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# Breaking the Limits – Redefining the Instantaneous Frequency

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## Abstract

The Carson and Fry (1937) introduced the concept of variable frequency, required to the theory of frequency modulation (FM), as a generalization of the definition of constant frequency. Moreover, the nonstationary nature of the signals and nonlinear systems require the idea of instantaneous frequency (IF). It is also the basis of the time-frequency-energy (TFE) analysis of a signal. The IF is the time derivative of the instantaneous phase and it is well-defined only when this derivative is positive. If this derivative is negative, the IF creates problem because it does not provide any physical significance. This study proposes a mathematical solution and eliminate this problem by redefining the definition of IF such that it is valid for all monocomponent and multicomponent signals which can be nonlinear and nonstationary in nature. This is achieved by using the property of the *multivalued* inverse tangent function. The efforts and understating of all the methods, which are using the IF, would improve significantly by using this definition. Simulations and numerical results demonstrate the efficacy and superiority of the proposed method, for the TFE analysis of a signal, as compared to other existing methods in the literature.

## INTRODUCTION

The time-domain representation and the frequency-domain representation are two classical representations of a signal. In both domains, the time ( $t$ ) and frequency ( $f$ ) variables are mutually exclusive [2]. The Time-Frequency Distributions (TFD) representation on the other hand, provides localized signal information in time and frequency domain. The TFD provides insight into the complex structure of a signal consisting of several components. There exist many types of Time-Frequency (TF) analysis methods such as short-time Fourier transform, Gabor transform, Wavelet transforms, Wigner-Ville distribution.

The instantaneous frequency (IF) is the basis of the TFD or time-frequency-energy (TFE) analysis of a sig-

nal. The IF is a practically important parameter of a signal which can reveal the underlying process and provides explanations for physical phenomenon in many applications such as vibration, acoustic, speech signal analysis [13], meteorological and atmospheric applications [1], seismic [13], radar, sonar, solar physics, structural engineering, communications, health monitoring, biomedical and medical applications [14], cosmological gravity wave and financial market data analysis.

The IF is the time derivative of the instantaneous phase and it is well-defined only when this derivative is positive. If this derivative is negative, it creates problem because it does not provide any physical significance. In order to avoid this problem, recently many nonlinear and nonstationary signal representation, decomposition and analysis methods, e.g. empirical mode decomposition (EMD) algorithms [1, 3–8], synchrosqueezed wavelet transforms (SSWT) [9], variational mode decomposition (VMD) [10], eigenvalue decomposition (EVD) [11] and Fourier decomposition methods (FDM) [12, 13, 15, 16, 18] based on the Fourier theory, are proposed. The Fourier theory is the only tool for spectrum analysis of a signal and the FDM has established that it is a superior tool for nonlinear and nonstationary time series analysis. The main objective of all these methods is to obtain the signal representation such that the IF of a signal under study is always positive.

Unlike these decomposition methods, the method proposed in this study does not necessitate to decompose the signal into a set of narrow band components, which satisfy certain properties, to generate the TFE distribution of any signal. That is, without any decomposition we can generate TFE distribution of a signal. It also provide freedom and potential to decompose the signal into desired (preferably orthogonal) bands by a (preferably zero-phase) filtering approach to obtain TFE distribution of a signal. All these features are obtained by redefining the IF when it is negative and thus defining the IF for both monocomponent and multicomponent signals. In order to redefine the IF, we use the fact that inverse tangent is a *multivalued* (i.e. one-to-many mapping) function. So defined IF of

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a signal is always positive and valid for any signal.

## METHODS

The concept of variable frequency with application to the theory of frequency modulation (FM) is proposed in [19], and it is argued that the notion of IF is a generalization of the definition of constant frequency. In order to formulate a definition of the IF by analyzing an expression for simple harmonic motion (SHM) is considered in [20] as

$$x_{\text{SHM}}(t) = a \cos \left[ \int_0^t 2\pi f(t) dt + \theta \right] = a \cos(\psi(t)), \quad (1)$$

where the argument of the cosine function is the phase  $\psi(t) = [\int_0^t 2\pi f(t) dt + \theta]$ . This leads to the definition of instantaneous frequency

$$f(t) = \frac{1}{2\pi} \frac{d\psi(t)}{dt}. \quad (2)$$

The concept of instantaneous frequency was enhanced in [21] where a method for generating a unique complex signal  $z(t)$  from a real signal  $x(t)$  and its Hilbert transform  $\hat{x}(t)$  was proposed. This method obtains

$$z(t) = x(t) + j\hat{x}(t) = a(t)e^{j\phi(t)}, \quad (3)$$

where

$$\left. \begin{aligned} a(t) &= [x^2(t) + \hat{x}^2(t)]^{1/2} \geq 0, \\ \phi(t) &= \tan^{-1}[\hat{x}(t)/x(t)] \end{aligned} \right\} \quad (4)$$

and  $z(t)$  is the Gabor's complex signal (well-known as the analytic signal) and  $\hat{x}(t)$  is the Hilbert Transform (HT) of  $x(t)$ , defined as

$$\hat{x}(t) = \text{p.v.} \int_{-\infty}^{\infty} \frac{x(\tau)}{\pi(t-\tau)} d\tau = \text{p.v.} \left[ x(t) * \frac{1}{\pi(t)} \right], \quad (5)$$

where p.v. denotes the Cauchy principal value of the integral [22] and  $*$  denotes convolution operation. The work done in [19] and [21] was unified in [23] to define the IF of a signal  $x(t) = a(t) \cos(\phi(t))$  as

$$\omega(t) = 2\pi f(t) = \frac{d}{dt} (\arg[z(t)]), \quad (6)$$

where  $z(t)$  is the analytic signal given by (3) and  $\arg[z(t)] = \phi(t)$  as defined in (4). The IF, as defined in (6), provides physical meaning only when it is positive and it becomes meaningless when it is negative [1, 3–5, 7, 9–13, 24, 25]. This is where we improve and provide the solution to make the IF positive for all time  $t$ . From (4) and (6), one can obtain the TFE distribution by 3-D plot of  $\{t, f(t), a^2(t)\}$ . However, it is to be noted that the frequencies, of signal  $x(t) = a(t) \cos(\phi(t))$ , obtained by the Fourier transform and Hilbert spectrum (6) are same only if  $a(t)$  is a constant, otherwise Hilbert spectrum gives frequencies of signal  $\cos(\phi(t))$  and not of signal  $x(t)$ .

Before evaluating the time derivative of phase in (6), phase unwrapping is necessary to ensure that all appropriate multiples of  $2\pi$  have been included in phase  $\phi(t)$ . Phase unwrap operation corrects the radian phase angles by adding multiples of  $\pm 2\pi$  when absolute jumps between consecutive elements of a phase vector are greater than or equal to the default jump tolerance of  $\pi$  radians [26]. This is being done for phase delay and IF determination by all the methods available in literature.

As is well-known that the tangent is a surjective (many-to-one mapping) function. The domain, range and period of  $\tan(x)$  are  $\{x|x \neq \frac{\pi}{2} + n\pi, \forall n \in \mathbb{Z}\}$ , all real numbers  $\mathbb{R} = (-\infty, \infty)$  and  $\pi$ , respectively. The inverse tangent is the *multivalued* function. The domain of  $\tan^{-1}(x)$  is all real numbers,  $\mathbb{R}$ , and range is  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . If  $z = x + jy$ , then the range of  $\tan^{-1}(y/x)$  is  $(-\pi, \pi]$ , sign of  $x$  and  $y$  is used to determine the specific quadrant.

In order to obtain the IF positive for all the time, we consider the discrete signal processing which is the only practical way to process data by a processor. The discrete time version of the equations (3), (4), (5) and (6) are

$$z[n] = x[n] + j\hat{x}[n] = a[n]e^{j\phi[n]}, \quad (7a)$$

$$a[n] = [x^2[n] + \hat{x}^2[n]]^{1/2} \geq 0, \quad (7b)$$

$$\phi[n] = \tan^{-1}(\hat{x}[n]/x[n]), \quad (7c)$$

$$\hat{x}[n] = x[n] * \left( \frac{1 - \cos(\pi n)}{\pi n} \right), \quad (7d)$$

$$\text{and } \omega[n] = (\phi[n] - \phi[n-1]), \quad (7e)$$

respectively.

By considering the phase unwrapping fact and *multivalued* nature of the inverse tangent function, we redefine the IF  $\omega[n]$ , defined in (7e) as a discrete version of (6), as

$$\omega[n] = \begin{cases} (\phi[n] - \phi[n-1]) & \text{if difference is } \geq 0, \\ (\phi[n] - \phi[n-1]) + \pi & \text{otherwise,} \end{cases} \quad (8)$$

which makes the IF positive for all time  $n$ . This small but extremely important fact has been illusive for many decades. The efforts and understating of any method, which uses the IF, would improve significantly by using this definition. The mathematical validity of this solution can easily be seen by the fact that  $\phi[n] = \tan^{-1}(\hat{x}[n]/x[n])$  and  $\phi[n] + kn\pi = \tan^{-1}(\hat{x}[n]/x[n]), \forall k, n \in \mathbb{Z}$  (due to periodicity of tangent function, i.e.  $\tan(\phi[n]) = \tan(\phi[n] + kn\pi)$ ). In solution (8), we have taken  $k = 0$  if  $(\phi[n] - \phi[n-1]) \geq 0$ , otherwise  $k = 1$ , thus we can write  $\omega[n] = (\phi[n] - \phi[n-1]) + k\pi$ .

A dual to IF (8), the group delay (GD) is defined as the negative frequency derivative of the phase in the Fourier domain. It measures the relative delay of

different frequencies from the input to the output in a system. Thus, similar to (8), we can modify the GD definition to make it always positive and valid for all signals.

A MATLAB implementation code of the proposed method is outlined in Algorithm 1. In order to avoid unnecessary variations in the IF, before evaluating the IF, the mean and dominating low frequency component like trend present in the signal can be removed. These can be easily removed by any zero-phase low pass filtering operation [13]. Moreover, depending upon the requirements, signal can be decomposed into a set of desired frequency bands.

## RESULTS AND DISCUSSION

In this section, we consider number of examples that are mostly discussed in literature to validate the efficacy of method under study.

**Example 1:** Here, we consider three nonstationary signals (a) linear chirp (unit amplitude and [1000–2000] Hz), (b) frequency modulated (FM) sinusoid (unit amplitude, carrier  $f_c = 780$  Hz and frequency deviation 200 Hz) and (c) mixture of a linear chirp and frequency modulated (FM) signals, i.e sum of signals of cases (a) and (b). Figure 1 (top to bottom) shows the time-frequency-energy (TFE) estimates of these three nonstationary signals [(a), (b) and (c)] obtained by using the proposed method without decomposition. We observe that the frequencies in case (c), Figure 1 (bottom one), are average frequencies of the first two cases (a) and (b). Figure 2 shows the IF estimates of nonstationary signal, which is sum of linear chirp and FM signals i.e. case (c), by this proposed method without decomposition: (top figure) with (7e) and (bottom figure) with (8). This, clearly, demonstrate that the proposed modified definition of the IF is able to obtain correct values of frequencies at all times. Figure 3 shows the TFE estimates of nonstationary signal, case (c), by this proposed method with decomposition: (top figure) into 2 bands of [0–1000, 1000–4000] Hz and (bottom figure) into 100 bands of equal frequencies. The proposed method is, clearly, able to track TFE distribution present in all these signals.

**Example 2:** We obtain a nonstationary signal by

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**Algorithm 1:** A MATLAB implementation code of the proposed method to estimate amplitude and frequency of data  $x$ . Here,  $F_s$  is the sampling frequency.

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```

z=hilbert(x);ampltd=abs(z);phi=unwrap(angle(z));
diffPhase=diff(phi);index=find(diffPhase<0);
diffPhase(index)=diffPhase(index)+pi;
frequency=[diffPhase;diffPhase(end)]*(Fs/(2*pi));

```

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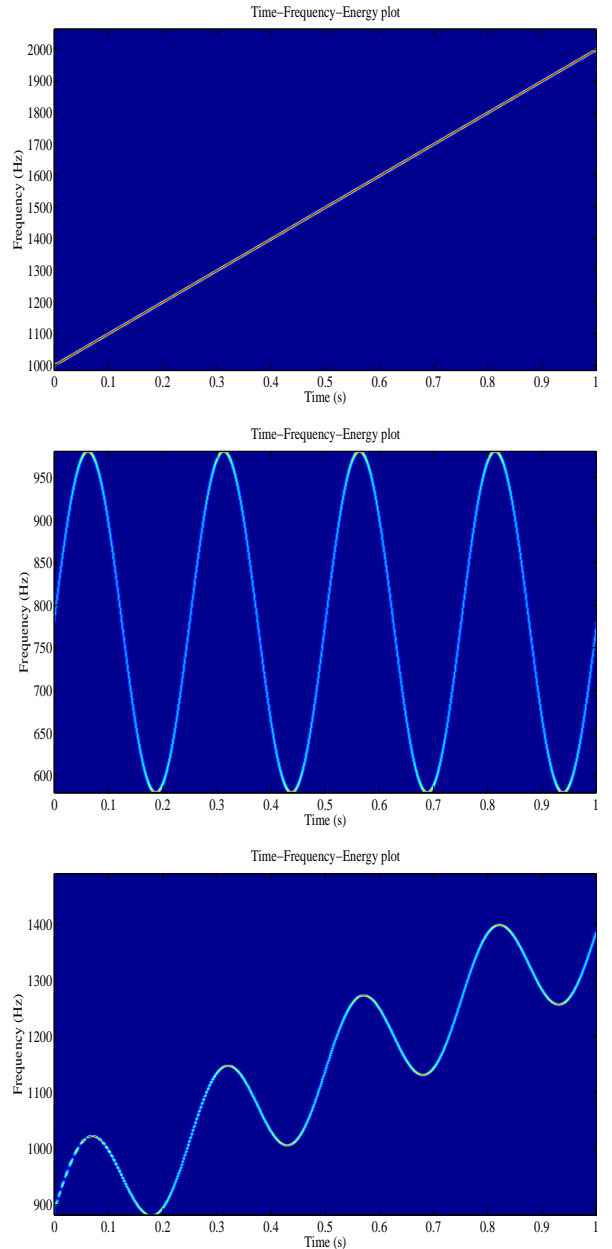


Figure 1: The TFE analysis of nonstationary signals by this proposed method without decomposition: (top) linear chirp, (middle) frequency modulated signal and (bottom) sum of linear chirp and FM signals.

adding five linear chirps of frequencies [500–1500] Hz, [1000–2000] Hz, [1500–2500] Hz, [2000–3000] Hz and [2500–3500] Hz. Figure 4 shows the TFE analysis of this nonstationary signal, which is sum of five linear chirp signals, by this proposed method: (top figure) without decomposition that presents average frequencies [1500–2500] Hz, which are average of frequencies present in five chirp signals; (middle figure) with decomposition into 10 bands and (bottom figure) with decomposition into 20 bands of equal frequencies. These two (middle and bottom one) figures clearly reveal the five chirp signals present in the signal under analysis.

**Example 3:** In Figure 5, we present the TFE analysis of a nonstationary signal, which is sum of a linear chirp signal [500–1500] Hz ([0–1] s chirp signal and [1–1.5] s no signal) and its delayed version ([0–0.5] s

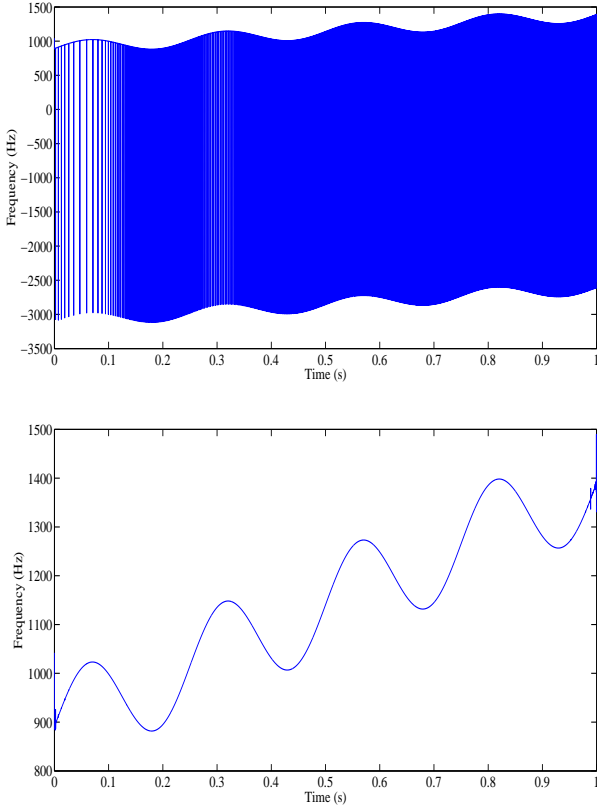


Figure 2: The instantaneous frequency (IF) estimate of nonstationary signal, which is sum of linear chirp and FM signals, by this proposed method without decomposition: (top) with (7e) and (bottom) with (8).

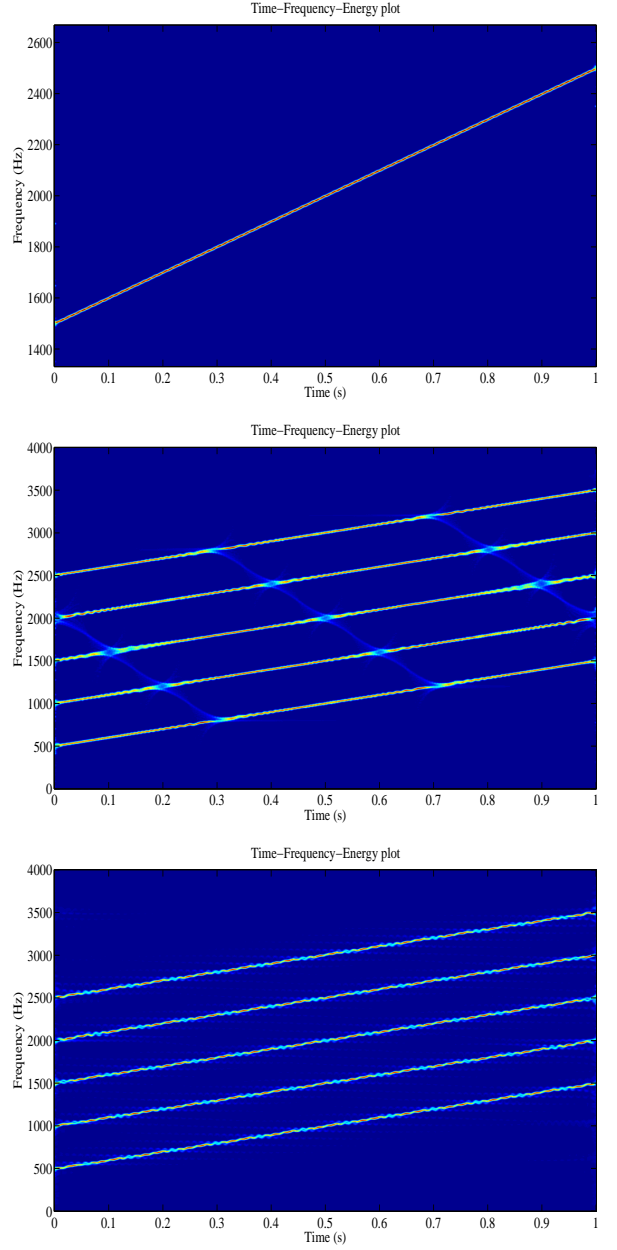


Figure 4: The TFE analysis of a nonstationary signal, which is sum of five linear chirp signals, by this proposed method: (top) without decomposition, (middle) with decomposition into 10 bands and (bottom) with decomposition into 20 bands of equal frequencies.

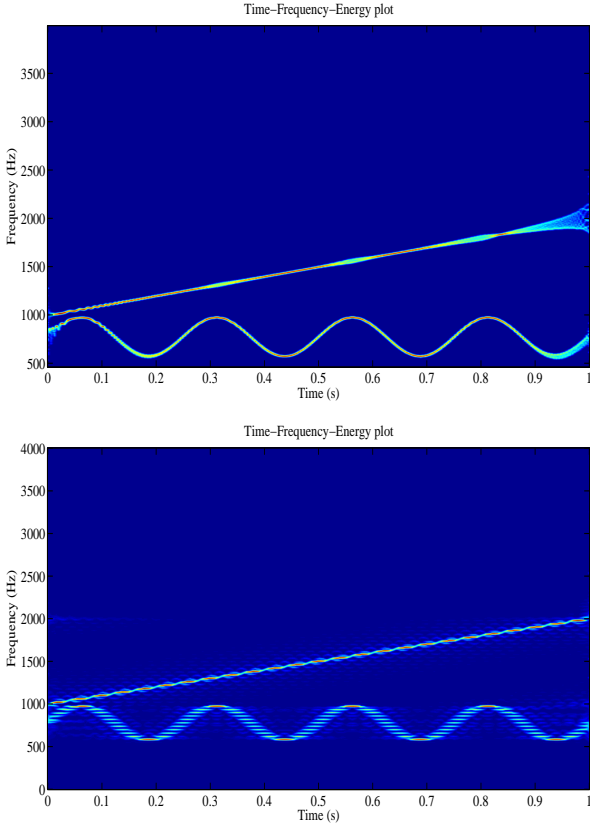


Figure 3: The TFE analysis of nonstationary signal, which is sum of linear chirp and FM signals, by this proposed method with decomposition: (top) into 2 bands of [0–1000, 1000–4000] Hz and (bottom) into 100 bands of equal frequencies.

no signal and [0.5–1.5] s same chirp signal), by this

proposed method. The top figure is obtained without decomposition that has three parts: (1) [0–0.5] s chirp signal frequencies, (2) [0.5–1] s average of overlapped chip signals frequencies and (3) [1–1.5] s chirp signal frequencies of delayed version. The bottom figure is obtained by decomposition of signal into 10 bands of equal frequencies, which clearly reveals that the signal under study is sum of a chirp signal and its delayed version.

**Example 4:** The unit sample sequence (delta function) is defined as  $x[n] = \delta[n - n_0] = 1$  at  $n = n_0$  and zero otherwise. Using the discrete-time Fourier transform (DTFT)  $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\omega n)$ , one can obtain the DTFT of unit sample sequence as  $X(\omega) = \exp(-j\omega n_0) \Rightarrow |X(\omega)| = 1$ . Using the inverse DTFT (IDTFT),  $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) \exp(j\omega n) d\omega$ , one can represent the unit sample sequence as  $x[n] =$

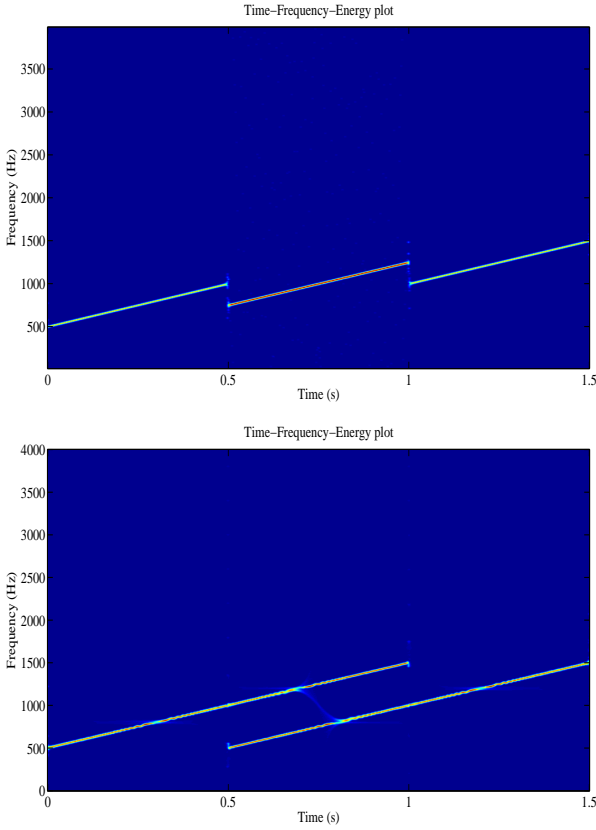


Figure 5: The TFE analysis of a nonstationary signal, which is sum of a linear chirp signal and its delayed version, by this proposed method: (top) without decomposition, and (bottom) with decomposition into 10 bands of equal frequencies.

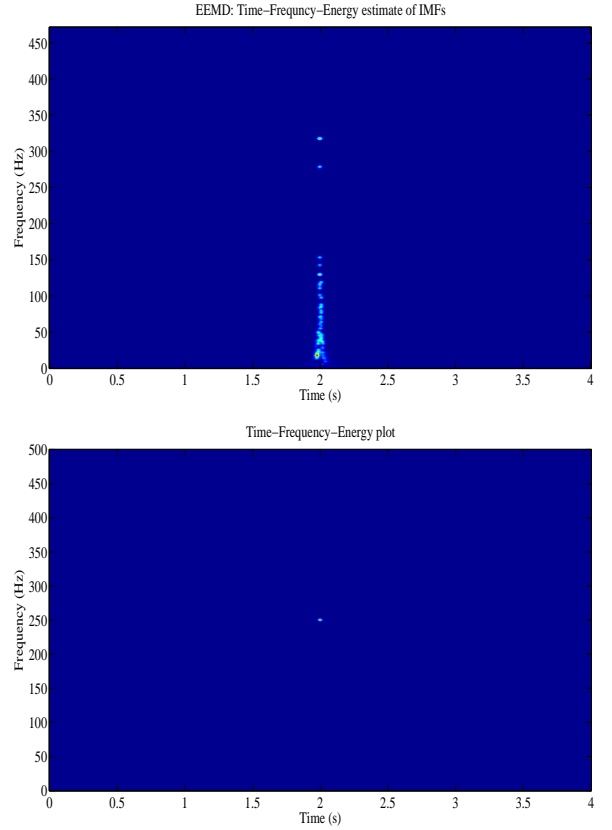


Figure 6: The TFE analysis of unit sample sequence  $\delta[n - n_0]$  (with,  $n_0 = 1999$ , sampling frequency  $F_s = 1000$  Hz, length  $N = 4000$ ) by ensemble EMD (EEMD) (top) and this proposed method without decomposition (bottom).

$\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\omega(n-n_0)) d\omega$ . This representation demonstrate that it is a superposition of equal amplitude sinusoidal functions of all frequencies  $[0-\pi)$ . The Nyquist frequency ( $F_s/2$ ) is the highest frequency that can be present at a given sampling rate,  $F_s$ , in a discrete-time signal. The analytic representation of this signal is given by [13]  $z[n] = \frac{\sin(\pi(n-n_0)) + j[1 - \cos(\pi(n-n_0))]}{\pi(n-n_0)} = a[n] \exp(j\phi[n])$ , where  $a[n] = \left| \frac{\sin(\frac{\pi}{2}(n-n_0))}{\frac{\pi}{2}(n-n_0)} \right|$ , and  $\phi[n] = \frac{\pi}{2}(n - n_0)$ . Thus, frequency  $\omega[n] = \phi[n + 1] - \phi[n] = \frac{\pi}{2}$ , which corresponds to half of the Nyquist frequency, i.e.  $F_s/4$ . Figure 6 shows the TFE estimates of this signal (with  $n_0 = 1999$ ,  $F_s = 1000$  Hz and length  $N = 4000$ ) using the EMD (top figure) and proposed method without decomposition (bottom figure). We again observe that the frequency present in this TFE plot is a average frequency (because delta function contains equal amplitude sinusoids of all frequencies 0 to  $F_s/2$ ). This example also demonstrate that the TFE plot, obtained by the Hilbert spectrum, is not limited by uncertainty principle and signal can be highly concentrated in time and frequency plane. However, it is to be noted that this TFE plot is not providing the true frequencies of delta function.

**Discussion:** In order to explain a average frequency effect in Figure 1, Figure 4, Figure 5 and Figure 6, let us consider a sum of sinusoids of equal amplitudes  $x(t) = \sum_{k=1}^N A \cos(\omega_0 kt)$ . Its analytic repre-

sentation is given by  $z(t) = \sum_{k=1}^N A \exp(j\omega_0 kt) = \frac{A \sin(\omega_0 \frac{N}{2} t)}{\sin(\omega_0 t/2)} \exp(j\omega_0 \frac{N+1}{2} t) = a(t) \exp(j\phi(t))$  which implies phase  $\phi(t) = (\omega_0 \frac{N+1}{2} t)$  and hence IF  $f(t) = \frac{1}{2\pi} \omega_0 \frac{N+1}{2}$ . This is what we observe in these figures (especially Figure 6 bottom one). Here, it is to be noted that if amplitudes of constituent sinusoids are not equal, then resultant IF would not be a constant (average) frequency but it would be a variable one.

**Example 5:** Figure 7 shows the TFE analysis of a white Gaussian noise (with zero mean, unit variance, 10240 samples and sampling frequency  $F_s = 100$  Hz) obtained from the EMD (top one) and this proposed method without decomposition (bottom one), which shows that the energy is randomly distributed across all the frequency range.

**Example 6:** An Earthquake time series signal is a nonlinear and nonstationary data. The Elcentro Earthquake data (sampled at  $F_s = 50$  Hz) has been taken from [27] and is shown in Figure 8 (top one). The critical frequency range that matter in the structural design is less than  $10$  Hz, and the Fourier based power spectral density (PSD), Figure 8 (bottom one), show that almost all the energy in this data is within  $10$  Hz. The TFE distributions by the (a) continuous wavelet transform (CWT) (b) EMD and (c) FDM methods are shown in Figure 9. The TFE distributions by the proposed method (a) without decomposition (b) with decomposition to four bands  $[0-5, 5-10, 10-20, 20-25]$

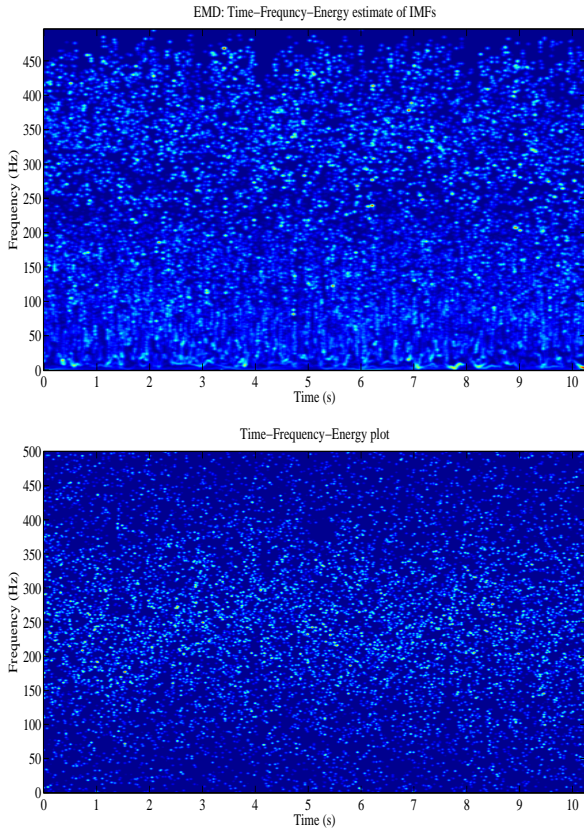


Figure 7: The TFE analysis of the Gaussian white noise with zero mean and unit variance (with sampling frequency  $F_s = 1000$  Hz, length  $N = 10240$ ) by the EMD (top figure) and this proposed method without decomposition (bottom figure).

Hz and (c) with decomposition to 25 bands of 1 Hz are shown in Figure 10. These TFE distribution indicate that the maximum energy concentration is around  $1.7Hz$  and 2 second. The TFE plot provide details of how the different waves arrive from the epicenter to the recording station, e.g. the compression waves of small amplitude but higher frequency range of 10 to  $20Hz$ , the shear and surface waves of strongest amplitude and lower frequency range of below  $5Hz$  which does most of the damage, and other body shear waves which are present over the full duration of the data span.

**Discussion:** It is clear from the above examples that different methods are producing the different TFE distributions of a signal. So, before concluding anything one needs to be careful while doing the analysis. For example, frequencies present in the Fourier spectrum is telling that these frequencies are present all the time in a signal under analysis, which may not be true. Consider another example, Figure 6 (bottom), of unit sample sequence which is telling that this signal is concentrated in time (which is true) and also concentrated in frequency (which is not true). If this were true, then we could transmit delta function in almost zero time with zero bandwidth through any system or channel. Thus, we conclude that the TFE representation depends on the number of bands in which data has been divided. In order to further illustrate this point

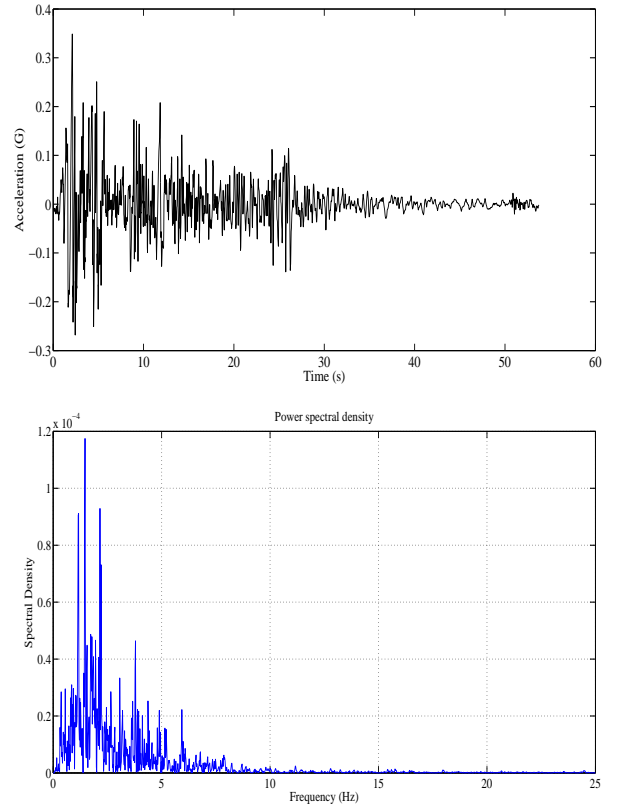


Figure 8: The Elcentro Earthquake May 18, 1940 North-South Component data (top), Fourier based power spectral density (PSD) (bottom).

we divided the same data, which is used for Figure 6, by zero-phase filtering into five and ten bands to obtain top and bottom figures in Figure 11, respectively, which reveal many those frequencies of data which are not present in Figure 6.

From the Figures (1, 4, 5, 6 and 11), it is clear that the proposed method using the Hilbert spectrum produces average frequencies and good time resolution if envelope of signal is smooth. However, if envelope of a signal is fluctuating randomly or rapidly, e.g. consider the case of Gaussian white noise and Earthquake time series, the TFE plot has good time and frequency resolution as shown in Figure 7 and Figure 10 (top one). It is also clear, from Figures (3, 4, 5, 6 and 11), that as the signal is decomposed to more number of narrow bands, true frequencies present in the signal under analysis are revealed, frequency resolution is also increasing while the time resolution is reducing marginally.

## CONCLUSION

The instantaneous frequency (IF) is an important parameter for the analysis of nonstationary signals and nonlinear systems. It is the basis of the time-frequency-energy (TFE) analysis of a signal. The IF is the time derivative of the instantaneous phase (IP) and, originally, it is well-defined only when this derivative is positive. That is, the IF is valid only for mono-component signals. If time derivative of IP is negative,

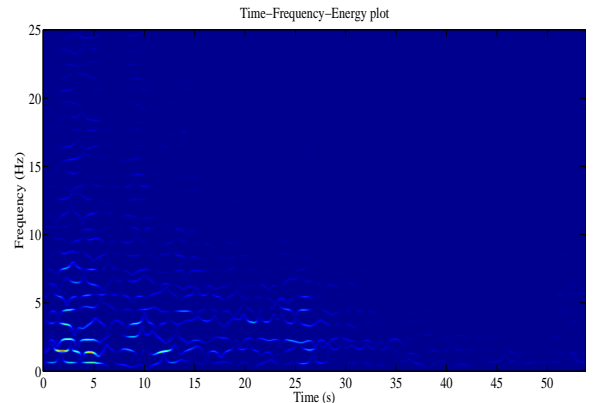
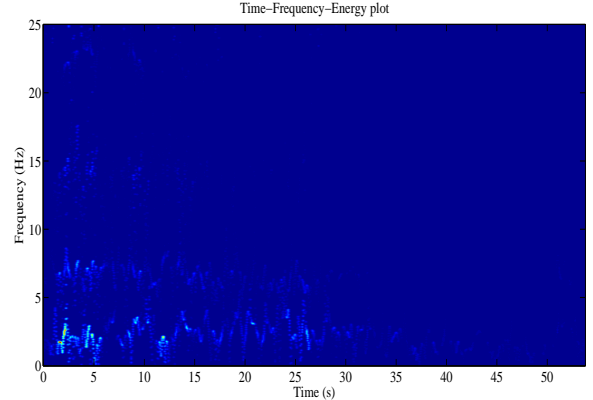
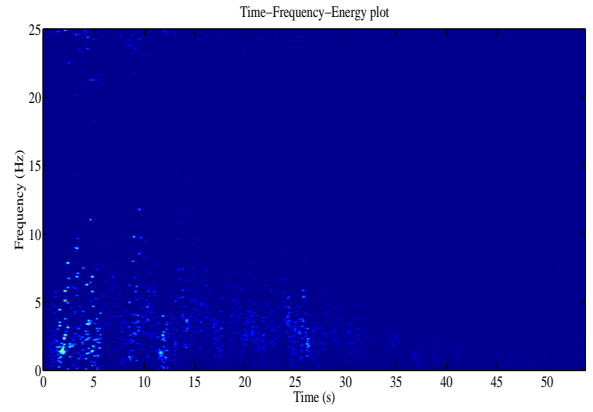
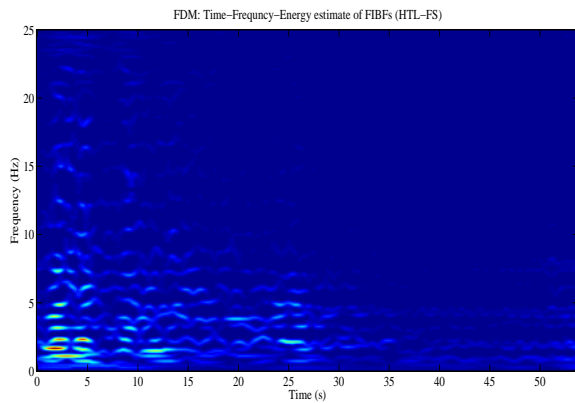
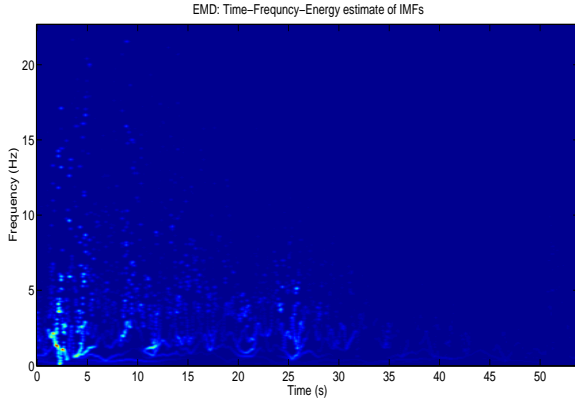
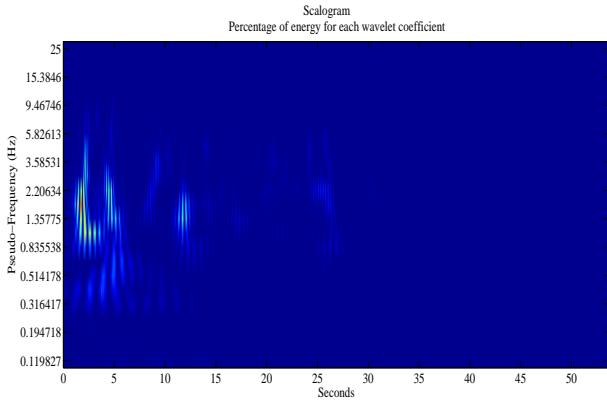


Figure 9: The TFE plot of the Elcentro Earthquake data (top to bottom) using the: (a) CWT (b) EMD and (c) FDM.

Figure 10: The TFE plot of the Elcentro Earthquake data (top to bottom) using the: (a) proposed method without decomposition and (b) proposed method with decomposition into four bands [0–5, 5–10, 10–20, 20–25] Hz and (c) decomposition into 25 bands of 1 Hz each.

i.e. the IF is negative, then it does not provide any physical significance. This study proposes a mathematical solution and eliminate this problem by modifying the present definition of IF. This is achieved by using the property of the *multivalued* inverse tangent function.

The fundamental and important conceptual innovation of the this study is the extension of the present definition of IF by redefining it such that it is always positive. This redefined IF is valid for all types of signals such as monocomponent and multicomponent, narrowband and wideband, stationary and nonstationary, linear and nonlinear signals. The understating of the time-frequency-energy representation, by all the methods which are using the IF, would improve significantly by using this definition.

Simulations and numerical results demonstrated the validity, efficacy and superiority of the proposed IF

for the TFE analysis of a signal as compared to other existing methods available in the literature.

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## References

- [1] Huang N. E., Shen Z., Long S., Wu M., Shih H., Zheng Q., Yen N., Tung C., and Liu H., The empirical mode decomposition and Hilbert spectrum



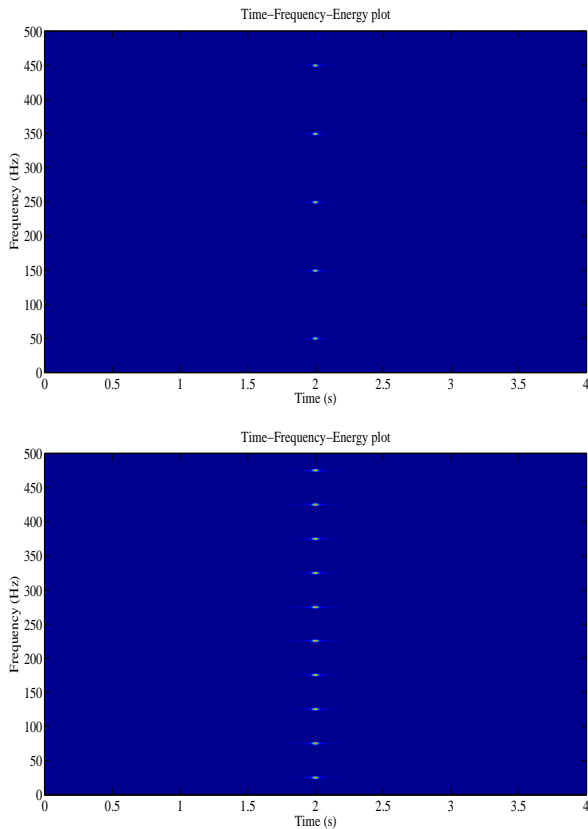


Figure 11: The TFE analysis of unit sample sequence  $\delta[n - n_0]$  (with,  $n_0 = 1999$ , sampling frequency  $F_s = 1000$  Hz, length  $N = 4000$ ) by proposed method with data divided into five (top) and ten (bottom) bands of equal frequency.

for non-linear and non-stationary time series analysis, *Proc. R. Soc. A*, 454 (1988) 903-995.

- [2] Boashash B., Time Frequency Signal Analysis and Processing: A Comprehensive Reference, Elsevier, Boston, 2003.
- [3] Wu Z. and Huang N. E., Ensemble Empirical Mode Decomposition: a noise-assisted data analysis method, *Adv. Adapt. Data Anal.*, 1 (1) (2009) 1-41.
- [4] Rehman N. and Mandic D. P., Multivariate empirical mode decomposition, *Proc. R. Soc. A*, 466 (2010) 1291-1302.
- [5] Singh P., Patney R. K., Joshi S.D. and Saha K., The Hilbert spectrum and the Energy Preserving Empirical Mode Decomposition, *arXiv:1504.04104v1 [cs.IT]*.
- [6] Singh P., Patney R. K., Joshi S. D. and Saha K., Some studies on nonpolynomial interpolation and error analysis, *Appl. Math. Comput.*, 244 (2014) 809-821.
- [7] P. Singh, P. K. Srivastava, R. K. Patney, S. D. Joshi, K. Saha, Nonpolynomial spline based empirical mode decomposition, *2013 International Conference on Signal Processing and Communication*, 435-440 (2013).
- [8] Singh P., Joshi S. D., Patney R. K., and Saha K., The Linearly Independent Non Orthogonal yet Energy Preserving (LINOEP) vectors, (2014) *arXiv:1409.5710 [math.NA]*.
- [9] Daubechies I., Lu J., Wu H. T., Synchrosqueezed Wavelet Transforms: an Empirical Mode Decomposition-like Tool, *Appl. Comput. Harmon. Anal.*, 30 (2011) 243-261.
- [10] Dragomiretskiy K., Zosso D., Variational Mode Decomposition, *IEEE Transactions on Signal Processing*, 62(3) (2014) 531-544.
- [11] P. Jain, R. B. Pachori, An iterative approach for decomposition of multi-component non-stationary signals based on eigenvalue decomposition of the Hankel matrix, *Journal of the Franklin Institute* (2015), <http://dx.doi.org/10.1016/j.jfranklin.2015.05.038>.
- [12] P. Singh, Some studies on a generalized fourier expansion for nonlinear and nonstationary time series analysis, *PhD thesis*, department of electrical engineering, IIT Delhi, India, 2016.
- [13] Singh P., Joshi S. D. Patney R. K., and Saha K., The Fourier Decomposition Method for nonlinear and non-stationary time series analysis, (2015) *arXiv:1503.06675 [stat.ME]*.
- [14] Cummings D. A., Irizarry R. A., Huang N.E., Endy T. P., Nisalak A., Ungchusak K., Burke D. S., Travelling waves in the occurrence of dengue haemorrhagic fever in Thailand, *Nature*, 427, 344-347, 2004.
- [15] Singh P., Joshi S.D., Some studies on multidimensional Fourier theory for Hilbert transform, analytic signal and space-time series analysis, (2015) *arXiv:1507.08117 [cs.IT]*.
- [16] Singh P., LINOEP vectors, spiral of Theodorus, and nonlinear time-invariant system models of mode decomposition, (2015) *arXiv:1509.08667v1 [cs.IT]*.
- [17] Gabor D., Theory of communication, *Proc. IEE*, 93 (III), 429-457, 1946.
- [18] P. Singh, S.D. Joshi, R.K. Patney, K. Saha, Fourier-based Feature Extraction for Classification of EEG Signals Using EEG Rhythms, *Circuits, Systems, and Signal Processing*, DOI 10.1007/s00034-015-0225-z, 2015.
- [19] Carson J., Fry T., Variable frequency electric circuit theory with application to the theory of frequency modulation, *Bell System Tech. J.*, 16, 513-540, 1937.

- [20] Van der Pol B., The fundamental principles of frequency modulation, *Proc. IEE*, 93 (111), 153–158, 1946.
- [21] Gabor D., Theory of communication, *Proc. IEE*, 93 (III), 429–457, 1946.
- [22] Hildebrand F. B., *Advanced Calculus for Engineers*, Englewood Cliffs, NJ: Prentice-Hall, 1949.
- [23] Ville J., Theorie et application de la notion de signal analytic, *Cables et Transmissions*, 2A (1), 61-74, Paris, France, 1948. Translation by I. Selin, Theory and applications of the notion of complex signal, Report T-92, RAND Corporation, Santa Monica, CA.
- [24] Boashash B., Estimating and interpreting the instantaneous frequency of a signal—Part 1: Fundamentals, *Proc. IEEE*, 80 (4), 520–538, 1992.
- [25] Boashash B., Estimating and interpreting the instantaneous frequency of a signal—Part 2: Algorithms and Applications, *Proc. IEEE*, 80 (4), 540–568, 1992.
- [26] [Online:] <http://in.mathworks.com/help/matlab/ref/unwrap.html>.
- [27] [Online]. <http://www.vibrationdata.com/elcentro.htm>.