(i) = \big(\forall l \in \mathbb{Z}, l \leq n \Rightarrow H_\theta^i(x)_l = 0\big)$ is true.

> $S(0)$ is clearly true. For an arbitrary $i \in [0, k-1]$, assume that $S(j)$ holds $\forall j \in [0, i]$. By Lemma 11, $H_\theta^i(x)_{[n+1,m]} = \sum_{j=0}^{i} u_\psi(\alpha_{i-j})_j$ and then, by minimality of k, it holds that
>
> $$H_\theta^i(x)_{[n+1,n+r]} = \sum_{j=0}^{i} (u_\psi(\alpha_{i-j})_j)_{[0,r-1]} = 0^r.$$
>
> Hence, for all integers $l \leq n + r$, $H_\theta^i(x)_l = 0$ and so, for all integers $l \leq n$, $H_\theta^{i+1}(x)_l = 0$, *i.e.*, $S(i+1)$ is true.

Since $S(i)$ is true for all $i \in [0, k]$, again by Lemma 11 and minimality of k, we obtain

$$H_\theta^k(x)_{[n+1,n+r]} = \sum_{j=0}^{k} (u_\psi(\alpha_{k-j})_j)_{[0,r-1]} = (u_\psi(v)_k)_{[0,r-1]} \neq 0^r \quad .$$

Thus, for all configuration y, we have $y_{[-m,m]} = (x + y)_{[-m,m]}$ but

$$H_\theta^k(y)_{[-n-r,n+r]} \neq H_\theta^k(x)_{[-n-r,n+r]} + H_\theta^k(y)_{[-n-r,n+r]} = H_\theta^k(x+y)_{[-n-r,n+r]},$$

which means that H_θ is sensitive with sensitivity constant 2^{-n-r}.

As to the converse, assume now that neither condition 1. nor 2. holds and let us prove that H_θ is equicontinuous. Let $n \in \mathbb{N}$, there exists $m_1 \geq n + r$ and $m_2 \leq -n-r$ such that $\theta_{[n+1,m_1]}$ is a right-wall and $\theta_{[m_2,-n-1]}$ is a left-wall. Let $m = \max(m_1, -m_2)$. By Proposition 10, $\theta_{[n+1,m]}$ is a right-wall and $\theta_{[-m,-n-1]}$ is a left-wall. For any configuration z, let z^-, $\tilde{z}$ and z^+ denote the configurations such that $z^-_i = z_i$ for $i < -m$, 0 otherwise; $\tilde{z}_i = z_i$ for $i \in [-m, m]$, 0 otherwise; $z^+_i = z_i$ for $i > m$, 0 otherwise. Let z be a configuration, we now prove that $\forall k \in \mathbb{N}$, the statement $S'(k) = \big(\forall j \leq n, H^k_\theta(z^+)_j = 0\big)$ is true.

> Clearly $S'(0)$ is true. For an arbitrary $k \in \mathbb{N}$, assume that $\forall i \in [0, k]$, $S'(i)$ holds. Let $\psi = \theta_{[n+1,m]}$ and $\alpha_i = H^i_\theta(z^+)_{[m+1,m+r]}$. By Lemma 11, $H^k_\theta(z^+)_{[n+1,m]} = \sum_{i=0}^{k} u_\psi(\alpha_{k-i})_i$ and, since ψ is a right-wall, we obtain
>
> $$H^k_\theta(z^+)_{[n+1,n+r]} = \sum_{i=0}^{k} (u_\psi(\alpha_{k-i})_i)_{[0,r-1]} = 0^r \ .$$
>
> Therefore, for all integers $j \leq n + r$, $H^k_\theta(z^+)_j = 0$ and so $\forall j \leq n$, $H^{k+1}_\theta(z^+)_j = 0$, *i.e.*, $S'(k + 1)$ holds.

Similarly, $\forall k \in \mathbb{N}$, the following statement holds: $\forall j \geq -n$, $H^k_\theta(z^-)_j = 0$. To conclude, let x, y be two arbitrary configurations such that $y_{[-m,m]} = x_{[-m,m]}$. Then, since both the above statements S and S' are true, it holds that $\forall k \in \mathbb{N}$

$$\begin{aligned}
H^k_\theta(y)_{[-n,n]} &= H^k_\theta(y^-)_{[-n,n]} + H^k_\theta(\tilde{y})_{[-n,n]} + H^k_\theta(y^+)_{[-n,n]} \\
&= 0^{2n+1} + H^k_\theta(\tilde{x})_{[-n,n]} + 0^{2n+1} \\
&= H^k_\theta(x^-)_{[-n,n]} + H^k_\theta(\tilde{x})_{[-n,n]} + H^k_\theta(x^+)_{[-n,n]} \\
&= H^k_\theta(x)_{[-n,n]} \ .
\end{aligned}$$

Thus, H_θ is equicontinuous and, by Proposition 7, it is not sensitive. □

In the following results of this section, we assume that $\mathcal{R}$ is a finite set of linear rules of radius 1. In this case, any rule $f \in \mathcal{R}$ will be expressed in the following form: $\forall a, b, c \in A, f(a, b, c) = \lambda^-_f \cdot a + \tilde{\lambda}_f \cdot b + \lambda^+_f \cdot c$ for some $\lambda^-_f, \tilde{\lambda}_f, \lambda^+_f \in A$.

Proposition 13. *A finite distribution $\psi \in \mathcal{R}^n$ is a right-wall (resp., left-wall) if and only if $\prod_{i=0}^{n-1} \lambda^+_{\psi_i} = 0$ (resp., $\prod_{i=0}^{n-1} \lambda^-_{\psi_i} = 0$).*

Proof. Assume that $\prod_{i=0}^{n-1} \lambda^+_{\psi_i} = 0$ and let $v \in A$. We prove that $\forall k \in \mathbb{N}$ the statement

$$S(k) = \left(\forall i \in [0, n), \exists \alpha_i \in A, (u_\psi(v)_k)_i = \alpha_i \cdot \prod_{j=i}^{n-1} \lambda^+_{\psi_j}\right)$$

is true, that immediately implies that for all $k \in \mathbb{N}$, $(u_\psi(v)_k)_0 = 0$, *i.e.*, and ψ is a right-wall. We proceed by induction. Taking $\alpha_i = 0$ for all $i \in [0, n)$, we have that $S(0)$ is true. Assume now that $S(k)$ is true for $k \in \mathbb{N}$. With $i = 0$ we can write

$$(u_\psi(v)_{k+1})_0 = \psi_0(0(u_\psi(v)_k)_{[0,1]}) = \tilde{\lambda}_{\psi_0} \cdot \alpha_0 \cdot \prod_{j=0}^{n-1} \lambda^+_{\psi_j} + \lambda^+_{\psi_0} \cdot \alpha_1 \cdot \prod_{j=1}^{n-1} \lambda^+_{\psi_j}$$
$$= \left(\tilde{\lambda}_{\psi_0} \cdot \alpha_0 + \alpha_1\right) \cdot \prod_{j=0}^{n-1} \lambda^+_{\psi_j} \ .$$

For all integer $i \in [1, n-2]$, we obtain

$$\begin{aligned}(u_\psi(v)_{k+1})_i &= \psi_i((u_\psi(v)_k)_{[i-1,i+1]}) \\ &= \lambda^-_{\psi_i} \cdot \alpha_{i-1} \cdot \prod_{j=i-1}^{n-1} \lambda^+_{\psi_j} + \tilde{\lambda}_{\psi_i} \cdot \alpha_i \cdot \prod_{j=i}^{n-1} \lambda^+_{\psi_j} + \lambda^+_{\psi_i} \cdot \alpha_{i+1} \cdot \prod_{j=i+1}^{n-1} \lambda^+_{\psi_j} \\ &= \left(\lambda^-_{\psi_i} \cdot \lambda^+_{\psi_{i-1}} \cdot \alpha_{i-1} + \tilde{\lambda}_{\psi_i} \cdot \alpha_i + \alpha_{i+1}\right) \cdot \prod_{j=i}^{n-1} \lambda^+_{\psi_j} \ .\end{aligned}$$

while for $i = n-1$, we have

$$\begin{aligned}(u_\psi(v)_{k+1})_{n-1} &= \psi_{n-1}((u_\psi(v)_k)_{[n-2,n-1]}\beta) \\ &= \lambda^-_{\psi_{n-1}} \cdot \alpha_{n-2} \cdot \lambda^+_{\psi_{n-2}} \cdot \lambda^+_{\psi_{n-1}} + \tilde{\lambda}_{\psi_{n-1}} \cdot \alpha_{n-1} \cdot \lambda^+_{\psi_{n-1}} + \lambda^+_{\psi_{n-1}} \cdot \beta \\ &= \left(\lambda^-_{\psi_{n-1}} \cdot \lambda^+_{\psi_{n-2}} \cdot \alpha_{n-2} + \tilde{\lambda}_{\psi_{n-1}} \cdot \alpha_{n-1} + \beta\right) \cdot \lambda^+_{\psi_{n-1}}\end{aligned}$$

where $\beta = v$ if $k = 1$, and $\beta = 0$ otherwise. Hence, $S(k+1)$ holds.

Concerning the converse, assume now that $\prod_{i=0}^{n-1} \lambda^+_{\psi_i} \neq 0$. It is easy to see that for all $k \in [1, n]$, $(u_\psi(1)_k)_{n-k} = \prod_{i=n-k}^{n-1} \lambda^+_{\psi_i}$. Hence, $(u_\psi(1)_n)_0 = \prod_{i=0}^{n-1} \lambda^+_{\psi_i} \neq 0$ and ψ is not a right-wall. The proof for left-walls is similar. □

For any set $\mathcal{R}$ of linear rules of radius $r = 1$, a finite automaton $\mathcal{A} = (Q, Z, T, I, F)$ recognizing walls can be constructed as follows. The alphabet Z is $\mathcal{R}$, the set of states Q is $\{-, +\} \times A$, $I = \{(-, 0)\}$, $F = \{(+, 0)\}$ and the set T of transitions is as follows

1. $((-, a), f, (-, \lambda^-_f \cdot a))$, $\forall a \in A \setminus \{0\}, \forall f \in \mathcal{R}$ (minimal left-wall detection).
2. $((-, 0), f, (-, 1))$, $\forall f \in \mathcal{R}$ (end of detection).
3. $((-, 1), f, (-, 1))$, $\forall f \in \mathcal{R}$ (waiting).
4. $((-, 1), f, (+, 1))$, $\forall f \in \mathcal{R}$ (transition from left part to right part).
5. $((+, 1), f, (+, 1))$, $\forall f \in \mathcal{R}$ (waiting).
6. $((+, 1), f, (+, 0))$, $\forall f \in \mathcal{R}$ (beginning of detection).
7. $((+, \lambda^+_f \cdot a), f, (+, a))$, $\forall a \in A \setminus \{0\}, \forall f \in \mathcal{R}$ (minimal right-wall detection).

Practically speaking, $\mathcal{A}$ consists of two components, the left and the right part, with a non-deterministic transition from left to right. Each component has

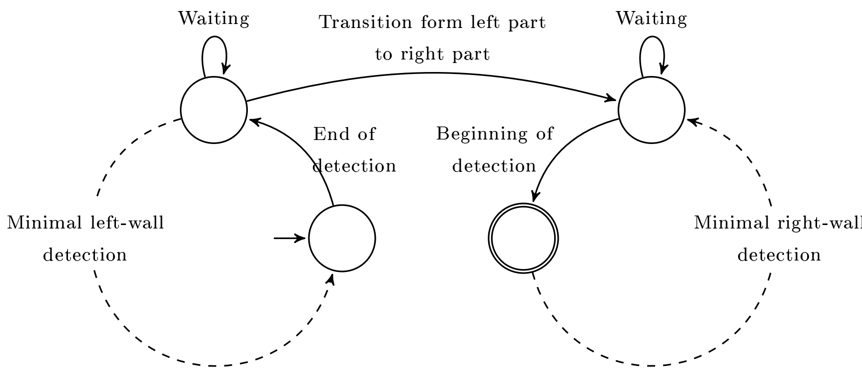

Figure 5: Conceptual structure of the automaton $\mathcal{A}$ for walls detection.

two special states: the first one (the state $(-,1)$ for the left part or $(+,1)$ for the right part) on which $\mathcal{A}$ loops waiting for the detection of a minimal (*w.r.t.* the length) wall, the second one on which $\mathcal{A}$ starts ($(+,0)$ for the right part) or ends ($(-,0)$ for the left part) the detection of such a wall. The graph structure of $\mathcal{A}$ is schematized in Figure 5.

Theorem 14. *Given a finite set of linear local rules $\mathcal{R}$ of radius $r=1$, let $\mathcal{L}=\{\theta\in\Theta : H_\theta$ is equicontinuous$\}$ and $\mathcal{L}'=\{\theta\in\Theta : H_\theta$ is sensitive$\}$. Then $\mathcal{L}$ and $\mathcal{L}'$ are ζ-rational languages.*

Proof. We are going to prove that $\mathcal{L}^\zeta(\mathcal{A})=\mathcal{L}$ where $\mathcal{A}$ is the automaton above introduced for the set $\mathcal{R}$. This permits to immediately state that $\mathcal{L}$ is ζ-rational, and that, by Proposition 7, $\mathcal{L}'$ is ζ-rational too.

Let $\theta\in\mathcal{L}^\zeta(\mathcal{A})$. We show that for all $n\in\mathbb{N}$, there exists $m\leq -n-1$ such that $\theta_{[m,-n-1]}$ is a left-wall. Let $n\in\mathbb{N}$. There is a successful path $p=\ldots\xrightarrow{\theta_{-1}}(s_0,a_0)\xrightarrow{\theta_0}(s_1,a_1)\ldots$ in $\mathcal{A}$ and integers i,j with $i<j<-n$ such that $(s_i,a_i)=(s_j,a_j)=(-,0)$ are two successive initial states. Let $m\in(i,j)$ be the greatest integer such that $(s_m,a_m)=(-,1)$ (m exists because $(s_{i+1},a_{i+1})=(-,1)$), the finite path $(s_m,a_m)\xrightarrow{\theta_m}(s_{m+1},a_{m+1})\xrightarrow{\theta_{m+1}}\ldots\xrightarrow{\theta_{j-1}}(s_j,a_j)$ is obtained by transitions of $\mathcal{A}$ from item 1.. Then, $0=a_j=a_m.\prod_{l=m}^{j-1}\lambda^-_{\theta_l}$, and, by Proposition 13, $\theta_{[m,j-1]}$ is a left-wall. By Proposition 10, $\theta_{[m,-n-1]}$ is a left-wall too. Similarly, it holds that for all $n\in\mathbb{N}$, there exists $m\geq n+1$ such that $\theta_{[n+1,m]}$ is a right-wall. Hence, by Propositions 12, H_θ is equicontinuous, *i.e.*, $\theta\in\mathcal{L}$. Therefore, $\mathcal{L}^\zeta(\mathcal{A})\subseteq\mathcal{L}$.

Let $\theta\in\mathcal{L}$. By Proposition 12, the sequence $(i_k)_{k\in\mathbb{Z}}$ such that $i_0=0$ and

$$\forall k\leq 0, i_{k-1}=\max\{j\in\mathbb{Z} : j<i_k \text{ and } \theta_{[j,i_k-2]} \text{ is a left-wall}\}$$

$$\forall k\geq 0, i_{k+1}=\min\{j\in\mathbb{Z} : j>i_k \text{ and } \theta_{[i_k+2,j]} \text{ is a right-wall}\}$$

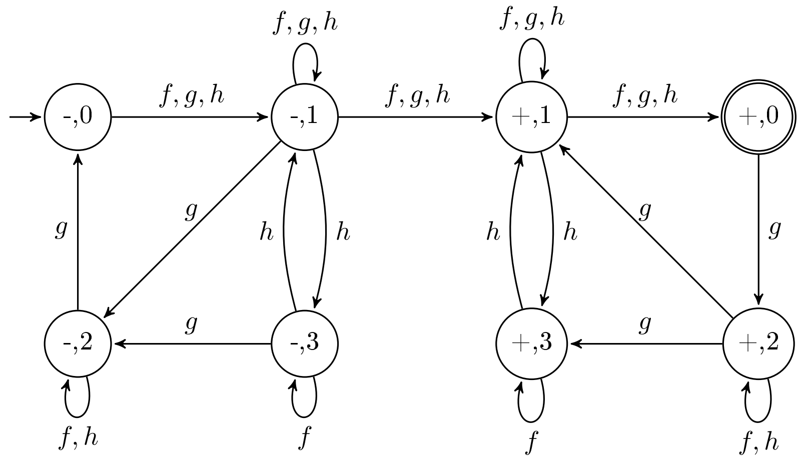

Figure 6: The automaton which recognizes the distributions over the rules $\{f, g, h\}$ inducing equicontinuous ν-CA (see Example 5).

is well-defined. For all $k < 0$, $\theta_{[i_k,i_{k+1}-2]}$ is a left-wall and then $\prod_{j=i_k}^{i_{k+1}-2} \lambda^-_{\theta_j} = 0$. So, for all $k < 0$, setting $n = \min\{l \in \mathbb{Z} : \prod_{j=i_k}^{l} \lambda^-_{\theta_j} = 0\}$,

$$p_k = (-,1) \xrightarrow{\theta_{i_k}} (-,\lambda^-_{\theta_{i_k}}) \xrightarrow{\theta_{i_k+1}} \dots \xrightarrow{\theta_n} (-,\prod_{j=i_k}^{n} \lambda^-_{\theta_j}) \xrightarrow{\theta_{n+1}} (-,1) \xrightarrow{\theta_{n+2}} \dots \xrightarrow{\theta_{i_{k+1}-1}} (-,1)$$

is a finite path in $\mathcal{A}$ from $(-,1)$ to $(-,1)$ with label $\theta_{[i_k,i_{k+1}-1]}$ which visits an initial state. Similarly, for all $k \geq 0$, there exists a finite path p_k in $\mathcal{A}$ from $(+,1)$ to $(+,1)$ with label $\theta_{[i_k+1,i_{k+1}]}$ which visits a final state. Then, $p = (p_k)_{k\in\mathbb{N}}$ is a successful bi-infinite path in $\mathcal{A}$ with label θ. Hence, $\theta \in \mathcal{L}^\zeta(\mathcal{A})$ and so $\mathcal{L} \subseteq \mathcal{L}^\zeta(\mathcal{A})$. □

Example 5. *Let $A = \{0, 1, 2, 3\}$ and $\mathcal{R} = \{f, g, h\}$, where f, g, h are the rules defined by*

$$\begin{array}{rrcll} \forall x,y,z \in A, & f(x,y,z) & = & x+z & \pmod 4 \\ & g(x,y,z) & = & 2\cdot(x+z) & \pmod 4 \\ & h(x,y,z) & = & 3\cdot(x+z) & \pmod 4 \end{array}$$

The automaton which recognizes the distributions inducing equicontinuous ν-CA is depicted on Figure 6. Due to the symmetry of the rules in $\mathcal{R}$, both the left and right walls are the finite distributions in $\mathcal{R}^*g\mathcal{R}^*g\mathcal{R}^*$, i.e. *the finite distributions containing at least two occurrences of the rule g.*

Remark 5. *Remark that the automaton $\mathcal{A}$, built in Theorem 14 to recognize distributions of equicontinous additive (radius 1) ν-CA has, in general, a huge number of states. However, it is possible to greatly reduce the number of states by* *considering the relation $\sim$ on A defined by $a \sim b$ if and only if there exists an invertible element c of A such that $a = b.c$. This is clearly an equivalence relation. Moreover the relation $\sim$ is compatible with the addition and the multiplication on A,* i.e., *for all $a, b, c \in A, a \sim b \Rightarrow a + c \sim b + c$ and $a.c \sim b.c$. Let $[a]$ denote the equivalence class of a and $A|_\sim$ the set of all equivalence classes. For $f \in \mathcal{R}$, let $[f]$ be the local rule of radius 1 on $A|_\sim$ defined by $[f]([x], [y], [z]) = [f(x, y, z)]$ and let $\mathcal{R}_\sim$ be the set of all those local rules. If ψ is a finite distribution on $\mathcal{R}$, $[\psi]$ denotes the finite distribution on $\mathcal{R}_\sim$ such that $|[\psi]| = |\psi|$ and for all integer i, $0 \leq i < |\psi|$, $[\psi]_i = [\psi_i]$. Similar notation is used for distributions. Consider now the automaton $\mathcal{A}'$ which recognizes the distributions inducing equicontinuous ν-CA on $\mathcal{R}_\sim$. Since $[0] = \{0\}$ and $\sim$ is compatible with multiplication, by Proposition 13, a finite distribution ψ on $\mathcal{R}$ is a left-wall (resp. a right-wall) if and only if $[\psi]$ is a left-wall (resp. a right-wall). Then, θ is recognized by $\mathcal{A}$ if and only if $[\theta]$ is recognized by $\mathcal{A}'$. Looking back at Example 6, the above remark means that $(-, 1) \sim (-, 3)$ and $(+, 1) \sim (+, 3)$.*

The following example witnesses the usefulness of the previous construction.

Example 6. *Let $A = \mathbb{Z}/2^n\mathbb{Z}$ for some integer $n > 0$ and $\mathcal{R}$ be some set of linear local rules of radius 1. Then, $\mathcal{A}$ has 2^{n+1} states but, using the previous remark, one finds $A|_\sim = \{[0], [1], [2], [2^2], [2^3], \ldots, [2^{n-1}]\}$ and hence $\mathcal{A}'$ has $2(n + 1)$ states. Indeed, for all integer $k \in [0, 2^n - 1]$, $k = 2^i k'$ for some $i \in [0, n]$ and some odd integer k'. Since k' is odd, it is invertible and $k \in [2^i]$. In other words, for all integers i and j such that $0 \leq i < j \leq n$, 2^i and 2^j are in different equivalence classes, otherwise we could find some k such that $2^i = 2^j k$ and multiplying by 2^{n-j}, we get $2^{n-j+i} = 0$ which is false.*

6. Conclusions

This paper investigates the complexity classes associated to languages characterizing distributions of local rules in ν-CA. Several interesting research directions should be explored.

First, we have proved that the language associated with distributions of equicontinuous or sensitive ν-CA is ζ-rational for the class of linear ν-CA with radius 1. It would be interesting to extend this result to sets of local rule distributions with higher radius. This seems quite difficult because this problem reduces to the study of the equicontinuity of ν-CA of radius 1 on a non-commutative ring, loosing in this way "handy" results like Proposition 13.

Second, there is no complexity gap between sets of distributions which give injective and sensitive (plus the previously mentioned constraints) ν-CA. This is contrary to intuition. Indeed, we suspect that the characterization of distributions giving injective ν-CA might be strengthened to deterministic ζ-rational languages.

As a third research direction, it would be interesting to study which dynamical property of ν-CA is associated with languages of complexity higher than

ζ-rational. We believe that sensitivity to initial conditions (with no further constraints) is a good candidate.

A further research direction would diverge from ν-CA and investigate the topological structure of languages as the one given the previous sections which use some non-standard acceptance condition for finite automata in the vein of [21]. The authors have just started investigating this last subject.

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