Generalized Satisfaction Equilibrium: A Model for Service-Level Provisioning in Networks

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Abstract—**This paper presents a generalization of the existing notion of satisfaction equilibrium (SE) for games in satisfaction form. The new equilibrium, which is referred to as the generalized SE (GSE), is particularly adapted for modeling problems such as service-level provisioning in decentralized self-configuring networks. Existence theorems for GSEs are provided for particular classes of games in satisfaction form and the problem of finding a pure strategy GSEs with a given number of satisfied players is shown to be NP-hard. Interestingly, for certain games there exist a dynamic, analogous to the best response of games in normal form, that is shown to efficiently converge to a pure strategy GSE under the given sufficient conditions. These contributions form a more flexible framework for studying self-configuring networks than the existing SE framework. This paper is concluded by a set of examples in wireless communications in which classical equilibrium concepts are shown to be not sufficiently adapted to model service-level provisioning. This reveals the relevance of the new solution concept of GSE.**

I. Introduction

Game theory has played a fundamental role in the analysis of decentralized self-configuring networks (DSCNs), e.g., sensor networks, body area networks, small cells, law-enforcement networks. See for instance [1]–[3] and references therein. A DSCN is an infrastructure-less network in which transmitters communicate with their respective receivers without the control of a central authority, for instance, a base station. Therefore, radio devices must autonomously tune their own transmit-receive configuration to meet a required quality-of-service (QoS) or quality-of-experience (QoE), as well as efficiently exploit the available radio resources. The underlying difficulty of this individual task is that meeting a given QoS/QoE depends also on the transmit-receive configuration adopted by all other counterparts. This suggests that communications networks can be modeled by games as first suggested in [4], which justifies the central role of game theory.

An object of central attention within this context is the notion of equilibrium. In particular, the notion of Nash equilibrium (NE) [5], [6] is probably the most popular solution to games arising from DSCNs. An NE is reminiscent to notions used in mechanics, for instance, a small perturbation to a system at a stable (mechanical) equilibrium induces the system to spontaneously go back to the equilibrium point. Similarly, within a communication network operating at an NE, any transmitter unilaterally deviating from the equilibrium point degrades its own individual performance and thus, backs down to the initial equilibrium configuration. The relevance of the notion of equilibrium is that it sets up the rules under which a DSCN can be considered stable, and thus exploitable. In any other state, the network cannot be fruitfully exploited as there always exists at least one radio device aiming to change its individual transmit-receive configuration. Aside from NE, there are other notions of equilibria particularly adapted to DSCN. Each solution concept has advantages and disadvantages, as described in [7].

A major disadvantage that is common to most of equilibrium concepts is that stability depends on whether or not each radio device achieves the highest performance possible. This does not necessarily meet the original problem in which radio devices must only ensure a QoS or QoE condition [8]. To overcome this constraint, a new solution concept known as satisfaction equilibrium (SE) was suggested in [9] and formally introduced in the realm of wireless communications in [10], [11]. The SE notion relaxes the condition of individual optimality and defines an equilibrium in which all radio devices satisfy the QoS or QoE constraints. From this perspective, radio devices are not anymore modeled by players that maximize their individual benefit but by players that aim at satisfying some individual constraints. This new approach was adopted to model the problem of dynamic spectrum access in [12]–[14] and small cells in [15]. Other applications of SE are reported for instance in the case of collaborative filtering in [16]. In [17], it is shown that the games in normal form discussed in [18], [19] have satisfaction form representations, such that their pure strategy NEs coincide with the SEs. However, the notion of SE as introduced in [10] presents several limitations. As pointed out in [19] and [20], the notion of SE is too restrictive. Simultaneously satisfying the QoS/QoE constraints of all radio devices might not always be feasible, and thus an SE cannot be achieved, even if some of the radio devices can be satisfied. Hence, existence of an SE is highly constrained, which limits its application to wireless communications. These limitations are more evident in the case of mixed-strategies. In mixed-strategies, an SE corresponds to a probability distribution that assigns positive probability to actions that satisfy the individual constraints for any action profile that might be adopted by all the other players.

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A. Contributions

In this paper, the notion of SE presented in [10] is generalized to embrace the case in which only a subset of the radio devices can satisfy their QoS/QoE individual constraints. This new notion of equilibrium is referred to as *generalized satisfaction equilibrium (GSE)*. At a GSE, there are two groups of players: satisfied and unsatisfied player set. The former is the set of players that meet their own QoS/QoE conditions. The latter is the set of players that are unable to meet their own QoS/QoE, given the actions adopted by all the other players. The key point is that at a GSE, none of the actions of a given unsatisfied player allows meeting the individual QoS/QoE constraints and thus, none of the players unilaterally deviates from an equilibrium point. Note that if all players can be satisfied, then the notion of SE and GSE are identical.

The existence of GSEs in games in satisfaction form is studied and general existence results are presented for some classes of games. Interestingly, these existence conditions are less restrictive than those observed for the case of SE in [10]. Nonetheless, the existence of a GSE is shown to be not ensured even in the case of mixed-strategies. This contrasts with other game formulations, such as the normal-form, for which there always exists an NE in mixed-strategies. Finally, the relevance of SF games in the realm of wireless communications is highlighted by several examples.

The rest of the paper is organized as follows. Sec. II introduces games in satisfaction form and presents the definition of GSE. The complexity of the problem of finding a pure strategy GSEs of a finite game in satisfaction form is also studied in this section. In particular, it is shown that this problem is NP-hard. This section concludes by presenting a satisfaction-response dynamic and identifies sufficient conditions under which this dynamic converges to a pure strategy GSE. Finally, Sec. IV concludes this paper.

II. Satisfaction Form and Generalized Satisfaction Equilibrium

This section introduces games in satisfaction form and generalizes the notion of equilibrium presented in [10].

A. Games in Satisfaction Form

A game $\mathtt{G}$ in satisfaction form is defined by the triplet

$$\mathtt{G} \triangleq \left(\mathcal{N}, \left\{\mathcal{A}_i\right\}_{i\in\mathcal{N}}, \left\{g_i\right\}_{i\in\mathcal{N}}\right), \tag{1}$$

where $\mathcal{N} = \{1, \ldots, N\}$ is a finite set containing the indices of all players. The set $\mathcal{A}_i$ is finite and contains all the pure strategies (actions) of player $i \in \mathcal{N}$. Let $\triangle(\mathcal{A}_i)$ denote the set of all probability distributions over $\mathcal{A}_i$. The *correspondence* $g_i : \triangle(\mathcal{A}_1) \times \ldots \times \triangle(\mathcal{A}_{i-1}) \times \triangle(\mathcal{A}_{i+1}) \times \ldots \times \triangle(\mathcal{A}_N) \to 2^{\triangle(\mathcal{A}_i)}$ determines the set of strategies that satisfy the individual constraints of player i. The notation $2^{\triangle(\mathcal{A}_i)}$ denotes the power-set of the set $\triangle(\mathcal{A}_i)$. More specifically, given a strategy profile $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_N) \in \triangle(\mathcal{A}_1) \times \ldots \times \triangle(\mathcal{A}_N)$, player i is said to be satisfied if $\boldsymbol{\pi}_i \in g_i(\boldsymbol{\pi}_{-i})$, with $\boldsymbol{\pi}_{-i} = (\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_{i-1}, \boldsymbol{\pi}_{i+1}, \ldots, \boldsymbol{\pi}_N)$.

The correspondence g_i should not be confused with a constraint on feasible strategies, as in the case of games with coupled actions [21]. Player i can choose any $\boldsymbol{\pi}_i \in \triangle(\mathcal{A}_i)$ as a response to a given $\boldsymbol{\pi}_{-i}$, however, only the strategies in $g_i(\boldsymbol{\pi}_{-i})$ satisfy its individual constraints. When only pure strategies are considered, with a slight abuse of notation, the correspondence in pure strategies is denoted by $g_i : \mathcal{A}_1 \times \ldots \times \mathcal{A}_{i-1} \times \mathcal{A}_{i+1} \times \ldots \times \mathcal{A}_N \to 2^{\mathcal{A}_i}$. Then, given $\boldsymbol{a}_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$, the set $g_i(\boldsymbol{a}_{-i})$ denotes the set of pure strategies that satisfies the individual constraints of player i.

B. Generalized Satisfaction Equilibrium

Each strategy profile $\boldsymbol{\pi}$ of the game (1) induces a partition over the set $\mathcal{N}$ of players formed by the sets $\mathcal{N}_{\rm s}$ and $\mathcal{N}_{\rm u}$. Players in the set $\mathcal{N}_{\rm s}$ are said to be satisfied, that is, $\forall i \in \mathcal{N}_{\rm s}$, $\boldsymbol{\pi}_i \in g_i(\boldsymbol{\pi}_{-i})$. Alternatively, players in the set $\mathcal{N}_{\rm u}$ are said to be unsatisfied, that is, $\forall i \in \mathcal{N}_{\rm u}$, $\boldsymbol{\pi}_i \notin g_i(\boldsymbol{\pi}_{-i})$. The players in $\mathcal{N}_{\rm s}$ are satisfied and thus, they do not possess any interest in changing their own strategy. Conversely, players in $\mathcal{N}_{\rm u}$ are unsatisfied and thus, to guarantee an equilibrium, it must hold that none of their strategies can be used to satisfy their individual constraints. This notion of equilibrium, namely generalized satisfaction equilibrium, is introduced by the following definition.

Definition 1. *Generalized Satisfaction Equilibrium (GSE): A strategy profile $\boldsymbol{\pi}$ is a GSE of the game in (1) if there exists a partition of $\mathcal{N}$, e.g., $\mathcal{N}_{\rm s}$ and $\mathcal{N}_{\rm u}$, such that $\forall i \in \mathcal{N}_{\rm s}$, it holds that $\pi_i \in g_i(\boldsymbol{\pi}_{-i})$ and $\forall j \in \mathcal{N}_{\rm u}$, it holds that $g_j(\boldsymbol{\pi}_{-j}) = \emptyset$.*

At a GSE strategy profile $\boldsymbol{\pi}$, either a player i satisfies its individual constraints or it is unable to satisfy its individual constraints since $g_i(\boldsymbol{\pi}_{-i}) = \emptyset$. From Def. 1 it follows that a pure strategy GSE of (1) is a profile $\boldsymbol{a} \in \mathcal{A}$, where $\forall i \in \mathcal{N}_{\rm s}$, $a_i \in g_i(\boldsymbol{a}_{-i})$ and $\forall j \in \mathcal{N}_{\rm u}$, $g_j(\boldsymbol{a}_{-j}) = \emptyset$. This equilibrium notion generalizes previously proposed solution concepts to games in satisfaction form. An SE, as introduced in [10], is a special case of a pure strategy GSE of Def. 1. Specifically, every GSE in which all players are satisfied in pure strategies is an SE, as suggested in [10]. An ϵ-SE, as defined in [10], is a GSE in which $\mathcal{N}_{\rm u} = \emptyset$ and $\forall i \in \mathcal{N}$, $g_i(\boldsymbol{\pi}_{-i}) = \{\boldsymbol{\pi} \in \triangle(\mathcal{A}_i) : \mathbb{E}\left(\mathbb{1}_{g_i(\boldsymbol{a}_{-i})}(a_i)\right) = 1 - \epsilon\}$, where the expectation is taken over the mixed strategy profile. Finally when $\epsilon = 0$, the SE in mixed strategies as introduced in [10], also follows as a special case of the GSE in Def. 1.

The set of all GSEs of a game can be categorized by the number of players that are satisfied. An $N_{\rm s}$-GSE denotes a GSE in which $N_{\rm s} \leq N$ players are satisfied. An N-GSE satisfies all players and thus, it is referred to as an SE in this paper. The qualifiers mixed- and pure- for the set of strategies may be omitted when the meaning is clear from the context.

C. Existence of Generalized Satisfaction Equilibria

The existence of a GSE in (1) depends on the properties of the correspondences $g_1, \ldots, g_N$. Let $\boldsymbol{g} : \triangle(\mathcal{A}_1) \times \ldots \times$

$\triangle(\mathcal{A}_N) \rightarrow 2^{\triangle(\mathcal{A}_1)\times\ldots\times\triangle(\mathcal{A}_N)}$ be such that for a given strategy profile $\boldsymbol{\pi}$, it follows that

$$\boldsymbol{g}(\boldsymbol{\pi}) \triangleq (\boldsymbol{\pi}'_1, \ldots, \boldsymbol{\pi}'_N), \tag{2}$$

with $\boldsymbol{\pi'}_i \in g_i(\boldsymbol{\pi'}_{-i})$, for all $i \in \mathcal{N}$. Then, an SE is a fixed point of $\boldsymbol{g}$, i.e.,

$$\boldsymbol{\pi} \in \boldsymbol{g}\left(\boldsymbol{\pi}\right), \tag{3}$$

and thus, the tools of fixed-point equations [22] can be used to state existence theorems of SEs. This is not the case for GSEs. Note that at a GSE action profile $\boldsymbol{\pi}$, where $N_{\mathrm{s}} < N$ there exists an $i \in \mathcal{N}$ for which $g_i\left(\boldsymbol{\pi}_{-i}\right) = \emptyset$ and thus, a fixed point is not properly defined. This observation highlights the difficulty of providing a general existence result for a GSE. It also emphasizes the key difference between GSE and NE. By definition, an NE is a fixed point of the special case when the correspondences of (1) are *best response mappings* with respect to individual utility functions and therefore, for finite games there always exists at least one NE [5], [6]. Thus, the satisfaction form in (1) is a more general formulation that the normal form [10].

Existence results can be given for very particular classes of correspondences $g_1, \ldots, g_N$. Consider for instance a game in which player i obtains an expected reward given by the function $u_i : \triangle(\mathcal{A}_1) \times \ldots \times \triangle(\mathcal{A}_N) \rightarrow \mathbb{R}$ and it is satisfied only if the expected reward is higher than a given threshold τ_i (the expectation is over the mixed strategies). That is, the set of mixed strategies that satisfies the individual constraints of player i is given by:

$$g_i\left(\boldsymbol{\pi}_{-i}\right) = \left\{\boldsymbol{\pi}_i \in \triangle(\mathcal{A}_i) : u_i\left(\boldsymbol{\pi}\right) \geq \tau_i\right\}. \tag{4}$$

Examples of games in satisfaction form following this construction are used in [10] to describe several dynamic spectrum access problems. In this case, the game in satisfaction form possesses at least one GSE. This observation is formalized by the following proposition.

Proposition 1. *The finite game in satisfaction form in (1) for which* $\forall i \in \mathcal{N}$, $g_i\left(\boldsymbol{\pi}_{-i}\right) = \left\{\boldsymbol{\pi}_i \in \triangle(\mathcal{A}_i) : u_i\left(\boldsymbol{\pi}\right) \geq \tau_i\right\}$, *possesses at least one GSE.*

Proof: The proof of Prop. 1 is presented in [23]. ■

The statement of Prop. 1 is only for games with the specified correspondences. Prop. 1 does not hold if the correspondence is modified for instance to $g_i\left(\boldsymbol{\pi}_{-i}\right) = \left\{\boldsymbol{\pi}_i \in \triangle(\mathcal{A}_i) : \underline{\tau}_i \leq u_i\left(\boldsymbol{\pi}\right) \leq \overline{\tau}_i\right\}$, with $\underline{\tau}_i$ and $\overline{\tau}_i$, any two reals. In general, the existence of an GSE in games in SF is not guaranteed.

An interesting example of a game in satisfaction form that does not possess a GSE in mixed strategies is presented hereunder. Define a two player game in which each player i has two actions $\left\{a_i^1, a_i^2\right\}$, $i \in \{1, 2\}$. The probability that the strategy of player i assigns to action a_i^j is $\pi_i(a_i^j)$, $j \in \{1, 2\}$. The correspondence of player 1 is

$$g_1\left(\boldsymbol{\pi}_2\right) = \begin{cases} \left\{\boldsymbol{\pi}_1 \in \triangle(\mathcal{A}_1) : \pi_1\left(a_1^1\right) < \pi_1\left(a_1^2\right)\right\} & \text{if } \pi_2\left(a_2^1\right) \geq \pi_2\left(a_2^2\right) \\ \left\{\boldsymbol{\pi}_1 \in \triangle(\mathcal{A}_1) : \pi_1\left(a_1^1\right) \geq \pi_1\left(a_1^2\right)\right\} & \text{othewise} \end{cases} \tag{5}$$

and the correspondence of player 2 is

$$g_2\left(\boldsymbol{\pi}_1\right) = \begin{cases} \left\{\boldsymbol{\pi}_2 \in \triangle(\mathcal{A}_i)_2 : \pi_2\left(a_2^1\right) < \pi_2\left(a_2^2\right)\right\} & \text{if } \pi_1\left(a_1^1\right) < \pi_1\left(a_1^2\right). \\ \left\{\boldsymbol{\pi}_2 \in \triangle(\mathcal{A}_i)_2 : \pi_2\left(a_2^1\right) \geq \pi_2\left(a_2^2\right)\right\} & \text{othewise} \end{cases} \tag{6}$$

Let $\boldsymbol{\pi} \in \Pi$ be an arbitrary strategy profile. Then, one of the following cases holds $\pi_2\left(a_2^1\right) \geq \pi_2\left(a_2^2\right)$ or $\pi_2\left(a_2^1\right) < \pi_2\left(a_2^2\right)$. Consider the case $\pi_2\left(a_2^1\right) \geq \pi_2\left(a_2^2\right)$. Then, player 1 is either in the case in which $\pi_1\left(a_1^1\right) < \pi_1\left(a_1^2\right)$ or else it is in the case $\pi_1\left(a_1^1\right) \geq \pi_1\left(a_1^2\right)$. In the former, i.e., $\pi_1\left(a_1^1\right) < \pi_1\left(a_1^2\right)$, player 1 is satisfied. In the latter, i.e., $\pi_1\left(a_1^1\right) \geq \pi_1\left(a_1^2\right)$, player 1 deviates to π'_1, with $\pi'_1\left(a_1^1\right) < \pi'_1\left(a_1^2\right)$. Either way when player 2 has $\pi_2\left(a_2^1\right) \geq \pi_2\left(a_2^2\right)$, player 1 converges to a strategy in which $\pi_1\left(a_1^1\right) < \pi_1\left(a_1^2\right)$. However, when player 1 is in this case, player 2 is unsatisfied and it deviates to a strategy $\pi'_2\left(a_2^1\right) < \pi'_2\left(a_2^2\right)$. This causes player 1 to be unsatisfied in its current strategy $\pi_1\left(a_1^1\right) < \pi_1\left(a_1^2\right)$ and it deviates to a strategy $\pi_1\left(a_1^1\right) \geq \pi_1\left(a_1^2\right)$. Since the above cases cover the entire mixed-strategy space, this game does not possess a GSE.

D. Complexity of Generalized Satisfaction Equilibria in Pure Strategies

This section establishes the complexity of the GSE search problem in pure strategies. The problem is stated as follows: given the game in satisfaction form in (1), if there is a pure strategy SE find it, otherwise indicate that it does not exist. The following proposition asserts its complexity. The method to establish the time complexity of a problem is the polynomial-time Karp reduction [24].

Proposition 2. *Pure strategy SE search problem is NP-hard.*

Proof: The proof of Prop. 1 is presented in [23]. ■

The pure strategy N_{s}-GSE search problem is: given the game in satisfaction form in (1) and a natural number N_{s}, with $1 \leq N_{\mathrm{s}} \leq N$, if there is an N_{s}-GSE, in pure strategies find it, with the highest possible N, otherwise, indicate that it does not exist.

Corollary 1. *Pure strategy* N_{s}*-GSE problem is NP-hard.*

Proof: The proof of Prop. 1 is presented in [23]. ■

Finding the complexity of the mixed strategy GSE search problem is left as an open problem.

E. Satisfaction Response Algorithm

Solving for a pure strategy GSE of the game in (1) is a hard problem in general, see [23]. However, it is possible to identify games in satisfaction form that have a special structure and thus, a pure strategy equilibrium can be efficiently found. Suppose $\mathcal{Y}$ is an ordered set so that $\forall (y, y') \in \mathcal{Y}^2$, either $y \leq y'$ or $y' > y$ holds. Define finite action spaces $\mathcal{A}_i \subset \mathcal{Y}$, $\forall i \in \mathcal{N}$, so that $\mathcal{A}_i$ is totally ordered as well. For all pairs $(\boldsymbol{a}, \boldsymbol{a}') \in \mathcal{A}^2$, the relation $\boldsymbol{a} \leq \boldsymbol{a}'$ holds if $\forall i \in \mathcal{N}$, $a_i \leq a'_i$. Alternatively, the relation $\boldsymbol{a} < \boldsymbol{a}'$ holds if $\forall i \in \mathcal{N}$ $a_i \leq a'_i$ and for at least one $j \in \mathcal{N}$, it holds that $a_j < a'_j$. The smallest and largest elements of $\mathcal{A}_i$ are denoted by $\underline{a}_i$ and $\overline{a}_i$ respectively. Define the following vectors,

$$\underline{\boldsymbol{a}} \triangleq (\underline{a}_1, \ldots, \underline{a}_N) \text{ and } \quad (7)$$

$$\overline{\boldsymbol{a}} \triangleq (\overline{a}_1, \ldots, \overline{a}_N). \quad (8)$$

Consider the following mappings:

$$\underline{\phi}_i : \mathcal{A}_{-i} \to \mathcal{Y} \text{ and } \quad (9)$$

$$\overline{\phi}_i : \mathcal{A}_{-i} \to \mathcal{Y}. \quad (10)$$

Given the condition $\boldsymbol{a}_{-i} \leq \boldsymbol{a}'_{-i}$, the mapping $\overline{\phi}_i$ is called order-preserving if

$$\overline{\phi}_i(\boldsymbol{a}_{-i}) \leq \overline{\phi}_i(\boldsymbol{a}'_{-i}) \quad (11)$$

and is called order-reversing if

$$\overline{\phi}_i(\boldsymbol{a}_{-i}) \geq \overline{\phi}_i(\boldsymbol{a}'_{-i}). \quad (12)$$

Then consider the game in satisfaction form in (1) and let the correspondence g_i, $\forall i \in \mathcal{N}$, be defined by

$$g_i(\boldsymbol{a}_{-i}) = \{a_i : \underline{\phi}_i(\boldsymbol{a}_{-i}) \leq a_i \leq \bar{\phi}_i(\boldsymbol{a}_{-i})\} \quad (13)$$

in which both $\underline{\phi}_i$ and $\overline{\phi}_i$ are order-preserving.

For $\boldsymbol{a} \in \mathcal{A}$, if $a_i \notin g_i(\boldsymbol{a}_{-i})$ and if $g_i(\boldsymbol{a}_{-i}) \neq \emptyset$, then there always exists an $a'_i \in g_i(\boldsymbol{a}_{-i})$ that player i can use to satisfy its individual constraints. This deviation a'_i is called a satisfaction response and is denoted by $\mathrm{SR}_i(\boldsymbol{a}_{-i}) \in g_i(\boldsymbol{a}_{-i})$. Let $\mathcal{N}'_{\mathrm{u}} \subseteq \mathcal{N}$ be the subset of unsatisfied players with nonempty correspondence, i.e., $i \in \mathcal{N}'_{\mathrm{u}}$, if $a_i \notin g_i(\boldsymbol{a}_{-i})$ and $g_i(\boldsymbol{a}_{-i}) \neq \emptyset$. Then, consider the discrete time asynchronous update sequence in which at each instance a subset $\mathcal{N}^\star \subseteq \mathcal{N}'_{\mathrm{u}}$, performs satisfaction response. This update process is called *asynchronous*, as opposed to *synchronous*, in which all players in $\mathcal{N}'_{\mathrm{u}}$ simultaneously perform the response and as opposed to *sequential*, in which only one of those players at a time performs the response. Algorithm 1 provides the pseudo code for asynchronous satisfaction response and Prop. 3 states its convergence properties.

Algorithm 1 Asynchronous Satisfaction Response

```
Initialize a = a̲
While a is not a GSE:
  Select N* ⊆ N'_u
  a := ((SR_j(a_{-j}))_{j∈N*}, (a_i)_{i∈N\N*})
```

$\boldsymbol{a} := \left((\mathrm{SR}_j(\boldsymbol{a}_{-j}))_{j \in \mathcal{N}^\star}, (a_i)_{i \in \mathcal{N} \setminus \mathcal{N}^\star} \right)$

Proposition 3. *Consider a game in satisfaction form* (1) *with* g_i *given by* (13), $\forall i \in \mathcal{N}$. *Then, starting at* $\underline{\boldsymbol{a}} \in \mathcal{A}$ *the asynchronous satisfaction response algorithm converges to a pure strategy GSE.*

Proof: The proof of Prop. 1 is presented in [23]. ■

Note that in Prop. 3, there exists an implicit assumption that every player that finds itself in $\mathcal{N}_{\mathrm{u}}$ with a nonempty correspondence performs satisfaction response within a finite number of future steps. If $\forall i \in \mathcal{N}$ and $\forall \boldsymbol{a}_{-i} \in \mathcal{A}_{-i}$, $\underline{\phi}_i, \overline{\phi}_i$ are order-reversing, then Algorithm 1 converges initialized at $\overline{\boldsymbol{a}} \in \mathcal{A}$. Worst case iterations for sequential satisfaction response is $O(N \max\{|\mathcal{A}_i| : i \in \mathcal{N}\})$ which occurs when all players are initially in $\mathcal{N}_{\mathrm{u}}$ and each player advances to $\mathcal{N}_{\mathrm{s}}$ with $\mathrm{SR}_i(\boldsymbol{a}_{-i}) = \underline{\phi}_i(\boldsymbol{a}_{-i})$ only to be found back in $\mathcal{N}_{\mathrm{u}}$ at the beginning of its next chance to respond. Simultaneous satisfaction response is bounded by $O(\max\{|\mathcal{A}_i| : i \in \mathcal{N}\})$. Convergence time of the more general asynchronous cases can be bounded between the sequential and simultaneous limits, with the minor condition that every player in $\mathcal{N}_{\mathrm{u}}$ has to perform a response at least once in a predetermined time interval lower than N.

Algorithm 1 applies to infinite action spaces that are closed intervals in the real line. However in that case convergence time may depend on the minimum step size. Power control in continuous domain to achieve a required rate is an example and is discussed later. Sequential satisfaction response up to a predefined fixed number of iterations is discussed in [9] as a possible learning algorithm however, conditions for convergence are not identified.

III. Applications of GSEs

This section presents a particular application of games in satisfaction form in wireless networks. The objective is to demonstrate the applicability of GSE into simple but relevant problems. Power control and channel allocation are the main focus.

A. Uplink Power Control Game

Power control under per user rate requirements has been well studied for its feasible region and Pareto optimal solutions [25]. The possibly infeasible case in which a subset of the transmitters may not be satisfied has received less attention. In [26], the over constrained SINR targets are handled by introducing multiple SINR targets such that the infeasible users switch to lower targets.

The single-input-single-output (SISO) power control game in the interference channel is presented in [27] as a generalized Nash equilibrium problem. The following development considers single-input-multiple-output (SIMO) case as a game in satisfaction form. The baseband equivalent signal at the destination of transmitter i is

$$\boldsymbol{y}_i = \sqrt{p_i}\boldsymbol{h}_{ii}s_i + \sum_{j \in \mathcal{N} \setminus \{i\}} \sqrt{p_j}\boldsymbol{h}_{ji}s_j + \boldsymbol{z}_i, \quad (14)$$

where $\boldsymbol{y}_i \in \mathbb{C}^{n_i}$ is the received symbol vector at the receiver of i^{th} transmitter, n_i is the number of receiver antennas, $s_i \in \mathbb{C}$ is the transmitted symbol of i, $\boldsymbol{h}_{ji} \in \mathbb{C}^{n_i}$ is the channel between transmitter j and destination of i, and $\boldsymbol{z}_i \sim \mathcal{CN}(\boldsymbol{0}, \sigma \boldsymbol{I})$ is the circular symmetric complex additive white Gaussian noise. The payoff of transmitter i is the achievable rate $u_i(p_i, \boldsymbol{p}_{-i}) = \log(1 + p_i \boldsymbol{h}_{ii}^H \boldsymbol{R}_{-i}^{-1} \boldsymbol{h}_{ii})$ bits/sec/Hz, where $\boldsymbol{R}_{-i} = \sum_{j \in \mathcal{N} \setminus \{i\}} p_j \boldsymbol{h}_{ji} \boldsymbol{h}_{ji}^H + \sigma \boldsymbol{I}$ is the interference plus noise covariance matrix. The transmit power is $p_i \in \mathcal{P}_i$, where $\mathcal{P}_i = [\underline{p}_i, \overline{p}_i]$, $\underline{p}_i, \overline{p}_i \in \mathbb{R}_{\geq 0}$. The game in satisfaction form played by the transmitters is

$$\mathtt{G}_{\mathrm{PC}} \triangleq \left(\mathcal{N}, \{\mathcal{P}_i\}_{i \in \mathcal{N}}, \{g_i\}_{i \in \mathcal{N}} \right), \quad (15)$$

in which $\forall i \in \mathcal{N}$, $g_i(\boldsymbol{p}_{-i}) = \{p_i \in \mathcal{P}_i : \underline{\tau}_i \leq u_i(\boldsymbol{p}) \leq \overline{\tau}_i\}$, where $0 \leq \underline{\tau}_i \leq \overline{\tau}_i$. The upper bound $\overline{\tau}_i$ is considered for the sake of generality. For instance the transmitter or receiver may have a maximum operational rate. This model is valid for $\bar{\tau}_i = +\infty$, which corresponds to rate unbounded from above.

Define $\forall i \in \mathcal{N}$ $\underline{\phi}_i\left(\boldsymbol{p}_{-i}\right) = \inf_{p_i \in \mathbb{R}} \left\{p_i : u_i\left(p_i, \boldsymbol{p}_{-i}\right) \geq \underline{\tau}_i\right\}$ and $\bar{\phi}_i\left(\boldsymbol{p}_{-i}\right) = \sup_{p_i \in \mathbb{R}} \left\{p_i : u_i\left(p_i, \boldsymbol{p}_{-i}\right) \leq \bar{\tau}_i\right\}$. Then restate the correspondence $g_i\left(\boldsymbol{p}_{-i}\right) \equiv \left\{p_i \in \mathcal{P}_i : \underline{\phi}_i\left(\boldsymbol{p}_{-i}\right) \leq p_i \leq \bar{\phi}_i\left(\boldsymbol{p}_{-i}\right)\right\}$. From the properties of positive (semi-)definite matrices [28], $\boldsymbol{p}_{-i} \leq \boldsymbol{p}'_{-i}$ implies $\boldsymbol{R}_{-i}^{-1}\left(\boldsymbol{p}_{-i}\right) \preceq \boldsymbol{R}_{-i}^{-1}\left(\boldsymbol{p}'_{-i}\right)$ which in turn implies $u_i\left(p_i, \boldsymbol{p}'_{-i}\right) \leq u_i\left(p_i, \boldsymbol{p}_{-i}\right)$ and therefore concludes that $\underline{\phi}_i\left(\boldsymbol{p}_{-i}\right) \leq \underline{\phi}_i\left(\boldsymbol{p}'_{-i}\right)$ and $\bar{\phi}_i\left(\boldsymbol{p}_{-i}\right) \leq \bar{\phi}_i\left(\boldsymbol{p}'_{-i}\right)$. The inequalities hold strictly if $\boldsymbol{p}_{-i} < \boldsymbol{p}'_{-i}$. Thus by extension of Prop. 3 to action spaces that are closed intervals in the real line Algorithm 1 converges to a GSE in the game (15). If the upper threshold is removed, by setting $\bar{\tau}_i = +\infty$, the stronger condition $\boldsymbol{p}_{-i} < \boldsymbol{p}'_{-i}$ implies $g_i\left(\boldsymbol{p}'_{-i}\right) \subset g_i\left(\boldsymbol{p}_{-i}\right)$.

The standard power control game is to minimize the transmit power with per-user rate constraints and one solution can be a generalized NE. However, a generalized NE might not always exist when the problem is over constrained. Interestingly, when this problem is modeled as a game in satisfaction form, there always exists a GSE.

B. *Efficient-GSEs and Admission Control*

At a pure strategy GSE $\boldsymbol{p} \in \mathcal{P}$ of (15), an unsatisfied player $i \in \mathcal{N}_{\mathrm{u}}$ obtains $u_i\left(\boldsymbol{p}\right) < \underline{\tau}_i$, but may have $p_i > \underline{p}_i$. If a player in $\mathcal{N}_{\mathrm{u}}$ lowers its power, then it is possible that another in $\mathcal{N}_{\mathrm{u}}$ can deviate to satisfaction and thus disrupt the equilibrium. In some applications it is desirable that at a GSE $\forall i \in \mathcal{N}_{\mathrm{u}}$ $p_i = \underline{p}_i$. Such profiles are called *efficient-GSEs* as the $\mathcal{N}_{\mathrm{u}}$ poses the least interference to $\mathcal{N}_{\mathrm{s}}$. Efficient-GSEs do not necessarily exist.

IV. Conclusion

This paper presents a generalization of the notion of satisfaction equilibrium, namely the generalized satisfaction equilibrium, (GSE) for games in satisfaction form. When players attempt to satisfy a required service level, rather than maximize their utility, the GSE is a more appealing solution. At a GSE, unsatisfied players are unable to unilaterally deviate to meet their individual constraints. GSE bridges constraint satisfaction problems and games in satisfaction form as the two problems can be transformed into each other.

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