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# Uniqueness in the Elastic Bounce Problem

DANILO PERCIVALE

*Scuola Normale Superiore, 56100 Pisa, Italy*

## INTRODUCTION

In a recent paper (see [4]) the elastic bounce problem for a material point constrained to move on a Riemannian manifold  $(V, \langle \cdot, \cdot \rangle)$  elastically bouncing against a submanifold  $V' = \{q \in V: f(q) = 0\}$ , where  $f \in C^3(V)$  was studied. Given  $T > 0$  we say that a pair  $(q, U) \in \text{Lip}(0, T; V) \times L^1(0, T; C^3(V))$  solves the elastic bounce problem if

- (i)  $f(q(t)) \geq 0$ ,
- (ii) there exists a bounded positive measure  $\mu$  on  $[0, T]$  such that  $q$  is an extremal for the functional

$$F(v) = \int_0^T \left\{ \frac{1}{2} \langle \dot{v}, \dot{v} \rangle - U(t, v) \right\} dt + \int_0^T f(v) d\mu \quad (\text{P})$$

and  $\text{spt } \mu \subseteq \{t \in [0, T]: f(q(t)) = 0\}$ ,

- (iii) the function  $E: t \rightarrow \langle \dot{q}(t), \dot{q}(t) \rangle$  is continuous on  $[0, T]$ .

Actually such a formulation applies to more complicated problems. For example, we may consider a rigid disk of radius  $R$  and unit mass which moves on the  $(x, y)$  plane and is elastically reflected by the "wall"  $\{y = 0\}$ . In this case we define  $V = S^1 \times \mathbb{R}^2$  (where  $S^1$  is the one-dimensional torus) and  $a_{ij} \equiv 0$  if  $i \neq j$ ,  $a_{11} = R^2/2$ ,  $a_{22} = a_{33} = 1$ ; denoting by  $(\xi_1, \xi_2) \in \mathbb{R}^2$  the coordinates of the center of mass of the disk and by  $\theta$  the coordinate on  $S^1$ , we may define  $V' = \{(\theta, \xi_1, \xi_2): \xi_2 - R = 0\}$  and so this problem can be put in the previous form.

In many recent papers uniqueness for the problem (P) has been studied and in [7] it was pointed out that even in the simple case  $V = \mathbb{R}$  with the Euclidean metric,  $f(q) = q$ ,  $U(t, q) = -u(t)q$  with  $u \in C^\infty[0, T]$ , assigned the initial data  $q(0), \dot{q}_+(0)$ , the Cauchy problem for (P) does not admit in general a unique solution.

In [7] a counterexample to uniqueness was given when  $\text{grad } U(t, q) = 0$ ,

$V = \mathbb{R}^2$  with the Euclidean metric,  $f \in C^\infty(\mathbb{R}^2)$  and the set  $\Omega = \{q: f(q) \geq 0\}$  is convex.

Again in [7] a (remarkably simple) sufficient condition ensuring uniqueness for the problem (P) was proved:

Let  $V = \mathbb{R}^n$  with the Euclidean metric,  $f \in C^\infty(\mathbb{R}^n)$  and  $\text{grad } 0(t, q) = 0$  on  $\mathbb{R}^n$ . If the boundary of the set  $\Omega = \{q: f(q) \geq 0\}$  has strictly negative Gaussian curvature, then when assigned the initial Cauchy data, the problem (P) admits a unique solution.

The aim of this paper is to find some conditions, weaker than the preceding ones, sufficient to ensure uniqueness for the problem (P). In Section 2 we deal with the general case, i.e., we generalize, developing different techniques, the previous theorem (due to M. Schatzman) to Riemannian manifolds with less restrictive conditions on the potential  $U$ . In Section 3 we confine ourselves to considering the case  $V = \mathbb{R}$  with the Euclidean metric,  $f(q) = q$ , and  $U(t, q) = u(t)q$  with  $u \in L^1(0, T)$ ; we look for a class of forces  $\mathcal{F} \subset L^1(0, T)$  such that for every  $u \in \mathcal{F}$  and for every choice of initial data the Cauchy problem admits a unique solution.

## 1. NOTATIONS AND STATEMENT OF THE PROBLEM

Let  $(V, \langle, \rangle)$  be a  $C^3$  Riemannian manifold and let  $d$  be the metric induced on  $V$  by the scalar product. If  $X$  and  $Y$  are vector fields on  $V$  we denote by  $\nabla_x Y$  the covariant differentiation of  $Y$  with respect to  $X$ . Given  $f \in C^3(V)$  such that  $df(q) \neq 0$  on the set  $V' = \{q \in V: f(q) = 0\}$ , we want to study the bounce problem for a material point whose position at time  $t$  will be indicated by  $q(t)$ . This point is subjected to a potential  $U(t, q)$  and moves in the region  $\{q \in V: f(q) \geq 0\}$ , bouncing against the submanifold  $V' = \{q \in V: f(q) = 0\}$ . Following [4] we shall say that a pair  $(U, q) \in L^1(0, T; C^2(V)) \times \text{Lip}(0, T; V)$  solves the bounce problem if

(i)  $f(q(t)) \geq 0$ ,

(ii) there exists a positive bounded measure  $\mu$  on  $(0, T)$  such that  $q$  is an extremal for the functional

$$F(V) = \int_0^T \left\{ \frac{1}{2} \langle \dot{v}, \dot{v} \rangle - U(t, v) \right\} dt + \int_0^T f(v) d\mu \quad (1.1)$$

and  $\text{spt } \mu \subseteq \{t \in (0, T): f(q(t)) = 0\}$ ,

(iii) the function  $E: t \rightarrow \langle \dot{q}(t), \dot{q}(t) \rangle$  is continuous on  $[0, T]$ ,

where  $\dot{q}_+$ ,  $\dot{q}_-$  denote, respectively, the right and left derivatives of  $q$ .

In fact it can be easily checked that  $q$  is an extremal for the functional  $F$

if and only if we have in local coordinates (the summation convention is adopted),

$$\frac{d}{dt} (a_{ij}(q) \dot{q}^j) = \frac{\partial}{\partial q^i} \left( \frac{1}{2} a_{rs}(q) \dot{q}^r \dot{q}^s - U(t, q) \right) + \mu \frac{\partial f}{\partial q^i}(q), \quad (1.2)$$

where  $a_{ij}$  denote the components of the metric tensor and the equality holds in the sense of distributions. Then (1.2) implies that  $\dot{q} \in BV(0, T; V)$  and so  $q$  has right and left derivatives at each  $t \in [0, T]$ . Moreover, by (iii) the functions  $\langle \dot{q}_+, \dot{q}_+ \rangle$  and  $\langle \dot{q}_-, \dot{q}_- \rangle$  must coincide and we denote their common value by  $\langle \dot{q}, \dot{q} \rangle$ .

Following [2–4] we introduce now the initial trace for the problem (P). Set<sup>1</sup>

$$\begin{aligned} E &= \{(U, q) \in L^1 \times \text{Lip}: (U, q) \text{ solves (P)}\}, \\ Y &= \{q \in \text{Lip}: U \in L^1 \text{ with } (U, q) \in E\}, \\ \mathcal{B} &= \mathbb{R} \times TV \times TV \times TV \times TV, \end{aligned}$$

we define the trace  $\mathcal{F}: [0, T] \times Y \rightarrow \mathcal{B}$  as

$$\mathcal{F}(t, q) = (\frac{1}{2} \langle \dot{q}(t), \dot{q}(t) \rangle, q(t), \dot{q}_\tau(t_0), f(q(t)) \dot{q}(t), 0),$$

where

$$\dot{q}_\tau = \langle \text{grad } f(q), \text{grad } f(q) \rangle \dot{q} - \langle \dot{q}, \text{grad } f(q) \rangle \text{grad } f(q).$$

It is easy to verify that  $\mathcal{F}$  is continuous with respect to  $t$  for every  $q \in Y$  and we remark that, if  $f(q(t)) > 0$ , then to assign  $\mathcal{F}(t, q)$  is equivalent to assigning the Cauchy data at the point  $t$ . We put for every  $t \in [0, T]$

$$\mathcal{A}(t) = \{(b, U, q) \in \mathcal{B} \times L^1 \times \text{Lip}: (U, q) \in E, \mathcal{F}(t, q) = b\};$$

in [3, 4] it was pointed out that the relation  $\mathcal{A}(t)$  does not uniquely characterize  $q$  as a function of  $(b, U)$ , i.e., for a suitable choice of the potential  $U$  and of the initial trace  $b$ , uniqueness for the problem (P) fails. Our goal is to find some sufficient conditions on  $U$  and on  $V'$  in such a way that  $q$  is completely determined as a function of the pair  $(b, U)$ .

For example, in Section 3, dealing with the particular case  $V = \mathbb{R}$  with the Euclidean metric,  $f(q) = q$  and  $U(t, q) = -u(t)q$ , we obtain a very simple result based on the regularity of  $u$ , namely,

If  $u \in C^k([0, T])$ ,  $k \geq 1$  and  $\sum_{j=0}^k |u^{(j)}(t)| \neq 0$  on  $[0, T]$ , then there exists a unique  $q \in \text{Lip}(0, T)$  such that  $(b, U, q) \in \mathcal{A}(t_0)$  for every  $t_0 \in [0, T]$  and for every  $b \in \mathcal{B}$ .

<sup>1</sup> Here we have denoted by  $L^1$  the space  $L^1(0, T; C^2(V))$  and by  $\text{Lip}$  the space  $\text{Lip}(0, T; V)$ .

From this result we shall argue that there exists a class of functions  $\mathcal{F} \subset L^1(0, T)$ , namely the analytic functions on  $[0, T]$ , such that for every  $u \in \mathcal{F}$  and for every choice of initial data the problem (P) admits a unique solution.

## 2. THE GENERAL CASE

Let  $t_0 \in [0, T]$ ,  $b \in \mathcal{B}$ ,  $U \in L^1(0, T; C^2(V))$ ,  $q \in \text{Lip}(0, T; V)$ ; in the sequel we choose local coordinates  $(q_1, \dots, q_n)$  on  $V$  so that to say  $(b, U, q) \in \mathcal{A}(t_0)$  will be equivalent to saying that the following conditions are fulfilled:

- (i)  $f(q(t)) \geq 0$ ,
- (ii) there exists a positive bounded measure  $\mu$  on  $(0, T)$  such that

$$\frac{d}{dt} (a_{ij}(q) \dot{q}^j) = \frac{\partial}{\partial q^i} \left\{ \frac{1}{2} a_{rs}(q) \dot{q}^r \dot{q}^s - U(t, q) \right\} + \mu \frac{\partial f}{\partial q^i}$$

and  $\text{spt } \mu \subseteq \{t \in [0, T]: f(q(t)) = 0\}$ ,

- (iii) for every  $t_1, t_2 \in [0, T]$  the energy relation

$$\frac{1}{2} a_{ij}(q) \dot{q}^i \dot{q}^j \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} - \frac{\partial U}{\partial q^i}(t, q) \dot{q}^i(t) dt$$

- (iv)  $\mathcal{F}(t_0, q) = b$ ,

where, as in the previous section,  $a_{ij}(q)$  denote the components of the metric tensor. In addition to the hypothesis stated in Section 1 we shall assume throughout this section that the following properties hold:

(H<sub>1</sub>) There exist  $\lambda_1, \lambda_2 > 0$  and a collection of open charts  $(X_\alpha, \varphi_\alpha)_{\alpha \in \mathcal{A}}$  such that for every  $x, y \in X_\alpha$  and for every  $\alpha \in \mathcal{A}$  we have

$$\lambda_1 \|\varphi_\alpha(x) - \varphi_\alpha(y)\|_{\mathbb{R}^n} < d(x, y) < \lambda_2 \|\varphi_\alpha(x) - \varphi_\alpha(y)\|_{\mathbb{R}^n}.$$

(H<sub>2</sub>)  $U \in H^1(0, T; C^2(V))$  and there exists  $k \in L^1(0, T)$  such that

$$\sup_{q \in V} \langle \text{grad } U(t, q), \text{grad } U(t, q) \rangle < k(t).$$

Now, set

$$G(t_0, U, b) = \{q \in \text{Lip}: (b, U, q) \in \mathcal{A}(t_0)\},$$

we prepare our main theorem with

**LEMMA 2.1.** *If  $G(t_0, U, b) \neq \emptyset$  and for every  $q \in G(t_0, U, b)$  the set*

$\{t \in [0, T]: f(q(t))=0\}$  has a finite number of connected components then  $G(t_0, U, b)$  is a singleton.

*Proof.* Without loss of generality we may assume that for every  $q \in G(t_0, U, b)$  there exists a unique interval  $I_q \subset [0, T]$  on which  $f(q(t))=0$  and  $t_0 \in I_q$ . Using standard arguments we can easily check that if  $q_1$  and  $q_2$  belong to  $G(t_0, U, b)$  then  $I_{q_1} = I_{q_2} = I$ . If  $I$  is reduced to a point the thesis follows easily; on the contrary, since  $f(q(t)) \equiv 0$  on  $I$ , we have

$$\ddot{q}^i \frac{\partial f}{\partial q^i} + \frac{\partial^2 f}{\partial q^h \partial q^k} \dot{q}^h \dot{q}^k = 0. \quad (2.1.11)$$

for every  $t \in I$ . It is not difficult to verify that the following equality holds on  $I$  in the sense of measures:

$$\ddot{q}^i = - \left\{ \begin{matrix} i \\ hk \end{matrix} \right\} \dot{q}^h \dot{q}^k - a^{ij}(q) \frac{\partial U}{\partial q^j} + \mu a^{ij}(q) \frac{\partial f}{\partial q^j}, \quad i = 1, \dots, n, \quad (2.1.12)$$

where  $\left\{ \begin{matrix} i \\ hk \end{matrix} \right\}$  denote the Christoffel symbols of second kind defined by

$$\left\{ \begin{matrix} i \\ hk \end{matrix} \right\} = \frac{1}{2} a^{li} \left\{ \frac{\partial a_{kl}}{\partial q^h} + \frac{\partial a_{hl}}{\partial q^k} - \frac{\partial a_{hk}}{\partial q^l} \right\}. \quad (2.1.13)$$

Substituting (2.1.12) into (2.1.11) we obtain

$$\mu = \left( a^{ij} \frac{\partial f}{\partial q^i} \frac{\partial f}{\partial q^j} \right)^{-1} \left( - \frac{\partial^2 f}{\partial q^h \partial q^k} + \left\{ \begin{matrix} i \\ hk \end{matrix} \right\} \frac{\partial f}{\partial q^i} \right) \dot{q}^h \dot{q}^k + a^{ij} \frac{\partial U}{\partial q^j} \frac{\partial f}{\partial q^i} \quad (2.1.14)$$

in the sense of measures (i.e., on  $I$  the measure  $\mu$  can be identified with an integrable function). From the above equality and from (2.1.12) we argue that  $q(t)$  solves, in local coordinates, the following system of ordinary differential equations on  $I$ ,

$$\begin{aligned} \ddot{q}^i = & - \left\{ \begin{matrix} i \\ hk \end{matrix} \right\} \dot{q}^h \dot{q}^k - a^{ij} \frac{\partial U}{\partial q^j} + \left( a^{ij} \frac{\partial f}{\partial q^i} \frac{\partial f}{\partial q^j} \right)^{-1} \left[ \left( \frac{\partial f}{\partial q^j} \left\{ \begin{matrix} i \\ hk \end{matrix} \right\} \right. \right. \\ & \left. \left. - \frac{\partial^2 f}{\partial q^h \partial q^k} \right) \dot{q}^h \dot{q}^k + a^{hk} \frac{\partial U}{\partial q^h} \frac{\partial f}{\partial q^k} \right] a^{ij} \frac{\partial f}{\partial q^j} \end{aligned} \quad (2.1.15)$$

and  $\mathcal{F}(t, b) = b$ . This fact is enough to guarantee that  $q$ , if it exists, is unique and so the lemma is proved.

Now we can state

**THEOREM 2.2.** *If  $H_1$  and  $H_2$  hold then  $G(t, U, b) \neq \emptyset$  for every  $t \in [0, T]$  and  $b \in \mathcal{B}$ . Suppose that one of the following conditions holds:*

(i)  $\nabla_v(df)(v) - df(q)(\text{grad } U(t, q)) \geq 0$  for every  $t \in [0, T]$ ,

$$q \in V' \quad \text{and} \quad v \in T_q V.$$

(ii)  $\nabla_v(df)(v) - df(q)(\text{grad } U(t, q)) < 0$  for every  $t \in [0, T]$ ,

$$q \in V' \quad \text{and} \quad v \in T_q V.$$

(iii)  $\text{grad } U(t, q) \equiv 0$  and  $\nabla_v(df)_q(v) < 0$  for every  $q \in V'$ ,

$$v \in T_q V, \quad v \neq 0.$$

Then  $G(t_0, U, b)$  is a singleton.

*Proof.* The first part of this theorem can be proved easily by using the methods developed in [4, Proposition 1.3, p. 11) taking into account  $(H_1)$  and  $(H_2)$ . To prove uniqueness we may consider only the case  $f(b_2) = 0$  and  $\langle b_3, b_3 \rangle - \langle \text{grad } f(b_2), \text{grad } f(b_2) \rangle^2 b_1 = 0$ , where we have put  $b = (b_1, b_2, b_3, b_4, 0)$ . Now let  $q \in G(t_0, U, b)$ ; set  $z(t) = f(q(t))$ . We observe that

$$\begin{aligned} \ddot{z}(t) &= \left( \frac{\partial^2 f}{\partial q^i \partial q^j} - \frac{\partial f}{\partial q^r} \left\{ \begin{matrix} r \\ ij \end{matrix} \right\} \right) \dot{q}^i \dot{q}^j - a^{\bar{y}}(q) \frac{\partial U}{\partial q^j} \frac{\partial f}{\partial q^i} \\ &\quad + \mu a^{\bar{y}}(q) \frac{\partial f}{\partial q^i} \frac{\partial f}{\partial q^j}. \end{aligned} \quad (2.2.1)$$

By using Proposition 1.5 of [4] we can build sequences  $q_h \rightarrow q$  in  $L^\infty$ ,  $U_h \rightarrow U$  in  $L^1$ , and  $b_h \rightarrow b$  in  $\mathcal{B}$  such that

$$\begin{aligned} \frac{d^2}{dt^2} (f(q_h(t))) &= \left( \frac{\partial^2 f}{\partial q^i \partial q^j} - \frac{\partial f}{\partial q^r} \left\{ \begin{matrix} r \\ ij \end{matrix} \right\} \right) \dot{q}_h^i \dot{q}_h^j - a^{\bar{y}}(q_h) \frac{\partial U}{\partial q^j} \frac{\partial f}{\partial q^i} \\ &\quad + \psi_h(f(q_h)) a^{\bar{y}}(q_h) \frac{\partial f}{\partial q^i} \frac{\partial f}{\partial q^j}, \end{aligned} \quad (2.2.3)$$

$$\begin{aligned} b_h &= \left( \frac{1}{2} a_{ij}(q_h(t_0)) \dot{q}_h^i(t_0) \dot{q}_h^j(t_0) \right. \\ &\quad \left. + \alpha_h(f(q_h(t_0)), \dot{q}_h(t_0), q_h(t_0), f(q_h(t_0))) \dot{q}_h(t_0), \frac{1}{h} \dot{q}_h(t_0) \right) \end{aligned} \quad (2.2.4)$$

where  $\psi_h: \mathbb{R} \rightarrow \mathbb{R}_+$ ,  $\alpha_h(x) = \int_x^0 \psi_h(y) dy$  and

- (i)  $\psi_h$  is continuous,  $\psi_h \geq 0$ ,  $\psi_h(x) = 0$  if  $x \geq 0$ ,
- (ii)  $\psi_h \rightarrow +\infty$  uniformly on any compact subset of  $(-\infty, 0)$ ,
- (iii)  $\lim_{h \rightarrow +\infty, x \rightarrow 0} \psi_h(x)/\alpha_h(x) = +\infty$ .

From (2.2.3) we argue

$$\begin{aligned} \left| \frac{d}{dt} f(q_h(t)) \right|^2 &= \left| \frac{d}{dt} f(q_h(t_0)) \right|^2 + \int_{t_0}^t \left\{ \left( \frac{\partial^2 f}{\partial q^i \partial q^j} - \frac{\partial f}{\partial q^r} \left\{ \begin{matrix} r \\ ij \end{matrix} \right\} \right) \dot{q}_h^i \dot{q}_h^j \right. \\ &\quad + \psi_h(f(q_h)) a^{ij}(q_h) \frac{\partial f}{\partial q^i} \frac{\partial f}{\partial q^j} \\ &\quad \left. - a^{ij}(q_h) \frac{\partial U}{\partial q^j} \frac{\partial f}{\partial q^i} \right\} \frac{d}{ds} f(q_h) ds. \end{aligned} \quad (2.2.5)$$

And so passing to the limit as  $h \rightarrow +\infty$  we have

$$|\dot{z}(t)|^2 = 2 \int_{t_0}^t \left( \frac{\partial^2 f}{\partial q^i \partial q^j} - \frac{\partial f}{\partial q^r} \left\{ \begin{matrix} r \\ ij \end{matrix} \right\} \right) \dot{q}^i \dot{q}^j - a^{ij}(q) \frac{\partial U}{\partial q^j} \frac{\partial f}{\partial q^i} \dot{z}(s) ds \quad (2.2.6)$$

Now we examine the two cases which may occur. If  $(\nabla_v(df))_q(v) - df(q)$  ( $\text{grad } U(t, q) \geq 0$ , i.e., in local coordinates

$$\left( \frac{\partial^2 f}{\partial q^i \partial q^j} - \frac{\partial f}{\partial q^r} \left\{ \begin{matrix} r \\ ij \end{matrix} \right\} \right) \dot{q}^i \dot{q}^j - a^{ij}(q) \frac{\partial U}{\partial q^j} \frac{\partial f}{\partial q^i} \geq 0,$$

we can easily see that there exists a neighborhood  $I$  of  $t_0$  such that the set  $\{t \in I: f(q(t)) = 0\}$  has a finite number of connected components. This fact implies local uniqueness and then global uniqueness as we have proved in Lemma 2.1.

Now let us suppose that hypothesis (ii) is fulfilled. Going back to (2.2.6) and integrating by parts we obtain

$$\begin{aligned} |\dot{z}(t)|^2 &= 2 \left( \lambda_{i,j}(q(t)) \dot{q}_-^i(t) \dot{q}_-^j(t) - a^{ij}(q(t)) \frac{\partial U}{\partial q^j} \frac{\partial f}{\partial q^i} \right) z(t) \\ &\quad - \int_{t_0}^t h(s) z(s) ds - \int_{t_0}^t z(s) d\gamma, \end{aligned} \quad (2.2.8)$$

where

$$\lambda_{i,j}(q(t)) = \left( \frac{\partial^2 f}{\partial q^i \partial q^j} - \frac{\partial f}{\partial q^r} \left\{ \begin{matrix} r \\ ij \end{matrix} \right\} \right) (q(t)),$$

$$h(t) = \frac{d}{dt} \left( a^{ij}(q) \frac{\partial U}{\partial q^j} \frac{\partial f}{\partial q^i} \right),$$

$$\gamma = \frac{d}{dt} (\lambda_{i,j}(q) \dot{q}^i \dot{q}^j) \quad \text{in the sense of distributions.}$$



By a standard argument we can check that  $\gamma = \gamma_1 + \gamma_2$ , where  $\gamma_1$  is absolutely continuous with respect to the Lebesgue measure with density  $g \in L^\infty(0, T)$  and  $\gamma_2 \ll \mu$ . Therefore (2.2.8) yields

$$|\dot{z}(t)|^2 = 2 \left( \lambda_{ij}(q(t)) \dot{q}^i_-(t) \dot{q}^j_-(t) - a^{ij}(q(t)) \frac{\partial U}{\partial q^i} \frac{\partial f}{\partial q^j} \right) - \int_{t_0}^t 2(g(s) + h(s)) z(s) ds. \quad (2.2.9)$$

We claim that  $z(t)$  is identically zero in some neighborhood of  $t_0$ . In fact two cases may occur:

(A) there exists  $\delta > 0$  such that in  $[t_0, t_0 + \delta]$  the following inequality holds:

$$z^2(t) \leq M \int_{t_0}^t z^2(s) ds \quad (2.2.8)$$

for a suitable constant  $M > 0$ . Then Gronwall's lemma implies that  $z^2(t) \equiv 0$  in  $[t_0, t_0 + \delta]$

(B) For every  $\delta > 0$  we have

$$\sup_{(t_0, t_0 + \delta)} \frac{z^2(t)}{\int_{t_0}^t z^2(s) ds} = +\infty, \quad (2.2.9)$$

then we may construct a sequence  $t_n$  converging to  $t_0$  such that

$$\lim_{n \rightarrow \infty} \int_{t_0}^{t_n} z^2(t) dt / z^2(t_n) = 0. \quad (2.2.10)$$

Taking into account (2.2.10), from (2.2.7) we argue

$$|\dot{z}(t_n)|^2 \leq 2 \left( \lambda_{ij}(q(t_n)) \dot{q}^i_-(t_n) \dot{q}^j_-(t_n) - a^{ij}(q(t_n)) \frac{\partial U}{\partial q^i} \frac{\partial f}{\partial q^j} \right) \times z(t_n) + \|h + g\|_{L^2} \int_{t_0}^{t_n} z^2(s) ds \quad (2.2.11)$$

which is a contradiction since  $\lambda_{ij}(q(t_n)) \dot{q}^i_-(t_n) \dot{q}^j_-(t_n) - a^{ij}(q(t_n)) (\partial U / \partial q^i) (\partial f / \partial q^j) < 0$  for  $n$  large enough.

Since the same arguments hold on the left of  $t_0$  we may conclude that there exists a neighborhood  $I_{t_0}$  of  $t_0$  such that  $f(q(t)) \equiv 0$  on it and so, by using Lemma 2.1 the thesis easily follows in case (ii).

To prove the theorem in case (iii) we observe at first that if  $b_1 = 0$  and  $\text{grad } U(t, q) \equiv 0$ , then uniqueness is obvious. On the contrary, if  $b_1 \neq 0$  the proof proceeds as in the previous case.

### 3. THE CASE $n = 1$

In this section we deal with the one-dimensional bounce problem (see [2, 5]) whose formulation can be deduced from (1.1) setting  $V = \mathbb{R}$  with the Euclidean metric,  $f(q) = q$  and so  $V' = \{0\}$ . We confine ourselves to considering the case  $U(t, q) = -u(t)q$ , where  $u \in L^1(0, T)$ , so that problem (P) can be rewritten as follows:

- (i)  $q(t) \geq 0$ ,
- (ii)  $\tilde{q} - u \geq 0$  in the sense of distributions,
- (iii)  $\text{spt}(\tilde{q} - u) \subset \{t \in [0, T]: q(t) = 0\}$ ,
- (iv) for every  $t_1, t_2 \in [0, T]$  we have

$$|\dot{q}(t_2)|^2 = |\dot{q}(t_1)|^2 + 2 \int_{t_1}^{t_2} u(t) \dot{q}(t) dt.$$

In what follows we shall say that  $(b, u, q) \in \mathcal{A}(t_0)$  instead of  $(b, U, q) \in \mathcal{A}(t_0)$  when the pair  $(q, u)$  solves (P) and  $\mathcal{F}(t_0, q) = b$ . In [5] it is proved that if  $u \in L^1(0, T)$  and  $b \in \mathcal{B}$  then there exists (at least one)  $q \in \text{Lip}([0, T])$  such that  $(b, u, q) \in \mathcal{A}(t_0)$ ; our aim is to show that some regularity assumptions on  $u$  are sufficient to ensure that this solution is unique. To this aim we recall the following results (see [5]):

**LEMMA 3.1.** *Let  $b \in \mathcal{B}$ ,  $t_0 \in [0, T]$ ,  $u \in L^1(0, T)$  and  $u \geq 0$  a.e. on  $[0, T]$ . Then there exists a unique  $q \in \text{Lip}([0, T])$  such that  $(b, u, q) \in \mathcal{A}(t_0)$ .*

**LEMMA 3.2.** *Let  $b \in \mathcal{B}$ ,  $t_0 \in [0, T]$ ,  $u \in L^1(0, T)$ . If there exists  $q \in \mathcal{A}(t_0)$  such that the set  $\{t \in [0, T]: q(t) = 0\}$  is finite, then  $q$  is unique.*

Using Lemma 3.2 we shall show that it is possible to generalize Lemma 3.1 assuming more regularity on  $u$ . Indeed we have

**LEMMA 3.3.** *Let  $t_0 \in [0, T]$ ,  $b \in \mathcal{B}$ , and  $u \in H^1(0, T)$ . If  $u \geq 0$  or  $u < 0$  on  $[0, T]$ , then there exists a unique  $q \in \text{Lip}([0, T])$  such that  $(b, u, q) \in \mathcal{A}(t_0)$ .*

The proof of this lemma is omitted, since it can be easily obtained by using the same techniques developed in the proof of Theorem 2.2. Now we state

**THEOREM 3.4.** *Let  $t_0 \in [0, T]$ ,  $b \in \mathcal{B}$ , and  $u \in C^k([0, T])$   $k \geq 1$ . If  $\sum_{j=0}^k |u^{(j)}(t)| \neq 0$  on  $[0, T]$ , then there exists a unique  $q \in \text{Lip}([0, T])$  such that  $(b, u, q) \in \mathcal{A}(t_0)$ .*

*Proof.* We may consider only the case  $b_1 = b_2 = 0$ ; since  $\sum_{j=0}^k |u^{(j)}(t)| \neq 0$  then there exists  $k$  such that  $|u^{(k)}(t_0)| \neq 0$ . If  $k_0 = 0$  we may apply

Lemma 3.3; if  $k_0 \neq 0$  we may confine ourselves to considering only the case  $u^{(k_0)}(t_0) < 0$ . We have only to show that there exists  $\delta > 0$  such that  $q(t) \equiv 0$  on  $[t_0, t_0 + \delta]$ . Let us choose  $\delta > 0$  such that  $u(t) < 0$  on  $(t_0, t_0 + \delta]$  and suppose that there exists a sequence  $\xi_n \in (t_0, t_0 + \delta]$  converging to  $t_0$  and such that  $q(\xi_n) > 0$ ; then there exists a decreasing sequence  $t_n \in (t_0, t_0 + \delta]$  converging to  $t_0$  such that  $q(t_n) = 0$ ,  $|\dot{q}(t_n)| > 0$  and  $q(t) > 0$  on  $(t_{n+1}, t_n)$  for every  $n \in \mathbb{N}$ . We have

$$\ddot{q}(t) = u(t) \quad \text{on} \quad (t_{n+1}, t_n)$$

and therefore

$$\frac{t_{n+1} - t_{n+2}}{t_n - t_{n+1}} = \frac{u(\sigma_n)}{u(\sigma_{n+1})}$$

for suitable  $\sigma_n \in (t_{n+1}, t_n)$  and  $\sigma_{n+1} \in (t_{n+2}, t_{n+1})$ . Since  $u^{(k_0)}(t_0) < 0$  we have  $u(\sigma_n) < u(\sigma_{n+1}) < 0$  for  $n$  large enough and so  $t_{n+1} - t_{n+2} > t_n - t_{n+1}$  which is a contradiction. ■

Combining Theorem 3.4 with Lemma 3.1 we easily obtain

**THEOREM 3.7.** *Let  $u$  be real-analytic on an open interval  $I \supset [0, T]$ . Then for every  $t_0 \in [0, T]$  and for every  $b \in \mathcal{B}$  there exists a unique  $q \in \text{Lip}(0, T)$  such that  $(b, u, q) \in \mathcal{A}(t_0)$ . ■*

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