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SINGULARITIES OF VARIETIES ADMITTING AN ENDOMORPHISM

AMAËL BROUSTET AND ANDREAS HÖRING

ABSTRACT. Let X be a normal variety such that K_X is \mathbb{Q} -Cartier, and let $f: X \rightarrow X$ be a finite surjective morphism of degree at least two. We establish a close relation between the irreducible components of the locus of singularities that are not log-canonical and the dynamics of the endomorphism f . As a consequence we prove that if X is projective and f polarised, then X has at most log-canonical singularities.

1. INTRODUCTION

1.A. Main result. Let X be a normal variety and let $f: X \rightarrow X$ be an endomorphism, i.e. a finite surjective morphism of degree $\deg(f) > 1$. If X is projective, an abundant literature [Bea01, Fuj02, Ame03, FN07, Nak08, AKP08, NZ10, Zha10] shows that the existence of an endomorphism imposes strong restrictions on the global geometry of X . In this paper we address the question if the existence of an endomorphism also imposes restrictions on the local geometry, i.e. restrictions on the nature of the singularities. In a recent paper Boucksom, de Fernex and Favre introduce the volume $\text{Vol}(X, x)$ of an isolated singularity. Using this invariant they give a precise answer to our question for isolated singularities.

1.1. Theorem. [BdFF12, Thm.B] *Let X be a normal variety with isolated singularities, and let $f: (X, x) \rightarrow (X, x)$ be an endomorphism of degree $\deg(f) > 1$. Then we have $\text{Vol}(X, x) = 0$.*

If K_X is \mathbb{Q} -Cartier then X has log-canonical singularities, and it furthermore has klt singularities if f is not étale in codimension one.

Fulger [Ful11] introduces a different invariant $\text{Vol}_F(X, x)$ associated to an isolated singularity and proves the analogue of Theorem 1.1 for $\text{Vol}_F(X, x)$. Let us note that

$$\text{Vol}(X, x) \geq \text{Vol}_F(X, x)$$

and equality holds if K_X is \mathbb{Q} -Cartier.

In this paper we will consider varieties such that K_X is \mathbb{Q} -Cartier, but the singularities are not isolated. In this case X is not necessarily log-canonical: if Y is any normal variety such that K_Y is \mathbb{Q} -Cartier and E an elliptic curve, then $X := Y \times E$

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admits the endomorphism $f := \text{id}_Y \times g$ with g the multiplication by $m \in \mathbb{N}$. However we can establish a close relation between the irreducible components of the non-lc locus and the dynamics of the endomorphism:

1.2. Theorem. *Let X be a normal variety such that K_X is \mathbb{Q} -Cartier, and let $f: X \rightarrow X$ be an endomorphism of degree $\deg(f) > 1$.*

Let Z be an irreducible component of $\text{Nlc}(X)$. Then (up to replacing f by some iterate) Z is totally invariant. In this case Z is not contained in the ramification divisor R , and the induced endomorphism $f|_Z: Z \rightarrow Z$ satisfies

$$\deg(f|_Z) = \deg(f).$$

Since we suppose $\deg(f) > 1$ the last part of this statement shows that Z cannot be a point, so we recover the \mathbb{Q} -Cartier case of Theorem 1.1. If X is projective we can consider the particularly interesting class of polarised endomorphisms, i.e. those endomorphisms such that there exists an ample divisor H satisfying $f^*H \simeq mH$. In this case the statement becomes much stronger:

1.3. Corollary. *Let X be a normal projective variety such that K_X is \mathbb{Q} -Cartier, and let $f: X \rightarrow X$ be a polarised endomorphism of degree $\deg(f) > 1$.*

Then X has at most log-canonical singularities. Moreover X is klt near the ramification divisor R .

1.B. Technique and generalisations. The proof of our main result comes in two steps. In the first step we use a classical computation describing the behaviour of log-discrepancies under finite morphisms [KM98, Prop.5.20] to prove that all the irreducible components of the non-lc locus are totally invariant. In the second step we use an idea introduced by Nakayama in his inspiring preprint [Nak08] on endomorphisms of normal surfaces: if $\mu: Y \rightarrow X$ is the log-canonical model (cf. Definition 2.2), the endomorphism f lifts to a (rational) endomorphism g of Y . We can then study the geometry of the ramification divisors along certain μ -exceptional divisors to deduce our result.

Our proof actually works more generally for log pairs (X, Δ) such that $K_X + \Delta$ is \mathbb{Q} -Cartier and a logarithmic ramification formula holds. In this paper we focus on the geometrically most interesting case where the boundary Δ is a totally invariant Weil divisor.

1.4. Theorem. *Let X be a normal variety, and let $f: X \rightarrow X$ be an endomorphism of degree $\deg(f) > 1$. Let Δ be a reduced effective totally invariant Weil divisor such that $K_X + \Delta$ is \mathbb{Q} -Cartier.*

Let Z be an irreducible component of $\text{Nlc}(X, \Delta)$. Then (up to replacing f by some iterate) Z is totally invariant. In this case we have $Z \not\subset R_\Delta$ where R_Δ is the logarithmic ramification divisor, and the induced endomorphism $f|_Z: Z \rightarrow Z$ satisfies

$$\deg(f|_Z) = \deg(f).$$

Theorem 1.4 is simply the case $\Delta = \emptyset$ in the preceding statement.

Let us note that the existence of log-canonical models has been proven recently by Odaka and Xu [OX12] for pairs (X, Δ) such that $K_X + \Delta$ is \mathbb{Q} -Cartier. If

log-canonical models exist in general¹, it seems plausible that our results can be generalised to arbitrary normal varieties.

1.5. Conjecture. *Let X be a normal variety, and let $f: X \rightarrow X$ be an endomorphism of degree $\deg(f) > 1$. Suppose that X admits a log-canonical model $\mu: Y \rightarrow X$. Let Z be an irreducible component of $\mu(E_\mu^{\text{lc}})$, where E_μ^{lc} is the sum of all the μ -exceptional prime divisors taken with coefficient one.*

Then (up to replacing f by some iterate) Z is totally invariant. In this case Z is not contained in the ramification divisor R , and the induced endomorphism $f|_Z: Z \rightarrow Z$ satisfies

$$\deg(f|_Z) = \deg(f).$$

If moreover X is projective and f is polarised, then μ is an isomorphism in codimension one.

This statement would also generalise Theorem 1.1 since we can prove that an isolated singularity has volume zero if and only if the log-canonical model (if it exists) is an isomorphism in codimension one, cf. Proposition 2.4.

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2. NOTATION AND BASIC RESULTS

We work over the complex field \mathbb{C} , topological notions always refer to the Zariski topology. For general definitions we refer to Hartshorne's book [Har77]. We will use standard terminology and results of the minimal model program (MMP) as explained in [KM98] or [HK10]. A variety is an integral scheme of finite type over \mathbb{C} . For D a \mathbb{Q} -Weil divisor on a normal variety X , we denote by $\text{supp}(D)$ its support.

2.A. Singularities of pairs. Let X be a normal variety, and let $\mu: X' \rightarrow X$ be a proper birational morphism from a normal variety X' . If $\Delta \subset X$ is a \mathbb{Q} -Weil divisor, we denote by $\mu_*^{-1}(\Delta)$ its strict transform.

A log-pair is a tuple (X, Δ) where X is a normal variety and $\Delta = \sum_i d_i \Delta_i$ is a \mathbb{Q} -Weil divisor on X with $d_i \leq 1$ for all i . We say that the pair (X, Δ) is lc (resp. klt)⁵ if $K_X + \Delta$ is \mathbb{Q} -Cartier and for every proper birational morphism $\mu: X' \rightarrow X$ from a normal variety X' we can write

$$K_{X'} + \mu_*^{-1}(\Delta) = \mu^*(K_X + \Delta) + \sum_j a(E_j, X, \Delta) E_j,$$

¹The existence of log-canonical models would be a consequence of the MMP, including the abundance conjecture.

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⁵Note that we do not assume that the boundary divisor Δ is effective, so some authors would say that such a pair is sub-lc (resp. sub-klt). We follow the notation of [KM98].

where the divisor E_j are μ -exceptional and $a(E_j, X, \Delta) \geq -1$ (resp. $a(E_j, X, \Delta) > -1$) for all j . If the pair (X, Δ) is log-canonical, we say that a subvariety $Z \subset X$ is an lc centre if there exists a morphism $\mu: X' \rightarrow X$ as above and a μ -exceptional divisor E such that $E \rightarrow Z$ and $a(E, X, \Delta) = -1$.

2.1. Definition. Let (X, Δ) be a log-pair such that $K_X + \Delta$ is \mathbb{Q} -Cartier. The non-lc locus $\text{Nlc}(X, \Delta)$ is the smallest closed set $W \subset X$ such that $(X \setminus W, \Delta|_{X \setminus W})$ is lc.

2.2. Definition. Let (X, Δ) be a log-pair such that $\Delta \geq 0$. A log-canonical model of the pair (X, Δ) is a proper birational morphism

$$\mu: Y \rightarrow X$$

such that if we set

$$\Delta_Y := \mu_*^{-1}(\Delta) + E_\mu^{lc},$$

where E_μ^{lc} is the sum of all the μ -exceptional prime divisors taken with coefficient one, the pair (Y, Δ_Y) is log-canonical and $K_Y + \Delta_Y$ is μ -ample.

2.3. Remark.

- a) If a pair (X, Δ) admits a log-canonical model, it is unique up to isomorphism [OX12, Prop.2.3].
- b) Suppose now that $\Delta \geq 0$ and $K_X + \Delta$ is \mathbb{Q} -Cartier. Then (X, Δ) admits a log-canonical model [OX12, Thm.1.1]. Moreover the μ -exceptional locus has pure codimension one [OX12, Lemma 2.4]. If we write

$$(1) \quad K_Y + \Delta_Y = \mu^*(K_X + \Delta) + \Delta_Y^{\geq 1},$$

then $\Delta_Y^{\geq 1}$ is antieffective and $\text{supp } \Delta_Y^{\geq 1} = \text{Exc}(\mu)$ (ibid). By the definition of Δ_Y we have $\text{supp } \Delta_Y^{\geq 1} \subset \Delta_Y$. Note also that since $K_X + \Delta$ and $K_Y + \Delta_Y$ are \mathbb{Q} -Cartier, the divisor $\Delta_Y^{\geq 1}$ is \mathbb{Q} -Cartier.

The following proposition establishes the link between Conjecture 1.5 and Theorem 1.1.

2.4. Proposition. Let X be a normal variety with singular locus a point x . Assume that X has a log-canonical model $\mu: (Y, \Delta_Y) \rightarrow X$.

Then $\text{Vol}(X, x) = 0$ if and only if μ is an isomorphism in codimension 1.

For the proof of this statement we will use the tools and terminology of [BdFF12]: given a canonical divisor K_X on X , there is a unique canonical divisor K_{X_π} , for each birational model $\pi: X_\pi \rightarrow X$, with the property that $\pi_* K_{X_\pi} = K_X$. Thus we obtain a canonical b -divisor $K_{\mathfrak{X}}$ over X . Boucksom, de Fernex and Favre define the nef envelope $\text{Env}_X(-K_X)$ of the Weil divisor $-K_X$ as the largest nef Weil b -divisor Z that is both relatively nef over X and satisfies $Z_X \leq -K_X$. The log-discrepancy b -divisor $A_{\mathfrak{X}/X}$ is then defined by

$$(2) \quad A_{\mathfrak{X}/X} = K_{\mathfrak{X}} + 1_{\mathfrak{X}/X} + \text{Env}_X(-K_X),$$

where the trace of $1_{\mathfrak{X}/X}$ in any model is equal to the reduced exceptional divisor over X .

Proof of Proposition 2.4. Suppose that $\text{Vol}(X, x) = 0$. We will argue by contradiction and suppose that the divisor Δ_Y is not zero. Let $\nu: Z \rightarrow Y$ be a dlt-model of the log-canonical pair (Y, Δ_Y) [Fuj11, Thm.10.4], i.e. ν is a birational morphism from a normal \mathbb{Q} -factorial variety Z such that if we denote by B the ν -exceptional divisors taken with coefficient one and set

$$\Delta_Z := \nu_*^{-1}(\Delta_Y) + B,$$

then the pair (Z, Δ_Z) is dlt and we have

$$K_Z + \Delta_Z = \nu^*(K_Y + \Delta_Y).$$

Set $\varphi := \mu \circ \nu$. Then the divisor $K_Z + \Delta_Z$ is φ -nef and its restriction to any irreducible component of $\nu_*^{-1}(\Delta_Y)$ is nef and big.

The trace of the equation (2) on Z is

$$\nu_*^{-1}((A_{\mathfrak{X}/X})_Y) = K_Z + \Delta_Z + (\text{Env}_X(-K_X))_Z.$$

Indeed Δ_Z is the union of all the φ -exceptional divisors taken with multiplicity one, so $(1_{\mathfrak{X}/X})_Z = \Delta_Z$. Moreover all the ν -exceptional divisors have log-discrepancy 0, so $(A_{\mathfrak{X}/X})_Z$ is just equal to the strict transform of $(A_{\mathfrak{X}/X})_Y$.

By [BdFF12, Lemma 2.10] the restriction of $(\text{Env}_X(-K_X))_Z$ to any φ -exceptional divisor is pseudoeffective, so the restriction of $\nu_*^{-1}(A_{\mathfrak{X}/X})_Y$ to any irreducible component of $\nu_*^{-1}(\Delta_Y)$ is big. Since Δ_Y is not zero, this implies that $\nu_*^{-1}(A_{\mathfrak{X}/X})_Y$ is not the zero divisor. Since we have

$$\text{supp } \nu_*^{-1}(A_{\mathfrak{X}/X})_Y \subset \text{supp } \nu_*^{-1}(\Delta_Y),$$

we see that the restriction of $\nu_*^{-1}(A_{\mathfrak{X}/X})_Y$ to any irreducible component of its support is big. By the negativity lemma (in its big version [Gra12, Prop.4.1]) this implies that $\nu_*^{-1}(A_{\mathfrak{X}/X})_Y$ is not effective. Thus the log-discrepancy b -divisor $A_{\mathfrak{X}/X}$ is not effective, a contradiction to [BdFF12, Prop.4.19].

Suppose that μ is an isomorphism in codimension 1. The variety Y has log-canonical singularities, so all the log-discrepancies are non-negative. Since μ is an isomorphism in codimension one we see that $A_{\mathfrak{X}/X}$ is effective, hence $\text{Vol}(X, x) = 0$ by [BdFF12, Prop.4.19]. \square

2.B. Logarithmic ramification formula. Let $f: X_1 \rightarrow X_2$ be a finite surjective morphism between normal varieties. For every Weil divisor $D \subset X_2$ we define the pull-back f^*D as the unique Weil divisor obtained by completing $f^*D|_{X_{2,\text{reg}}}$. If D is \mathbb{Q} -Cartier of Cartier index m , then f^*D is \mathbb{Q} -Cartier of index m . The ramification divisor is defined by

$$(3) \quad R := \sum_{D \subset X_2} f^*D - \text{supp}(f^*D),$$

where the sum runs over all prime divisors in X_2 . By generic smoothness the sum is finite, so R is an effective Weil divisor. Its image $B := f(R)$ is the branch divisor of f . By the ramification formula we have

$$K_{X_1} = f^*K_{X_2} + R.$$

2.5. Lemma. *Let $f: X_1 \rightarrow X_2$ be a finite surjective morphism between normal varieties. Let Δ_2 be a reduced effective Weil divisor, and set $\Delta_1 := \text{supp } f^* \Delta_2$. Then we have the logarithmic ramification formula*

$$(4) \quad K_{X_1} + \Delta_1 = f^*(K_{X_2} + \Delta_2) + R_\Delta,$$

where R_Δ is an effective divisor. Moreover Δ_1 and R_Δ do not have any common component. We call R_Δ the logarithmic ramification divisor.

Proof. Adding $\Delta_1 = f^* \Delta_2 - (f^* \Delta_2 - \Delta_1)$ to the ramification formula we obtain

$$K_{X_1} + \Delta_1 = f^*(K_{X_2} + \Delta_2) + R - (f^* \Delta_2 - \Delta_1).$$

We claim that $R_\Delta := R - (f^* \Delta_2 - \Delta_1)$ is an effective divisor such that Δ_1 and R_Δ do not have any irreducible components in common. Indeed if $W \subset \Delta_2$ is an irreducible component, we have

$$f^* W = \sum m_i W_i,$$

with W_i the irreducible components of $f^* W$ and m_i the ramification index along W_i . In particular if W is not in the branch divisor B , then $f^* W = \sum W_i$ so

$$\text{mult}_{W_i} f^* \Delta_2 = \text{mult}_{W_i} \Delta_1 = 1,$$

and obviously no W_i is contained in R . If $W \subset B$, then by (3) we have

$$\text{mult}_{W_i} R = \text{mult}_{W_i} f^* W - 1 = \text{mult}_{W_i} f^* \Delta_2 - \text{mult}_{W_i} \Delta_1.$$

Thus we have $\text{mult}_{W_i} R_\Delta = 0$. □

2.6. Remark. If $K_{X_1} + \Delta_1$ and $K_{X_2} + \Delta_2$ are \mathbb{Q} -Cartier, then R_Δ is \mathbb{Q} -Cartier.

We will also use a weak generalisation of the logarithmic ramification formula (4) to morphisms which are only generically finite.

2.7. Lemma. *Let $g: V \rightarrow Y$ be a generically finite, projective, surjective morphism between normal varieties. Let Δ_Y be a reduced effective Weil divisor on Y such that $K_Y + \Delta_Y$ is \mathbb{Q} -Cartier. Let $\eta: V \rightarrow V_{St}$ and $h: V_{St} \rightarrow Y$ be the Stein factorisation of g . Set*

$$\Delta_V := \eta_*^{-1}(\text{supp } h^* \Delta_Y).$$

Then we have

$$K_V + \Delta_V = g^*(K_Y + \Delta_Y) + R_g$$

where R_g is a \mathbb{Q} -Weil divisor. Moreover Δ_V and R_g do not have any common component.

Proof. The morphism h is finite, so by (4) we have

$$K_{V_{St}} + \text{supp } h^* \Delta_Y = h^*(K_Y + \Delta_Y) + R_{\Delta_{St}},$$

where $R_{\Delta_{St}}$ is an effective Weil divisor that has no common component with $\text{supp } h^* \Delta_Y$. The divisor $K_{V_{St}} + \text{supp } h^* \Delta_Y - R_{\Delta_{St}}$ is \mathbb{Q} -Cartier, so we can write

$$K_V + \eta_*^{-1}(\text{supp } h^* \Delta_Y - R_{\Delta_{St}}) = \eta^*(K_{V_{St}} + \text{supp } h^* \Delta_Y - R_{\Delta_{St}}) + E = f^*(K_Y + \Delta_Y) + E$$

where E is an η -exceptional divisor. Set now

$$R_g := E + \eta_*^{-1}(R_{\Delta_{St}}).$$

Since every irreducible component of E is η -exceptional and $R_{\Delta_{St}}$ has no common component with $\text{supp } h^* \Delta_Y$, it is clear that $\eta_*^{-1}(\text{supp } h^* \Delta_Y)$ has no common component with R_g . \square

2.C. Endomorphisms and Nlc-locus.

2.8. Definition. Let X be a normal variety, and let $f: X \rightarrow X$ be an endomorphism of degree $\deg(f) > 1$. We say that a closed subset $Z \subset X$ is totally invariant if we have a set-theoretical equality $f^{-1}(Z) = Z$.

2.9. Remark. Let $f: X_1 \rightarrow X_2$ be a finite surjective morphism between normal varieties. By [Gro66, Cor.14.4.] the morphism f is universally open. In particular if $Z \subset X_2$ is any subvariety, the induced morphism $X_1 \times_{X_2} Z \rightarrow Z$ is open. Hence every irreducible component of $X_1 \times_{X_2} Z$ dominates Z .

2.10. Lemma. Let X be a normal variety, and let $f: X \rightarrow X$ be an endomorphism of degree $\deg(f) > 1$. Let Δ be a reduced effective totally invariant Weil divisor such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let $Z \subset X$ be an irreducible component of $\text{Nlc}(X, \Delta)$. Then (up to replacing f by some power) we have

$$f^{-1}(Z) = Z.$$

If (X, Δ) has at most log-canonical singularities, let Z be an lc centre. Then (up to replacing f by some power) we have

$$f^{-1}(Z) = Z.$$

In this case we have $Z \not\subset R_\Delta$ where R_Δ is the logarithmic ramification divisor.

Proof. By (4) and Remark 2.6 we have

$$K_X + \Delta = f^*(K_X + \Delta) + R_\Delta,$$

with R_Δ an effective Weil divisor that is \mathbb{Q} -Cartier.

Let us recall a computation from [KM98, Prop.5.20]: let $W \subset X$ be any subvariety, and let $\mu: X' \rightarrow X$ be a proper birational morphism from a normal variety X' such that

$$K_{X'} + \mu_*^{-1}(\Delta) = \mu^*(K_X + \Delta) + R + a(E, X, \Delta)E$$

with R a μ -exceptional divisor and E a μ -exceptional prime divisor such that $\mu(E) = W$. Let X'' be the normalisation of the fibre product $X \times_X X'$ and consider the following commutative diagram

$$\begin{array}{ccc} X'' & \xrightarrow{f'} & X' \\ \mu' \downarrow & & \downarrow \mu \\ X & \xrightarrow{f} & X \end{array}$$

Let W_1, \dots, W_r be the irreducible components of $f^{-1}(W)$. By Remark 2.9 every W_i dominates W via f . Thus for every $i \in \{1, \dots, r\}$ the fibre product

$$W_i \times_W E \subset X \times_X X'$$

contains an irreducible divisorial component that surjects onto W_i . Let $E'_i \subset X''$ be a prime divisor that maps onto this divisor, then we have $f'(E'_i) = E$. Denote by r_i the ramification index of f' along E'_i . By [KM98, p.160, last line] we have

$$a(E'_i, X, \Delta - R_\Delta) + 1 = r(a(E, X, \Delta) + 1).$$

Since R_Δ is effective and \mathbb{Q} -Cartier, we have

$$a(E'_i, X, \Delta) \leq a(E'_i, X, \Delta - R_\Delta)$$

with equality holding if and only if $W_i \not\subset R_\Delta$. Thus we see that if $a(E, X, \Delta) < 1$ (resp. $a(E, X, \Delta) \leq 1$) then we have $a(E'_i, X, \Delta) < 1$ (resp. $a(E'_i, X, \Delta) \leq 1$). Moreover we have the following implication:

(5) If $a(E, X, \Delta) = 1$ and $a(E'_i, X, \Delta) = 1$, then $W_i \not\subset R_\Delta$.

Proof of the first statement. We will argue by descending induction on the dimension of the irreducible components of $\text{Nlc}(X, \Delta)$. The start of the induction is trivial since there is no irreducible component of $\text{Nlc}(X, \Delta)$ of dimension $\dim X$. Suppose now that every irreducible component of $\text{Nlc}(X, \Delta)$ of dimension at least $m + 1$ is totally invariant, and let Z_1, \dots, Z_k be the irreducible components of $\text{Nlc}(X, \Delta)$ of dimension m .

Fix a $j \in \{1, \dots, k\}$, and let Z'_j be an irreducible component of $f^{-1}(Z_j)$. By what precedes we have $Z'_j \subset \text{Nlc}(X, \Delta)$. We claim that Z'_j is actually an irreducible component of $\text{Nlc}(X, \Delta)$: if this was not the case there would be an irreducible component W of $\text{Nlc}(X, \Delta)$ such that $Z'_j \subset W$ and $\dim W \geq m + 1$. Yet by our induction hypothesis W is totally invariant, so $Z'_j \subset W$ implies that $Z_j \subset W$. Thus Z_j is not an irreducible component of $\text{Nlc}(X, \Delta)$, a contradiction.

Hence every irreducible component of $f^{-1}(Z_j)$ is an irreducible component of dimension m of $\text{Nlc}(X, \Delta)$. Since there are only finitely many such components, namely Z_1, \dots, Z_k , we see that f^{-1} induces a bijection on the irreducible components of dimension m of $\text{Nlc}(X, \Delta)$. Thus some power of f induces the identity.

Proof of the second statement. Since X is log-canonical there exist only finitely many lc centres. We can now repeat the proof of the first statement to see that f^{-1} acts by permutation on the lc centres, so some power induces the identity. An lc centre Z that is totally invariant and contained in R_Δ contradicts the statement (5), so it does not exist. \square

2.11. Lemma. *Let X_1 and X_2 be normal varieties, and let $f: X_1 \rightarrow X_2$ be a finite morphism. Let Δ_1 and Δ_2 be reduced effective Weil divisors on X_1 and X_2 such that $\Delta_1 = \text{supp } f^* \Delta_2$ and we have*

$$K_{X_1} + \Delta_1 = f^*(K_{X_2} + \Delta_2).$$

Suppose that the pair (X_2, Δ_2) has a log-canonical model $\mu_2: (Y_2, \Delta_{Y,2}) \rightarrow (X_2, \Delta_2)$.

Then the pair (X_1, Δ_1) has a log-canonical model $\mu_1: (Y_1, \Delta_{Y,1}) \rightarrow (X_1, \Delta_1)$, moreover f lifts to a finite morphism $g: Y_1 \rightarrow Y_2$ such that

$$K_{Y_1} + \Delta_{Y,1} = g^*(K_{Y_2} + \Delta_{Y,2})$$

and $\mu_2 \circ g = f \circ \mu_1$.

Our proof follows Nakayama's argument in the surface case [Nak08, Lemma 2.7.6].

Proof. Let Y_1 the normalization of the fiber product $X_1 \times_{X_2} Y_2$. Then we have a commutative diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{p_2} & Y_2 \\ \downarrow p_1 & & \downarrow \mu_2 \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

where the morphisms p_i are induced by the projections from the fibre product. Recall that by Definition 2.2 one has

$$\Delta_{Y,2} = (\mu_2)_*^{-1}(\Delta_2) + E_{\mu_2}^{lc},$$

where $E_{\mu_2}^{lc}$ is the sum of all the μ_2 -exceptional prime divisors taken with coefficient one. Since f and p_2 are finite we see that

$$\text{supp}(p_2^* E_{\mu_2}^{lc})$$

is the sum of all the p_1 -exceptional prime divisors taken with coefficient one. We set

$$\Delta_{Y,1} := (p_1^{-1})_* \Delta_1 + \text{supp}(p_2^* E_{\mu_2}^{lc})$$

and claim that the ramification formula

$$K_{Y_1} + \Delta_{Y,1} = p_2^*(K_{Y_2} + \Delta_{Y,2})$$

holds. Assuming this for the time being, let us see how to conclude: by [KM98, Prop.5.20] the pair $(Y_1, \Delta_{Y,1})$ is log-canonical. Since the morphism p_1 is obtained by base-changing μ_2 and normalising, the pull-back of the μ_2 -ample divisor $K_{Y_2} + \Delta_{Y,2}$ is p_1 -ample. By uniqueness of the log-canonical model (cf. Remark 2.3) we see that $(Y_1, \Delta_{Y,1})$ is the log-canonical model of (X_1, Δ_1) . The finite morphism $g := p_2$ gives the lifting of f .

Proof of the claim. We have $\text{supp } f^* \Delta_2 = \Delta_1$, hence by our definition of $\Delta_{Y,1}$

$$\text{supp } p_2^* \Delta_{Y,2} = \Delta_{Y,1}.$$

Thus by the logarithmic ramification formula (4) we have

$$K_{Y_1} + \Delta_{Y,1} = p_2^*(K_{Y_2} + \Delta_{Y,2}) + R_\Delta$$

with R_Δ an effective divisor that has no common component with $\Delta_{Y,1}$. Since by hypothesis $K_{X_1} + \Delta_1 = f^*(K_{X_2} + \Delta_2)$ it is clear that R_Δ is p_1 -exceptional. Since $\Delta_{Y,1}$ contains every p_1 -exceptional prime divisors with coefficient one, the divisor R_Δ is zero. \square

3. PROOFS OF THE MAIN RESULTS

3.1. Proposition. *Let X be a normal variety, and let $f: X \rightarrow X$ be an endomorphism of degree $\deg(f) > 1$. Let Δ be a reduced effective totally invariant Weil divisor such that $K_X + \Delta$ is \mathbb{Q} -Cartier.*

Let Z be an irreducible component of $\text{Nlc}(X, \Delta)$ that is totally invariant. Then $Z \not\subset R_\Delta$ where R_Δ is the logarithmic branch divisor.

3.2. Remark. If $\Delta = 0$ and X is a surface this follows from a theorem of Wahl [Wah90], cf. also Favre [Fav10]. More generally if $\Delta = 0$ and X has at most isolated singularities, we can apply [BdFF12, Thm.B] or [Ful11, Cor.]. Our strategy is inspired by Nakayama's proof of the surface case [Nak08, Lemma 2.7.9].

Proof. Let $\mu: (Y, \Delta_Y) \rightarrow (X, \Delta)$ be the log-canonical model of (X, Δ) . By Remark 2.3 we have

$$(6) \quad K_Y + \Delta_Y = \mu^*(K_X + \Delta) + \Delta_Y^{>1},$$

where $\Delta_Y^{>1}$ is an antieffective divisor such that $\text{supp } \Delta_Y^{>1} = \text{Exc}(\mu)$. Since Z is an irreducible component of $\text{Nlc}(X, \Delta)$ there exists at least one prime divisor E_1 in Y that surjects onto Z . Denote by E_1, \dots, E_k the irreducible components of $\text{supp}(\Delta_Y^{>1})$ that surject onto Z . Then we can write

$$(7) \quad \Delta_Y^{>1} = \sum_{i=1}^k a_i E_i + E',$$

where the a_i are the log-discrepancies with respect to (X, Δ) . Since Z is an irreducible component of $\text{Nlc}(X, \Delta)$ the antieffective divisor E' has the property $Z \not\subset \mu(\text{supp}(E'))$.

We will argue by contradiction and suppose that $Z \subset R_\Delta$.

Step 1. An estimate of the discrepancies. Let

$$K_X + \Delta = f^*(K_X + \Delta) + R_\Delta$$

be the logarithmic ramification formula. By Remark 2.6 the divisor R_Δ is \mathbb{Q} -Cartier and we denote by m its Cartier index. Thus the pull-back $\mu^* R_\Delta$ is well-defined and since $Z \subset R_\Delta$ we have

$$\text{mult}_{E_i}(\mu^* R_\Delta) \geq \frac{1}{m}$$

for every $i = 1, \dots, k$. Note moreover that for all $l \in \mathbb{N}$ the logarithmic ramification divisor $R_{\Delta, l}$ of the l -th iterate f^l satisfies

$$R_{\Delta, l} = \sum_{j=0}^{l-1} (f^j)^*(R_\Delta).$$

Since $Z \subset R_\Delta$ and $f^{-1}(Z) = Z$ we see that $Z \subset (f^j)^*(R_\Delta)$, hence

$$\text{mult}_{E_i}(\mu^*(f^j)^* R_\Delta) \geq \frac{1}{m}$$

for all i and j . Thus for l sufficiently high we have $\text{mult}_{E_i}(\mu^* R_{\Delta, l}) + a_i \geq 0$. Since our statement does not depend on the iterate of f we can suppose without loss of generality that these inequalities holds for $l = 1$. Thus we have

$$(8) \quad \text{mult}_{E_i}(\mu^* R_\Delta) + a_i \geq 0$$

for all $i \in \{1, \dots, k\}$.

Step 2. Comparing the discrepancies. The endomorphism f induces a rational map $Y \dashrightarrow Y$, we choose a resolution of the indeterminacies of $\nu: V \rightarrow Y$ such that V is smooth. Then we obtain a generically finite, projective, surjective morphism $g: V \rightarrow Y$ such that we have a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\nu} & Y & \xrightarrow{\mu} & X \\ & \searrow g & & & \downarrow f \\ & & Y & \xrightarrow{\mu} & X \end{array}$$

Using the notation of Lemma 2.7 we have

$$(9) \quad K_V + \Delta_V = g^*(K_Y + \Delta_Y) + R_g.$$

Note that by the definition of Δ_V we have $\Delta_Y = g(\Delta_V)$.

The pair (Y, Δ_Y) is log-canonical, so we can write

$$K_V = \nu^*(K_Y + \Delta_Y) + N'$$

where N' is a divisor such that all coefficients are at least -1 . Thus if we set $N := N' + \Delta_V$, then

$$(10) \quad K_V + \Delta_V = \nu^*(K_Y + \Delta_Y) + N$$

and for every irreducible component $D \subset \Delta_V$ we have

$$(11) \quad \text{mult}_D N \geq 0.$$

By (9) and (10) we have

$$\nu^*(K_Y + \Delta_Y) + N = g^*(K_Y + \Delta_Y) + R_g.$$

Plugging in (6) on both sides we get

$$\nu^*(\mu^*(K_X + \Delta) + \Delta_Y^{\geq 1}) + N = g^*(\mu^*(K_X + \Delta) + \Delta_Y^{\geq 1}) + R_g$$

By the logarithmic ramification formula $K_X + \Delta = f^*(K_X + \Delta) + R_\Delta$ we can simplify to

$$(12) \quad \nu^*(\mu^*R_\Delta + \Delta_Y^{\geq 1}) + N = g^*\Delta_Y^{\geq 1} + R_g$$

Since $g(\Delta_V) = \Delta_Y$ and $\text{supp } \Delta_Y^{\geq 1} \subset \Delta_Y$ (cf. Remark 2.3) there exists a prime divisor $D \subset \Delta_V$ such that $g(D) = E_1$. Let us first observe that

$$(13) \quad \mu(\nu(D)) = Z.$$

Indeed by our commutative diagram

$$f(\mu(\nu(D))) = \mu(g(D)) = \mu(E_1) = Z,$$

hence $\mu(\nu(D))$ is contained in $f^{-1}(Z)$ which by hypothesis is Z . Since Z is irreducible and $\mu(\nu(D))$ has dimension at least $\dim Z$ (it surjects via f on Z), we get the equality (13).

By Lemma 2.7 we know that Δ_V and R_g do not have common components, so $\text{mult}_D R_g = 0$. Since $\Delta_Y^{\geq 1}$ is antieffective and its support contains E_1 , we obtain

$$(14) \quad \text{mult}_D(g^*\Delta_Y^{\geq 1} + R_g) < 0.$$

Consider now the decomposition $\Delta_Y^{\geq 1} = \sum_{i=1}^k a_i E_i + E'$ introduced in (7). We have $Z \not\subset \mu(\text{supp}(E'))$ and $\mu(\nu(D)) = Z$ by (13), so we see that $\nu(D) \not\subset \text{supp}(E')$. Since $\mu^*R_\Delta + \Delta_Y^{\geq 1}$ is \mathbb{Q} -Cartier this implies that $\mu^*R_\Delta + \sum a_i E_i$ is \mathbb{Q} -Cartier in the generic point of $\nu(D)$. By the inequalities (8) we know that

$$\mu^*R_\Delta + \sum a_i E_i$$

is an effective divisor, so we obtain

$$\text{mult}_D \nu^*(\mu^*R_\Delta + \Delta_Y^{\geq 1}) = \text{mult}_{\nu(D)_{\text{gen}}} (\mu^*R_\Delta + \sum a_i E_i) \geq 0.$$

Yet by (11) this implies that

$$\text{mult}_D (\nu^*(\mu^*R_\Delta + \Delta_Y^{\geq 1}) + N) \geq 0,$$

so by (12) we have a contradiction to (14). \square

Proof of Theorem 1.4. By Lemma 2.10 we can suppose (up to replacing f by some iterate) that all the irreducible components of $\text{Nlc}(X, \Delta)$ are totally invariant. Let Z be such an irreducible component, then by Proposition 3.1 we have $Z \not\subset R_\Delta$, where R_Δ is the logarithmic branch divisor. We will now argue by contradiction and suppose that there exists an irreducible component $Z \subset \text{Nlc}(X, \Delta)$ such that the induced endomorphism $f|_Z: Z \rightarrow Z$ satisfies

$$(15) \quad \deg(f|_Z) < \deg(f).$$

Let $(\tilde{X}, Z_{\text{gen}})$ be the germ of the normal variety X in the generic point $Z_{\text{gen}} \subset X$, and denote by

$$\tilde{f}: (\tilde{X}, Z_{\text{gen}}) \rightarrow (\tilde{X}, Z_{\text{gen}})$$

the induced endomorphism. Set $\tilde{\Delta} := \Delta|_{\tilde{X}}$, then the finite morphism \tilde{f} étale in codimension one, i.e. we have

$$(16) \quad K_{\tilde{X}} + \tilde{\Delta} = (\tilde{f})^*(K_{\tilde{X}} + \tilde{\Delta}).$$

Let $\tilde{\mu}: (\tilde{Y}, \Delta_{\tilde{Y}}) \rightarrow (\tilde{X}, \tilde{\Delta})$ be the log-canonical model. By Lemma 2.11 the finite morphism \tilde{f} lifts to a finite morphism $g: \tilde{Y} \rightarrow \tilde{Y}$ such that

$$(17) \quad K_{\tilde{Y}} + \Delta_{\tilde{Y}} = g^*(K_{\tilde{Y}} + \Delta_{\tilde{Y}})$$

and $\mu \circ g = \tilde{f} \circ \mu$.

Since Z_{gen} is an irreducible component of $\text{Nlc}(\tilde{X}, \tilde{\Delta})$ and the μ -exceptional locus has pure codimension one (cf. Remark 2.3), there exists at least one prime divisor E_1 in \tilde{Y} that surjects onto Z_{gen} . Let E_1, \dots, E_k be the prime divisors in $\mu^{-1}(Z_{\text{gen}})$ that surject onto Z_{gen} , then g^{-1} acts by permutation on the set of divisors $\{E_1, \dots, E_k\}$. Thus (up to replacing \tilde{f} and hence g by some iterate) we can assume that g^{-1} acts as the identity. Let now

$$g|_{E_1}: E_1 \rightarrow E_1$$

be the induced endomorphism. We claim that we have

$$\deg(g|_{E_1}) = \deg(f|_Z).$$

Assuming this for the time being, let us see how to conclude: since $\deg(f) = \deg(g)$ our claim and (15) implies that $\deg(g|_{E_1}) < \deg(g)$. Thus E_1 is contained in the branch divisor of g and we have

$$(18) \quad g^*E_1 = rE_1$$

with $r > 1$. By Remark 2.3 we have

$$K_{\tilde{Y}} + \Delta_{\tilde{Y}} = \mu^*(K_{\tilde{X}} + \tilde{\Delta}) + \Delta_{\tilde{Y}}^{\geq 1},$$

where $\Delta_{\tilde{Y}}^{\geq 1}$ is an antieffective divisor such that $\text{supp } \Delta_{\tilde{Y}}^{\geq 1} = \text{Exc}(\mu)$. Plugging this into (17) we obtain

$$\mu^*(K_{\tilde{X}} + \tilde{\Delta}) + \Delta_{\tilde{Y}}^{\geq 1} = g^*\mu^*(K_{\tilde{X}} + \tilde{\Delta}) + g^*\Delta_{\tilde{Y}}^{\geq 1}.$$

Yet by (16) this simplifies to

$$\Delta_{\tilde{Y}}^{\geq 1} = g^*\Delta_{\tilde{Y}}^{\geq 1}.$$

Since $\text{supp } \Delta_{\tilde{Y}}^{\geq 1} = \text{Exc}(\mu)$ it contains the divisor E_1 . Thus by restricting the equation above to E_1 we obtain $g^*E_1 = E_1$, a contradiction to (18).

Proof of the claim. We have a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{g|_{E_1}} & E_1 \\ \mu|_{E_1} \downarrow & & \downarrow \mu|_{E_1} \\ Z_{\text{gen}} & \xrightarrow{f|_{Z_{\text{gen}}}} & Z_{\text{gen}} \end{array}$$

Let F_1 be a general fibre of $\mu|_{E_1}$ and set $F_2 := g|_{E_1}(F_1)$. Then F_2 is a general $\mu|_{E_1}$ -fibre, in particular F_1 and F_2 are homologous. Set $\tilde{g}: F_1 \rightarrow F_2$. By (17) we have

$$(K_{\tilde{Y}} + \Delta_{\tilde{Y}})^{\dim F_1} \cdot F_1 = (g^*(K_{\tilde{Y}} + \Delta_{\tilde{Y}}))^{\dim F_1} \cdot F_1 = \deg(\tilde{g})(K_{\tilde{Y}} + \Delta_{\tilde{Y}})^{\dim F_1} \cdot F_2.$$

Since F_1 and F_2 are homologous we have

$$(K_{\tilde{Y}} + \Delta_{\tilde{Y}})^{\dim F_1} \cdot F_2 = (K_{\tilde{Y}} + \Delta_{\tilde{Y}})^{\dim F_1} \cdot F_1.$$

Moreover $K_{\tilde{Y}} + \Delta_{\tilde{Y}}$ is ample on F_1 , so these intersection numbers are not zero. Thus we obtain that

$$\deg \tilde{g} = 1.$$

By the commutative diagram above this implies the claim. \square

3.3. Corollary. *Let X be a normal projective variety, and let $f: X \rightarrow X$ be a polarised endomorphism of degree $\deg(f) > 1$. Let Δ be a reduced effective totally invariant Weil divisor such that $K_X + \Delta$ is \mathbb{Q} -Cartier.*

Then the pair (X, Δ) is log-canonical. Moreover if Z is an lc centre of (X, Δ) , then (up to replacing f by some iterate) Z is totally invariant. In this case we have $Z \not\subset R_\Delta$ where R_Δ is the logarithmic ramification divisor.

Note that the case $\Delta = 0$ of this statement corresponds to Corollary 1.3.

Proof. The endomorphism f is polarised, so there exists an ample divisor H such that $f^*H \simeq mH$ with $m > 1$. Thus if $Z \subset X$ is a totally invariant subvariety, the endomorphism $f|_Z: Z \rightarrow Z$ is polarised by $H|_Z$. In particular we have

$$\deg(f|_Z) = m^{\dim Z} < m^{\dim X} = \deg(f).$$

By Theorem 1.4 this implies that $\text{Nlc}(X, \Delta)$ is empty. The second part of the statement follows from Lemma 2.10. \square

For inductive purposes the following non-normal version should be useful.

3.4. Corollary. *Let X be a projective variety that is S_2 and whose codimension one points are either regular points or ordinary nodes⁶. Let $f: X \rightarrow X$ be a polarised endomorphism of degree $\deg(f) > 1$. Let Δ be a reduced effective totally invariant Weil divisor such that $K_X + \Delta$ is \mathbb{Q} -Cartier and no irreducible component of Δ is contained in the non-normal locus.*

Then the pair (X, Δ) is semi-log-canonical.

⁶ X is demi-normal in the sense of Kollár.

Proof. Let $\nu: \tilde{X} \rightarrow X$ be the normalisation. Let $D \subset X$ be the divisor defined by the conductor of the normalisation, and let $\tilde{\Delta}$ be the divisorial part of $\nu^{-1}(\Delta)$. Then we have

$$K_{\tilde{X}} + \tilde{\Delta} + D = \nu^*(K_X + \Delta),$$

so $K_{\tilde{X}} + \tilde{\Delta} + D$ is \mathbb{Q} -Cartier. Note that D is reduced since X has ordinary nodes in codimension one.

By the universal property of the normalisation, the endomorphism f lifts to an endomorphism $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$. Moreover the divisor D is totally invariant (cf. Prop.5.4. in the arXiv version of [NZ10]). By Corollary 3.3 the pair $(\tilde{X}, \tilde{\Delta} + D)$ is log-canonical. Thus (X, Δ) is semi-log-canonical. \square

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