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Global spectra, polytopes and stacky invariants

ANTOINE DOUAI *

Laboratoire J.A Dieudonné, UMR CNRS 7351,
Université de Nice, Parc Valrose, F-06108 Nice Cedex 2, France
Email address: Antoine.DOUAI@unice.fr

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Abstract

Given a convex polytope, we define its geometric spectrum, a stacky version of Batyrev's stringy E-functions, and we prove a stacky version of a formula of Libgober and Wood about the E-polynomial of a smooth projective variety. As an application, we get a closed formula for the variance of the geometric spectrum and, as a consequence, for the variance of the spectrum at infinity of tame Laurent polynomials. This gives an explanation and positive answers to Hertling's conjecture about the variance of the spectrum of tame regular functions, but also a Noether's formula for two dimensional Fano polytopes (polytopes whose vertices are primitive lattice points).

1 Introduction

Let X be a smooth projective variety of dimension n with Hodge numbers $h^{p,q}(X)$. It follows from Hirzebruch-Riemann-Roch theorem that

$$\frac{d^2}{du^2}E(X; u, 1)|_{u=1} = \frac{n(3n-5)}{12}c_n(X) + \frac{1}{6}c_1(X)c_{n-1}(X) \quad (1)$$

where $E(X; u, v) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q$, see Libgober and Wood [18] and also Borisov [5, Proposition 2.2]. By duality, we get

$$\sum_{p,q} (-1)^{p+q} h^{p,q}(X) \left(p - \frac{n}{2}\right)^2 = \frac{n}{12}c_n(X) + \frac{1}{6}c_1(X)c_{n-1}(X) \quad (2)$$

More generally, if X is a n -dimensional projective variety with at most log-terminal singularities (we will focus on the toric case), Batyrev [2] has proved a stringy version of formula (1)

$$\frac{d^2}{du^2}E_{st}(X; u, 1)|_{u=1} = \frac{n(3n-5)}{12}e_{st}(X) + \frac{1}{6}c_{st}^{1,n}(X)$$

where E_{st} is the stringy E -function of X , e_{st} is the stringy Euler number and $c_{st}^{1,n}(X)$ is a stringy version of $c_1(X)c_{n-1}(X)$.

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On the singularity theory side, the expected mirror partners of toric varieties are the Givental-Hori-Vafa models [13], [16], a class of Laurent polynomial. One associates to such functions their *spectrum at infinity*, a sequence $\alpha_1, \dots, \alpha_\mu$ of rational numbers, suitable logarithms of the eigenvalues of the monodromy at infinity of the function involved (see [22]; the main features are recalled in section 5). A specification of mirror symmetry is that the spectrum at infinity of a given Givental-Hori-Vafa model is related to the degrees of the (orbifold in the singular case) cohomology groups of its mirror variety X . So one can expect a formula similar to (2) involving the spectrum at infinity of any regular tame function: the aim of this text is to look for such a counterpart. The key observation is that the spectrum at infinity of a Laurent polynomial can be described (under a tameness condition due to Kouchnirenko [17], see section 5) with the help of the Newton filtration of its Newton polytope. Since a polytope determines a stacky fan [3], one is lead to define a stacky version of the E -polynomial. Given a Laurent polynomial f with Newton polytope P , global Milnor number μ and spectrum at infinity $\alpha_1, \dots, \alpha_\mu$, the program is thus as follows:

- to construct a stacky version of the E -polynomial, a *geometric spectrum* $Spec_P^{geo}(z)$: we define $Spec_P^{geo}(z) := (z-1)^n \sum_{v \in N} z^{-\nu(v)}$ where ν is the Newton function of the polytope P , see section 4. This geometric spectrum is closely related to the Ehrhart series and δ -vector of the polytope P , more precisely to their twisted versions studied by Stapledon [23] after the work of Mustata-Payne [19]; this function is also an orbifold Poincaré series, thanks to the description of orbifold cohomology given by Borisov, Chen and Smith [3, Proposition 4.7],
- to show that this geometric spectrum is equal to the (generating function of the) spectrum at infinity of f : this would give a close formula for the latter and the expected identification between the spectrum at infinity and orbifold degrees, see section 6 and corollary 4.2.6,
- to show a formula

$$\frac{d^2}{dz^2} Spec_P^{geo}(z)|_{z=1} = \frac{n(3n-5)}{12} \mu + \frac{1}{6} \widehat{\mu}$$

where $\widehat{\mu}$ is a linear combination of intersection numbers (see theorem 7.1.5).

At the end one would get a version of (2) for the spectrum at infinity of Laurent polynomials:

$$\sum_{i=1}^{\mu} \left(\alpha_i - \frac{n}{2}\right)^2 = \frac{n}{12} \mu + \frac{1}{6} \widehat{\mu} \quad (3)$$

This is achieved in section 7.

In order to enlighten this formula, assume that $N = \mathbb{Z}^2$ and that P is a full dimensional reflexive lattice polytope in $N_{\mathbb{R}}$. Then we have the following well-known Noether's formula

$$12 = \mu_P + \mu_{P^\circ} \quad (4)$$

where P° is the polar polytope of P and μ_P (*resp.* μ_{P°) is the normalized volume of P (*resp.* P°), see equation (10) (by Pick's formula, $\mu_P = |\partial P \cap N|$ if P is reflexive). We show in section 8 that if P is a Fano lattice polytope (a polytope is Fano if its vertices are primitive lattice points) we have $\widehat{\mu}_P = \mu_{P^\circ}$. From formula (3), we then get

$$\sum_{i=1}^{\mu_P} (\alpha_i - 1)^2 = \frac{1}{6} \mu_P + \frac{1}{6} \mu_{P^\circ} \quad (5)$$

which is a generalization of formula (4): a reflexive polytope P in $N_{\mathbb{R}}$ is Fano and its (algebraic and/or geometric) spectrum satisfies $\sum_{i=1}^{\mu_P} (\alpha_i - 1)^2 = 2$.

Last, if $\widehat{\mu} \geq 0$, it follows from (3) that, with obvious notations,

$$\frac{1}{\mu} \sum_{i=1}^{\mu} \left(\alpha_i - \frac{n}{2}\right)^2 \geq \frac{\alpha_{max} - \alpha_{min}}{12} \quad (6)$$

because $\alpha_{max} - \alpha_{min} = n$ for Laurent polynomials and this inequality is expected to be true for any tame regular function: this is the global version of Hertling's conjecture about the variance of the spectrum, see section 9. For instance, formula (5) show that this will be the case in the two dimensional case if the Newton polytope of f is Fano. We also give positive answers to this conjecture in the two dimensional case in theorem 7.2.2. We test these results on (fake) weighted projective spaces.

This paper is organized as follows: in section 2 we recall the basic facts on polytopes and toric varieties that we will use. In section 3, we discuss of what should be the spectrum of a polytope. The geometric spectrum is defined in section 4 and the algebraic spectrum is defined in section 5: both are compared in section 6. The previous results are used in section 7 in order to get formula (3). We show Noether's formula for Fano polytopes in section 8. Last, we use our results in order to motivate (and partly show) the conjecture about the variance of the algebraic spectrum in section 9.

This text owes much to Batyrev's work [1], [2]. The starting point was [1, Remark 3.13] and its close resemblance with Hertling's conjecture about the variance of the spectrum of an isolated singularity [14]: this link is previously alluded to in [15]. Last, formula (3) is in essence produced by Hirzebruch-Riemann-Roch theorem and the contribution of this theorem to singularity theory has to be probably (?) further explored.

2 Polytopes and toric varieties (framework)

We give in this section an overview of the results that we will use and we set the notations.

2.1 Polytopes and reflexive polytopes

Let N be the lattice \mathbb{Z}^n , M its dual lattice, $\langle \cdot, \cdot \rangle$ the pairing between $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. A full dimensional lattice polytope $P \subset N_{\mathbb{R}}$ is the convex hull of a finite set of N such that $\dim P = n$. If P is a full dimensional lattice polytope containing the origin in its interior, there exists, for each facet (face of dimension $n - 1$) F of P , $u_F \in M_{\mathbb{Q}}$ such that

$$P \subset \{n \in N_{\mathbb{R}}, \langle u_F, n \rangle \leq 1\} \text{ and } F = P \cap \{n \in N_{\mathbb{R}}, \langle u_F, n \rangle = 1\} \quad (7)$$

This gives the hyperplane presentation

$$P = \cap_F \{n \in N_{\mathbb{R}}, \langle u_F, n \rangle \leq 1\} \quad (8)$$

We define, for $v \in N_{\mathbb{R}}$, $\nu_F(v) := \langle u_F, v \rangle$ and $\nu(v) := \max_F \nu_F(v)$ where the maximum is taken over the facets of P .

Definition 2.1.1 *The function $\nu : N_{\mathbb{R}} \rightarrow \mathbb{R}$ is the Newton function of P .*

If P is a full dimensional lattice polytope in $N_{\mathbb{R}}$ containing the origin, the polytope

$$P^{\circ} = \{m \in M_{\mathbb{R}}, \langle m, n \rangle \leq 1 \text{ for all } n \in P\}$$

is the *polar polytope* of P . A lattice polytope P is *reflexive* if it contains the origin and if P° is a lattice polytope. The vertices of P° are in correspondence with the facets of P via

$$u_F \text{ vertice of } P^{\circ} \leftrightarrow F = P \cap \{x \in N_{\mathbb{R}}, \langle u_F, x \rangle = 1\} \quad (9)$$

The polytopes considered in this paper are full dimensional lattice polytopes containing the origin in their interior. We will write

$$\mu_P := n! \text{Vol}(P) \quad (10)$$

where the volume $\text{Vol}(P)$ is normalized such that the volume of the unit cube is equal to 1.

2.2 Ehrhart polynomial and Ehrhart series

Let Q be a full dimensional lattice polytope. The function $\ell \mapsto \text{Ehr}_Q(\ell) := \text{Card}((\ell Q) \cap M)$ is a polynomial of degree n , the *Ehrhart polynomial*. We have

$$F_Q(z) := \sum_{m \geq 0} \text{Ehr}_Q(m) z^m = \frac{\delta_0 + \delta_1 z + \cdots + \delta_n z^n}{(1-z)^{n+1}} \quad (11)$$

where the δ_j 's are positive integers [4, Theorem 3.12]: F_Q is the *Ehrhart series* and the vector

$$\delta = (\delta_0, \dots, \delta_n) \in \mathbb{N}^{n+1} \quad (12)$$

is the δ -vector of the polytope Q . We have

$$\delta_0 = 1, \delta_1 = \text{Card}(Q \cap M) - (n+1), \delta_n = \text{Card}(\text{Int}(Q) \cap M) \quad (13)$$

and

$$\delta_0 + \cdots + \delta_n = n! \text{Vol}(Q) \quad (14)$$

see [4, Chapter 3]. The δ -vector gives a characterization of reflexive polytopes, see for instance [4, Theorem 4.6]:

Proposition 2.2.1 *The polytope Q is reflexive if and only if $\delta_i = \delta_{n-i}$ for $i = 0, \dots, n$.*

2.3 Toric varieties

Let Δ be a fan in $N_{\mathbb{R}}$ and denote by $\Delta(i)$ the set of its cones of dimension i . The *rays* of Δ are its one-dimensional cones. Let $X := X_{\Delta}$ be the toric variety of the fan Δ : X is *simplicial* if each cone Δ is generated by independent vectors of $N_{\mathbb{R}}$, *complete* if the support of its fan (the union of its cones) is $N_{\mathbb{R}}$. The variety X is smooth if each cone is generated by a part of a basis of N . In general the Euler characteristic of a complete toric variety X_{Δ} is the number of maximal cones in the fan. If moreover X_{Δ} is simplicial, its cohomology groups vanish in odd degrees [6, Theorem 12.3.11] and $b_2(X_{\Delta}) = |\Delta(1)| - n$ [6, Theorem 12.3.12]. *Otherwise stated, all toric varieties that we will consider are complete and simplicial.*

One can get toric varieties from polytopes in the following ways:

- *The toric variety of a polytope in $M_{\mathbb{R}}$* : a full dimensional lattice polytope Q in $M_{\mathbb{R}}$ yields a toric variety X_Q , associated with the normal fan Σ_Q of Q , which is a fan in N .
- *The toric variety of a polytope in $N_{\mathbb{R}}$* : let $P \subset N_{\mathbb{R}}$ be a full dimensional lattice polytope containing the origin in its interior; we get a complete fan Δ_P in $N_{\mathbb{R}}$ by taking the cones over the proper faces of P . We get in this way a toric variety X_{Δ_P} .

Both constructions are dual, see for instance [6, Exercise 2.3.4]: if P° is the polar polytope of the polytope P in $N_{\mathbb{R}}$ then Δ_P is the normal fan of ℓP° where ℓ is an integer such that ℓP° is a lattice polytope and $X_{\Delta_P} = X_{\ell P^\circ}$. In particular, $X_{\Delta_P} = X_{P^\circ}$ if P is reflexive.

Recall that a projective normal toric variety X is (Gorenstein if not smooth) *Fano* (*resp. weak Fano*) if the anticanonical divisor $-K_X$ is Cartier and ample (*resp. nef and big*). (Weak) Fano toric varieties play an important role in our vision of mirror symmetry, see section 5.2. We will say that a full dimensional lattice polytope P containing the origin in its interior is

- *Fano* if the origin is contained in the strict interior of P and if each of its vertex is a primitive lattice point of N ,
- *smooth Fano* if the origin is contained in the strict interior of P and if each of its facets has exactly n vertices forming a basis of the lattice N .

2.4 Stacky fans and orbifold cohomology

Let Δ be a complete simplicial fan, ρ_1, \dots, ρ_r be its rays generated respectively by the primitive vectors v_1, \dots, v_r of N . Choose $b_1, \dots, b_r \in N$ whose images in $N_{\mathbb{Q}}$ generate ρ_1, \dots, ρ_r : the data $\mathbf{\Delta} = (N, \Delta, \{b_i\})$ is a *stacky fan*, see [3]. One associates to a stacky fan a Deligne-Mumford stack $\mathcal{X}(\mathbf{\Delta})$ and its orbifold cohomology $H_{orb}^\bullet(\mathcal{X}(\mathbf{\Delta}), \mathbb{Q})$: by [3, Proposition 4.7] we have

$$H_{orb}^{2i}(\mathcal{X}(\mathbf{\Delta}), \mathbb{Q}) = \bigoplus_{\sigma \in \Delta(n)} \bigoplus_{v \in \square(\sigma) \cap N} H^{2(i-\varphi(v))}(X_{Star(\sigma)}, \mathbb{Q})$$

where $\square(\sigma) := \{\sum_{\rho_i \subset \sigma} \lambda_i b_i, \lambda_i \in [0, 1]\}$ and $\varphi : N_{\mathbb{R}} \rightarrow \mathbb{R}$ is the linear function on each cone of Δ such that $\varphi(b_i) = 1$ for $i = 1, \dots, r$. If P is a lattice polytope containing the origin, there are integers a_i such that $b_i := a_i v_i \in \partial P \cap N$: the polytope P defines a stacky fan $\mathbf{\Delta} = (N, \Delta_P, \{b_i\})$, where the toric variety Δ_P is assumed to be simplicial: this is the *stacky fan of P* .

2.5 Batyrev's stringy functions

Let X_{Δ} be a normal \mathbb{Q} -Gorenstein toric variety and $\rho : Y \rightarrow X_{\Delta}$ be a toric (log-)resolution defined by a refinement Δ' of Δ , see [6, Proposition 11.2.4]. The irreducible components of the exceptional divisor of ρ are in one-to-one correspondence with the primitive generators v'_1, \dots, v'_q of the rays of $\Delta'(1)$ of Y that do not belong to $\Delta(1)$ and in the formula

$$K_Y = \rho^* K_{X_{\Delta}} + \sum_{i=1}^q a_i D_i \tag{15}$$

we have $a_i = \varphi(v'_i) - 1$ where φ is the support function of the divisor $K_{X_{\Delta}}$, see for instance [6, Lemma 11.4.10]. In our toric situation we have $a_i > -1$ because $\varphi(v'_i) > 0$.

Recall the E -polynomial of a smooth variety X defined by

$$E(X, u, v) := \sum_{p,q}^n (-1)^{p+q} h^{p,q}(X) u^p v^q \quad (16)$$

where the $h^{p,q}(X)$'s are the Hodge numbers of X . Notice that $E(X, 1, 1) = \chi(X)$, the Euler characteristic of X . It is possible to extend this definition to singular spaces having log-terminal singularities and to get *stringy* invariants that extend topological invariants of smooth varieties. Here is the construction: let $\rho : Y \rightarrow X$ be a resolution of $X := X_\Delta$ as above, $I = \{1, \dots, q\}$ and put, for any subset $J \subset I$,

$$D_J := \cap_{j \in J} D_j \text{ if } J \neq \emptyset, \quad D_J := Y \text{ if } J = \emptyset \text{ and } D_J^\circ = D_J - \bigcup_{j \in I-J} D_j$$

The following definition is due to Batyrev [1] (we assume that the product over \emptyset is 1; recall that $a_i > -1$):

Definition 2.5.1 *Let X be a toric variety. The function*

$$E_{st}(X, u, v) := \sum_{J \subset I} E(D_J^\circ, u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \quad (17)$$

is the stringy E -function of X . The number

$$e_{st}(X) := \lim_{u,v \rightarrow 1} E_{st}(X, u, v) \quad (18)$$

is the stringy Euler number.

The stringy E -function can be defined using motivic integrals, see [1] and [24]. By [1, Theorem 3.4], $E_{st}(X, u, v)$ do not depend on the resolution.

In our setting, E_{st} depends on the variable $z := uv$, and we will write $E_{st}(X, z)$ instead of $E_{st}(X, u, v)$.

3 The spectrum of a polytope

Let P be a full dimensional lattice polytope in $N_{\mathbb{R}}$. In this text, a spectrum $Spec_P$ of P is *a priori* an ordered sequence of rational numbers

$$\alpha_1 \leq \dots \leq \alpha_\mu$$

that we will identify with the generating function $Spec_P(z) := \sum_{i=1}^{\mu} z^{\alpha_i}$. The specifications are the following ($d(\alpha_i)$ denotes the multiplicity of α_i in the $Spec_P$):

- *Rationality* : the α_i 's are rational numbers,
- *Positivity* : the α_i 's are positive numbers,
- *Poincaré duality* : $Spec_P(z) = z^n Spec_P(z^{-1})$,

- *Volume* : $\lim_{z \rightarrow 1} \text{Spec}_P(z) = n! \text{Vol}(P) := \mu_P$
- *Normalisation* : $d(\alpha_1) = 1$
- *Modality (Lefschetz)* : $d(\alpha_1) \leq d(\alpha_2) \leq \dots \leq d(\alpha_\ell)$ if $\alpha_\ell \leq \lfloor \frac{n}{2} \rfloor$

In particular Spec_P is contained in $[0, n]$ and $\sum_{i=1}^{\mu} \alpha_i = \frac{n}{2} \mu_P$.

4 Geometric spectrum of a polytope

We define here the geometric spectrum of a polytope and we give several methods in order to compute it. Recall that the toric varieties considered here (and in the sequel) are assumed to be simplicial.

4.1 The geometric spectrum

Let P be a full dimensional lattice polytope in $N_{\mathbb{R}}$, containing the origin in its interior. Recall the Newton function ν of P of definition 2.1.1.

Definition 4.1.1 *The function*

$$\text{Spec}_P^{\text{geo}}(z) := (z - 1)^n \sum_{v \in N} z^{-\nu(v)}$$

is the geometric spectrum of the polytope P . The number $e_P := \lim_{z \rightarrow 1} \text{Spec}_P^{\text{geo}}(z)$ is the geometric Euler number of P .

We shall see (corollary 4.2.2) that $\text{Spec}_P^{\text{geo}}(z) = \sum_i d(c_i) z^{c_i}$ where the c_i 's are non-negative rational numbers and the $d(c_i)$'s are non-negative integers such that $\sum_i d(c_i) = e_P$. Let $\beta_1, \beta_2, \dots, \beta_{e_P}$ be the sequence (arranged by increasing order)

$$c_1, \dots, c_1, c_2, \dots, c_2, \dots, c_\ell, \dots, c_\ell$$

each c_i being counted $d(c_i)$ -times so that

$$\text{Spec}_P^{\text{geo}}(z) = z^{\beta_1} + z^{\beta_2} + \dots + z^{\beta_{e_P}}$$

We shall also say that the sequence $\beta_1, \beta_2, \dots, \beta_{e_P}$ is the *geometric spectrum* of the polytope P .

Remark 4.1.2 *Let X be a toric variety, v_1, \dots, v_k be the primitive generators of the rays of its fan. Assume that the v_i 's are the vertices of a convex polytope P . Then $\text{Spec}_P^{\text{geo}}(z) = E_{\text{st}}(X, z)$ (this follows from the definition of ν and [1, Theorem 4.3]).*

4.2 Various interpretations

We give three methods to compute $\text{Spec}_P^{\text{geo}}$, showing that it yields finally a spectrum of P in the sense of section 3. The first one and the third one are inspired by the works of Mustata-Payne [19] and Stapledon [23]. The second one is inspired by Batyrev's stringy E -functions.

4.2.1 First interpretation: fundamental domains

Let P be a full dimensional lattice polytope in $N_{\mathbb{R}}$, containing the origin in its interior, $\Delta := \Delta_P$ be the complete fan whose maximal cones are built on the facets of P . As in section 2.4, we identify each vertex of P with an element $b_i \in N$. If $\sigma \in \Delta(r)$ is generated by b_1, \dots, b_r , set

$$\square(\sigma) := \left\{ \sum_{i=1}^r q_i b_i, q_i \in [0, 1[, i = 1, \dots, r \right\},$$

and

$$\text{Box}(\sigma) := \left\{ \sum_{i=1}^r q_i b_i, q_i \in]0, 1[, i = 1, \dots, r \right\}$$

Proposition 4.2.1 *We have*

$$\text{Spec}_P^{\text{geo}}(z) = \sum_{r=0}^n (z-1)^{n-r} \sum_{\sigma \in \Delta(r)} \sum_{v \in \square(\sigma) \cap N} z^{\nu(v)} \quad (19)$$

and $e_P = n! \text{Vol}(P)$.

Proof. Let $\sigma \in \Delta(r)$. A lattice element $v \in \overset{\circ}{\sigma}$ has one of the following decomposition:

- $v = w + \sum_{i=1}^r \lambda_i b_i$ with $w \in \text{Box}(\sigma) \cap N$ and $\lambda_i \geq 0$ for all i ,
- $v = w + \sum_{i=1}^r \lambda_i b_i$ with $w \in \text{Box}^c(\sigma) \cap N - \{0\}$, $\lambda_i \geq 0$ for all $i \geq 2$ and $\lambda_1 > 0$,
- $v = \sum_{i=1}^r \lambda_i b_i$ where $\lambda_i > 0$ for all i

where $\text{Box}^c(\sigma)$ is the complement of $\text{Box}(\sigma)$ in $\square(\sigma)$. We get

$$(z-1)^r \sum_{v \in \overset{\circ}{\sigma} \cap N} z^{-\nu(v)} = \sum_{v \in \text{Box}(\sigma) \cap N} z^{r-\nu(v)} + \sum_{v \in \text{Box}^c(\sigma) \cap N - \{0\}} z^{r-1-\nu(v)} + 1 \quad (20)$$

because

- $\sum_{\lambda_1, \dots, \lambda_r \geq 0} z^{-\nu(w)} z^{-\lambda_1} \dots z^{-\lambda_r} = \frac{z^{r-\nu(w)}}{(z-1)^r}$ if $w \in \text{Box}(\sigma) \cap N$,
- $\sum_{\lambda_1 > 0, \lambda_2, \dots, \lambda_r \geq 0} z^{-\nu(w)} z^{-\lambda_1} \dots z^{-\lambda_r} = \frac{z^{r-1-\nu(w)}}{(z-1)^r}$ if $w \in \text{Box}^c(\sigma) \cap N - \{0\}$,
- $\sum_{\lambda_1, \dots, \lambda_r > 0} z^{-\lambda_1} \dots z^{-\lambda_r} = \frac{1}{(z-1)^r}$

(and we use the fact that $\nu(b_i) = 1$). Moreover,

- $\alpha \in \nu(\text{Box}(\sigma)) := \{\nu(v), v \in \text{Box}(\sigma)\}$ if and only if $r - \alpha \in \nu(\text{Box}(\sigma))$,
- $\alpha \in \nu(\text{Box}^c(\sigma)) := \{\nu(v), v \in \text{Box}^c(\sigma)\}$ if and only if $r - 1 - \alpha \in \nu(\text{Box}^c(\sigma))$.

because $q_i \in]0, 1[$ if and only if $1 - q_i \in]0, 1[$. We then deduce from (20) that

$$(z - 1)^n \sum_{v \in \overset{\circ}{\sigma} \cap N} z^{-\nu(v)} = (z - 1)^{n-r} \sum_{v \in \square(\sigma) \cap N} z^{\nu(v)} \quad (21)$$

for any $\sigma \in \Delta(r)$. The expected equality follows because the relative interiors of the cones of the complete fan Δ give a partition of its support. For the assertion about the Euler number, notice that

$$\lim_{z \rightarrow 1} \text{Spec}_P^{geo}(z) = \sum_{\sigma \in \Delta(n)} \sum_{v \in \square(\sigma) \cap N} 1 = n! \text{Vol}(P)$$

because the volume of $\sigma \cap \{v \in N_{\mathbb{R}}, \nu(v) \leq 1\}$ is equal to the number of lattice points in $\square(\sigma)$. \square

Corollary 4.2.2 *We have*

$$\text{Spec}_P^{geo}(z) = \sum_{\sigma \in \Delta} \sum_{v \in \text{Box}(\sigma) \cap N} h_{\sigma}(z) z^{\nu(v)}$$

where $h_{\sigma}(z) := \sum_{\tau \subseteq \sigma} (z - 1)^{n - \dim \tau}$. In particular, $\text{Spec}_P^{geo}(z) = \sum_i d(c_i) z^{c_i}$ where the c_i 's are non-negative rational numbers and the $d(c_i)$'s are non-negative natural numbers such that $\sum_i d(c_i) = e_P$.

Proof. The expected equality follows from (19). For the remaining assertions, notice that h_{σ} is the E -polynomial of the orbit closure $V(\sigma)$ (as defined for instance in [6, page 121]), hence the E -polynomial of a toric variety: it follows that its coefficients are non-negative natural numbers. \square

4.2.2 Second interpretation: resolution of singularities

Let P be a full dimensional lattice polytope in $N_{\mathbb{R}}$, containing the origin in its interior. Let $\rho : Y \rightarrow X$ be a resolution of $X := X_{\Delta_P}$ as in section 2.5, ρ_1, \dots, ρ_r be the rays of Y with primitive generators v_1, \dots, v_r and associated divisors D_1, \dots, D_r and I . Put, for any subset $J \subset I := \{1, \dots, r\}$,

$$D_J := \bigcap_{j \in J} D_j \text{ if } J \neq \emptyset, \quad D_J := Y \text{ if } J = \emptyset$$

and define

$$E_{st,P}(z) := \sum_{J \subset I} E(D_J, z) \prod_{j \in J} \frac{z - z^{\nu_j}}{z^{\nu_j} - 1} \quad (22)$$

where $\nu_j = \nu(v_j)$ and ν is the Newton function of P , see definition 2.1.1.

Proposition 4.2.3 *We have $\text{Spec}_P^{geo}(z) = E_{st,P}(z)$. In particular, $E_{st,P}(z)$ does not depend on the resolution ρ .*

Proof. Using the notations of section 2.5 we have $E(D_J^{\circ}, z) = \sum_{J' \subset J} (-1)^{|J| - |J'|} E(D_{J'}, z)$ and

$$E_{st,P}(z) = \sum_{J \subset I} E(D_J^{\circ}, z) \prod_{j \in J} \frac{z - 1}{z^{\nu_j} - 1}$$

as in [1, Proof of theorem 3.7]. Let σ be a smooth cone of Δ' , the fan of Y , generated by v_{i_1}, \dots, v_{i_r} and $v \in \overset{\circ}{\sigma}$: we have $v = a_1 v_{i_1} + \dots + a_r v_{i_r}$ for $a_1, \dots, a_r > 0$ and $\nu(v) = a_1 \nu(v_{i_1}) + \dots + a_r \nu(v_{i_r})$. Thus

$$\sum_{v \in \overset{\circ}{\sigma} \cap N} z^{-\nu(v)} = \frac{1}{z^{\nu(v_{i_1})} - 1} \cdots \frac{1}{z^{\nu(v_{i_r})} - 1}$$

With these two observations in mind, the proof of the proposition is similar to the one of [1, Theorem 4.3]. \square

Corollary 4.2.4 *We have $z^n \text{Spec}_P^{geo}(z^{-1}) = \text{Spec}_P^{geo}(z)$.*

Proof. Applying Poincaré duality to the smooth subvarieties D_J , we get

$$E(D_J, z^{-1}) \prod_{j \in J} \frac{z^{-1} - z^{-\nu_j}}{z^{-\nu_j} - 1} = z^{|J|-n} E(D_J, z) z^{-|J|} \prod_{j \in J} \frac{z - z^{\nu_j}}{z^{\nu_j} - 1}$$

and the assertion follows. \square

4.2.3 Third interpretation: twisted δ -vector

Let P be a full dimensional lattice polytope in $N_{\mathbb{R}}$, containing the origin in its interior. Following [23], we define

$$F_P^0(z) = \sum_{m \geq 0} \sum_{v \in mP \cap N} z^{\nu(v) - \lceil \nu(v) \rceil + m}$$

which is a twisted version of the Ehrhart series $F_P(z)$ defined in section 2.2.

Proposition 4.2.5 *We have $\text{Spec}_P^{geo}(z) = (1 - z)^{n+1} F_P^0(z)$.*

Proof. Notice first that $v \in mP$ if and only if $\nu(v) \leq m$: this follows from the presentation (8) and the definition of the Newton function ν . We thus have

$$F_P^0(z^{-1}) = \sum_{m \geq 0} \sum_{\nu(v) \leq m} z^{-\nu(v) + \lceil \nu(v) \rceil - m} = \sum_{v \in N} \sum_{\lceil \nu(v) \rceil \leq m} z^{-\nu(v) + \lceil \nu(v) \rceil - m} = \frac{1}{1 - z^{-1}} \sum_{v \in N} z^{-\nu(v)}$$

It follows that

$$(z - 1)^n (1 - z^{-1}) F_P^0(z^{-1}) = \text{Spec}_P^{geo}(z)$$

and

$$(1 - z)^{n+1} F_P^0(z) = z^n \text{Spec}_P^{geo}(z^{-1}) = \text{Spec}_P^{geo}(z)$$

by corollary 4.2.4. \square

Corollary 4.2.6 *Assume that X is simplicial. The coefficient of z^{α_i} in $\text{Spec}_P^{geo}(z)$ is equal to $\dim_{\mathbb{C}} H_{orb}^{2\alpha_i}(\mathcal{X}(\Delta), \mathbb{C})$ where $\mathcal{X}(\Delta)$ is the stack defined by P (see section 2.4).*

Proof. Follows from proposition 4.2.5 and [23, Theorem 4.3]. □

Corollary 4.2.7 *We have*

$$\text{Spec}_P^{geo} \cap [0, 1[= \{\nu(v), v \in \text{Int } P \cap N\}$$

where ν is the Newton function of P , the left hand side denoting the part of the spectrum contained in $[0, 1[$. Moreover the multiplicity of 1 in Spec_P^{geo} is equal to $|\partial P \cap N| - n$.

Proof. Scrutinization of the coefficients of z^a , $a \leq 1$, in $\text{Spec}_P^{geo}(z)$ and proposition 4.2.5. See also [23, Lemma 3.13]. □

If P is reflexive, we have the following link between the δ -vector of P from section 2.2 and its geometric spectrum (see also [19]):

Corollary 4.2.8 *Let P be a reflexive full dimensional lattice polytope containing the origin in its interior. Then*

$$\text{Spec}_P^{geo}(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n \tag{23}$$

where $\delta = (\delta_0, \dots, \delta_n)$ is the δ -vector of P .

Proof. By (9) we have $\nu(v) \in \mathbb{N}$ for all $v \in N$ because P is reflexive. We thus get $F_P^0(z) = F_P(z)$ where $F_P(z)$ is the Ehrhart series of P of section 2.2 because

$$F_P(z) = \sum_{m \geq 0} |mP \cap N| z^m = \sum_{m \geq 0} \sum_{v \in mP \cap N} z^m$$

By proposition 4.2.5 we have $\text{Spec}_P^{geo}(z) = (1 - z)^{n+1} F_P(z)$ and we use formula (11). □

4.2.4 Résumé

The geometric spectrum is thus a spectrum in the sense of section 3. Rationality, positivity and the volume property are given by proposition 4.2.1, symmetry (Poincaré duality) by corollary 4.2.4 and modality by corollary 4.2.6.

5 Algebraic spectrum of a polytope

Singularity theory associates to a (tame) Laurent polynomial function a *spectrum at infinity*, see [22]. We recall its definition and its main properties in section 5.3. We can shift this notion to the Newton polytope P of f and get in this way the *algebraic spectrum* of P . In order to motivate the next sections, we describe the Givental-Hori-Vafa models [13], [16] which are the expected mirror partners of toric varieties. This will also emphasize the link between regular functions and polytopes and we will test our results on this class of examples. In order to make the text as self-contained as possible, we first recall Kouchnirenko's results.

5.1 Preliminaries: Kouchnirenko's framework

We briefly recall the setting of [17]. Let $f : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ be a Laurent polynomial, $f(\underline{u}) = \sum_{a \in \mathbb{Z}^n} c_a \underline{u}^a$ where $\underline{u}^a := u_1^{a_1} \cdots u_n^{a_n}$. The Newton polytope P of f is the convex hull of the multi-indices a such that $c_a \neq 0$. We say that f is *convenient* if P contains the origin in its interior, *nondegenerate* if, for any face F of P , the system

$$u_1 \frac{\partial f_F}{\partial u_1} = \cdots = u_n \frac{\partial f_F}{\partial u_n} = 0$$

has no solution on $(\mathbb{C}^*)^n$ where $f_F(\underline{u}) = \sum_{a \in F \cap P} c_a \underline{u}^a$ and the sum is taken over the multi-indices a such that $c_a \neq 0$. A convenient and nondegenerate Laurent polynomial f has only isolated critical points and its global Milnor number μ_f (the number of critical points with multiplicities) is $\mu_P := n! \text{Vol}(P)$. Moreover, f is tame in the sense that the set outside which f is a locally trivial fibration is made from critical values of f , and these critical values belong to this set only because of the critical points at finite distance.

5.2 Givental-Hori-Vafa models and mirror symmetry

Let $N = \mathbb{Z}^n$, M be the dual lattice, Δ be a complete and simplicial fan and v_1, \dots, v_r be the primitive generators of its rays. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z}^{r-n} \xrightarrow{\psi} \mathbb{Z}^r \xrightarrow{\varphi} \mathbb{Z}^n \longrightarrow 0$$

where $\varphi(e_i) = v_i$ for $i = 1, \dots, r$ ((e_i) denotes the canonical basis of \mathbb{Z}^r) and ψ describes the relations between the v_i 's. Applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to this exact sequence, we get

$$1 \longrightarrow (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^r \xrightarrow{\pi} (\mathbb{C}^*)^{r-n} \longrightarrow 1$$

where

$$\pi(u_1, \dots, u_r) = (q_1, \dots, q_{r-n}) = (u_1^{a_{1,1}} \cdots u_r^{a_{r,1}}, \dots, u_1^{a_{1,r-n}} \cdots u_r^{a_{r,r-n}}) \quad (24)$$

and the integers $a_{i,j}$ satisfy $\sum_{j=1}^r a_{j,i} v_j = 0$ for $i = 1, \dots, r-n$. The *Givental-Hori-Vafa model* of X_{Δ} is the function

$$u_1 + \cdots + u_r \text{ restricted to } U := \pi^{-1}(q_1, \dots, q_{r-n})$$

We will denote it by f_{Δ} . The stacky version of this construction is straightforward (replace the v_i 's by the b_i 's).

Proposition 5.2.1 *Assume that (v_1, \dots, v_n) is the canonical basis of N . Then f_{Δ} is the Laurent polynomial defined on $(\mathbb{C}^*)^n$ by*

$$f_{\Delta}(u_1, \dots, u_n) = u_1 + \cdots + u_n + \sum_{i=n+1}^r q_i u_1^{v_1^i} \cdots u_n^{v_n^i}$$

if $v_i = (v_1^i, \dots, v_n^i) \in \mathbb{Z}^n$ for $i = n+1, \dots, r$.

Proof. We have $v_i = \sum_{j=1}^n v_j^i v_j$ for $i = n+1, \dots, r$ and we use presentation (24). \square

Above f we make grow a differential system and we say that f is a mirror partner of a variety X if this differential system is isomorphic to the one associated with the (small quantum) cohomology of X , see for instance [8], [9], [21]. If f is the mirror partner of a smooth variety X the following properties are in particular expected (non-exhaustive list):

- the Milnor number of f is equal to the rank of the cohomology of X ,
- the spectrum at infinity of f (see section 5.3 below) is equal to half of the degrees of the cohomology groups of X ,
- multiplication by f on its Jacobi ring yields the cup-product by $c_1(X)$ on the cohomology algebra of X .

(the ulterior motive of this text was to discuss the singular case, where cohomology should be replaced by orbifold cohomology). The first thing to do is to compare the dimension of the Jacobi ring of f_Δ , hence its Milnor number μ_{f_Δ} , and the rank of the cohomology algebra of X_Δ . In the smooth case, we have equality if (and only if) X_Δ is weak Fano. Indeed, let P be the convex hull the primitive generators of the rays of Δ :

- If X_Δ is Fano (P is a smooth Fano polytope, see section 2.3), the Givental-Hori-Vafa model of X_Δ on the fiber $\pi^{-1}(1, \dots, 1)$ is $f_\Delta(\underline{u}) = \sum_{a \in P \cap \mathbb{Z}^n} \underline{u}^a$; f_Δ is convenient and nondegenerate, thanks to the smooth Fano condition, and its Milnor number is $\mu_{f_\Delta} = n! \text{Vol}(P) = \chi(X_\Delta)$.
- If X_Δ is weak Fano (P is reflexive), we can argue as above, decomposing P into simplices: f_Δ is convenient and (generically with respect to the parameter q) nondegenerate.

If X_Δ is smooth, complete, but not weak Fano we have $\mu_{f_\Delta} > \chi(X_\Delta)$: see section 5.5 for a picture of this phenomenon.

5.3 The spectrum at infinity of a tame Laurent polynomial

We assume in this section that f is a convenient and nondegenerate Laurent polynomial, defined on $U := (\mathbb{C}^*)^n$, with global Milnor number μ . For the (very small) D -module part, we use the notations of [10, 2.c]. Let G be the Fourier-Laplace transform of the Gauss-Manin system of f , G_0 be its Brieskorn lattice (G_0 is indeed a free $\mathbb{C}[\theta]$ -module because f is convenient and nondegenerate and $G = \mathbb{C}[\theta, \theta^{-1}] \otimes G_0$, see [10, Remark 4.8]) and V_\bullet be the V -filtration of G at infinity, that is along $\theta^{-1} = 0$. From these data we get by projection a V -filtration on the μ -dimensional vector space $\Omega_f := \Omega^n(U)/df \wedge \Omega^{n-1}(U) = G_0/\theta G_0$, see [10, Section 2.e].

Definition 5.3.1 *The spectrum at infinity Spec_f of f is the spectrum of the V -filtration defined on Ω_f .*

Thus, the spectrum at infinity of f is the (ordered) sequence $\alpha_1, \alpha_2, \dots, \alpha_\mu$ of rational numbers with the following property: the frequency of α in the spectrum is equal to $\dim_{\mathbb{C}} \text{gr}_V^\alpha \Omega_f$. We will write $\text{Spec}_f(z) = \sum_{i=1}^\mu z^{\alpha_i}$. Recall the following facts, see for instance [10]:

- Spec_f is positive : $\alpha_i \geq 0$ for all i ,

- $Spec_f(z) = z^n Spec_f(z^{-1})$

In particular, $Spec_f \subset [0, n]$.

In the convenient and nondegenerate case, the spectrum at infinity of f can be computed using the Newton function of the Newton polytope of f : let us define the Newton filtration ν_\bullet on $\Omega^n(U)$ by

$$\nu_\alpha \Omega^n(U) := \{\omega \in \Omega^n(U), \nu(\omega) \leq \alpha\}$$

where

$$\nu(\omega) := \nu(v) \text{ if } \omega = u_1^{v_1} \cdots u_n^{v_n} \frac{du_1 \wedge \cdots \wedge du_n}{u_1 \cdots u_n} \text{ and } v = (v_1, \dots, v_n) \in N$$

(notice the normalization $\nu(\frac{du_1 \wedge \cdots \wedge du_n}{u_1 \cdots u_n}) = 0$). This filtration induces a filtration on Ω_f by projection and the spectrum at infinity of f is equal to the spectrum of this filtration, see [10, Corollary 4.13].

Proposition 5.3.2 *Let f be a convenient and nondegenerate Laurent polynomial. Then $Spec_f$ depends only on the Newton polytope P of f and*

$$Spec_f \cap [0, 1[= \{\nu(v), v \in \text{Int } P \cap N\}$$

where ν is the Newton function of P and $Spec_f \cap [a, b[$ denotes the part of $Spec_f$ contained in $[a, b[$. In particular, the multiplicity of 0 in $Spec_f$ is equal to one.

Proof. Let f and g be two convenient and nondegenerate Laurent polynomials having the same Newton polytope P . By [17] we have $\mu_f = \mu_g$ and it follows from [20] that $Spec_f(z) = Spec_g(z)$. This gives the first assertion. The second then follows from [10, Lemma 4.6], as in [10, Example 4.17]. \square

5.4 The algebraic spectrum of a polytope

We define the algebraic spectrum $Spec_P^{alg}$ of a full dimensional lattice polytope P containing the origin in its interior to be the spectrum at infinity of the convenient and nondegenerate Laurent polynomial $f(\underline{u}) = \sum_{b \in \mathcal{V}(P)} \underline{u}^b$ where $\mathcal{V}(P)$ denotes the set of the vertices of P . As usual, we will identify it with its generating function $Spec_P^{alg}(z) = \sum_{i=1}^{\mu} z^{\alpha_i}$.

In the two dimensional case, this algebraic spectrum of is easily described:

Proposition 5.4.1 *Let P be a full dimensional lattice polytope in $N_{\mathbb{R}} = \mathbb{R}^2$. Then*

$$Spec_P^{alg}(z) = (\text{Card}(\partial P \cap N) - 2)z + \sum_{v \in \text{Int } P \cap N} (z^{\nu(v)} + z^{2-\nu(v)})$$

where ν is the Newton function of P .

Proof. Let f be a convenient and nondegenerate Laurent polynomial whose Newton polytope is P . From proposition 5.3.2 and the symmetry of the spectrum we get

$$Spec_P^{alg}(z) = Spec_f(z) = (\text{Card}(\partial P \cap N) - 2)z + \sum_{v \in \text{Int } P \cap N} (z^{\nu(v)} + z^{2-\nu(v)})$$

The coefficient of z is computed using Pick's formula because $\mu_f = 2! \text{Vol}(P)$. \square

We also have the following description for reflexive polytopes:

Proposition 5.4.2 *Let P be a full dimensional reflexive polytope in $N_{\mathbb{R}} = \mathbb{R}^n$. Then:*

- $Spec_P^{alg}(z) = \sum_{i=0}^n d(i)z^i$ where $d(i) \in \mathbb{N}$,
- $d(i) = d(n-i)$ for $i = 0, \dots, n$ with $d(0) = d(n) = 1$,
- $\sum_{i=0}^n d(i) = \mu_P := n! \text{Vol}(P)$

Proof. Because P is reflexive, the Newton function takes integer values at the lattice points, see (9). This gives the first point because $Spec_P^{alg} \subset [0, n]$. For the second one, use the symmetry and the fact that 0 is in the spectrum with multiplicity one because the origin is in the interior of P . The third assertion follows from [17]. \square

5.5 Example: Hirzebruch surfaces and their Givental-Hori-Vafa models

Let m be a positive integer. The fan $\Delta_{\mathbb{F}_m}$ of the Hirzebruch surface \mathbb{F}_m is the one whose rays are generated by the vectors $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, m)$, $v_4 = (0, -1)$, see for instance [12]. The surface \mathbb{F}_m is Fano if $m = 1$, weak Fano if $m = 2$. Its Givental-Hori-Vafa model is the Laurent polynomial

$$f_m(u_1, u_2) = u_1 + u_2 + \frac{q_1}{u_2} + q_2 \frac{u_2^m}{u_1}$$

defined on $(\mathbb{C}^*)^2$, where q_1 and q_2 are two non zero parameters. We have

1. $\mu_{f_1} = 4$ and $Spec_{f_1}(z) = 1 + 2z + z^2$,
2. $\mu_{f_2} = 4$ if $q_2 \neq \frac{1}{4}$ and $Spec_{f_2}(z) = 1 + 2z + z^2$,
3. $\mu_{f_m} = m + 2$ if $m \geq 3$ and

$$Spec_{f_m}(z) = 1 + 2z + z^2 + z^{\frac{1}{p}} + z^{\frac{2}{p}} + \dots + z^{\frac{p-1}{p}} + z^{2-\frac{1}{p}} + z^{2-\frac{2}{p}} + \dots + z^{2-\frac{p-1}{p}}$$

if $m = 2p$ and $p \geq 2$,

$$Spec_{f_m}(z) = 1 + z + z^2 + z^{\frac{2}{m}} + z^{\frac{4}{m}} + \dots + z^{\frac{2p}{m}} + z^{2-\frac{2}{m}} + z^{2-\frac{4}{m}} + \dots + z^{2-\frac{2p}{m}}$$

if $m = 2p + 1$ and $p \geq 1$.

Indeed, for $m \neq 2$ we have $\mu_{f_m} = 2! \text{Vol}(P)$, where P is the Newton polytope of f_m , because f_m is convenient and nondegenerate for all non zero value of the parameters, see section 5.1. For $m = 2$, f_2 is nondegenerate if and only if $q_2 \neq \frac{1}{4}$ and the previous argument applies in this case (if $q_2 = 1/4$ the Milnor number is 2). The spectrum is given by proposition 5.4.1.

The function f_2 is a guenuine mirror partner of the surface \mathbb{F}_2 , see [8], [21]. If $m \geq 3$, we have $\mu_{f_m} > 4$ and the model f_m has too many critical points: because \mathbb{F}_m is not weak Fano in this case, this is consistent with the results of section 5.2.

6 Geometric spectrum vs algebraic spectrum

We show the equality $Spec_P^{alg}(z) = Spec_P^{geo}(z)$ in some cases. We expect it to be true for any full dimensional lattice polytope P containing the origin in its interior.

6.1 First cases of equality

A first step towards equality:

Proposition 6.1.1 *Let P be a full dimensional lattice polytope containing the origin in its interior. Then*

$$\text{Spec}_P^{\text{alg}} \cap [0, 1[= \text{Spec}_P^{\text{geo}} \cap [0, 1[= \{\nu(v), v \in \text{Int } P \cap N\}$$

where ν is the Newton function of P .

Proof. Corollary 4.2.7 and proposition 5.3.2. □

In the two dimensional case we get:

Corollary 6.1.2 *Let P be a full dimensional polytope in $N_{\mathbb{R}} = \mathbb{R}^2$, containing the origin in its interior. Then $\text{Spec}_P^{\text{alg}}(z) = \text{Spec}_P^{\text{geo}}(z)$.*

Proof. Use proposition 6.1.1, the symmetry and the fact that $\text{Spec}_P^{\text{alg}}(1) = \text{Spec}_P^{\text{geo}}(1) = \mu_P$ where the last equality follows from proposition 4.2.1. □

Let now P be a full dimensional reflexive polytope in $N_{\mathbb{R}}$ containing the origin, and $X := X_{\Delta_P}$, see section 2.3.

Proposition 6.1.3 *Assume that X has a crepant resolution¹. Then $\text{Spec}_P^{\text{alg}}(z) = \text{Spec}_P^{\text{geo}}(z)$. In particular, the multiplicity of 1 in $\text{Spec}_P^{\text{alg}}$ is equal to $\text{Card}(\partial P \cap N) - n$.*

Proof. Let us denote by $\rho : Y \rightarrow X$ the crepant resolution alluded to. Because P is reflexive, Y is smooth and weak Fano and thus has a mirror partner f_Y , see for instance [21, Proposition 3.9]. We have

$$\begin{aligned} \text{Spec}_{f_Y}(z) &= h^0(Y) + \cdots + h^{2n}(Y)z^n = h_{st}^0(Y) + \cdots + h_{st}^{2n}(Y)z^n \\ &= h_{st}^0(X) + \cdots + h_{st}^{2n}(X)z^n = E_{st}(z) = \text{Spec}_P^{\text{geo}}(z) \end{aligned}$$

The first equality follows from the fact that f_Y is the mirror partner of Y , the second one follows from [1, corollary 3.6] because Y is smooth, the third one by [1, Theorem 3.12] because ρ is crepant, the fourth one by [1, Proposition 4.4] because P is reflexive and the last one by remark 4.1.2. We also have $\text{Spec}_{f_Y} = \text{Spec}_P^{\text{alg}}$ by proposition 5.3.2 and the result follows. For the assertion about the multiplicity of 1, use corollary 4.2.8 and equations (13). □

¹This will always be the case if $n = 2$.

6.2 A significant class of examples: weighted projective spaces

Let $(\lambda_0, \dots, \lambda_n) \in (\mathbb{N}^*)^{n+1}$ such that $\gcd(\lambda_0, \dots, \lambda_n) = 1$ and X be the weighted projective space $\mathbb{P}(\lambda_0, \dots, \lambda_n)$. The (stacky) fan of X is the simplicial complete fan whose rays are generated by vectors e_0, \dots, e_n in N such that

1. $\lambda_0 e_0 + \dots + \lambda_n e_n = 0$
2. the e_i 's generate N

Such a family is unique, up to isomorphism. We have $\lambda_0 = 1$ if and only if (e_1, \dots, e_n) is a basis of N and this will be our favorite situation: we assume from now on that $\lambda_0 = 1$.

Definition 6.2.1 *The polytope of $\mathbb{P}(1, \lambda_1, \dots, \lambda_n)$ is the convex hull P of e_0, \dots, e_n .*

Notice that $\mu_P = 1 + \lambda_1 + \dots + \lambda_n$ and $\mu_{P^\circ} = \frac{(1 + \lambda_1 + \dots + \lambda_n)^n}{\lambda_1 \dots \lambda_n}$ (recall that $\mu_Q := n! \text{Vol}(Q)$ if Q is a polytope in $N_{\mathbb{R}}$, see equation (10)). Moreover, P is reflexive if and only if λ_i divides μ_P for all i .

The Givental-Hori-Vafa model of $\mathbb{P}(1, \lambda_1, \dots, \lambda_n)$ (a mirror theorem is shown in [9]) is the Laurent polynomial defined on $(\mathbb{C}^*)^n$ by

$$f(u_1, \dots, u_n) = u_1 + \dots + u_n + \frac{q}{u_1^{\lambda_1} \dots u_n^{\lambda_n}}$$

where $q \in \mathbb{C}^*$, see proposition 5.2.1 and its Milnor number is $\mu_f = 1 + \lambda_1 + \dots + \lambda_n = \mu_P$. Let

$$F := \left\{ \frac{\ell}{\lambda_i} \mid 0 \leq \ell \leq \lambda_i - 1, 0 \leq i \leq n \right\}.$$

and f_1, \dots, f_k the elements of F arranged by increasing order. We then define

$$S_{f_i} := \{j \mid \lambda_j f_i \in \mathbb{Z}\} \subset \{0, \dots, n\} \text{ and } d_i := \text{Card } S_{f_i}$$

Let $c_0, c_1, \dots, c_{\mu-1}$ be the sequence

$$\underbrace{f_1, \dots, f_1}_{d_1}, \underbrace{f_2, \dots, f_2}_{d_2}, \dots, \underbrace{f_k, \dots, f_k}_{d_k}$$

arranged by increasing order. The following result can be found in [11, Theorem 1]:

Lemma 6.2.2 *The spectrum of f is the sequence $\alpha_0, \alpha_1, \dots, \alpha_{\mu-1}$ where $\alpha_k := k - \mu c_k$ for $k = 0, \dots, \mu - 1$.*

Notice that the spectrum of f is integral if and only if the polytope of $\mathbb{P}(1, \lambda_1, \dots, \lambda_n)$ is reflexive.

Proposition 6.2.3 *Let P be the polytope of the weighted projective space $\mathbb{P}(1, \lambda_1, \dots, \lambda_n)$. Then $\text{Spec}_P^{\text{alg}}(z) = \text{Spec}_P^{\text{geo}}(z)$.*

Proof. Use corollary 4.2.6 and [9, Lemma 3.4.2]. □

Example 6.2.4 We test proposition 6.2.3 (the computation of the geometric spectrum is done using proposition 4.2.3).

1. Let a be a positive integer, Δ be the fan whose rays are $(1, 0)$, $(-1, -a)$ and $(0, 1)$ and P their convex hull: P is the polytope of $\mathbb{P}(1, 1, a)$. Using the fan Δ' whose rays are $(1, 0)$, $(0, -1)$, $(-1, -a)$ and $(0, 1)$ we get

$$\begin{aligned} \text{Spec}_P^{\text{geo}}(z) &= E(\mathbb{F}_a, z) + E(\mathbb{P}^1, z) \frac{z - z^{2/a}}{z^{2/a} - 1} = 1 + 2z + z^2 + (1+z) \left(\frac{z-1}{z^{2/a} - 1} - 1 \right) \\ &= 1 + z + z^2 + z^{2/a} + z^{4/a} + \dots + z^{2(a-1)/a} = \text{Spec}_P^{\text{alg}}(z) \end{aligned}$$

where \mathbb{F}_a is the Hirzebruch surface. We have $e_P = a + 2 = \mu_P$.

2. Let Δ be the fan whose rays are $(1, 0)$, $(-\ell, -\ell)$ for $\ell \in \mathbb{N}^*$ and $(0, 1)$ and P their convex hull: P is the polytope of $\mathbb{P}(1, \ell, \ell)$. The variety $X := X_{\Delta_P}$ is \mathbb{P}^2 , generated by the rays $v_1 = (-1, -1)$, $v_2 = (0, 1)$ and $v_3 = (1, 0)$, with $\nu(v_1) = \frac{1}{\ell}$ and we get

$$\begin{aligned} \text{Spec}_P^{\text{geo}}(z) &= E(\mathbb{P}^2, z) + E(\mathbb{P}^1, z) \frac{z - z^{1/\ell}}{z^{1/\ell} - 1} = 1 + z + z^2 + (1+z) \frac{z - z^{1/\ell}}{z^{1/\ell} - 1} \\ &= z^2 + z + 1 + z^{1/\ell} + \dots + z^{(\ell-1)/\ell} + z^{1+1/\ell} + \dots + z^{1+(\ell-1)/\ell} = \text{Spec}_P^{\text{alg}}(z) \end{aligned}$$

We also have $e_P = 1 + 2\ell = \mu_P$.

3. Let Δ be the fan whose rays are $(1, 0)$, $(-2, -5)$ and $(0, 1)$ and P their convex hull: P is the polytope of $\mathbb{P}(1, 2, 5)$. Using the fan Δ' whose rays are $(1, 0)$, $(0, -1)$, $(-1, -3)$, $(-2, -5)$, $(-1, -2)$ and $(0, 1)$ we get

$$\text{Spec}_P^{\text{geo}}(z) = z^2 + 4z + 1 + (z+1) \frac{z - z^{3/5}}{z^{3/5} - 1} + (z+1) \frac{z - z^{4/5}}{z^{4/5} - 1} + \frac{z - z^{3/5}}{z^{3/5} - 1} \cdot \frac{z - z^{4/5}}{z^{4/5} - 1}$$

Gathering the different terms we also get

$$\text{Spec}_P^{\text{geo}}(z) = z^2 + 2z + 1 + z^{3/5} + z^{4/5} + z^{6/5} + z^{7/5} = \text{Spec}_P^{\text{alg}}(z)$$

We have $e_P = 8 = \mu_P$.

7 Libgober-Wood's formula for the spectra

We are now ready to prove formula (3) of the introduction. We first show it for the geometric spectrum.

7.1 Spectra and Hirzebruch-Riemann-Roch

In order to get first a stacky version of the Libgober-Wood formula (1), we give the following definition, inspired by Batyrev's stringy number $c_{st}^{1, n-1}(X)$, see [2, Definition 3.1]:

Definition 7.1.1 Let P be a full dimensional lattice polyope in N containing the origin in its interior and let $\rho : Y \rightarrow X$ be a resolution of $X := X_{\Delta_P}$. We define

$$\begin{aligned} \widehat{\mu}_P &:= c_1(Y)c_{n-1}(Y) + \sum_{J \subset I, J \neq \emptyset} c_1(D_J)c_{n-|J|-1}(D_J) \prod_{j \in J} \frac{1-\nu_j}{\nu_j} \\ &\quad - \sum_{J \subset I, J \neq \emptyset} \left(\sum_{j \in J} \nu_j \right) c_{n-|J|}(D_J) \prod_{j \in J} \frac{1-\nu_j}{\nu_j} \end{aligned} \quad (25)$$

where the notations in the right hand term are the ones of section 4.2.2 (convention: $c_r(D_J) = 0$ if $r < 0$).

The number $\widehat{\mu}_P$ is a rational number and we shall see in corollary 7.1.4 that it does not depend on the resolution ρ .

Remark 7.1.2 We have $\widehat{\mu}_P = c_1(Y)c_{n-1}(Y)$ if $\nu_i = 1$ for all i (crepant resolutions) and $\widehat{\mu}_P = c_1(X)c_{n-1}(X)$ if X is smooth.

Theorem 7.1.3 Let P be a full dimensional lattice polyope in N containing the origin in its interior and $\text{Spec} c_P^{geo}(z) = \sum_{i=1}^{e_P} z^{\beta_i}$ be its geometric spectrum. Then

$$\sum_{i=1}^{e_P} \left(\beta_i - \frac{n}{2} \right)^2 = \frac{n}{12} \mu_P + \frac{1}{6} \widehat{\mu}_P \quad (26)$$

where $\widehat{\mu}_P$ is defined by formula (25) and $\mu_P := n! \text{Vol}(P)$.

Proof. Recall the stacky E -function $E_{st,P}(z) := \sum_{J \subset I} E(D_J, z) \prod_{j \in J} \frac{z-z^{\nu_j}}{z^{\nu_j}-1}$, see (22). Then we have

$$E''_{st,P}(1) = \frac{n(3n-5)}{12} e_P + \frac{1}{6} \widehat{\mu}_P \quad (27)$$

where e_P is the geometric Euler number of P , see definition 4.1.1. The proof of this formula is a straightforward computation and is similar to the one of [2, Theorem 3.8]: if V is a smooth variety of dimension n , we have the Libgober-Wood formula

$$E''(V, 1) = \frac{n(3n-5)}{12} c_n(V) + \frac{1}{6} c_1(V)c_{n-1}(V) \quad (28)$$

where E is the E -polynomial of V , see [18, Proposition 2.3]; in order to get (27), apply this formula to the components $E(D_J, z)$ of $E_{st,P}(z)$ and use the equalities

$$E(D_J, 1) = c_{n-|J|}(D_J), \quad \frac{d}{dz} (E(D_J, z))|_{z=1} = \frac{n-|J|}{2} c_{n-|J|}(D_J)$$

(the first one follows from the fact that the value at $z = 1$ of the Poincaré polynomial is the Euler characteristic and we get the second one using Poincaré duality for D_J) and

$$\frac{d}{dz} \left(\frac{z-z^\nu}{z^\nu-1} \right) |_{z=1} = \frac{1-\nu}{2\nu}, \quad \frac{d^2}{dz^2} \left(\frac{z-z^\nu}{z^\nu-1} \right) |_{z=1} = \frac{(\nu-1)(\nu+1)}{6\nu}$$

if ν a positive rational number. By proposition 4.2.3, we have $\text{Spec}_P^{geo}(z) = E_{st,P}(z)$. Finally, we get

$$\frac{d^2}{dz^2}(\text{Spec}_P^{geo}(z))|_{z=1} = \frac{n(3n-5)}{12}e_P + \frac{1}{6}\widehat{\mu}_P \quad (29)$$

We have $\frac{d}{dz}(\text{Spec}_P^{geo}(z))|_{z=1} = \frac{n}{2}e_P$, because the geometric spectrum is symmetric with respect to $\frac{n}{2}$ (see corollary 4.2.4), and we deduce that

$$\frac{d^2}{dz^2}(\text{Spec}_P^{geo}(z))|_{z=1} = \sum_{i=1}^{e_P} (\beta_i - \frac{n}{2})^2 + \frac{n(n-2)}{4}e_P \quad (30)$$

Now, formulas (29) and (30) give equality (26) because $e_P = \mu_P$ by proposition 4.2.1. \square

Corollary 7.1.4 *The number $\widehat{\mu}_P$ does not depend on the resolution ρ .*

Proof. By proposition 4.2.1, the number e_P doesn't depend on ρ . The assertion then follows from equation (26). \square

The version for singularities is given by the following result:

Theorem 7.1.5 *Let f be a convenient and nondegenerate Laurent polynomial with global Milnor number μ and spectrum at infinity $\text{Spec}_f(z) = \sum_{i=1}^{\mu} z^{\alpha_i}$. Let P be its Newton polytope and assume that $\text{Spec}_P^{alg}(z) = \text{Spec}_P^{geo}(z)$. Then*

$$\sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 = \frac{n}{12}\mu_P + \frac{1}{6}\widehat{\mu}_P \quad (31)$$

where $\widehat{\mu}_P$ is defined by formula (25) and $\mu_P := n! \text{Vol}(P)$.

Proof. We have $\text{Spec}_P^{alg}(z) = \text{Spec}_f(z)$ and the assertion thus follows from theorem 7.1.3. \square

The smooth case is not surprising and is described by the following result (mirror symmetry is discussed in section 5.2):

Corollary 7.1.6 *Assume that f is the mirror partner of a projective, smooth, weak Fano toric variety X of dimension n . Then*

$$\sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 = \mu \frac{n}{12} + \frac{1}{6}c_1(X)c_{n-1}(X) \quad (32)$$

In particular $\sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 \geq \mu \frac{n}{12}$.

Proof. Let P be the convex hull of the primitive generators of the rays of the fan of X . Then P is reflexive because X is weak Fano. By proposition 6.1.3, $\text{Spec}_P^{alg}(z) = \text{Spec}_P^{geo}(z)$ and (32) follows from theorem 7.1.5 and remark 7.1.2. Last, $c_1(X)c_{n-1}(X) \geq 0$ because X is weak Fano. \square

7.2 The two dimensional case

In this section, P denotes a full dimensional polytope in $N_{\mathbb{R}} = \mathbb{R}^2$, containing the origin, and Δ_P denotes the complete fan in $N_{\mathbb{R}}$ obtained by taking the cones over the proper faces of P . Let $\rho : Y \rightarrow X$ be a resolution of $X := X_{\Delta_P}$ as in section 4.2.2. We assume that the v_i 's (primitive generators of the rays of Y) are numbered clockwise. In what follows, indices are considered as integers modulo r (r is the cardinality of $\Delta'(1)$) so $\nu_{r+1} := \nu_1$ (recall that $\nu_i := \nu(v_i)$ where ν is the Newton function of P).

Proposition 7.2.1 *We have*

1. $\widehat{\mu}_P = c_1^2(Y) - 2r + \sum_{i=1}^r (\frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i})$,
2. $\widehat{\mu}_P = (\sum_{i=1}^r \nu_i D_i) (\sum_{j=1}^r \frac{1}{\nu_j} D_j)$.

Proof. By definition, we have

$$\begin{aligned} \widehat{\mu}_P &:= c_1(Y)c_1(Y) + \sum_{J \subset I, J \neq \emptyset} c_1(D_J)c_{1-|J|}(D_J) \prod_{j \in J} \frac{1-\nu_j}{\nu_j} \\ &\quad - \sum_{J \subset I, J \neq \emptyset} (\sum_{j \in J} \nu_j) c_{2-|J|}(D_J) \prod_{j \in J} \frac{1-\nu_j}{\nu_j} \end{aligned} \tag{33}$$

and thus

$$\widehat{\mu}_P = c_1^2(Y) + 2 \sum_{i=1}^r \frac{(1-\nu_i)^2}{\nu_i} - \sum_{i=1}^r (\nu_i + \nu_{i+1}) \frac{(1-\nu_i)}{\nu_i} \cdot \frac{(1-\nu_{i+1})}{\nu_{i+1}}$$

It follows that

$$\begin{aligned} \widehat{\mu}_P - c_1^2(Y) &= \sum_{i=1}^r \left(\frac{1}{\nu_i} + \nu_i - \frac{1}{\nu_{i+1}} - \nu_{i+1} + \frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i} - 2 \right) \\ &= -2r + \sum_{i=1}^r \left(\frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i} \right) \end{aligned}$$

and this gives the first point. For the second, notice that

$$\begin{aligned} \left(\sum_{i=1}^r \nu_i D_i \right) \left(\sum_{j=1}^r \frac{1}{\nu_j} D_j \right) &= \sum_{i=1}^r \left(D_i^2 + \frac{\nu_i}{\nu_{i+1}} D_i D_{i+1} + \frac{\nu_i}{\nu_{i-1}} D_i D_{i-1} \right) \\ &= \sum_{i=1}^r \left(D_i^2 + \frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i} \right) = c_1^2(Y) - 2r + \sum_{i=1}^r \left(\frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i} \right) = \widehat{\mu}_P \end{aligned}$$

where the last equality is given by the first point. \square

Theorem 7.2.2 *Let f be a convenient and nondegenerate Laurent polynomial on $(\mathbb{C}^*)^2$, with spectrum at infinity $\sum_{i=1}^{\mu} z^{\alpha_i}$, and P be its Newton polytope. Then*

$$\sum_{i=1}^{\mu} (\alpha_i - 1)^2 = \frac{\mu P}{6} + \frac{\widehat{\mu} P}{6} \quad (34)$$

Moreover, $\widehat{\mu} P \geq c_1^2(Y)$ for any resolution $\rho : Y \rightarrow X_{\Delta_P}$. In particular, $\sum_{i=1}^{\mu} (\alpha_i - 1)^2 \geq \frac{\mu P}{6}$ if there exists a resolution ρ such that $c_1^2(Y) \geq 0$.

Proof. If $n = 2$, we have $\text{Spec}_P^{\text{alg}}(z) = \text{Spec}_P^{\text{geo}}(z)$ by corollary 6.1.2 and the first equality follows from theorem 7.1.5. By proposition 7.2.1 we have $\widehat{\mu} P \geq c_1^2(Y)$ because $\nu + \frac{1}{\nu} \geq 2$ for all real positive number ν . \square

7.3 Examples

We test theorem 7.1.5 weighted projective spaces: given $X = \mathbb{P}(1, \lambda_1, \dots, \lambda_n)$, f will denote its Givental-Hori-Vafa model (see section 6.2), with spectrum at infinity $\sum_{i=1}^{\mu} z^{\alpha_i}$ given by lemma 6.2.2, and P will denote the polytope of X , see definition 6.2.1. In what follows, we put $V(\alpha) := \sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2$.

- The polytope P is Fano (see section 2.3):

X	μ	$V(\alpha)$	$\mu n/12$	$\widehat{\mu} P$
$\mathbb{P}(1, 1, a)$	$a + 2$	$(2a^2 + 6a + 4)/6a$	$(a + 2)/6$	$\frac{(a+2)^2}{a}$
$\mathbb{P}(1, 2, 5)$	8	12/5	4/3	32/5

For $\mathbb{P}(1, 1, a)$, the polytope P is the convex hull of $(1, 0)$, $(0, 1)$ and $(-1, -a)$ and we consider the resolution obtained by adding the ray generated by $(0, -1)$. Using the notations of theorem 7.2.2 we have $\nu_1 = 1$, $\nu_2 = \frac{2}{a}$, $\nu_3 = 1$ and $\nu_4 = 1$.

For $\mathbb{P}(1, 2, 5)$, the polytope P is the convex hull of $(1, 0)$, $(0, 1)$ and $(-2, -5)$ and we consider the resolution obtained by adding the ray generated by $(0, -1)$, $(-1, -3)$ and $(-1, -2)$: we have $\nu_1 = 1$, $\nu_2 = \frac{3}{5}$, $\nu_3 = \frac{4}{5}$, $\nu_4 = 1$, $\nu_5 = 1$ and $\nu_6 = 1$.

Notice that in these examples we have $\widehat{\mu} P = \mu_{P^\circ}$ where μ_{P° is the volume of the polar polytope: this is not a coincidence, see section 8 below (this will be no longer true if P is not Fano, see the example $\mathbb{P}(1, \ell, \ell)$ below).

- The polytope P is not Fano:

Example	μ	$V(\alpha)$	$\mu n/12$	$\widehat{\mu} P$
$\mathbb{P}(1, \ell, \ell)$	$1 + 2\ell$	$2 + \frac{(\ell-1)(2\ell-1)}{3\ell}$	$(2\ell + 1)/6$	$9 + 2\frac{(\ell-1)^2}{\ell}$
$\mathbb{P}(1, 2, 2, 2)$	7	7	7/4	63/2

For $\mathbb{P}(1, \ell, \ell)$, $\ell \geq 2$, the polytope P is the convex hull of $(1, 0)$, $(0, 1)$ and $(-\ell, -\ell)$ and we have $\text{Spec}_P^{\text{geo}}(z) = \text{Spec}_P^{\text{alg}}(z)$ by 6.2.4, example (2). Formula (25) gives

$$\widehat{\mu}_P = c_1(\mathbb{P}^2)c_1(\mathbb{P}^2) + c_1(\mathbb{P}^1)(\ell - 1) - \frac{1}{\ell}c_1(\mathbb{P}^1)(\ell - 1)$$

For $\mathbb{P}(1, 2, 2, 2)$, P is the convex hull of $(-2, -2, -2)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. We have $X := X_{\Delta_P} = \mathbb{P}^3$, generated by the rays $v_1 = (-1, -1, -1)$, $v_2 = (1, 0, 0)$, $v_3 = (0, 1, 0)$ and $v_4(0, 0, 1)$, with $\nu(v_1) = \frac{1}{2}$. Thus,

$$\text{Spec}_P^{\text{geo}}(z) = E(\mathbb{P}^3, z) - E(\mathbb{P}^2, z) + E(\mathbb{P}^2, z) \frac{z - 1}{z^{1/2} - 1} = \text{Spec}_P^{\text{alg}}(z)$$

because $\text{Spec}_P^{\text{alg}}(z) = z^3 + z^2 + z + 1 + z^{1/2} + z^{3/2} + z^{5/2}$ by lemma 6.2.2. Formula (25) gives

$$\widehat{\mu}_P = c_1(\mathbb{P}^3)c_2(\mathbb{P}^3) + c_1(\mathbb{P}^2)c_1(\mathbb{P}^2) - \frac{1}{2}c_2(\mathbb{P}^2)$$

Let us consider now a more general situation:

- let $P_{1,2,2}$ be the polytope with vertices $b_1 = (1, 0)$, $b_2 = (0, 2)$ and $b_3 = (-2, -2)$. It gives the stacky fan $\Delta = (N, \Delta, \{b_1, b_2, b_3\})$ where Δ is the fan whose rays are $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, -1)$.
- Let $P_{\ell, \ell, \ell}$ be the polytope with vertices $b_1 = (\ell, 0)$, $b_2 = (0, \ell)$ and $b_3 = (-\ell, -\ell)$ where ℓ is a positive integer. It gives the stacky fan $\Delta = (N, \Delta, \{b_1, b_2, b_3\})$ where Δ is the fan whose rays are $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, -1)$.

Example	μ	$V(\alpha)$	$\mu n/12$	$\widehat{\mu}_P$
$P_{1,2,2}$	8	3	4/3	10
$P_{\ell, \ell, \ell}$	$3\ell^2$	$(\ell^2 + 3)/2$	$\ell^2/2$	9

This agrees with formula (31)

8 A Noether's formula for two dimensional Fano polytopes

In this section, we still focus on the two dimensional case: P denotes a full dimensional polytope in $N_{\mathbb{R}} = \mathbb{R}^2$, containing the origin. Recall that a convex lattice polytope is *Fano* if the origin is contained in the strict interior of P and if each of its vertex is a primitive lattice point of N , see section 2.3.

If P is a reflexive polytope, X_{Δ_P} has a crepant resolution and $\widehat{\mu}_P = c_1^2(Y)$ by remark 7.1.2. The anticanonical divisor of Y is nef and $c_1^2(Y) = \mu_{P^\circ}$ by [6, Theorem 13.4.3]. Hence $\widehat{\mu}_P = \mu_{P^\circ}$. On the other side, the geometric spectrum (which is equal to the algebraic spectrum by corollary 6.1.2) of P satisfies $\sum_{i=1}^{\mu} (\beta_i - 1)^2 = 2$. Finally, we get from equation (34) the well-known formula

$$12 = \mu_P + \mu_{P^\circ}$$

We have the following generalization for Fano polytopes (a reflexive polytope is Fano):

Theorem 8.0.1 Assume that P is a Fano polytope in $N_{\mathbb{R}}$ with geometric spectrum $\text{Spec}_P^{geo}(z) = \sum_{i=1}^{\mu_P} z^{\beta_i}$. Then

$$\sum_{i=1}^{\mu_P} (\beta_i - 1)^2 = \frac{\mu_P}{6} + \frac{\mu_{P^\circ}}{6} \quad (35)$$

where P° is the polar polytope of P .

Proof. Notice first that, because of the Fano assumption, the support function of the \mathbb{Q} -Cartier divisor K_X is equal to the Newton function of P and thus $\rho^*(-K_X) = \sum_{i=1}^r \nu_i D_i$ since $\rho^*(-K_X)$ and $-K_X$ have the same support function. We shall show that

$$\widehat{\mu}_P = \rho^*(-K_X)\rho^*(-K_X) \quad (36)$$

Because $(\rho^*(-K_X))^2 = (-K_X)^2 = \mu_{P^\circ}$ (for the first equality see [6, Lemma 13.4.2] and for the second one see the \mathbb{Q} -Cartier version of [6, Theorem 13.4.3]), equation (35) will follow from theorem 7.1.3. By proposition 7.2.1, we have

$$\widehat{\mu}_P = \sum_{i=1}^r \left(D_i^2 + \frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i} \right) = \sum_{i=1}^r D_i^2 + \sum_{i=1}^r \left(\frac{\nu_{i-1}}{\nu_i} + \frac{\nu_{i+1}}{\nu_i} \right)$$

and, as noticed at the beginning of this proof,

$$\rho^*(-K_X)\rho^*(-K_X) = \sum_{i=1}^r (\nu_i^2 D_i^2 + 2\nu_i \nu_{i+1}) = \sum_{i=1}^r \nu_i^2 D_i^2 + \sum_{i=1}^r (\nu_i \nu_{i+1} + \nu_i \nu_{i-1})$$

so (36) reads

$$\sum_{i=1}^r (\nu_i^2 - 1) D_i^2 = \sum_{i=1}^r (\nu_i^2 - 1) \left(-\frac{\nu_{i-1}}{\nu_i} - \frac{\nu_{i+1}}{\nu_i} \right) \quad (37)$$

Now, observe the following:

- if ν_{i-1} , ν_i and ν_{i+1} are primitive generators of rays of Y inside a same cone of the fan of X , we have

$$\nu(\nu_{i-1} + \nu_{i+1}) = \left(\frac{\nu_{i-1}}{\nu_i} + \frac{\nu_{i+1}}{\nu_i} \right) \nu(\nu_i)$$

because $\nu(\nu_{i-1}) = \nu_{i-1}$, $\nu(\nu_i) = \nu_i$ and $\nu(\nu_{i+1}) = \nu_{i+1}$ and the Newton function is linear on each cone of the fan of X . Because Y is smooth and complete, it follows that

$$\nu_{i-1} + \nu_{i+1} = \left(\frac{\nu_{i-1}}{\nu_i} + \frac{\nu_{i+1}}{\nu_i} \right) \nu_i$$

and we get $D_i^2 = -\frac{\nu_{i-1}}{\nu_i} - \frac{\nu_{i+1}}{\nu_i}$.

- otherwise, $\nu_i = 1$ due to the Fano condition.

Equation (37), hence equation (36), follows from these two observations. \square

Corollary 8.0.2 *Let f be a nondegenerate and convenient Laurent polynomial on $(\mathbb{C}^*)^2$, $\alpha_1, \dots, \alpha_\mu$ be its spectrum at infinity. Assume that the Newton polytope of f is Fano. Then*

$$\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - 1)^2 \geq \frac{1}{6}$$

Proof. By corollary 6.1.2 both spectra, algebraic and geometric, coincide. The assertion thus follows from theorem 8.0.1. \square

Remark 8.0.3 *Theorem 8.0.1 is not true if we forget the assumption Fano, see examples 7.3. Moreover, it follows from proposition 7.2.1 that $\widehat{\mu}_{\ell P} = \widehat{\mu}_P$ if ℓ is an integer greater or equal than one, and thus $\widehat{\mu}_P$ can't be seen as a volume in general.*

Example 8.0.4 *Let P be the polytope of $\mathbb{P}(1, \lambda_1, \lambda_2)$ (the convex hull of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix}$, see definition 6.2.1) where λ_1 and λ_2 are relatively prime. Then,*

$$\sum_{i=1}^{\mu} (\alpha_i - 1)^2 = \frac{1}{6} \left((1 + \lambda_1 + \lambda_2) + \frac{(1 + \lambda_1 + \lambda_2)^2}{\lambda_1 \lambda_2} \right)$$

9 Hertling's conjecture for regular functions

Corollary 7.1.6, theorem 7.2.2, examples 7.3 and corollary 8.0.2 motivate, and partly show, the following conjecture which has been already stated, without any further comments, in [10, Remark 4.15] as a global counterpart of C. Hertling's conjecture for germs of holomorphic functions (see [14], where the equality is inversed). The tameness assumption is discussed in section 5.1.

Conjecture on the variance of the spectrum (global version) *Let f be a regular, tame function on a smooth n -dimensional affine variety U . Then*

$$\frac{1}{12} (\alpha_\mu - \alpha_1) \leq \frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 \tag{38}$$

where $(\alpha_1, \dots, \alpha_\mu)$ is the (ordered) spectrum of f at infinity. \square

Some observations:

- If f is a Laurent polynomial equality (38) becomes $\frac{n}{12} \leq \frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2$ because $\alpha_1 = 0$ and $\alpha_\mu = n$, see proposition 5.3.2. If true, this inequality for Laurent polynomials is the best possible, see example 8.0.4.
- This is a different story, but, as suggested by corollary 7.1.6, one should expect $\sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 = \mu \frac{\alpha_\mu - \alpha_1}{12}$ if f belongs to the ideal generated by its partial derivatives (recall that the multiplication by f on its Jacobi ring corresponds to the cup-product by $c_1(X)$ on the cohomology algebra), see [7] and [14] for quasi-homogeneous polynomials. Example: $f(x, y) = xy(x - 1)$ for which we have $\mu = 2$ and $\alpha_1 = \alpha_2 = 1$.

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