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Abstract

This paper deals with a novel formulation of a nonlinear reduced-order computational model for analyzing the nonlinear vibrations of a linear viscoelastic structure with weak nonlinear geometrical effects, coupled with a linear acoustic liquid with sloshing and capillarity on the free surface. The model proposed is derived from the one used in fluid-structure interaction for linear systems, for which the analysis of the acoustic-sloshing-capillarity phenomena is efficient thanks to the use of a projection on the linear modes of the linear acoustic liquid and on the sloshing modes with capillarity. Concerning the construction of the vector basis for the structure, it is proposed to use a POD approach for the viscoelastic structure with weak nonlinear geometrical effects and taking into account the added mass induced by the liquid. The methodology for constructing the vector bases of the admissible sets and for obtaining the nonlinear ROM are detailed. The computational nonlinear ROM that is presented can directly be used for analyzing the vibrations of such fluid-structure systems using commercial finite element softwares for computing the vector bases and the projections.

Keywords: Nonlinear vibration, nonlinear reduced-order model, ROM, structure coupled with liquid, FSI, geometrical nonlinearity, sloshing, capillarity, surface

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tension,

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1. Introduction

This paper is the continuation of the work published in [1] for which the linear behavior of the structure is replaced by a weak geometrical nonlinear behavior (a requirement being that if the structural nonlinear term goes to zero, the linear formulation is retrieved). The structure contains a linear dissipative acoustic liquid (no flow), with linear sloshing and capillarity, and for which the effects of internal gravity waves are neglected. A linear formulation for the acoustic liquid is kept in order to preserve the vibration analysis expressed in terms of acoustic and sloshing modes for small (infinitesimal) fluid displacements (consequently, the ALE formulation that requires a mesh deformation is not well adapted to such a requirement). Since the structure can undergo small finite displacements, the model proposed consists in writing the equations of the linear acoustic fluid on the undeformed configuration and in transporting the actual configuration of the structure on its undeformed configuration. In addition, we also want to preserve a slip condition for the fluid on the fluid-structure interface, which is facilitated with the formulation proposed (such a condition is of course not valid in micro- and nano-fluidics). It should be noted that the underlying hypotheses of such a formulation (small finite displacements for the structure and linear behavior of the acoustic liquid in presence of sloshing and capillarity effects) are justified in many applications (see for instance the experimental observations described in [2]). The simplifications introduced in the formulation proposed (such as the non-inclusion of the finite fluctuations of the wetting fluid-structure interface) introduce uncertainties induced by modeling errors. Such uncertainties could be taken into account in a probabilistic framework of uncertainty quantification. Since these types of uncertainties are not relevant of the usual parametric probabilistic approach [3], an open way of research could be the nonparametric probabilistic approach of modeling errors [4, 5]. But, for the present fluid-structure interaction problem, novel nonparametric approaches should be developed in order to take into account modeling errors in computational fluid dynamics.

The novel feature of the paper is to propose the construction of a computational nonlinear reduced-order model (ROM) derived from the one used in linear fluid-structure interaction (for which the analysis of the acoustic-sloshing-capillarity phenomena is efficient thanks to the use of a projection on the linear modes of the acoustic liquid and on the sloshing modes with capillarity), in taking into account weak geometric nonlinearities in the structure. Furthermore, for the structure with geometric nonlinearities, a linear viscoelastic constitutive equation without memory effects is used for generating damping in the structure instead of the use of an empirical damping model that is arbitrary added in the reduced-order equations (as frequently done).

Concerning the linear formulation for the compressible liquid with free surface, the sloshing is taken into account with surface tension (capillarity) effects. We refer the reader to [6, 7] for the classical theory on capillarity, to [8, 9, 10] for developments of the behavior of liquids in microgravity environment, to [11, 12] for general analyzes of sloshing problems for incompressible liquids in rigid structures, to [13, 14, 15, 16, 17, 18] for sloshing problems of incompressible liquids without capillarity effects in elastic structures, to [13, 19, 20, 21] for sloshing problems of incompressible liquids with capillarity effects in rigid structures, to [22, 23, 24, 25, 26, 27, 28, 13, 29] for the conditions of contact angle between the free surface and the structure, to [30] for sloshing problems of incompressible liquids with capillarity effects in elastic structures, to [31, 32, 33] for sloshing problems of compressible liquids with capillarity effects in rigid structures, to [1] for linear dissipative acoustic liquids with sloshing and capillarity effects in linear elastic structures. Concerning nonlinear sloshing and capillarity for incompressible liquids in rigid tanks submitted to rigid body motions, see [34, 35, 36, 37, 38]. Concerning computational reduced-order models for the linear vibration of structures containing compressible liquids without surface tension and without sloshing effects, we refer the reader for instance to [13, 39, 40, 41, 5]. Note that the case of a structure with weak geometrical nonlinearities coupled with a linear acoustic fluid has been investigated in the high-frequency domain in [42].

General computational methods for strongly nonlinear fluid-structure interaction problems including sloshing and capillarity, without the introduction of reduced-order models (but generally using ALE-based approach for the fluid), can be found in [43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55]. A general discussion on elasto-capillary fluid-structure interaction including fluid-solid surface-tension effects and taking into account a transition layer made up a diffuse-interface model related to a phase separation between complex fluids and/or soft matters (Cahn-Hilliard model [56]) can be found in [57].

For constructing computational reduced-order model (ROM) in linear and nonlinear structural dynamics and in fluid-structure interaction, some methodologies have extensively been analyzed in literature. A lot of works have been published for constructing ROM and parametric ROM for linear and nonlinear computational models in dynamics in the area of computational structural dynamics and fluid-structure interaction among which [13, 58, 59, 60, 61, 62, 63, 64, 65] and in vibroacoustics [40, 66, 67, 5]. A hyper reduction is required for reducing the numerical cost of the ROM construction (see for instance [68, 69, 70, 71]). The construction of the reduced-order basis (ROB) is based on the computation of solution snapshots performed with the computational model. The Proper Orthogonal Decomposition (POD) method [72] is a possible approach for constructing the ROB as used in linear structural dynamics for all possible input forces [73] or for a given set of input forces [74, 75, 76, 77]. When the POD method is used for constructing the ROB for a nonlinear computational model [62, 78], a large number of snapshots is used (for time evolution problems, the time becomes a parameter) and the responses are compressed into a low-dimensional ROB using, for example, the singular value decomposition method and several ingredients such as those presented in [79, 80].

In Section 2, we give the notations and hypotheses used. Section 3 deals with the nonlinear equation for the structure expressed in the reference configuration. The boundary value problem for the nonlinear fluid-structure problem is presented in Section 4 for which the formulation retained is expressed using the pressure field in the liquid, the elevation of the free surface, and the displacement field of the structure. In Sections 5 and 6, we present the computational fluid-structure model. The decomposition of the admissible space of the discretized coupled problem and the construction of the associated vector bases are presented in Section 7. Finally, the nonlinear reduced-order computational model is detailed in Section 9.

2. Notations and hypotheses

We consider the fluid-structure system in its *reference configuration* defined in Figure 1. For this reference configuration, the structure is undeformed and occupies an open, bounded, and connected subset Ω_S of $\mathbb{R}^3$. The existing prestresses are not taken into account, which means that the undeformed configuration corresponds to a natural state. In the reference configuration, the undeformed structure contains a liquid, which occupies an open and bounded domain Ω_L. The geometry of domain Ω_L results from a pre-computation for finding the static equilibrium of the liquid in the structure considered as rigid, and submitted to the gravity effects and to the surface-tension effects (capillarity) of the free surface and of the fluid-structure interface. Since the liquid is linear and since we are interested in studying the dynamics around the static equilibrium, the fluid in domain Ω_L is assumed to be in a natural state.

The constitutive equation for the structure is assumed to be linear viscoelastic in finite displacements with instantaneous memory, described by a displacement field measured from the reference configuration, and can undergo weak finite displacements (small strain and moderate rotations). The liquid is a linear dissipative acoustic fluid, which means that the liquid is homogeneous, compressible, inviscid with a volumic additional dissipative term, described by a pressure field, and which is in irrotational infinitesimal motions with respect to the reference configuration. It should be noted that the liquid is inviscid on the fluid-structure interface on which a slip condition will be written.

The physical space is referred to a cartesian reference system. In the reference configuration, the boundary of Ω_L is assumed to be smooth enough and is written as $\partial\Omega_L = \Gamma_L \cup \gamma \cup \Gamma$. The open part Γ_L is the fluid-structure interface. The open part Γ is the free surface of the liquid in its reference configuration. The close part γ is a curve (the contact line), which is such that $\gamma = \partial\Gamma_L = \partial\Gamma$. The boundary of Ω_S (supposed smooth enough) is written as $\partial\Omega_S = \Gamma_E \cup \Gamma_L \cup \gamma \cup \Gamma_G$ (see Figure 1). The boundary of the open parts Γ_L and Γ_G is the curve γ. The domain whose boundary is the open set $\Gamma \cup \Gamma_G$ is empty or is filled by a gas whose effects are neglected for sake of brevity (this is the domain above Ω_L in Figure 1). The external unit normal to $\partial\Omega_S$ is denoted as $\mathbf{n}^S$ while the one to $\partial\Omega_L$ is denoted as $\mathbf{n}$. The external unit normal to γ belonging to the tangent plane to surface Γ is denoted as $\boldsymbol{\nu}$. Let $\Omega_S(t)$ be the actual domain occupied by the structure at time t and let $\partial\Omega_S(t)$ be its boundary.

We are interested in analyzing the nonlinear vibration (structural geometric nonlinearities) of the coupled system around its reference configuration. The structure is assumed to be free (free-free structure for which no prescribed Dirichlet boundary conditions are given on $\partial\Omega_S(t)$). The structure is submitted to given forces, which are applied to $\Omega_S(t)$, and which are assumed to be in equilibrium (in the actual configuration).

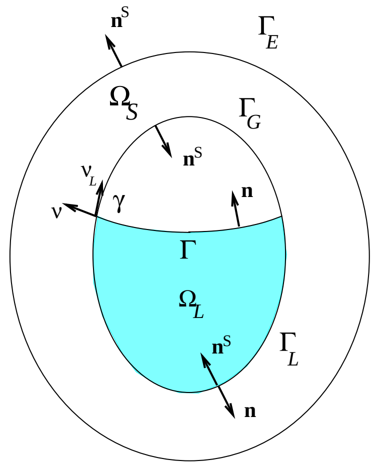

Figure 1: Reference configuration of the fluid-structure system

3. Nonlinear equation for the structure expressed in the reference configuration

In the cartesian reference system, let $\mathbf{x} = (x_1, x_2, x_3)$ be the position vector of a point in the reference configuration $\overline{\Omega}_S = \Omega_S \cup \partial\Omega_S$. In the actual (deformed) configuration $\Omega_S(t)$ at time t, the position vector $\mathbf{X} = (X_1, X_2, X_3)$ of the point that is transformed from $\mathbf{x}$, is written as

$$\mathbf{X}(\mathbf{x}, t) = \mathbf{x} + \mathbf{u}(\mathbf{x}, t)\,, \tag{1}$$

in which $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement field defined in $\overline{\Omega}_S$. The deformation gradient tensor $\mathbb{F}(\mathbf{x}, t)$ defined in Ω_S by

$$\mathbb{F}_{ij}(\mathbf{x}, t) = \frac{\partial X_i(\mathbf{x}, t)}{\partial x_j} = \delta_{ij} + \frac{\partial u_i(\mathbf{x}, t)}{\partial x_j}\,, \tag{2}$$

and is assumed to be invertible and orientation preserving for all $\mathbf{x} \in \overline{\Omega}_S$ and for all t:

$$J(\mathbf{x}, t) \stackrel{\text{def}}{=} \det \mathbb{F}(\mathbf{x}, t) > 0\,. \tag{3}$$

The dynamic equilibrium equations of the structure at time t, expressed with respect to the reference configuration, are written [81, 82] as

$$\rho_{\mathrm{S}}\, \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathrm{div}\,(\mathbb{F}\,\mathbb{S}) = \mathbf{b}\ ,\ \forall \mathbf{x} \in \Omega_S\,, \tag{4}$$

where $\rho_{\rm S}(\mathbf{x}) > 0$ is the mass density in the reference configuration, where $\mathbb{S}$ is the second Piola-Kirchhoff symmetric stress tensor in the reference configuration, where $\text{div}\,(\mathbb{F}\,\mathbb{S})$ denotes the divergence of the second-order tensor $\mathbb{F}\,\mathbb{S}$, and where $\mathbf{b} = (b_1, b_2, b_3)$ corresponds to the transport on Ω_S of the body force field $\mathbf{B}$ applied to deformed configuration $\Omega_S(t)$, which is written as

$$\mathbf{b}(\mathbf{x}, t) = J(\mathbf{x}, t)\, \mathbf{B}(\mathbf{X}(\mathbf{x}, t), t)\,. \tag{5}$$

At time t, let $\mathbf{F}$ be the surface force field applied to the deformed boundary $\partial\Omega_S(t)$ of the actual configuration. The transport of $\mathbf{F}$ on the undeformed boundary $\partial\Omega_S$ is denoted by $\mathbf{f} = (f_1, f_2, f_3)$ and is written as

$$\mathbf{f}(\mathbf{x}, t) = \{\det \mathbb{F}(\mathbf{x}, t)\} \|\mathbb{F}(\mathbf{x}, t)^{-T}\mathbf{n}^S(\mathbf{x})\|\, \mathbf{F}(\mathbf{X}(\mathbf{x}, t), t). \tag{6}$$

At time t, the boundary condition associated with surface force field $\mathbf{f}$ applied to the undeformed boundary $\partial\Omega_S$ is written as

$$\mathbb{F}\,\mathbb{S}\,\mathbf{n}^S = \mathbf{f}\,. \tag{7}$$

The constitutive equation of the structure is chosen in the framework of the theory of the linear viscoelasticity in finite displacements [83] and is written using the hypothesis of instantaneous memory (see [84], references included),

$$\mathbb{S}_{ij} = a_{ij\ell m}(\mathbf{x})\, \mathbb{E}_{\ell m} + b_{ij\ell m}(\mathbf{x})\, \frac{d\mathbb{E}_{\ell m}}{dt}\,, \tag{8}$$

in which the classical convention for summations over repeated Latin indices is used, where tensors $a_{ij\ell m}(\mathbf{x})$ and $b_{ij\ell m}(\mathbf{x})$ are fourth-order real tensors depending on $\mathbf{x}$, which verify symmetry and positiveness properties, and where $\mathbb{E}_{\ell m}$ is the Green-Lagrange strain tensor, which is defined by

$$\begin{aligned}\mathbb{E}_{\ell m}(\mathbf{x}, t) &= \frac{1}{2}(\mathbb{F}_{k\ell}(\mathbf{x}, t)\, \mathbb{F}_{km}(\mathbf{x}, t) - \delta_{\ell m}) \\ &= \frac{1}{2}(\frac{\partial u_\ell}{\partial x_m} + \frac{\partial u_m}{\partial x_\ell} + \frac{\partial u_k}{\partial x_\ell}\,\frac{\partial u_k}{\partial x_m})\,.\end{aligned} \tag{9}$$

4. Boundary value problem in $(p, \eta, \mathbf{u})$

For the linear dissipative acoustic liquid, the equations of motion are written in the reference configuration. Let $p(\mathbf{x}, t)$ be the acoustic pressure field defined in $\overline{\Omega}_L = \Omega_L \cup \partial\Omega_L$ (reference configuration), and let $\eta(\mathbf{x}, t)$ be the field defined on

Γ, which is the normal displacement to the free surface Γ along $\mathbf{n}$ at point $\mathbf{x}$ in the reference configuration. At time t, a surface force field and a body force field are applied to the structure in the actual configuration, which are transported on the reference configuration and give the force field $\mathbf{f}(\mathbf{x}, t)$ on Γ_E (see Eq. (6)) and the body force field $\mathbf{b}(\mathbf{x}, t)$ in Ω_S (see Eq. (5)).

At time t, and in the reference configuration, the boundary value problem is expressed in terms of the structural displacement field $\mathbf{u}(\mathbf{x}, t)$, the internal pressure field $p(\mathbf{x}, t)$, and the normal displacement $\eta(\mathbf{x}, t)$ of the free surface. Under Cauchy initial conditions at time t_0, the problem consists in finding, for $t > t_0$, the fields $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$, and $\eta(\mathbf{x}, t)$, such that

$$\frac{1}{\rho_0 c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\tau}{\rho_0}\boldsymbol{\nabla}^2\frac{\partial p}{\partial t} - \frac{1}{\rho_0}\boldsymbol{\nabla}^2 p = 0 \quad \text{in} \quad \Omega_L\,, \tag{10}$$

$$(1+\tau\frac{\partial}{\partial t})\frac{\partial p}{\partial \mathbf{n}} = -\rho_0\frac{\partial^2 \mathbf{u}}{\partial t^2}\cdot\mathbf{n} \quad \text{on} \quad \Gamma_L\,, \tag{11}$$

$$(1+\tau\frac{\partial}{\partial t})\frac{\partial p}{\partial \mathbf{n}} = -\rho_0\frac{\partial^2 \eta}{\partial t^2} \quad \text{on} \quad \Gamma\,, \tag{12}$$

$$p = \rho_0\,\eta\,\mathbf{g}\cdot\mathbf{n} - \sigma_\Gamma\{(\frac{1}{R_1^2}+\frac{1}{R_2^2})\eta + \boldsymbol{\nabla}_\Gamma^2\eta\} \quad \text{on} \quad \Gamma\,, \tag{13}$$

$$\frac{\partial\eta}{\partial\boldsymbol{\nu}} = c_\eta\,\eta + \mathcal{J}\mathbf{u} \quad \text{on} \quad \gamma\,, \tag{14}$$

$$\rho_s\frac{\partial^2\mathbf{u}}{\partial t^2} - \text{div}\,(\mathbb{F}\,\mathbb{S}) = \mathbf{b}\;,\;\forall\mathbf{x}\in\Omega_S\,, \tag{15}$$

$$\mathbb{F}\,\mathbb{S}\,\mathbf{n}^S = \mathbf{f} \quad \text{on} \quad \Gamma_E\,, \tag{16}$$

$$\mathbb{F}\,\mathbb{S}\,\mathbf{n}^S\,d\Gamma_L = -p\,\mathbf{n}^S\,d\Gamma_L + \sigma_\Gamma\,(\mathcal{J}'\eta)\,d\mu_\gamma \;\text{ on } \Gamma_L\,, \tag{17}$$

in which $\mathbb{F}$ is defined by Eq. (2) and where $\mathbb{S}$ is related to $\mathbb{E}$ (defined by Eq. (9)) by the constitutive equation defined by Eq. (8). In these equations, the different quantities are defined as follows:

• Eq. (10) is the linear dissipative acoustic equation, in which ρ_0 is the constant mass density of the homogeneous liquid at equilibrium, c_0 is the constant speed of sound, and τ is the constant coefficient that characterizes the dissipation in the internal liquid (as a function of the dynamic, kinematic, and second viscosities).

• Eq. (11) is the kinematic fluid-structure coupling condition on Γ_L, which expresses the continuity of the normal velocity field on the interface.

• Eq. (12) represents the kinematic equation for the free surface Γ.

• Eq. (13) corresponds to the free-surface constitutive equation of surface Γ, in which σ_Γ is the surface tension coefficient, **g** is the gravitational acceleration vector, R_1 and R_2 are the principal curvature radii, and where $\boldsymbol{\nabla}^2_\Gamma \eta$ denotes the surface Laplacian related to surface Γ (if capillarity is neglected, the classical sloshing free-surface boundary condition is retrieved). It can be seen that $(1/R_1^2 + 1/R_2^2)\,\eta + \boldsymbol{\nabla}^2_\Gamma \eta$ results from the change of the surface energy induced by an infinitesimal change in η, which yields a contribution to the surface pressure (classical Young-Laplace law) (see [13], Chapter 4).

• Eq. (14) correspond to a new boundary condition for the contact angle compatible with a deformable structure, considered here as linear viscoelastic with geometrical nonlinearities. In the right-hand side of Eq. (14), the first term corresponds to the classical contact angle condition on the contact line, γ, in which c_η is the contact angle coefficient (which is only valid for fixed rigid structure) while the additional second term allows the structure deformation to be taken into account. In this term, $\mathcal{J}$ is a differential operator on manifold Γ_L, which is (i) defined on a set of sufficiently differentiable $\mathbb{R}^3$-valued functions that are the traces on Γ_L of functions on Ω_S, (ii) with values in a set of $\mathbb{R}$-valued functions that are defined on γ. A particular case for differential operator $\mathcal{J}$ is the one given in [13] (Section 4.3, page 80):

$$\mathcal{J}\mathbf{u} = E\,\mathbf{u}\cdot\mathbf{n}^S - \frac{\partial(\mathbf{u}\cdot\mathbf{n}^S)}{\partial\boldsymbol{\nu}_L}\,, \tag{18}$$

in which E is a real coefficient and where $\boldsymbol{\nu}_L$ is external unit normal to γ belonging to the tangent plane to surface Γ_L. The physical interpretation of Eq. (18) corresponds to the change of the energy due to an infinitesimal change in the position of the contact line, which yields a change of the wetted surface.

• Eq. (17), which have been introduced for the first time in [1], corresponds to a new fluid-structure boundary condition on Γ_L in the presence of capillarity, which allows the condition of contact angle to be taken into account in a presence of a linear viscoelastic structure with geometrical nonlinearities. For the readability of the present paper, a short summary of the construction is recalled hereinafter. The first term of the right-hand side corresponds to the fluid-structure coupling condition for a linear acoustic fluid coupled with the weak geometrical nonlinear structure. Concerning the second term, let $\mathbf{x} \mapsto f(\mathbf{x})$ be any real function defined on Γ_L such that its trace on γ is integrable on γ with respect to the curvilinear measure $d\gamma$ on γ (the length of curve γ is $\int_\gamma d\gamma = |\gamma|$). Then, $d\mu_\gamma$ is a real measure on Γ_L such that $\int_{\Gamma_L} f(\mathbf{x})\, d\mu_\gamma(\mathbf{x}) = \int_\gamma f(\mathbf{x})\, d\gamma(\mathbf{x})$ (this means that the support of measure $d\mu_\gamma$ is γ). The term $(\mathcal{J}'\eta)\, d\mu_\gamma$ is defined on Γ_L by algebraic duality of the term $\mathcal{J}\mathbf{u}$ defined on γ (see Eq. (14)) using the duality brackets,

$$< \mathcal{J}\mathbf{u}\,,\eta >_{d\gamma} = \int_\gamma (\mathcal{J}\mathbf{u})\,\eta\, d\gamma\,, \tag{19}$$

$$\ll \mathbf{u}\,,\mathcal{J}'\eta \gg_{d\mu_\gamma} = \int_{\Gamma_L} (\mathcal{J}'\eta)\cdot \mathbf{u}\, d\mu_\gamma\,. \tag{20}$$

The term $(\mathcal{J}'\eta)\, d\mu_\gamma$ is then defined by

$$\ll \mathbf{u}\,,\mathcal{J}'\eta \gg_{d\mu_\gamma} = < \mathcal{J}\mathbf{u}\,,\eta >_{d\gamma}\ . \tag{21}$$

5. Computational fluid-structure model

Let $\mathbf{P}(t)$, $\mathbf{H}(t)$, and $\mathbf{U}(t)$ be the vectors corresponding to the spatial discretization of fields $p(\mathbf{x},t)$, $\eta(\mathbf{x},t)$, and $\mathbf{u}(\mathbf{x},t)$. The first time derivative is denoted by a dot and the second time derivative with a double dot. The discretization of the variational formulation of the boundary value problem in $(p,\eta,\mathbf{u})$ yields:

- for Eqs. (10) to (12),

$$[\mathcal{A}^L]\{\mathbf{P}(t)\} - [C_{p\eta}]^T\, \ddot{\mathbf{H}}(t) - [C_{p\mathbf{u}}]^T\, \ddot{\mathbf{U}}(t) = 0\,, \tag{22}$$

in which $[\mathcal{A}^L]$ is the symmetric matrix-valued linear differential operator defined by $[\mathcal{A}^L] = [M]d^2/dt^2 + [D]d/dt + [K]$.

- for Eqs. (13) and (14),

$$[C_{p\eta}]\,\mathbf{P}(t) + ([K_g] + [K_c])\,\mathbf{H}(t) + [C_{\eta\mathbf{u}}]\,\mathbf{U}(t) = 0\,. \tag{23}$$

- for Eqs. (15) to (17),

$$[C_{p\mathbf{u}}]\,\mathbf{P}(t) + [C_{\eta\mathbf{u}}]^T\mathbf{H}(t) + [M^S]\,\ddot{\mathbf{U}}(t) + \boldsymbol{\mathcal{A}}^S\{\mathbf{U}(t),\dot{\mathbf{U}}(t)\} = \mathbf{F}^S(U(t),t)\,, \tag{24}$$

in which $\boldsymbol{\mathcal{A}}^S$ is the vector-valued nonlinear differential operator that can be written as $\boldsymbol{\mathcal{A}}^S\{\mathbf{U}(t), \dot{\mathbf{U}}(t)\} = [D^S(\mathbf{U}(t))]\, \dot{\mathbf{U}}(t) + \mathbf{K}^S\{\mathbf{U}(t)\}$, where $[D^S(\mathbf{U}(t))]$, $\mathbf{K}^S\{\mathbf{U}(t)\}$ and $\mathbf{F}^S(\mathbf{U}(t), t)$ are nonlinear mappings of $\mathbf{U}(t)$. The different quantities introduced in Eqs. (22) to (24) are defined in Section 6.

It can be proved that, for all $t \geq t_0$, the problem defined by Eqs. (22) to (24) with the initial conditions, has a solution $(\mathbf{P}(t), \mathbf{H}(t), \mathbf{U}(t))$.

6. Vectors and matrices of the discretized problem presented in Section 5

In this section, we give the expressions of the vectors and matrices introduced in Section 5, which result from the discretization of the weak formulation of the boundary value problem defined by Eqs. (10) to (17). For such a construction, for fixed t, we consider the fields $(p(t), \eta(t), \mathbf{u}(t))$ and the test-function fields $(\delta p$, $\delta\eta$, $\delta\mathbf{u})$, which belong to the admissible spaces. In order to simplify the notations, parameter t will be removed if there is no possible confusion.

Matrices related to the equations in P for the linear dissipative acoustic liquid.

- Symmetric real matrix $[M]$ is positive definite, and corresponds to the discretization of the bilinear form $\frac{1}{\rho_0 c_0^2} \int_{\Omega_L} p\, \delta p\, d\mathbf{x}$.
- Symmetric real matrix $[K]$ is positive semidefinite with a kernel of dimension 1, and corresponds to the discretization of the bilinear form $\frac{1}{\rho_0} \int_{\Omega_L} \boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} \delta p\, d\mathbf{x}$.
- Symmetric real matrix $[D] = \tau\, [K]$ is positive semidefinite with a kernel of dimension 1.

Matrices related to the equations in H for the liquid free surface with capillarity.

- Symmetric real matrix $[K_g]$ is positive definite, and corresponds to discretization of the bilinear form $\rho_0 \int_\Gamma \mathbf{g} \cdot \mathbf{n}\, \eta\, \delta\eta\, d\Gamma$.
- Symmetric real matrix $[K_c]$ is positive definite, and corresponds to the discretization of the bilinear form $\sigma_\Gamma \int_\Gamma \boldsymbol{\nabla}_\Gamma \eta \cdot \boldsymbol{\nabla}_\Gamma \delta\eta\, d\Gamma - \sigma_\Gamma \int_\Gamma (\frac{1}{R_1^2} + \frac{1}{R_2^2})\, \eta\, \delta\eta\, d\Gamma - \sigma_\Gamma \int_\gamma c_\eta \eta\, \delta\eta\, d\gamma$.

Matrices related to the coupling terms.

- Rectangular real matrix $[C_{p\mathbf{u}}]$ corresponds to the discretization of the bilinear form $-\int_{\Gamma_L} p\, \mathbf{n} \cdot \delta\mathbf{u}\, d\Gamma_L$.

• Rectangular real matrix $[C_{p\eta}]$ corresponds to the discretization of the bilinear form $-\int_\Gamma p\,\delta\eta\,d\Gamma$.

• Rectangular real matrix $[C_{\eta\mathbf{u}}]$ corresponds to the discretization of the bilinear form $-\sigma_\Gamma \int_\gamma (\mathcal{J}\mathbf{u})\,\delta\eta\,d\gamma$.

Vectors and matrices related to the equations in ***u*** *for the structure with geometric nonlinearities and a linear viscoelastic constitutive equation without memory effects*.

• Symmetric real matrix $[M^S]$ is positive definite, and corresponds to the discretization of the bilinear form $\int_{\Omega_S} \rho_{\rm S}\,\mathbf{u}\cdot\delta\mathbf{u}\,d\mathbf{x}$.

• The vector $[D^S(U(t))]\,\dot U(t) + K^S(U(t))$ corresponds to the discretization (see [85, 86, 87, 88, 89]) of $\int_{\Omega_S} \mathbb{F}_{ij}\,\mathbb{S}_{jk}\,(\partial\delta u_i/\partial x_k)\,d\mathbf{x}$, in which $\mathbb{F}$ is a function of **u** (see Eq. (2)) and where $\mathbb{S}$ is also a function of **u** (see Eqs. (8) and (9)).

Vector of external forces.

• Vector $F^S(U(t),t)$ of external forces correspond to the discretization of $\int_{\Gamma_E} \mathbf{f}\cdot\delta\mathbf{u}\,d\Gamma_E + \int_{\Omega_S} \mathbf{b}\cdot\delta\mathbf{u}\,d\mathbf{x}$, in which **b** depends on **u** (see Eq. (5)) and where **f** also depends on **u** (see Eq. (6)).

7. Decomposition of the admissible space of the discretized coupled problem and associated vector bases

The method consists in constructing a decomposition of the admissible space $C_{P,H,U}$ of the discretized problem defined by Eqs. (22) to (24) and in constructing an adapted vector basis that spans each admissible space.

It can be shown (see [1]) that this admissible space can be decomposed in the following direct sum,

$$C_{P,H,U} = C_P \oplus C_H \oplus C_U\,, \tag{25}$$

in which each admissible space and its associated vector basis is defined hereinafter.

7.1. Admissible space C_P and vector basis

The space C_P is related to the discretized problem in **P** (see Eq. (22)) for which the liquid is a linear acoustic liquid (without the dissipative term) occupying domain Ω_L, and for which we have the boundary condition $\partial p/\partial\mathbf{n} = 0$ on Γ_L and

$p = 0$ on $\Gamma \cup \gamma$ (*i.e.* there are no sloshing and no capillarity). Therefore, C_P is the admissible space for the problem defined by $[M]\,\ddot{\mathbf{P}}(t) + [K]\,\mathbf{P}(t) = 0$ with $\mathbf{P}(t) = 0$ for all the degrees of freedom (DOFs) related to $\Gamma \cup \gamma$. A vector basis of C_P can then be constructed as the *acoustic modes*, which are the eigenvectors of the generalized eigenvalue problem with constraints,

$$[K]\,\mathbf{P} = \lambda_L\,[M]\,\mathbf{P}\,, \tag{26}$$

$$\text{with } \mathbf{P} = 0 \text{ for the DOF related to } \Gamma \cup \gamma\,. \tag{27}$$

Let $[\mathcal{P}] = [\mathbf{P}_1 \ldots \mathbf{P}_{N_L}]$ be the rectangular real matrix whose N_L columns are constituted of the eigenvectors associated with the N_L first smallest positive eigenvalues.

7.2. *Admissible space C_H and vector basis*

The space C_H is related to the discretized problem in $\mathbf{H}$ for which the liquid is considered as an inviscid incompressible liquid, with sloshing and capillarity, and for which the boundary condition is $\partial p/\partial \mathbf{n} = 0$ on Γ_L. Therefore, C_H is the admissible space of the problem, $[K]\,\mathbf{P}(t) - [C_{p\eta}]^T\ddot{\mathbf{H}}(t) = 0$ and $[C_{p\eta}]\mathbf{P}(t) + ([K_g] + [K_c])\mathbf{H}(t) = 0$. A vector basis of C_H can then be constructed as the *sloshing-capillarity modes* of liquid occupying domain Ω_L, which are the eigenvectors of the generalized eigenvalue problem with constraints,

$$[K]\,\mathbf{P} + \lambda_\Gamma [C_{p\eta}]^T\mathbf{H} = 0\,, \tag{28}$$

$$[C_{p\eta}]\mathbf{P} + ([K_g] + [K_c])\mathbf{H} = 0\,. \tag{29}$$

Since the kernel of $[K]$ is equal to 1, the elimination of $\mathbf{P}$ yields,

$$[K_{gc}]\,\mathbf{H} = \lambda_\Gamma\,[M_\Gamma]\,\mathbf{H}\,. \tag{30}$$

$$[L]\,\mathbf{H} = 0\,, \tag{31}$$

in which $[L]$ is a real row matrix, and where $[K_{gc}]$ and $[M_\Gamma]$, under the constraints $[L]\,\mathbf{H} = 0$, are positive-definite symmetric matrices that are constructed as a function of matrices $[K]$, $[C_{p\eta}]$, $[K_g]$ and $[K_c]$. For practical construction of these matrices, we refer the reader to Section 4.6 of Chapter 4 in Ref. [13].
Let $[\mathcal{H}] = [\mathbf{H}_1 \ldots \mathbf{H}_{N_\Gamma}]$ be the rectangular real matrix whose N_Γ columns are constituted of the eigenvectors associated with the N_Γ first smallest strictly positive eigenvalues.

7.3. *Admissible space C_U and vector basis*

Since the structure has a nonlinear behavior, the construction is presented in two steps: definition of the admissible space C_U and then construction of a vector basis.

7.3.1. *Definition of the admissible space C_U*

C_U is the admissible space of the discretized problem in **U** (see Eq. (24)), related to the viscoelastic structure Ω_S, coupled with the liquid considered as an inviscid incompressible liquid (without sloshing and capillarity) occupying domain Ω_L, with the boundary condition $p = 0$ on Γ. We then obtain the following discretized equations in $(\mathbf{P}, \mathbf{U})$ deduced from Eqs. (22) and (24),

$$[K]\,\mathbf{P}(t) - [C_{p\mathbf{u}}]^T \ddot{\mathbf{U}}(t) = 0 \text{ with } \mathbf{P}(t) = 0 \text{ on } \Gamma \cup \gamma, \tag{32}$$

$$[C_{p\mathbf{u}}]\,\mathbf{P}(t) + [M^S]\,\ddot{\mathbf{U}}(t) + [D^S(\mathbf{U}(t))]\,\dot{\mathbf{U}}(t) + \mathbf{K}^S\{\mathbf{U}(t)\} = \mathbf{F}^S(\mathbf{U}(t), t)\,. \tag{33}$$

Using Eq. (32), $\mathbf{P}(t)$ can be eliminated in order to obtain an equation in $\mathbf{U}(t)$. Let $[K_*^{-1}]$ be the matrix such that the solution of $[K]\,\mathbf{P} = \mathbf{F}$ (in which **F** is an arbitrary vector) under the constraint $\mathbf{P} = 0$ on $\Gamma \cup \gamma$, is written as $\mathbf{P} = [K_*^{-1}]\,\mathbf{F}$. Consequently, Eq. (32) yields $\mathbf{P}(t) = [K_*^{-1}]\,[C_{p\mathbf{u}}]^T\,\ddot{\mathbf{U}}(t)$. By substitution in (33), we obtain

$$([M^S] + [M_A])\,\ddot{\mathbf{U}}(t) + [D^S(\mathbf{U}(t))]\,\dot{\mathbf{U}}(t) + \mathbf{K}^S\{\mathbf{U}(t)\} = \mathbf{F}^S(\mathbf{U}(t), t)\,, \tag{34}$$

in which $[M_A]$ is a positive symmetric matrix (called the added mass matrix), which is written as

$$[M_A] = [C_{p\mathbf{u}}]\,[K_*^{-1}]\,[C_{p\mathbf{u}}]^T\,. \tag{35}$$

7.3.2. *Construction of a vector basis for C_U*

Two main approaches can be used for constructing the vector basis for C_U, represented by the rectangular matrix $[\mathcal{U}] = [\mathbf{U}_1 \ldots \mathbf{U}_{N_S}]$ in which the N_S columns are N_S algebraically independent vectors that span a subspace of dimension N_S.

- The first one is the proper orthogonal decomposition (POD) approach, which does not require expertise in the selection of the vector basis, but which yields a basis that depends on the selected excitation forces, and which requires solving a nonlinear dynamical problem in high dimension.

- The second one is the linear modes approach, which requires a very great expertise for selecting the linear modes that participate in the nonlinear response of the structure.

The reader will find in [90] an extensive, comprehensive, and detailed review concerning the construction of such a vector basis in structural dynamics for a structure with nonlinear geometric effects.

In the present paper, we propose to construct a vector basis of C_U in using the nonlinear structure coupled with the liquid considered as an incompressible inviscid liquid in order to take into account the effects of added mass. It should be noted that we do not recommend to use the structure in vacuo (no added mass effect taken into account). Let us recall that the added-mass effects corresponding to compressible and incompressible fluids are distinct (see Chapters 5 and 8 in [13]) and see [91, 92] in which it is shown that the added-mass of a compressible flow does not admit a simple local-in-time decomposition. This is the reason why the added-mass effects have been introduced using an associated incompressible liquid. Let us remark that solely for the construction of the basis of C_U, the use of the POD for the complete fluid-structure problem defined by Eqs. (22) to (24) would not be computationally efficient, and in addition is not in the spirit of the method proposed for constructing the reduced-order model for the fluid-structure problem under consideration.

(i) - POD-based approach. The POD approach requires to solve the following evolution problem

$$\begin{aligned}([M^S] + [M_A])\,\ddot{\mathbf{U}}(t) + [D^S(\mathbf{U}(t))]\,\dot{\mathbf{U}}(t)& \\ + \mathbf{K}^S\{\mathbf{U}(t)\} = \mathbf{F}^S(\mathbf{U}(t),t) \quad t > t_0\,, \quad (36)&\end{aligned}$$

with the initial conditions

$$\mathbf{U}(t_0) = \mathbf{U}_0 \quad , \quad \dot{\mathbf{U}}(t_0) = \mathbf{V}_0\,. \tag{37}$$

A displacement snapshot matrix $[U_{\text{snap}}]$ is formed using the responses $\mathbf{U}(t_1),\ldots,\mathbf{U}(t_n)$ of Eqs. (36) and (37) at n instants $t_1,\ldots,t_n$ (snapshots) with n sufficiently large, in forming the real symmetric matrix

$$[R_{\text{snap}}] = \frac{1}{n}[U_{\text{snap}}]^T[U_{\text{snap}}]\,, \tag{38}$$

and in computing the dominant eigensubspace (associated with the N_S largest positive eigenvalues $\mu_1 \geq \ldots \geq \mu_{N_S} > 0$), represented by the rectangular matrix $[\mathcal{U}] = [\mathbf{U}_1 \ldots \mathbf{U}_{N_S}]$, of the eigenvalue problem,

$$[R_{\text{snap}}]\,\mathbf{U}_\alpha = \mu_\alpha \mathbf{U}_\alpha \quad , \quad \alpha = 1, \ldots, N_S\,. \tag{39}$$

(ii) - Use of the linear modes. With the linear modes approach, the vector basis is constituted of the *elastic modes* of the underlying linear structural system with the added mass, and is then constructed by solving the generalized eigenvalue problem (deduced from Eq. (34)),

$$[K^{(1)}]\,\mathbf{U} = \lambda_S\,([M^S] + [M_A])\,\mathbf{U}\,, \tag{40}$$

in which

$$[K^{(1)}]_{ij} = \{\partial\{\mathbf{K}^S\{\mathbf{U}\}\}_i/\partial U_j\}|_{\mathbf{U}=0}\,, \tag{41}$$

which is the linear part of the nonlinear operator $\mathbf{U} \mapsto \mathbf{K}^S\{\mathbf{U}\}$. As explained before, the linear modes $\mathbf{U}_1 \ldots \mathbf{U}_{N_S}$ have to be carefully selected because all the participating modes in the nonlinear responses must be included [90]. This means that the associated eigenvalues $0 \leq \lambda_{S,1} \leq \ldots \leq \lambda_{S,N_S}$ (which are in increasing order) are not, in general, consecutive. It should be noted that the eigenvectors that correspond to zero eigenvalues are the rigid body modes of the underlying linear structural system with the added mass (and not, in general, the rigid body modes of the nonlinear fluid-structure system) .
Let $[\mathcal{U}\,] = [\mathbf{U}_1 \ldots \mathbf{U}_{N_S}]$ be the rectangular real matrix whose N_S columns are constituted of the selected eigenvectors.

8. Nonlinear reduced-order computational model

8.1. Methodology for constructing the nonlinear reduced-order computational model

The reduced-order model of order (N_L, N_Γ, N_S) is obtained by projecting $(\mathbf{P}(t), \mathbf{H}(t), \mathbf{U}(t))$ on the vector bases constructed in Section 7,

$$\mathbf{P}(t) = [\mathcal{P}]\,\mathbf{q}^P(t)\,, \tag{42}$$

$$\mathbf{H}(t) = [\mathcal{H}]\,\mathbf{q}^H(t)\,, \tag{43}$$

$$\mathbf{U}(t) = [\mathcal{U}]\,\mathbf{q}^U(t)\,, \tag{44}$$

and then in projecting Eqs. (22) to (24). The two linear matrix equations defined by Eqs. (22) and (23) can classically be projected without any particular difficulties. Concerning the nonlinear equation defined by Eq. (24), due to the nonlinearity in $\mathbf{U}(t)$, several approaches can be used for constructing the projection in a computational framework, that we briefly summarized hereinafter. The possible approaches can be classified in the class of the intrusive methods with respect to the commercial softwares and the class of the non-intrusive one.

(i) For the class of intrusive methods, two main approaches can be identified:

(i-1) The first one consists in implementing, at each time step t required by the time integration scheme used, the direct computation of the reduced nonlinear force $[\mathcal{U}]^T \boldsymbol{\mathcal{A}}^S\{[\mathcal{U}]\,\mathbf{q}^U(t), [\mathcal{U}]\,\dot{\mathbf{q}}^U(t)\}$ and in introducing a hyper reduction technique for reducing the numerical cost of the computation (see [68, 69, 70, 71]).

(i-2) The second one consists in implementing the direct computation of the reduced nonlinear terms that result from an algebraic calculation of the different contributions of the projection of $\boldsymbol{\mathcal{A}}^S\{\mathbf{U}(t), \dot{\mathbf{U}}(t)\}$,

$$\begin{aligned}
&[\mathcal{U}]^T \boldsymbol{\mathcal{A}}^S\{[\mathcal{U}]\,\mathbf{q}^U(t), [\mathcal{U}]\,\dot{\mathbf{q}}^U(t)\} \\
&\qquad = [\mathcal{U}\,]^T [D^S([\mathcal{U}]\,\mathbf{q}^U(t))]\,[\mathcal{U}]\,\dot{\mathbf{q}}^U(t) \\
&\qquad\quad + [\mathcal{U}\,]^T \mathbf{K}^S\{[\mathcal{U}]\,\mathbf{q}^U(t)\}\,.
\end{aligned} \tag{45}$$

The component α of the reduced conservative term can be written as

$$\begin{aligned}
\{[\mathcal{U}\,]^T \mathbf{K}^S\{[\mathcal{U}]\,\mathbf{q}^U(t)\}\}_\alpha &= \mathcal{K}^{(1)}_{\alpha\beta}\, q^U_\beta(t) \\
&+ \mathcal{K}^{(2)}_{\alpha\beta\gamma}\, q^U_\beta(t)\, q^U_\gamma(t) + \mathcal{K}^{(3)}_{\alpha\beta\gamma\delta}\, q^U_\beta(t)\, q^U_\gamma(t)\, q^U_\delta(t)\,.
\end{aligned} \tag{46}$$

with summation on the repeated Greek indices. The component α of the reduced dissipative term can be written as

$$\begin{aligned}
[\mathcal{U}]^T [D^S([\mathcal{U}]\,\mathbf{q}^U(t))]\,[\mathcal{U}]\,\dot{\mathbf{q}}^U(t) &= \frac{d}{dt}\Big\{\mathcal{D}^{(1)}_{\alpha\beta}\, q^U_\beta(t) \\
&+ \mathcal{D}^{(2)}_{\alpha\beta\gamma}\, q^U_\beta(t)\, q^U_\gamma(t) + \mathcal{D}^{(3)}_{\alpha\beta\gamma\delta}\, q^U_\beta(t)\, q^U_\gamma(t)\, q^U_\delta(t)\Big\}\,.
\end{aligned} \tag{47}$$

With such a direct intrusive approach, all the constant coefficients of the tensors in the right-hand sides of Eqs. (46) and (47) are calculated inside the software.

For the conservative term defined by Eq. (46), such a direct implementation can be found in [93, 94] and can be extended without difficulty to the dissipative term defined by Eq. (47).

(ii) For the class of non-intrusive methods related to the calculation of the conservative part defined by Eq. (46), the approach consists in numerically identifying all the coefficients of the tensors that appear in the right-hand side of Eq. (46) by solving an inverse problem based on the computation of the nonlinear static responses of the structure in vacuo submitted to appropriate Dirichlet conditions related to the selected vectors $\mathbf{U}_1, \ldots \mathbf{U}_{N_S}$ of the vector basis. In [90], the reader will find a detailed review concerning such a numerical identification for all the coefficients of the tensors relative to the elastic part (see the right-hand side of Eq. (46)). Some applications of such a methodology can be found, for instance, in [4, 95, 96]. This approach could certainly be extended for identifying the coefficients of the tensors related to the dissipative part (see the right-hand side of Eq. (47)).

8.2. Nonlinear reduced-order computational model for the fluid-struture system

Using the methodology presented in Section 8.1, the following nonlinear matrix equation is obtained for the reduced-order computational model,

$$
\begin{aligned}
[A_{\text{FSI}}] \begin{bmatrix} \ddot{\mathbf{q}}^P(t) \\ \ddot{\mathbf{q}}^H(t) \\ \ddot{\mathbf{q}}^U(t) \end{bmatrix} + [B_{\text{FSI}}] \begin{bmatrix} \dot{\mathbf{q}}^P(t) \\ \dot{\mathbf{q}}^H(t) \\ \dot{\mathbf{q}}^U(t) \end{bmatrix} & \\
+ [C_{\text{FSI}}] \begin{bmatrix} \mathbf{q}^P(t) \\ \mathbf{q}^H(t) \\ \mathbf{q}^U(t) \end{bmatrix} & \\
+ \begin{bmatrix} 0 \\ 0 \\ [\mathcal{U}]^T \boldsymbol{\mathcal{A}}^S\{[\mathcal{U}]\,\mathbf{q}^U(t), [\mathcal{U}]\,\dot{\mathbf{q}}^U(t)\} \end{bmatrix} & \\
= \begin{bmatrix} 0 \\ 0 \\ [\mathcal{U}]^T\,\mathbf{F}^S([\mathcal{U}]\,\mathbf{q}^U(t), t) \end{bmatrix} &\,, \quad (48)
\end{aligned}
$$

in which the matrices $[A_{\text{FSI}}]$, $[B_{\text{FSI}}]$, and $[C_{\text{FSI}}]$ are written as

$$[A_{\text{FSI}}] = \begin{bmatrix} [\mathcal{M}] & -[\,\mathcal{C}_{p\eta}]^T & -[\,\mathcal{C}_{pu}]^T \\ 0 & 0 & 0 \\ 0 & 0 & [\mathcal{M}^S] \end{bmatrix} , \tag{49}$$

$$[B_{\text{FSI}}] = \begin{bmatrix} [\mathcal{D}] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{50}$$

$$[C_{\text{FSI}}] = \begin{bmatrix} [\,\mathcal{K}] & 0 & 0 \\ [\,\mathcal{C}_{p\eta}] & [\,\mathcal{K}_{gc}] & [\,\mathcal{C}_{\eta u}] \\ [\,\mathcal{C}_{pu}] & [\,\mathcal{C}_{\eta u}]^T & 0 \end{bmatrix} , \tag{51}$$

with the following definitions for all the matrices in Eqs. (49) to (51):

- For the reduced matrices relative to the liquid, $[\mathcal{M}] = [\mathcal{P}]^T\,[M]\,[\mathcal{P}]$, $[\mathcal{D}] = [\mathcal{P}]^T\,[D]\,[\mathcal{P}]$, and $[\,\mathcal{K}] = [\mathcal{P}]^T\,[K]\,[\mathcal{P}]$.

- For the reduced matrix relative to the free surface, $[\,\mathcal{K}_{gc}] = [\mathcal{H}]^T\,([K_g] + [K_c])\,[\mathcal{H}]$.

- For the reduced matrix relative to the structure, $[\mathcal{M}^S] = [\mathcal{U}]^T\,[M^S]\,[\mathcal{P}]$.

- For the reduced matrices relative to the coupling, $[\,\mathcal{C}_{p\eta}] = [\mathcal{H}]^T\,[\,C_{p\eta}]\,[\mathcal{P}]$, $[\,\mathcal{C}_{pu}] = [\mathcal{U}]^T\,[C_{pu}]\,[\mathcal{P}]$, and $[\,\mathcal{C}_{\eta u}] = [\mathcal{H}]^T\,[\,C_{\eta u}]\,[\mathcal{U}]$.

9. Conclusion

In this paper, a nonlinear reduced-order computational model has been constructed for a linear viscoelastic structure with weak nonlinear geometrical effects, the structure being coupled with a linear acoustic liquid with sloshing and capillarity on the free surface. The model proposed is derived from the one used in fluid-structure interaction for linear systems, for which the analysis of the acoustic-sloshing-capillarity phenomena is efficient thanks to the use of a projection on the linear modes of the linear acoustic liquid and on the sloshing modes with capillarity. Concerning the construction of the vector basis for the structure, its is proposed to use a POD approach for the viscoelastic structure with weak nonlinear geometrical effects and taking into account the added mass induced by the liquid. The methodology for constructing the vector bases of the admissible sets and for obtaining the nonlinear ROM are detailed. The computational nonlinear ROM presented can directly be used for analysis the vibration of such fluid-structure systems using commercial finite element softwares for computing the vector bases and the projections.

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