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STRUWE'S DECOMPOSITION FOR A POLYHARMONIC OPERATOR ON A COMPACT RIEMANNIAN MANIFOLD WITH OR WITHOUT BOUNDARY

SAIKAT MAZUMDAR

ABSTRACT. Given a high-order elliptic operator on a compact manifold with or without boundary, we perform the decomposition of Palais-Smale sequences for a nonlinear problem as a sum of bubbles. This is a generalization of the celebrated 1984 result of Struwe [16]. Unlike the case of second-order operators, bubbles close to the boundary might appear. Our result includes the case of a smooth bounded domain of \mathbb{R}^n .

1. INTRODUCTION

Let (M, g) be a smooth, compact Riemannian manifold of dimension n with or without boundary. In the latter case we understand that \overline{M} is a compact, oriented submanifold of (\tilde{M}, g) which is itself a smooth, compact Riemannian manifold without boundary and with the same metric g . As one checks, this includes smooth bounded domains of \mathbb{R}^n . When the boundary $\partial M \neq \emptyset$, we let ν be its outward oriented normal vector in \tilde{M} . Let k be a positive integer such that $2k < n$. We define the Sobolev space $H_{k,0}^2(M)$ as the completion of $C_c^\infty(M)$ for the norm $u \mapsto \sum_{i=0}^k \|\nabla^i u\|_2$. This norm is equivalent (see Robert [14]) to the Hilbert norm $\|u\|_{H_k^2} := \left(\sum_{l=0}^k \int_M (\Delta_g^{l/2} u)^2 dv_g \right)^{1/2}$ where $\Delta_g := -\operatorname{div}_g(\nabla)$ is the Laplace-Beltrami operator and, for α odd, $\Delta_g^\alpha u \Delta_g^\alpha v := (\nabla \Delta_g^{\frac{\alpha-1}{2}} u, \nabla \Delta_g^{\frac{\alpha-1}{2}} v)_g$ for all $u, v \in H_k^2(M)$. For details we refer to Aubin [3] and Hebey [9].

We consider the functional

$$I(u) := \frac{1}{2} \int_M (\Delta_g^{k/2} u)^2 dv_g + \frac{1}{2} \sum_{l=0}^{k-1} \int_M A_l(\nabla^l u, \nabla^l u) dv_g - \frac{1}{2_k^\sharp} \int_M |u|^{2_k^\sharp} dv_g$$

where for all $l \in \{0, \dots, k-1\}$, A_l is a smooth T_{2l}^0 -tensor field on M and A_l is symmetric (that is $A_l(X, Y) = A_l(Y, X)$ for all T_0^l -tensors X, Y on M). Here, $2_k^\sharp := \frac{2n}{n-2k}$ is the critical Sobolev exponent such that $H_{k,0}^2(M) \hookrightarrow L^{2_k^\sharp}(M)$ is continuous, which makes the definition of I consistent for all $u \in H_{k,0}^2(M)$. Critical

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points $u \in H_{k,0}^2(M)$ for I are weak solutions to the pde

$$(1) \quad \begin{cases} Pu = |u|^{2_k^\sharp - 2}u & \text{in } M \\ \partial_\nu^\alpha u = 0 & \text{on } \partial M \text{ for } |\alpha| \leq k-1 \end{cases}$$

where for any $u \in C^{2k}(M)$, we define

$$Pu := \Delta_g^k u + \sum_{l=0}^{k-1} (-1)^l \nabla^{j_l \dots j_1} ((A_l)_{i_1 \dots i_l, j_1 \dots j_l} \nabla^{i_1 \dots i_l} u)$$

and where we say that $u \in H_{k,0}^2(M)$ is a weak solution to (1) if

$$\int_M \Delta_g^{k/2} u, \Delta_g^{k/2} \varphi \, dv_g + \sum_{l=0}^{k-1} \int_M A_l (\nabla^l u \nabla^l \varphi) \, dv_g = \int_M |u|^{2_k^\sharp - 2} u \varphi \, dv_g$$

for all $\varphi \in H_{k,0}^2(M)$. As shown by the regularity theorem in Mazumdar [13], a weak solution u to (1) is indeed a strong solution, $u \in C^{2k}(\overline{M})$.

Definition 1.1. Let $(X, \|\cdot\|)$ be a Banach space and $F \in C^1(X)$. A sequence (u_α) in X is said to be a Palais-Smale sequence for F if $(F(u_\alpha))_\alpha$ has a limit in \mathbb{R} when $\alpha \rightarrow +\infty$, while $DF(u_\alpha) \rightarrow 0$ strongly in X' as $\alpha \rightarrow +\infty$.

In this paper, we describe the lack of relative compactness of Palais-Smale sequences for I , which is due to the noncompact embedding $H_{k,0}^2(M) \hookrightarrow L^{2_k^\sharp}(M)$. For Ω any open domain of \mathbb{R}^n , we let $\mathcal{D}_k^2(\Omega)$ be the completion of $C_c^\infty(\Omega)$ for the norm $u \mapsto \|\Delta^{k/2} u\|_2$. The limiting equations of (1) are

$$(2) \quad \Delta^k u = |u|^{2_k^\sharp - 2} u \text{ in } \mathbb{R}^n, \quad u \in \mathcal{D}_k^2(\mathbb{R}^n)$$

$$(3) \quad \left\{ \begin{array}{l} \Delta^k u = |u|^{2_k^\sharp - 2} u \quad \text{in } \mathbb{R}_-^n \\ \partial_\nu^\alpha u = 0 \quad \quad \quad \text{on } \partial \mathbb{R}_-^n \end{array} \right\}, \quad u \in \mathcal{D}_k^2(\mathbb{R}_-^n)$$

where $\Delta := \Delta_{\text{Eucl}}$ is the Laplacian on \mathbb{R}^n endowed with the Euclidean metric Eucl. Associated to the functional I is the limiting functional

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^n} (\Delta u)^2 \, dx - \frac{1}{2_k^\sharp} \int_{\mathbb{R}^n} |u|^{2_k^\sharp} \, dx \text{ for all } u \in \mathcal{D}_k^2(\mathbb{R}^n).$$

Our main theorem below shows that the lack of convergence to a solution of equation (1) is described by a sum of Bubbles:

Theorem 1.1. Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. Then there exists $d \in \mathbb{N}$ bubbles $[(x_\alpha^{(j)}), (r_\alpha^{(j)}), u^{(j)}]$, $j = 1, \dots, d$, (see Definition 2.1 below) there exists $u_\infty \in H_{k,0}^2(M)$ a solution to (1) such that, up to a subsequence,

$$u_\alpha = u_\infty + \sum_{j=1}^d B_{x_\alpha^{(j)}, r_\alpha^{(j)}}(u^{(j)}) + o(1) \text{ where } \lim_{\alpha \rightarrow +\infty} o(1) = 0 \text{ in } H_{k,0}^2(M)$$

and

$$I(u_\alpha) = I(u_\infty) + \sum_{j=1}^d E(u^{(j)}) + o(1) \text{ as } \alpha \rightarrow +\infty.$$

In Section 2, Bubbles are defined up to a term going to 0 strongly, which is relevant here. As one checks, given $u \in \mathcal{D}_k^2(\mathbb{R}^n)$ a nontrivial weak solution to (2) or (3), then multiplying the equation by u and integrating by parts yields

$$(4) \quad E(u) \geq \beta^\sharp := \frac{k}{n} K_0(n, k)^{-n/2k}$$

where $K_0(n, k)$ be the best constant of the embedding $\mathcal{D}_k^2(\mathbb{R}^n) \hookrightarrow L^{2^\sharp_k}(\mathbb{R}^n)$, that is

$$(5) \quad K_0(n, k)^{-1} = \inf_{u \in \mathcal{D}_k^2(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} (\Delta^{k/2} u)^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\sharp_k} dx \right)^{\frac{2}{2^\sharp_k}}}$$

When the Palais-Smale sequence is nonnegative, the bubbles are positive and correspond to positive solutions to (2). As shown in Lions [12], Swanson [17], Ge-Wei-Zhou [7], these solutions are exactly the extremals for (5) and are of the form

$$(6) \quad u = U_{a, \lambda} := \alpha_{n, k} \left(\frac{\lambda}{1 + \lambda^2 |\cdot - a|^2} \right)^{\frac{n-2k}{2}} \quad a \in \mathbb{R}^n, \lambda > 0$$

where $\alpha_{n, k} > 0$ is explicit. We then get the following:

Theorem 1.2. *Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. We assume that $u_\alpha \geq 0$ for all $\alpha \in \mathbb{N}$. Then there exists $u_\infty \in H_{k,0}^2(M)$ a solution to (1), there exists $d \in \mathbb{N}$, there exist $(x_\alpha^{(1)}), \dots, (x_\alpha^{(d)}) \in M$, $(r_\alpha^{(1)}), \dots, (r_\alpha^{(d)}) \in (0, +\infty)$ such that $r_\alpha^{(j)} \rightarrow 0$ and $r_\alpha^{(j)} = o(d(x_\alpha^{(j)}, \partial M))$ as $\alpha \rightarrow +\infty$ for all $j = 1, \dots, d$, such that, up to a subsequence,*

$$u_\alpha = u_\infty + \sum_{j=1}^d \eta \left((\tilde{r}_\alpha^{(j)})^{-1} \exp_{x_\alpha^{(j)}}^{-1}(\cdot) \right) \alpha_{n, k} \left(\frac{r_\alpha^{(j)}}{(r_\alpha^{(j)})^2 + d_g(\cdot, x_\alpha^{(j)})^2} \right)^{\frac{n-2k}{2}} + o(1)$$

where $\lim_{\alpha \rightarrow +\infty} o(1) = 0$ in $H_{k,0}^2(M)$, and η and $(\tilde{r}_\alpha^{(j)})$'s are as in (8). Moreover,

$$I(u_\alpha) = I(u_\infty) + d\beta^\sharp + o(1) \quad \text{as } \alpha \rightarrow +\infty$$

where β^\sharp is as in (4).

When $k = 1$ and M is a smooth bounded domain of \mathbb{R}^n , Theorem 1.1 is the pioneering result of Struwe [16]. There have been several extensions. Without being exhaustive, we refer to Hebey-Robert [11] for $k = 2$ and manifolds without boundary, Saintier [15] for the p -Laplace operator, El-Hamidi-Vétois [5] for anisotropic operators and Almaraz [1] for nonlinear boundary conditions. A general reference for description as bubbles is the monograph by Fieseler-Tintarev [18]. Another possible description is in the sense of measures as in Lions [12]: a general result of this flavour for high order elliptic operators on manifolds is in Mazumdar [13].

Palais-Smale sequence are produced via critical point techniques, like the Mountain-Pass Lemma of Ambrosetti-Rabinowitz [2] or other topological methods (see for instance the monograph Ghoussoub [8] and the references therein). Concerning higher-order problems, we refer to Bartsch-Weth-Willem [4], Ge-Wei-Zhou [7], Mazumdar [13], the general monograph Gazzola-Grunau-Sweers [6] and the references therein. Theorem 1.1 is used by the author in [13] to get Coron-type solutions to equation (1).

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2. DEFINITION OF BUBBLES

In the spirit of the exponential map, we first cook up a chart around any boundary point. We fix $x_0 \in \partial M$. Since M is a smooth submanifold of \tilde{M} , there exist Ω an open subset of \tilde{M} with $x_0 \in \Omega$, there exists $U \subset \mathbb{R}^n$ open with $0 \in U$, such that for any $x \in \Omega \cap \partial M$ there exists $\mathcal{T}_x \in C^\infty(U, \tilde{M})$ having the following properties.

$$(7) \quad \left\{ \begin{array}{l} \bullet \mathcal{T}_x(0) = x \\ \bullet \mathcal{T}_x \text{ is a smooth diffeomorphism onto its image } \mathcal{T}_x(U). \\ \bullet \mathcal{T}_x(U \cap \{x_1 < 0\}) = \mathcal{T}_x(U) \cap M \\ \bullet \mathcal{T}_x(U \cap \{x_1 = 0\}) = \mathcal{T}_x(U) \cap \partial M \\ \bullet (x, z) \mapsto \mathcal{T}_x(z) \text{ is smooth from } \Omega \times U \text{ to } \tilde{M} \\ \bullet d\mathcal{T}_x(0) : \mathbb{R}^n \rightarrow T_x M \text{ is an isometry} \\ \bullet d\mathcal{T}_x(0)[e_1] = \nu_x \text{ where } \nu_x \text{ is the outer unit normal vector to } \partial M \\ \text{at the point } x. \end{array} \right.$$

This map is defined uniformly with respect to x in a neighborhood Ω of a fixed point $x_0 \in \partial M$. By a standard abuse of notation, we will always consider $x \mapsto \mathcal{T}_x$ without any reference to Ω or x_0 : this will make sense in the sequel since the relevant points will always be in the neighborhood of a fixed point.

Definition 2.1. A “Bubble” is a triplet $[(x_\alpha), (r_\alpha), u]$ where $x_\alpha \in \overline{M}$ is a convergent sequence, $r_\alpha > 0$ for all $m \in \mathbb{N}$ with $\lim_{\alpha \rightarrow +\infty} r_\alpha = 0$ and

$$\left. \begin{array}{l} \text{either } \left\{ x_\alpha \in M, \lim_{\alpha \rightarrow +\infty} \frac{d(x_\alpha, \partial M)}{r_\alpha} = +\infty \text{ and } u \in \mathcal{D}_k^2(\mathbb{R}^n) \text{ satisfies (2)} \right\} \\ \text{or } \left\{ x_\alpha \in \partial M \text{ and } u \in \mathcal{D}_k^2(\mathbb{R}_-^n) \text{ satisfies (3)} \right\} \end{array} \right\}$$

If $x_\alpha \in M$, we let $\tilde{r}_\alpha > 0$ be such that

$$(8) \quad \lim_{\alpha \rightarrow +\infty} \tilde{r}_\alpha = \tilde{r}_\infty \in \left[0, \frac{i_g(\tilde{M})}{2} \right), \quad \lim_{\alpha \rightarrow +\infty} \frac{r_\alpha}{\tilde{r}_\alpha} = 0 \text{ and } \tilde{r}_\alpha < \frac{d_g(x_\alpha, \partial M)}{2}$$

and we define

$$B_{x_\alpha, r_\alpha}(u) := \eta \left(\frac{\exp_{x_\alpha}^{-1}(x)}{\tilde{r}_\alpha} \right) r_\alpha^{-\frac{n-2k}{2}} u \left(\frac{\exp_{x_\alpha}^{-1}(x)}{r_\alpha} \right)$$

where $\eta \in C_c^\infty(B_0(i_g(\tilde{M})))$ is identically 1 in a neighborhood of 0. Here, the exponential map is taken on the ambient manifold (\tilde{M}, g) .

If $x_\alpha \in \partial M$, we let $x_0 := \lim_{\alpha \rightarrow +\infty} x_\alpha$, and we define

$$B_{x_\alpha, r_\alpha}(u) := \eta(\mathcal{T}_{x_\alpha}^{-1}(x)) r_\alpha^{-\frac{n-2k}{2}} u \left(\frac{\mathcal{T}_{x_\alpha}^{-1}(x)}{r_\alpha} \right)$$

where \mathcal{T}_x is as in (7), Ω is a neighborhood of $x_0 \in \partial M$ and $\eta \in C_c^\infty(U)$ is identically 1 in a neighborhood of 0.

Beside $[(x_\alpha), (r_\alpha), u]$, the definition of a bubble depends on the choice of the cut-off function η , the radius \tilde{r}_α and the chart \mathcal{T}_x . However, as shown in the proposition below, after quotienting by sequences going to 0, the class of a Bubble is independent of these later parameters.

Proposition 2.1. *The definition of Bubbles depend only on $[(x_\alpha), (r_\alpha), u]$, up to a sequence going to 0 strongly in $H_{k,0}^2(M)$.*

Proof of Proposition 2.1. We first assume that $u \in \mathcal{D}_k^2(\mathbb{R}^n)$ satisfies (2) and that

$$(9) \quad \lim_{\alpha \rightarrow +\infty} \frac{d_g(x_\alpha, \partial M)}{r_\alpha} = +\infty.$$

For $i = 1, 2$, we set the bubbles $B_\alpha^i := \eta^i \left((\tilde{r}_\alpha^i)^{-1} \exp_{x_\alpha}^{-1}(\cdot) \right) r_\alpha^{-\frac{n-2k}{2}} u \left(r_\alpha^{-1} \exp_{x_\alpha}^{-1}(\cdot) \right)$, where $\eta^i \in C_c^\infty(B_0(2a_i))$, $\eta^i \equiv 1$ in $B_0(a_i)$ with $0 < 2a_i \leq \iota_g(\tilde{M})$; $\tilde{r}_\alpha^i > 0$ are as in (8). We let $r_\alpha^{max} = \max\{a_1 \tilde{r}_\alpha^1, a_2 \tilde{r}_\alpha^2\}$ and $r_\alpha^{min} = \min\{a_1 \tilde{r}_\alpha^1, a_2 \tilde{r}_\alpha^2\}$, and let $\epsilon_\alpha^{max} = r_\alpha / r_\alpha^{max}$ and $\epsilon_\alpha^{min} = r_\alpha / r_\alpha^{min}$. Then $\lim_{\alpha \rightarrow 0} \epsilon_\alpha^{max} = 0$ and $\lim_{\alpha \rightarrow 0} \epsilon_\alpha^{min} = 0$. The comparison lemma 9.1 of [13] yields $C > 0$ such that for any $R > 0$ and α large

$$\begin{aligned} \sum_{l=0}^k \|\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2)\|_2^2 &\leq \sum_{l=0}^k \int_{B_{2r_\alpha^{max}}(x_\alpha) \setminus B_{r_\alpha^{min}}(x_\alpha)} \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g \\ &\leq \sum_{i=1,2} \sum_{l=0}^k \int_{M \setminus B_{Rr_\alpha}(x_\alpha)} \left(\Delta_g^{l/2} (B_{x_\alpha, r_\alpha}^i(u)) \right)^2 dv_g. \end{aligned}$$

Therefore, using (33), we get that $B_\alpha^1 - B_\alpha^2 = o(1)$ in $H_k^2(M)$ as $\alpha \rightarrow +\infty$.

Now we consider the case of a boundary bubble, that is $x_\alpha \in \partial M$ and $u \in \mathcal{D}_k^2(\mathbb{R}^n)$ satisfies (3). For $i = 1, 2$, we set $B_\alpha^i := \eta^i \left(\mathcal{T}_{x_\alpha}^{-1}(\cdot) \right) r_\alpha^{-\frac{n-2k}{2}} u \left(r_\alpha^{-1} \mathcal{T}_{x_\alpha}^{-1}(\cdot) \right)$ where \mathcal{T}_x , $i = 1, 2$, are as in (7), U is a neighborhood of $x_0 \in \partial M$ and $\eta^1, \eta^2 \in C_c^\infty(U)$ are identically 1 in a neighborhood of 0. One has

$$\begin{aligned} \sum_{l=0}^k \int_M \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g &\leq \\ \sum_{l=0}^k \int_{D_\alpha(R) \cap M} \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g &+ \sum_{l=0}^k \int_{M \setminus D_\alpha(R)} \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g \end{aligned}$$

where $D_\alpha(R) := \mathcal{T}_{x_\alpha}^{-1}(B_0(r_\alpha R)) \cup \mathcal{T}_{x_\alpha}^2(B_0(r_\alpha R))$. It follows as in the comparison Lemma 9.1 of [13] that there exists $C > 0$ such that for α large

$$\begin{aligned} \sum_{l=0}^k \int_{D_\alpha(R) \cap M} \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g &\leq \\ C \sum_{l=0}^k \int_{(B_0(r_\alpha R) \cup \Phi_\alpha^{-1}(B_0(r_\alpha R))) \cap \mathbb{R}_-^n} \left(\Delta^{l/2} \left((B_\alpha^1 \circ \mathcal{T}_{x_\alpha}^1) - (B_\alpha^2 \circ \mathcal{T}_{x_\alpha}^1) \right) \right)^2 dx &\leq \\ C \sum_{l=0}^k \int_{B_0(R) \cap \mathbb{R}_-^n} \left[\Delta^{l/2} (\eta^a(r_\alpha \cdot) u) - \Delta^{l/2} (\eta^b(\Phi_\alpha(r_\alpha \cdot)) u (r_\alpha^{-1} \Phi_\alpha(r_\alpha \cdot))) \right]^2 dx &= o(1) \end{aligned}$$

where $\Phi_\alpha := \mathcal{T}_{x_\alpha}^2{}^{-1} \circ \mathcal{T}_{x_\alpha}^1$ and $d(\Phi_\alpha)_0 = Id$. Similarly to the case (9), we get that

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \sum_{l=0}^k \int_{M \setminus D_\alpha(R)} \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g = 0.$$

This completes the proof of Proposition 2.1. \square

3. PRELIMINARY ANALYSIS

The proof of Theorem 1.1 goes through four steps. All results are up to a subsequence. We let $(u_\alpha)_\alpha \in H_{k,0}^2(M)$ be a Palais-Smale sequence for I .

Step 1: We claim that $(u_\alpha)_\alpha$ is bounded in $H_{k,0}^2(M)$.

Proof of the claim: Since (u_α) is a Palais-Smale sequence, we have that

$$\begin{aligned} \langle DI(u_\alpha), u_\alpha \rangle &= \int_M (\Delta_g^{k/2} u_\alpha)^2 dv_g + \sum_{\alpha=0}^{k-1} \int_M A_l(\nabla^l u_\alpha, \nabla^l u_\alpha) dv_g \\ &\quad - \int_M |u_\alpha|^{2_k^\sharp} dv_g = o\left(\|u_\alpha\|_{H_k^2}\right) \end{aligned}$$

Therefore

$$(10) \quad \int_M |u_\alpha|^{2_k^\sharp} dv_g = \frac{n}{k} I(u_\alpha) + o\left(\|u_\alpha\|_{H_k^2}\right) \leq C + o\left(\|u_\alpha\|_{H_k^2}\right)$$

Since $(I(u_\alpha))_\alpha$ is bounded, then putting together these equalities yields

$$\|u_\alpha\|_{H_k^2}^2 \leq C + C \|u_\alpha\|_{H_{k-1}^2}^2 + C \int_M |u_\alpha|^{2_k^\sharp} dv_g$$

Now since the embedding of $H_{k,0}^2(M)$ in $H_{0,k-1}^2(M)$ is compact, then for any $\varepsilon > 0$ there exists a $B_\varepsilon > 0$ such that $\|u\|_{H_{k-1}^2}^2 \leq \varepsilon \|u\|_{H_k^2}^2 + B_\varepsilon \|u\|_{2_k^\sharp}^2$ for all $u \in H_k^2(M)$. Therefore, taking $\varepsilon > 0$ small enough, we get that

$$\|u_\alpha\|_{H_k^2}^2 \leq C + C \int_M |u_\alpha|^{2_k^\sharp} dv_g$$

Then using (10) we get that $\|u_\alpha\|_{H_k^2}^2 \leq C + C \|u_\alpha\|_{H_k^2}^2$ for all α , and therefore the sequence (u_α) is bounded in $H_{k,0}^2(M)$. This proves the claim. \square

Since (u_α) is bounded in $H_{k,0}^2(M)$, there exists $u_\infty \in H_{k,0}^2(M)$ such that

$$(11) \quad \begin{cases} u_\alpha \rightharpoonup u_\infty & \text{weakly in } H_{k,0}^2(M) \text{ and } L^{2_k^\sharp}(M), \\ u_\alpha \rightarrow u_\infty & \text{strongly in } H_{l,0}^2(M) \text{ and in } L^q(M) \text{ for } l < k, q < 2_k^\sharp, \\ u_\alpha(x) \rightarrow u_\infty(x) & \text{a.e in } M \end{cases}$$

We define $v_\alpha := u_\alpha - u_\infty$.

Step 2: We claim that

- (1) $DI(u_\infty) = 0$
- (2) (v_α) is a Palais-Smale sequence for the functional J on the space $H_{k,0}^2(M)$,
- (3) $J(v_\alpha) = I(u_\alpha) - I(u_\infty) + o(1)$ as $\alpha \rightarrow +\infty$.

where

$$J(u) := \frac{1}{2} \int_M (\Delta_g^{k/2} u)^2 dv_g - \frac{1}{2_k^\sharp} \int_M |u|^{2_k^\sharp} dv_g \text{ for } u \in H_{k,0}^2(M)$$

Proof of the claim: We fix $\varphi \in H_{k,0}^2(M)$. We have that

$$\begin{aligned} \langle DI(u_\alpha), \varphi \rangle &= \int_M \Delta_g^{k/2} u_\alpha \Delta_g^{k/2} \varphi dv_g + \sum_{\alpha=0}^{k-1} \int_M A_l(g)(\nabla^l u_\alpha, \nabla^l \varphi) \\ &\quad - \int_M |u_\alpha|^{2_k^\sharp-2} u_\alpha \varphi dv_g = o(1) \end{aligned} \quad (12)$$

The following classical integration Lemma will be often used in the sequel (see Lemma 6.2.7 in Hebey [10] for a proof):

Lemma 3.1. *Let (M, g) be a Riemannian manifold. If (f_α) is a bounded sequence in $L^p(M)$, $1 < p < +\infty$, such that $f_\alpha \rightarrow f$ a.e in M , then $f \in L^p(M)$ and $f_\alpha \rightharpoonup f$ weakly in $L^p(M)$.*

Since $(|u_\alpha|^{2_k^\sharp-2} u_\alpha)_\alpha$ is bounded in $L^{\frac{2_k^\sharp}{2_k^\sharp-1}}$ and converges a.e., Lemma 3.1 yields

$$\int_M |u_\alpha|^{2_k^\sharp-2} u_\alpha \varphi dv_g = \int_M |u_\infty|^{2_k^\sharp-2} u_\infty \varphi dv_g + o(1) \quad (13)$$

Therefore, the weak convergence of (u_α) to u_∞ , (12) and (13) yield that u_∞ is a weak solution to (1). This proves point (1) of Step 2.

We now estimate $I(u_\alpha)$. From (11) we have

$$\begin{aligned} \int_M (\Delta_g^{k/2} u_\alpha)^2 dv_g - \int_M (\Delta_g^{k/2} u_\infty)^2 dv_g &= \int_M (\Delta_g^{k/2} v_\alpha)^2 dv_g + o(1), \\ \sum_{l=0}^{k-1} \int_M A_l(\nabla^l u_\alpha, \nabla^l u_\alpha) dv_g &= \sum_{l=0}^{k-1} \int_M A_l(\nabla^l u_\infty, \nabla^l u_\infty) dv_g + o(1) \end{aligned}$$

The following two inequalities will be of constant use in the sequel: for any $1 < p < +\infty$, there exists $C > 0$ such that

$$| |a+b|^p - |a|^p - |b|^p | \leq C (|a|^{p-1}|b| + |b|^{p-1}|a|) \quad (14)$$

$$| |a+b|^p(a+b) - |a|^p a - |b|^p b | \leq C (|a|^p|b| + |b|^p|a|) \quad (15)$$

for all $a, b \in \mathbb{R}$. It then follows from (14) that

$$\left| |u_\alpha|^{2_k^\sharp} - |u_\infty|^{2_k^\sharp} - |v_\alpha|^{2_k^\sharp} \right| \leq C \left(|v_\alpha|^{2_k^\sharp-1} |u_\infty| + |u_\infty|^{2_k^\sharp-1} |v_\alpha| \right),$$

and then using Lemma 3.1, we get that

$$\int_M |u_\alpha|^{2_k^\sharp} dv_g - \int_M |u_\infty|^{2_k^\sharp} dv_g = \int_M |v_\alpha|^{2_k^\sharp} dv_g + o(1)$$

Hence $I(u_\alpha) - I(u_\infty) = J(v_\alpha) + o(1)$ as $\alpha \rightarrow +\infty$, which proves point (3) of Step 2.

Next we show the sequence (v_α) is a Palais-Smale sequence for the functional J on $H_{k,0}^2(M)$. Let $\varphi \in H_{k,0}^2(M)$, we have

$$\langle DJ(v_\alpha), \varphi \rangle = \langle DI(u_\alpha), \varphi \rangle - \langle DI(u_\infty), \varphi \rangle + \int_M \Phi_\alpha \varphi dv_g + o(\|\varphi\|_{H_k^2}) \quad (16)$$

where

$$\Phi_\alpha := |v_\alpha + u_\infty|^{2_k^\sharp - 2} (v_\alpha + u_\infty) - |u_\infty|^{2_k^\sharp - 2} u_\infty - |v_\alpha|^{2_k^\sharp - 2} v_\alpha$$

Inequality (15) and Hölder's inequality yield

$$(17) \quad \left| \int_M \Phi_\alpha \varphi \, dv_g \right| \leq C \left(\left\| |v_\alpha|^{2_k^\sharp - 2} u_\infty \right\|_{\frac{2_k^\sharp}{2_k^\sharp - 1}} + \left\| |u_\infty|^{2_k^\sharp - 2} v_\alpha \right\|_{\frac{2_k^\sharp}{2_k^\sharp - 1}} \right) \|\varphi\|_{2_k^\sharp}$$

Since $v_\alpha \rightharpoonup 0$ in $L^{2_k^\sharp}(M)$, Lemma 3.1 yields

$$\left\| |v_\alpha|^{2_k^\sharp - 2} u_\infty \right\|_{\frac{2_k^\sharp}{2_k^\sharp - 1}} + \left\| |u_\infty|^{2_k^\sharp - 2} v_\alpha \right\|_{\frac{2_k^\sharp}{2_k^\sharp - 1}} = o(1)$$

Since (u_α) is a Palais-Smale for I , then (16), (17) and the continuous embedding $H_{k,0}^2(M) \hookrightarrow L^{2_k^\sharp}(M)$ yields $\langle DJ(v_\alpha), \varphi \rangle = o(\|\varphi\|_{H_k^2})$ as $\alpha \rightarrow +\infty$ uniformly wrt $\varphi \in H_{k,0}^2(M)$. This proves the claim and ends Step 2. \square

The next lemma addresses the compactness of a Palais-Smale sequence for small energy. It will be generalized to the case of small local energy in Proposition 4.1.

Step 3: Let (v_α) be a Palais-Smale sequence for J on $H_{k,0}^2(M)$. We assume that $v_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$, and that $J(v_\alpha) \rightarrow \beta$ with $\beta < \beta^\sharp$, where β^\sharp is as in (4). We claim that $v_\alpha \rightarrow 0$ strongly in $H_{k,0}^2(M)$.

Proof of the claim: Since (v_α) is bounded and $\langle DJ(v_\alpha), v_\alpha \rangle = o(\|v_\alpha\|_{H_k^2})$, we get that

$$(18) \quad J(v_\alpha) = \frac{k}{n} \int_M (\Delta_g^{k/2} v_\alpha)^2 \, dv_g + o(1) = \frac{k}{n} \int_M |v_\alpha|^{2_k^\sharp} \, dv_g + o(1) = \beta + o(1).$$

As a consequence, $\beta \geq 0$. It follows from Mazumdar [13] that for any $\varepsilon > 0$ there exists $B_\varepsilon > 0$ such that

$$(19) \quad \|u\|_{2_k^\sharp}^2 \leq (K_0(n, k) + \varepsilon) \int_{\tilde{M}} (\Delta_g^{k/2} u)^2 \, dv_g + B_\varepsilon \|u\|_{H_{k-1}^2}^2$$

for all $u \in H_k^2(\tilde{M})$. Applying this inequality to v_α , the strong convergence to 0 in H_{k-1}^2 and (18) yield

$$\left(\frac{n}{k} \beta \right)^{2/2_k^\sharp} \leq (K_0(n, k) + \varepsilon) \frac{n}{k} \beta$$

Letting $\varepsilon \rightarrow 0$ and using $0 \leq \beta < \beta^\sharp$, we get that $\beta = 0$, and then (18) yields $v_\alpha \rightarrow 0$ strongly in $H_{k,0}^2(M)$. This proves the claim and ends Step 3. \square

Step 4: Proof of Theorem 1.1. Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. By subtracting the weak limit u_∞ , we get a Palais-Smale sequence (v_α) for the functional J with energy $J(v_\alpha) = I(u_\alpha) - I(u_\infty) + o(1)$ as $\alpha \rightarrow +\infty$. If $v_\alpha \rightarrow 0$ strongly in $H_{k,0}^2(M)$, then we end the process. If not, we apply Lemma 4.1 to subtract a bubble modeled on $v \in \mathcal{D}_k^2(\mathbb{R}^n) \setminus \{0\}$ and we get a new Palais-Smale sequence for J , but with the energy decreased by $E(v)$. If the resulting sequence goes strongly to 0, we stop the process, if not, we iterate it again. This process must stop since the energy $E(v) \geq \beta^\sharp$ and after finitely many steps, the energy goes below the critical threshold β^\sharp and then the convergence is strong by Step 3. This proves Theorem 1.1. \square

The rest of the paper is devoted to the proof of Lemma 4.1.

4. EXTRACTION OF A BUBBLE

In the sequel, for any (M, g) as in the introduction, we let $H_k^2(M)$ be the completion of $\{u \in C^\infty(M) : \|u\|_{H_k^2} < +\infty\}$ for the norm $\|\cdot\|_{H_k^2}$. The space $H_{k,0}^2(M)$ is then a closed subspace of $H_k^2(M)$. The following lemma is the main ingredient in the proof of Theorem 1.1

Lemma 4.1. *Let (v_α) be a Palais-Smale sequence for the functional J on $H_{k,0}^2(M)$ such that $v_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$ but not strongly. Then there exists a bubble $(B_{x_\alpha, r_\alpha}(v))$ such that upto a subsequence, the following holds:*

- $w_\alpha := v_\alpha - B_{x_\alpha, r_\alpha}(v)$ is a Palais-Smale sequence for J ,
- $J(w_\alpha) = J(v_\alpha) - E(v) + o(1)$ as $\alpha \rightarrow +\infty$.

The proof of this lemma goes through 10 steps.

Step 1: We prove a strong convergence Lemma for small energies. This is a localized version of Step 3 of Section 3.

Proposition 4.1. *Let (N, g_∞) be a Riemannian manifold with positive injectivity radius.*

- Let $(g_i)_i$ be metrics on N such that $g_i \rightarrow g_\infty$ in C_{loc}^p as $i \rightarrow +\infty$ for all p .
- Let $(P_i)_i$ be a family of operators on $C^\infty(N)$ such that

$$P_i := \Delta_{g_i}^k + \sum_{l=0}^{k-1} (-1)^l \nabla^{i_1 \dots i_l} \left((A_l^i)_{i_1 \dots i_l j_1 \dots j_l} \nabla^{j_1 \dots j_l} \right)$$

with families of symmetric tensors $(A_l^i) \rightarrow A_l$ in C_{loc}^p as $i \rightarrow +\infty$ for all p .

- We fix $\Omega \subset N$ an open smooth domain, and we define

$$(20) \quad J_i(u) := \frac{1}{2} \int_{\Omega} u P_i u \, dv_{g_i} - \frac{1}{2_k^\sharp} \int_{\Omega} |u|^{2_k^\sharp} \, dv_{g_i} \text{ for } u \in H_k^2(\Omega),$$

such that J_i is C^1 . Here, the background metric is g_∞ .

- We let $(u_i) \in H_{k,0}^2(\Omega)$ and $u_\infty \in H_{k,0}^2(\Omega)$ be such that $u_i \rightharpoonup u_\infty$ weakly in $H_{k,0}^2(\Omega)$ as $i \rightarrow +\infty$.
- We assume that there exist a compact $K \subset N$ such that

$$\lim_{i \rightarrow +\infty} \sup_{u \in H_{k,0}^2(\Omega), \text{Supp } \varphi \subset K} \frac{\langle DJ_i(u_i), \varphi \rangle}{\|\varphi\|_{H_k^2(\Omega)}} = 0$$

- We assume that there exists $K_\infty > 0$ and $C \geq 0$ such that

$$(21) \quad \left(\int_N |u|^{2_k^\sharp} \, dv_{g_\infty} \right)^{\frac{2}{2_k^\sharp}} \leq K_\infty \int_N (\Delta_{g_\infty}^{k/2} u)^2 \, dv_{g_\infty} + C \|u\|_{H_{k-1}^2}^2 \text{ for all } u \in C_c^\infty(N).$$

We fix $x_0 \in \Omega$ and $\delta \in (0, i_{g_\infty}(N)/2)$. We assume that

$$(22) \quad \begin{cases} B_{x_0}(2\delta) \subset K \text{ (the ball is wrt } g_\infty), \\ \int_{B_{x_0}(2\delta) \cap \Omega} |u_i|^{2_k^\sharp} \, dv_{g_i} \leq \left(\frac{1}{2K_\infty} \right)^{\frac{2_k^\sharp}{2_k^\sharp - 2}} \text{ for all } i \in \mathbb{N}. \end{cases}$$

Then $u_i \rightarrow u_\infty$ strongly in $H_k^2(B_{x_0}(\delta) \cap \Omega)$.

Proof of Proposition 4.1: Up to extracting a subsequence, we assume that $u_i \rightarrow u_\infty$ strongly in $H_{k-1}^2(\omega)$ as $i \rightarrow +\infty$ for $\omega \subset \Omega$ relatively compact and $u_i(x) \rightarrow u_\infty(x)$ as $i \rightarrow +\infty$ for a.e. $x \in \Omega$. Let $\eta \in C^\infty(N)$ such that $\eta(x) = 1$ for $x \in B_{x_0}(\delta)$ and $\eta(x) = 0$ for $x \in N \setminus B_{x_0}(2\delta)$. Since η has compact support, we get that $\eta^2(u_i - u_\infty) \in H_{k,0}^2(\Omega)$ is uniformly bounded in $H_{k,0}^2(\Omega)$. Since $B_{x_0}(2\delta) \subset K$, it then follows from hypothesis (20) that

$$\langle DJ_i(u_i), \eta^2(u_i - u_\infty) \rangle = o(1) \text{ as } i \rightarrow +\infty.$$

Since $\eta^2(u_i - u_\infty) \rightarrow 0$ strongly in $H_{k-1}^2(\Omega)$, we then get that

$$(23) \quad \int_{\Omega} \Delta_{g_i}^{k/2} u_i \Delta_{g_i}^{k/2} (\eta^2(u_i - u_\infty)) dv_{g_i} = \int_{\Omega} |u_i|^{2_k^\sharp - 2} u_i \eta^2(u_i - u_\infty) dv_{g_i} + o(1)$$

as $i \rightarrow +\infty$. The weak convergence of u_i to u_∞ and the strong convergence of g_i to g_∞ on compact sets yields

$$(24) \quad \int_{\Omega} \Delta_{g_i}^{k/2} u_i \Delta_{g_i}^{k/2} (\eta^2(u_i - u_\infty)) dv_{g_i} = \int_{\Omega} \Delta_{g_i}^{k/2} (u_i - u_\infty) \Delta_{g_i}^{k/2} (\eta^2(u_i - u_\infty)) dv_{g_i} + o(1)$$

as $i \rightarrow +\infty$. As one checks, for any $\varphi \in H_k^2(\Omega)$, we have that $\Delta_{g_i}^{k/2} \varphi \Delta_{g_i}^{k/2} (\eta^2 \varphi) = \left(\Delta_{g_i}^{k/2} (\eta \varphi) \right)^2 + \sum_{p < k, l \leq k} \nabla^p \varphi \star \nabla^l \varphi$, where $A \star B$ denotes a linear combination of bilinear forms in A and B . Therefore, using again the strong convergence of $\eta^2(u_i - u_\infty)$ to 0 in H_{k-1}^2 , we get that

$$(25) \quad \int_{\Omega} \Delta_{g_i}^{k/2} u_i \Delta_{g_i}^{k/2} (\eta^2(u_i - u_\infty)) dv_{g_i} = \int_{\Omega} \left(\Delta_{g_i}^{k/2} (\eta(u_i - u_\infty)) \right)^2 dv_{g_i} + o(1)$$

as $i \rightarrow +\infty$. Moreover, since $|u_i|^{2_k^\sharp - 2} \eta^2(u_i - u_\infty)$ is uniformly bounded in $L^{2_k^\sharp / (2_k^\sharp - 1)}$ and goes to 0 almost everywhere as $i \rightarrow +\infty$, then it goes weakly to 0 in $L^{2_k^\sharp / (2_k^\sharp - 1)}$, and then $\int_{\Omega} |u_i|^{2_k^\sharp - 2} \eta^2(u_i - u_\infty) u_\infty dv_{g_i} \rightarrow 0$ as $i \rightarrow +\infty$. Therefore, plugging (24) and (25) into (23), we get that

$$\int_{\Omega} \left(\Delta_{g_i}^{k/2} (\eta(u_i - u_\infty)) \right)^2 dv_{g_i} = \int_{\Omega} |u_i|^{2_k^\sharp - 2} (\eta(u_i - u_\infty))^2 dv_{g_i} + o(1)$$

as $i \rightarrow +\infty$. Since $g_i \rightarrow g_\infty$ as $i \rightarrow +\infty$ in C^p locally on compact sets and $\eta(u_i - u_\infty)$ is uniformly bounded in $H_k^2(\Omega)$, we get that

$$\int_{\Omega} \left(\Delta_{g_\infty}^{k/2} (\eta(u_i - u_\infty)) \right)^2 dv_{g_\infty} = \int_{\Omega} |u_i|^{2_k^\sharp - 2} (\eta(u_i - u_\infty))^2 dv_{g_\infty} + o(1)$$

as $i \rightarrow +\infty$. Hölder's inequality, the Sobolev inequality (21), the convergence of (g_i) , the strong convergence in H_{k-1}^2 and (22) then yields

$$\begin{aligned} & \int_{\Omega} \left(\Delta_{g_{\infty}}^{k/2} (\eta(u_i - u_{\infty})) \right)^2 dv_{g_{\infty}} \\ & \leq \left(\int_{B_{x_0}(2\delta) \cap \Omega} |u_i|^{2_k^{\sharp}} dv_{g_{\infty}} \right)^{\frac{2_k^{\sharp} - 2}{2_k^{\sharp}}} \left(\int_N |\eta(u_i - u_{\infty})|^{2_k^{\sharp}} dv_{g_{\infty}} \right)^{\frac{2}{2_k^{\sharp}}} + o(1) \\ & \leq \frac{1}{2K_{\infty}} \left(K_{\infty} \int_N \left(\Delta_{g_{\infty}}^{k/2} (\eta(u_i - u_{\infty})) \right)^2 dv_{g_{\infty}} + C \|\eta(u_i - u_{\infty})\|_{H_{k-1}^2}^2 \right) + o(1) \\ & \leq \left(\int_{B_{x_0}(2\delta) \cap \Omega} |u_i|^{2_k^{\sharp}} dv_{g_i} \right)^{\frac{2_k^{\sharp} - 2}{2_k^{\sharp}}} K_{\infty} \int_{\Omega} \left(\Delta_{g_{\infty}}^{k/2} (\eta(u_i - u_{\infty})) \right)^2 dv_{g_{\infty}} + o(1) \end{aligned}$$

as $i \rightarrow +\infty$. Therefore, we get that $\|\Delta_{g_{\infty}}^{k/2} (\eta(u_i - u_{\infty}))\|_2 \rightarrow 0$ as $i \rightarrow +\infty$. Since $\eta(u_i - u_{\infty}) \rightarrow 0$ strongly in H_{k-1}^2 and η has compact support, we get that $\eta(u_i - u_{\infty}) \rightarrow 0$ strongly in $H_k^2(\Omega)$, and therefore $u_i \rightarrow u_{\infty}$ in $H_k^2(B_{x_0}(\delta) \cap \Omega)$. Note that this is up to a subsequence. Indeed, by uniqueness, the convergence holds for the initial sequence (u_i) . This proves Proposition 4.1. \square

Step 2: Since $\langle DJ(v_{\alpha}), v_{\alpha} \rangle = o(1)$, one has

$$J(v_{\alpha}) = \frac{k}{n} \int_M |v_{\alpha}|^{2_k^{\sharp}} dv_g + o(1) = \beta + o(1) \text{ as } \alpha \rightarrow +\infty$$

where $\beta := \lim_{\alpha \rightarrow +\infty} J(v_{\alpha})$. By Step 3 of Section 3, $\beta \geq \beta^{\sharp}$. Therefore, since \overline{M} is compact, for any $r_0 > 0$, there exists $y_0 \in \overline{M}$ and $\lambda_0 > 0$ such that

$$\int_{B_{y_0}(r_0) \cap M} |v_{\alpha}|^{2_k^{\sharp}} dv_g \geq \lambda_0$$

For any $r > 0$, we set

$$(26) \quad \mu_{\alpha}(r) := \max_{x \in \overline{M}} \int_{B_x(r) \cap M} |v_{\alpha}|^{2_k^{\sharp}} dv_g,$$

the Levy concentration function. In particular, $\mu_{\alpha}(r_0) \geq \lambda_0$ for all α . We fix

$$0 < \lambda < \epsilon_0 := \min \left\{ \lambda_0, \frac{1}{(2K_0(n, k))^{2_k^{\sharp}/(2_k^{\sharp}-2)}} \right\}$$

where $K_0(n, k)$ is the best constant in the Euclidean Sobolev inequality (5). Since $\mu_{\alpha}(0) = 0$, there exists $(r_{\alpha})_{\alpha} \in (0, r_0)$ and $(x_{\alpha})_{\alpha} \in \overline{M}$ such that:

$$(27) \quad \lambda = \mu_{\alpha}(r_{\alpha}) = \int_{B_{x_{\alpha}}(r_{\alpha}) \cap M} |v_{\alpha}|^{2_k^{\sharp}} dv_g$$

Step 3: We claim that $\lim_{\alpha \rightarrow +\infty} r_{\alpha} = 0$.

Proof of the claim. We argue by contradiction. If (r_{α}) does not go to 0 up to a subsequence, we get that there exists $\delta \in (0, i_g(\tilde{M})/2)$ such that for all $x \in M$, we have that $\int_{B_x(2\delta) \cap M} |v_{\alpha}|^{2_k^{\sharp}} dv_g \leq \lambda$ for all α . We apply Proposition 4.1 with $(N, g_{\infty}) = (\tilde{M}, g)$, $\Omega = M$, $P_{\alpha} = P$, $g_{\alpha} = g$, $J_{\alpha} = J$, and the Sobolev inequality

(19) of [13], and we get $v_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$ in $H_k^2(M \cap B_x(\delta))$ for all $x \in M$. With a finite covering, we get that $v_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$ strongly in $H_{k,0}^2(M)$, contradicting our initial hypothesis. This proves the claim and ends Step 3. \square

First assume that

$$(28) \quad \lim_{\alpha \rightarrow +\infty} \frac{d(x_\alpha, \partial M)}{r_\alpha} = +\infty.$$

We define

$$\tilde{v}_\alpha(x) := r_\alpha^{\frac{n-2k}{2}} u_\alpha(\exp_{x_\alpha}(r_\alpha x)) \text{ for } |x| < \frac{i_g(\tilde{M})}{r_\alpha} \text{ and } |x| < \frac{d(x_\alpha, \partial M)}{r_\alpha}$$

Step 4: Suppose that (28) holds. We claim that there exists $v \in \mathcal{D}_k^2(\mathbb{R}^n)$ such that for any $\eta \in C_c^\infty(\mathbb{R}^n)$, we have that

$$\eta \tilde{v}_\alpha \rightharpoonup \eta v \text{ weakly in } \mathcal{D}_k^2(\mathbb{R}^n) \text{ as } k \rightarrow +\infty.$$

Proof of the claim. Fix $\eta \in C_c^\infty(\mathbb{R}^n)$, and let $R_0 > 0$ be such that $\text{Supp } \eta \subset B_0(R_0)$. We define

$$\eta_\alpha(x) := \eta\left(\frac{\exp_{x_\alpha}^{-1}(x)}{r_\alpha}\right) \text{ for } x \in B_{x_\alpha}(R_0 r_\alpha), \text{ and } \eta_\alpha(x) := 0 \text{ outside.}$$

Up to a subsequence, there exists $x_0 \in \tilde{M}$ and $\tau > 0$ such that $B_{x_\alpha}(R_0 r_\alpha) \subset B_{x_0}(\tau) \subset \tilde{M}$. It then follows from the comparison Lemma 9.1 of Mazumdar [13] that there exists $C > 0$ such that

$$\int_{B_0(R_0 r_\alpha)} \left(\Delta^{k/2}[(\eta_\alpha v_\alpha) \circ \exp_{x_\alpha}]\right)^2 dx \leq C \int_{B_{x_\alpha}(R_0 r_\alpha)} \left(\Delta_g^{k/2}(\eta_\alpha v_\alpha)\right)^2 dv_g$$

for all α . With a change of variable, rough estimates of the differential terms and Hölder's inequality, we then get

$$(29) \quad \begin{aligned} & \int_{B_0(R_0)} \left(\Delta^{k/2}(\eta \tilde{v}_\alpha)\right)^2 dx \leq C \sum_{l=0}^k \int_{B_{x_\alpha}(R_0 r_\alpha)} |\nabla^l u_\alpha|_g^2 |\nabla^{k-l} \eta_\alpha|_g^2 dv_g \\ & \leq C \sum_{l=0}^k \int_{B_{x_\alpha}(R_0 r_\alpha)} r_\alpha^{2(l-k)} |\nabla^l v_\alpha|_g^2 dv_g \leq C \sum_{l=0}^k \|\nabla^l v_\alpha\|_{\frac{2n}{n-2(k-l)}}^2 \end{aligned}$$

It follows from Sobolev's embedding theorem that $H_{k-l}^2(M) \subset L^{\frac{2n}{n-2(k-l)}}(M)$ for all $l = 0, \dots, k$ and that this embedding is continuous. Since $(v_\alpha)_\alpha$ is bounded in H_k^2 , then $(\nabla^l v_\alpha)_\alpha$ is uniformly bounded in H_{k-l}^2 (with tensorial values), and then there exists $C > 0$ such that

$$(30) \quad \|\nabla^l v_\alpha\|_{\frac{2n}{n-2(k-l)}} \leq C \|v_\alpha\|_{H_k^2} \leq C'$$

for all $\alpha > 0$ and $l = 0, \dots, k$. It then follows from (29) that $(\eta \tilde{v}_\alpha)_\alpha$ is bounded in $\mathcal{D}_k^2(\mathbb{R}^n)$. Therefore, up to a subsequence, there exists $v_\eta \in \mathcal{D}_k^2(\mathbb{R}^n)$ such that $\eta \tilde{v}_\alpha \rightharpoonup v_\eta$ weakly in $\mathcal{D}_k^2(\mathbb{R}^n)$ as $\alpha \rightarrow +\infty$. A classical diagonal argument then yields the existence $v \in H_{k,loc}^2(\mathbb{R}^n)$ such that $\eta \tilde{v}_\alpha \rightharpoonup \eta v$ weakly in $\mathcal{D}_k^2(\mathbb{R}^n)$ as $\alpha \rightarrow +\infty$. We fix $R > 0$. For any $R' > R$, a change of variables and (30) yields

$$\int_{B_0(R)} |\nabla^l \eta_{R'} \tilde{v}_\alpha|_{g_\alpha}^{\frac{2n}{n-2(k-l)}} dv_{g_\alpha} \leq \int_{B_{x_\alpha}(R_0 r_\alpha)} |\nabla^l v_\alpha|_g^{\frac{2n}{n-2(k-l)}} dv_g \leq C$$

where $g_\alpha := \exp_{x_\alpha}^* g(r_\alpha \cdot)$. Using weak convergence and convexity, letting $\alpha \rightarrow +\infty$ and then $R \rightarrow +\infty$ yields $|\nabla^l v| \in L^{\frac{2n}{n-2(k-l)}}(\mathbb{R}^n)$. As one checks, we then have that the sequence $(\eta_R v)_R$ is a Cauchy sequence in $\mathcal{D}_k^2(\mathbb{R}^n)$, and then we get that $v \in \mathcal{D}_k^2(\mathbb{R}^n)$. This ends the proof of the claim, and ends Step 4. \square

Step 5: We assume that (28) holds. We let $v \in \mathcal{D}_k^2(\mathbb{R}^n)$ as in Claim 3. We claim that $v \neq 0$ is a weak solution to $\Delta^k v = |v|^{2_k^* - 2} v$ in $\mathcal{D}_k^2(\mathbb{R}^n)$.

Proof of the claim. We fix $R > 0$ and we apply Proposition 4.1 with $(N, g_\infty) := (\mathbb{R}^n, \text{Eucl})$ and $\Omega := \mathbb{R}^n$. As above, we define a family of smooth metrics $(g_\alpha)_\alpha$ such that $g_\alpha(x) := \exp_{x_\alpha}^* g(r_\alpha x)$ for $x \in B_0(3R)$, $g_\alpha(x) = \text{Eucl}$ for $x \in \mathbb{R}^n \setminus B_0(4R)$, and $g_\alpha \rightarrow \text{Eucl}$ in $C_{loc}^p(\mathbb{R}^n)$ as $\alpha \rightarrow +\infty$ for all p . Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\text{Supp } \varphi \subset B_0(R)$. We define

$$\varphi_\alpha(x) := r_\alpha^{-\frac{n-2k}{2}} \varphi\left(\frac{\exp_{x_\alpha}^{-1}(x)}{r_\alpha}\right)$$

for all $x \in M$. As one checks, φ_α is well-defined and has support in $B_{x_\alpha}(Rr_\alpha)$. Moreover, using the comparison Lemma 9.1 in Mazumdar [13] and arguing as in Step 4, we get that $\|\varphi_\alpha\|_{H_{k,0}^2(M)} \leq C(R)\|\varphi\|_{H_{k,0}^2(\mathbb{R}^n)}$ for all $\alpha > 0$. Since (u_α) is a Palais-Smale sequence, we have that

$$\langle DJ(v_\alpha), \varphi_\alpha \rangle = o(\|\varphi_\alpha\|_{H_{k,0}^2}) = o(\|\varphi\|_{H_{k,0}^2(\mathbb{R}^n)})$$

as $\alpha \rightarrow +\infty$ uniformly for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\text{Supp } \varphi \subset B_0(R)$. With a change of variable, we get $\langle DJ(v_\alpha), \varphi_\alpha \rangle = \langle DJ_\alpha(\eta_R \tilde{v}_\alpha), \varphi \rangle$ where

$$J_\alpha(u) := \frac{1}{2} \int_{\mathbb{R}^n} (\Delta_{g_\alpha}^{k/2} u)^2 dv_{g_\alpha} - \frac{1}{2_k^*} \int_{\mathbb{R}^n} |u|^{2_k^*} dv_{g_\alpha}$$

for all $u \in H_k^2(\mathbb{R}^n)$. Therefore, $\langle DJ_\alpha(\eta_R \tilde{v}_\alpha), \varphi \rangle = o(\|\varphi\|_{H_{k,0}^2(\mathbb{R}^n)})$ as $\alpha \rightarrow +\infty$ uniformly for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\text{Supp } \varphi \subset B_R(0)$.

We fix $x_0 \in \mathbb{R}^n$ such that $B_{x_0}(1/2) \subset B_0(R)$. A change of variable yields

$$\int_{B_{x_0}(1/2) \cap B_0(2R)} |\eta_R \tilde{v}_\alpha|^{2_k^*} dv_{g_\alpha} = \int_{\exp_{x_\alpha}(r_\alpha B_{x_0}(1/2))} |u_\alpha|^{2_k^*} dv_g.$$

For $\alpha > 0$ large enough, we have that $\exp_{x_\alpha}(r_\alpha B_{x_0}(1/2)) \subset B_{\exp_{x_\alpha}(x_0)}(r_\alpha)$. Therefore, it follows from the definition of μ_α that

$$\int_{B_{x_0}(1/2) \cap B_0(2R)} |\eta_R \tilde{v}_\alpha|^{2_k^*} dv_{g_\alpha} \leq \mu_\alpha(r_\alpha) = \lambda < \epsilon_0$$

for all α large enough and $x_0 \in \mathbb{R}^n$ such that $1/2 + |x_0| < R$. With the Sobolev inequality (5) on \mathbb{R}^n , we apply Proposition 4.1 to $(\eta_R \tilde{v}_\alpha)_\alpha$, and we get that

$$\lim_{\alpha \rightarrow +\infty} \eta_R \tilde{v}_\alpha = \eta_R v \text{ strongly in } H_k^2(B_{x_0}(1/4)).$$

Using a finite covering, we then have $\tilde{v}_\alpha \rightarrow v$ strongly in $H_k^2(B_0(R/2))$ as $\alpha \rightarrow +\infty$. Sobolev's embedding theorem yield the convergence in $L^{2_k^*}(B_0(1))$. Since

$$\int_{B_0(1)} |\tilde{v}_\alpha|^{2_k^*} dv_{g_\alpha} = \int_{B_{x_\alpha}(r_\alpha)} |v_\alpha|^{2_k^*} dv_g = \mu_\alpha(r_\alpha) = \lambda > 0,$$

passing to the limit $\alpha \rightarrow +\infty$ yields $\int_{B_0(1)} |v|^{2_k^*} dx = \lambda \neq 0$, and therefore $v \neq 0$. This proves the claim and ends Step 5. \square

Note that indeed, we have proved that

$$(31) \quad \lim_{\alpha \rightarrow +\infty} \tilde{v}_\alpha = v \text{ strongly in } H_k^2(B_0(R)) \text{ for all } R > 0.$$

We choose a sequence (\tilde{r}_α) of positive real numbers as in (8) with $\eta \in C_c^\infty(B_0(\delta))$ (with $\delta \in (0, i_g(\tilde{M}))$) identically 1 around 0. As in Definition 2.1, we set

$$V_\alpha(x) := B_{x_\alpha, r_\alpha}(v) := \eta \left(\frac{\exp_{x_\alpha}^{-1}(x)}{\tilde{r}_\alpha} \right) r_\alpha^{-\frac{n-2k}{2}} v \left(\frac{\exp_{x_\alpha}^{-1}(x)}{r_\alpha} \right)$$

We have that $V_\alpha \in H_{k,0}^2(M)$.

Step 6: We claim that

$$(32) \quad V_\alpha \rightarrow 0 \text{ in } H_{k,0}^2(M) \text{ as } \alpha \rightarrow +\infty.$$

Proof of the claim. We argue essentially as in [13]. We fix $0 \leq l \leq k$ and we define $\epsilon_\alpha := r_\alpha/\tilde{r}_\alpha$ such that $\lim_{\alpha \rightarrow +\infty} \epsilon_\alpha = 0$. We fix $R \geq 0$ (potentially 0). It follows from the comparison Lemma 9.1 of [13] that there exists $C > 0$ such that

$$\begin{aligned} \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} (\Delta_g^{l/2} V_\alpha)^2 dv_g &\leq C \int_{B_0(\delta\tilde{r}_\alpha) \setminus B_0(Rr_\alpha)} (\Delta^{l/2}(V_\alpha \circ \exp_{x_\alpha}))^2 dx \\ &\leq Cr_\alpha^{2(k-l)} \int_{B_0(\delta\epsilon_\alpha^{-1}) \setminus B_0(R)} \left(\Delta^{l/2}(\eta(\epsilon_\alpha \cdot)v) \right)^2 dx \\ &\leq Cr_\alpha^{2(k-l)} \int_{B_0(\delta\epsilon_\alpha^{-1}) \setminus B_0(R)} |\nabla^l(\eta(\epsilon_\alpha \cdot)v)|^2 dx \\ &\leq Cr_\alpha^{2(k-l)} \sum_{i=0}^l \int_{\mathbb{R}^n \setminus B_0(R)} |\nabla^{l-i}\eta(\epsilon_\alpha \cdot)| |\nabla^i v|^2 dx \\ &\leq Cr_\alpha^{2(k-l)} \sum_{i=0}^l \int_{\mathbb{R}^n \setminus B_0(R)} \epsilon_\alpha^{2(l-i)} |\nabla^i v|^2 dx \end{aligned}$$

Since $v \in \mathcal{D}_k^2(\mathbb{R}^n)$, we have that $\nabla^i v \in \mathcal{D}_{k-i}^2(\mathbb{R}^n)$, and therefore $|\nabla^i v| \in L^{2^\sharp(k-i)}(\mathbb{R}^n)$ where $2^\sharp(k-i) := \frac{2n}{n-2(k-i)}$. Therefore, Hölder's inequality yields

$$(33) \quad \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} (\Delta_g^{l/2} V_\alpha)^2 dv_g \leq C\tilde{r}_\alpha^{2(k-l)} \sum_{i=0}^l \left(\int_{\mathbb{R}^n \setminus B_0(R)} |\nabla^i v|^{2^\sharp(k-i)} dx \right)^{\frac{2}{2^\sharp(k-i)}}$$

Taking $R = 0$ and $l = 0, \dots, k$ yields the boundedness of $(V_\alpha)_\alpha$ in $H_{k,0}^2(M)$.

Arguing as in above, we get that for any $R > 0$ and any $l = 0, \dots, k$, we have that

$$(34) \quad \int_{B_{x_\alpha}(Rr_\alpha)} (\Delta_g^{l/2} V_\alpha)^2 dv_g \leq Cr_\alpha^{2(k-l)} \sum_{i=0}^l \int_{B_0(R)} \epsilon_\alpha^{2(l-i)} |\nabla^i v|^2 dx$$

Since $\nabla^i v \in L_{loc}^2(\mathbb{R}^n)$ for all $i = 0, \dots, k$, then taking $l = 0$ in (33) and (34), letting $\alpha \rightarrow +\infty$ and then $R \rightarrow +\infty$ yields $V_\alpha \rightarrow 0$ in $L^2(M)$. Then the weak compactness of bounded sequences yields (32). This proves the claim and ends Step 6. \square

Step 7: We claim that

$$(35) \quad DJ(V_\alpha) \rightarrow 0 \text{ strongly as } \alpha \rightarrow +\infty$$

Proof of the claim. We set $\varphi \in C_c^\infty(M)$. We have that

$$\langle DJ(V_\alpha), \varphi \rangle = \int_M \Delta_g^{k/2} V_\alpha \Delta_g^{k/2} \varphi \, dv_g - \int_M |V_\alpha|^{2_k^\sharp - 2} V_\alpha \varphi \, dv_g$$

We fix $R > 0$ and we define

$$I_{R,\alpha}(\varphi) := \int_{B_{x_\alpha}(Rr_\alpha)} \Delta_g^{k/2} V_\alpha \Delta_g^{k/2} \varphi \, dv_g - \int_{B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\sharp - 2} V_\alpha \varphi \, dv_g$$

and

$$II_{R,\alpha}(\varphi) := \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} \Delta_g^{k/2} V_\alpha \Delta_g^{k/2} \varphi \, dv_g - \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\sharp - 2} V_\alpha \varphi \, dv_g.$$

Step 7.1: we estimate $II_{R,\alpha}(\varphi)$. Via Hölder's and Sobolev inequality, we have that

$$(36) \quad |II_{R,\alpha}(\varphi)| \leq \left(\int_{D_\alpha(R)} (\Delta_g^{k/2} V_\alpha)^2 \, dv_g \right)^{\frac{1}{2}} \times \|\Delta_g^{k/2} \varphi\|_2 \\ + \left(\int_{D_\alpha(R)} |V_\alpha|^{2_k^\sharp} \, dv_g \right)^{\frac{2_k^\sharp - 1}{2_k^\sharp}} \times \|\varphi\|_{2_k^\sharp} \\ \leq \left(\int_{D_\alpha(R)} (\Delta_g^{k/2} V_\alpha)^2 \, dv_g \right)^{\frac{1}{2}} + \left(\int_{D_\alpha(R)} |V_\alpha|^{2_k^\sharp} \, dv_g \right)^{\frac{2_k^\sharp - 1}{2_k^\sharp}} \cdot \|\varphi\|_{H_k^2}$$

with $D_\alpha(R) := M \setminus B_{x_\alpha}(Rr_\alpha)$. Lemma 9.1 in [13] and $v \in L^{2_k^\sharp}(\mathbb{R}^n)$ yield

$$(37) \quad \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\sharp} \, dv_g \leq C \int_{\mathbb{R}^n \setminus B_0(Rr_\alpha)} |V_\alpha \circ \exp_{x_\alpha}|^{2_k^\sharp} \, dx \leq C \int_{\mathbb{R}^n \setminus B_0(R)} |v|^{2_k^\sharp} \, dx$$

Plugging (33) with $l = k$ and (37) into (36), letting $R \rightarrow +\infty$ and $\alpha \rightarrow +\infty$ yields

$$(38) \quad \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \frac{II_{R,\alpha}(\varphi)}{\|\varphi\|_{H_k^2}} = 0 \text{ uniformly wrt } \varphi \in H_{k,0}^2(M) \setminus \{0\}$$

Step 7.2: We now estimate $I_{R,\alpha}(\varphi)$. We define

$$\overline{\varphi}_\alpha(x) = \eta(\epsilon_\alpha x) r_\alpha^{\frac{n-2k}{2}} \varphi(\exp_{x_\alpha}(r_\alpha x))$$

where $\epsilon_\alpha := r_\alpha / \tilde{r}_\alpha$. As one checks, $\overline{\varphi}_\alpha \in C_c^\infty(\mathbb{R}^n)$. Using the comparison Lemma 9.1 in [13] and arguing as in (33)-(34), we get that

$$\|\overline{\varphi}_\alpha\|_{\mathcal{D}_k^2(\mathbb{R}^n)} \leq C \|\varphi\|_{H_k^2}$$

where $C > 0$ is independent of φ . As one checks,

$$I_{R,\alpha}(\varphi) = \int_{B_0(R)} \Delta_{g_\alpha}^{k/2} v \Delta_{g_\alpha}^{k/2} \overline{\varphi}_\alpha \, dv_{g_\alpha} - \int_{B_0(R)} |v|^{2_k^\sharp - 2} v \overline{\varphi}_\alpha \, dv_{g_\alpha}$$

Since $g_\alpha \rightarrow \text{Eucl}$ as $\alpha \rightarrow +\infty$ in $C_{loc}^p(\mathbb{R}^n)$ for all $p \geq 1$, we get

$$(39) \quad I_{R,\alpha}(\varphi) = \int_{B_0(R)} \Delta^{k/2} v \Delta^{k/2} \overline{\varphi}_\alpha \, dx - \int_{B_0(R)} |v|^{2_k^\sharp - 2} v \overline{\varphi}_\alpha \, dx + o\left(\|\overline{\varphi}_\alpha\|_{\mathcal{D}_k^2(\mathbb{R}^n)}\right)$$

where the convergence is uniform wrt $\bar{\varphi}_\alpha$. Since v is a weak solution to (1), then (39) yields

$$(40) \quad \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \frac{I_{R,\alpha}(\varphi)}{\|\varphi\|_{H_k^2}} = 0 \text{ uniformly wrt } \varphi \in H_{k,0}^2(M) \setminus \{0\}$$

The limits (38) and (40) yield $\langle DJ(V_\alpha), \varphi \rangle = o(\|\varphi\|_{H_k^2})$ as $\alpha \rightarrow +\infty$ uniformly wrt $\varphi \in C_c^\infty(M)$. The boundedness of (V_α) in $H_{k,0}^2(M)$ then yields $DJ(V_\alpha) \rightarrow 0$ strongly in $(H_{k,0}^2(M))'$ as $\alpha \rightarrow +\infty$. This proves (35) and ends Step 7. \square

We define $w_\alpha := v_\alpha - V_\alpha$. It follows from (32) that $w_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$.

Step 8: We claim that

$$(41) \quad DJ(w_\alpha) \longrightarrow 0 \text{ strongly}$$

Proof of the claim. For $\varphi \in H_{k,0}^2(M)$, we write

$$(42) \quad \langle DJ(w_\alpha), \varphi \rangle = \langle DJ(v_\alpha), \varphi \rangle - \langle DJ(V_\alpha), \varphi \rangle - \int_M \Phi_\alpha \varphi \, dv_g$$

where $\Phi_\alpha := |w_\alpha|^{2_k^\# - 2} w_\alpha - |v_\alpha|^{2_k^\# - 2} v_\alpha + |V_\alpha|^{2_k^\# - 2} V_\alpha$. Then by applying the Hölder and Sobolev inequalities we get

$$\left| \int_M \Phi_\alpha \varphi \, dv_g \right| \leq C \|\varphi\|_{H_k^2} \|\Phi_\alpha\|_{2_k^\# / (2_k^\# - 1)}$$

Step 8.1: We fix $R > 0$. Inequality (15) and Hölder's inequality yield

$$\begin{aligned} & \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2_k^\# / (2_k^\# - 1)} \, dv_g \\ & \leq C \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} \left(|v_\alpha|^{2_k^\# - 2} |V_\alpha| + |V_\alpha|^{2_k^\# - 2} |v_\alpha| \right)^{2_k^\# / (2_k^\# - 1)} \, dv_g \\ & \leq C \left(\int_M |v_\alpha|^{2_k^\#} \, dv_g \right)^{\frac{2_k^\# - 2}{2_k^\# - 1}} \left(\int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\#} \, dv_g \right)^{\frac{1}{2_k^\# - 1}} \\ & \quad + C \left(\int_M |v_\alpha|^{2_k^\#} \, dv_g \right)^{\frac{1}{2_k^\# - 1}} \left(\int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\#} \, dv_g \right)^{\frac{2_k^\# - 2}{2_k^\# - 1}} \end{aligned}$$

Since (v_α) is uniformly bounded in $H_k^2(M)$, then (37) yields

$$(43) \quad \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2_k^\# / (2_k^\# - 1)} \, dv_g = 0.$$

This ends Step 8.1.

Step 8.2: We fix $R > 0$. A change of variable and inequality (15) yield

$$\begin{aligned} & \int_{B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2_k^\sharp/(2_k^\sharp-1)} dv_g \\ &= \int_{B_0(R)} \left| |\tilde{v}_\alpha - v|^{2_k^\sharp-2} (\tilde{v}_\alpha - v) - |\tilde{v}_\alpha|^{2_k^\sharp-2} \tilde{v}_\alpha + |v|^{2_k^\sharp-2} v \right|^{2_k^\sharp/(2_k^\sharp-1)} dv_{g_\alpha} \\ &\leq C \int_{B_0(R)} \left(|\tilde{v}_\alpha - v|^{\frac{(2_k^\sharp-2)2_k^\sharp}{2_k^\sharp-1}} |v|^{\frac{2_k^\sharp}{2_k^\sharp-1}} + |v|^{\frac{(2_k^\sharp-2)2_k^\sharp}{2_k^\sharp-1}} |\tilde{v}_\alpha - v|^{\frac{2_k^\sharp}{2_k^\sharp-1}} \right) dx \end{aligned}$$

For any $\eta \in C_c^\infty(\mathbb{R}^n)$, we have that $\eta\tilde{v}_\alpha \rightharpoonup \eta v$ weakly in $\mathcal{D}_k^2(\mathbb{R}^n)$. Therefore, up to extracting a subsequence, $(\tilde{v}_\alpha)_\alpha$ is uniformly bounded in $L^{2_k^\sharp}(B_0(R))$ and goes to v almost everywhere as $\alpha \rightarrow +\infty$. Therefore Lemma 3.1 yields that for any $R > 0$,

$$(44) \quad \lim_{\alpha \rightarrow +\infty} \int_{B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2_k^\sharp/(2_k^\sharp-1)} dv_g = 0.$$

The limits (43)-(44) yield $\|\Phi_\alpha\|_{2_k^\sharp/(2_k^\sharp-1)} \rightarrow 0$ as $\alpha \rightarrow +\infty$. Then by (42) we get $DJ(w_\alpha) \rightarrow 0$ in $(H_{k,0}^2(M))'$ as $\alpha \rightarrow +\infty$. This proves (41) and ends Step 8. \square

Step 9: We claim that we have the following decomposition of energy.

$$(45) \quad J(w_\alpha) = J(v_\alpha) - E(v) + o(1) \text{ where } o(1) \rightarrow 0 \text{ as } \alpha \rightarrow +\infty.$$

Proof of the claim. As one checks,

$$\begin{aligned} & J(v_\alpha) - J(w_\alpha) - J(V_\alpha) = \langle DJ(w_\alpha), V_\alpha \rangle \\ & - \frac{1}{2_k^\sharp} \int_M \left(|w_\alpha + V_\alpha|^{2_k^\sharp} - |w_\alpha|^{2_k^\sharp} - 2_k^\sharp |w_\alpha|^{2_k^\sharp-2} w_\alpha V_\alpha - |V_\alpha|^{2_k^\sharp} \right) dv_g \end{aligned}$$

We fix $R > 0$. Arguing as in the proof of (44), we get that

$$\lim_{\alpha \rightarrow +\infty} \int_{B_{x_\alpha}(Rr_\alpha)} \left(|w_\alpha + V_\alpha|^{2_k^\sharp} - |w_\alpha|^{2_k^\sharp} - 2_k^\sharp |w_\alpha|^{2_k^\sharp-2} w_\alpha V_\alpha - |V_\alpha|^{2_k^\sharp} \right) dv_g = 0.$$

As one checks, there exists $C > 0$ such that

$$\left| |a+b|^{2_k^\sharp} - |a|^{2_k^\sharp} - 2_k^\sharp |a|^{2_k^\sharp-2} ab - |b|^{2_k^\sharp} \right| \leq C \left(|a|^{2_k^\sharp-2} |b|^2 + |a| \cdot |b|^{2_k^\sharp-1} \right)$$

for all $a, b \in \mathbb{R}$. As in the proof of (43), we get that

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{D_\alpha(R)} \left(|w_\alpha + V_\alpha|^{2_k^\sharp} - |w_\alpha|^{2_k^\sharp} - 2_k^\sharp |w_\alpha|^{2_k^\sharp-2} w_\alpha V_\alpha - |V_\alpha|^{2_k^\sharp} \right) dv_g = 0,$$

where $D_\alpha(R) := M \setminus B_{x_\alpha}(Rr_\alpha)$. These yield $J(v_\alpha) = J(w_\alpha) + J(V_\alpha) + o(1)$.

We now estimate $J(V_\alpha)$. The estimates (33) and (37) yield

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} \left((\Delta_g^{k/2} V_\alpha)^2 + |V_\alpha|^{2_k^\sharp} \right) dv_g = 0$$

For $R > 0$, we have that

$$\int_{B_{x_\alpha}(Rr_\alpha)} \left(\frac{(\Delta_g^{k/2} V_\alpha)^2}{2} - \frac{|V_\alpha|^{2_k^\sharp}}{2_k^\sharp} \right) dv_g = \int_{B_0(R)} \left(\frac{(\Delta_{g_\alpha}^{k/2} v)^2}{2} - \frac{|v|^{2_k^\sharp}}{2_k^\sharp} \right) dv_{g_\alpha}$$

Since $g_\alpha \rightarrow \text{Eucl}$ locally uniformly in C^p for all p and $v \in \mathcal{D}_k^2(\mathbb{R}^n)$, we get that

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{B_{x_\alpha}(Rr_\alpha)} \left(\frac{(\Delta_g^{k/2} V_\alpha)^2}{2} - \frac{|V_\alpha|^{2_k^\sharp}}{2_k^\sharp} \right) dv_g = \int_{\mathbb{R}^n} \left(\frac{(\Delta^{k/2} v)^2}{2} - \frac{|v|^{2_k^\sharp}}{2_k^\sharp} \right) dx$$

All these estimates yield (45). This ends Step 9. \square

Step 10: Next we deal with the case

$$d_g(x_\alpha, \partial M) = O(r_\alpha) \quad \text{as } \alpha \rightarrow +\infty$$

Since $r_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$, then there exists $x_\infty \in \partial M$ such that $x_\alpha \rightarrow x_\infty$ as $\alpha \rightarrow +\infty$. For any $\alpha \in \mathbb{N}$, we let $z_\alpha \in \partial M$ be such that

$$d_g(x_\alpha, z_\alpha) = d_g(x_\alpha, \partial M)$$

In particular, $\lim_{\alpha \rightarrow +\infty} z_\alpha = x_\infty$. We choose a family of charts $z \mapsto \mathcal{T}_z$ for $z \in \Omega \cap \partial M$ as in (7). Since the $d(\mathcal{T}_z)_0$ is an isometry, there exists $\mathcal{C}_1, \mathcal{C}_2 > 0$, $\tau_1, \tau_2 > 0$ such that for any $z \in \Omega \cap \partial M$, $r < \tau_1$ and $y \in \mathbb{R}_-^n \cap B_0(\tau_2)$, one has

$$B_{\mathcal{T}_z(y)}(\mathcal{C}_1 r) \cap M \subset \mathcal{T}_z(B_y(r) \cap \mathbb{R}_-^n) \subset B_{\mathcal{T}_z(y)}(\mathcal{C}_2 r) \cap M$$

For $x \in r_\alpha^{-1}U \cap \{x_1 < 0\}$, we define

$$\tilde{v}_\alpha(x) := r_\alpha^{\frac{n-2k}{2}} v_\alpha \circ \mathcal{T}_{z_\alpha}(r_\alpha x) \quad \text{and} \quad \tilde{g}_\alpha(x) := \mathcal{T}_{z_\alpha}^* g(r_\alpha x)$$

As one checks, for any $\eta \in C_c^\infty(\mathbb{R}^n)$, we have that $\eta \tilde{v}_\alpha \in \mathcal{D}_k^2(\mathbb{R}_-^n)$. Arguing as Step 4, we get that there exists $v \in \mathcal{D}_k^2(\mathbb{R}_-^n)$ such that

$$\eta \tilde{v}_\alpha \rightharpoonup \eta v \quad \text{weakly in } \mathcal{D}_k^2(\mathbb{R}_-^n) \quad \text{as } \alpha \rightarrow +\infty.$$

Moreover, using Proposition 4.1 and arguing as in Step 5, we get that $v \not\equiv 0$ is a weak solution to (3) and $\tilde{v}_\alpha \rightarrow v$ as $\alpha \rightarrow +\infty$ strongly in $H_k^2(B_0(R) \cap \mathbb{R}_-^n)$ for all $R > 0$. As in Definition 2.1, for $\alpha \in \mathbb{N}$ and $x \in \overline{M}$, we set

$$V_\alpha(x) := B_{z_\alpha, r_\alpha}(v)(x) = \eta(\mathcal{T}_{z_\alpha}^{-1}(x)) r_\alpha^{-\frac{n-2k}{2}} v(r_\alpha^{-1} \mathcal{T}_{z_\alpha}^{-1}(x))$$

We define $w_\alpha := v_\alpha - V_\alpha$. Arguing as in Steps 6 to 9, we get that

- $w_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$
- $DJ(w_\alpha) \rightarrow 0$ weakly in $(H_{k,0}^2(M))'$
- $J(w_\alpha) = J(v_\alpha) - E(v) + o(1)$

as $\alpha \rightarrow +\infty$. This completes the proof of Lemma 4.1.

5. NONNEGATIVE PALAIS-SMALE SEQUENCES

To prove Theorem 1.2, we first set the following property:

Proposition 5.1. *Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. Let $d \in \mathbb{N}$ and d bubbles $[(x_\alpha^{(j)}), (r_\alpha^{(j)}), u^{(j)}]$, $j = 1, \dots, d$, be as in Theorem 1.1. Then, for any $N \in \{1, \dots, d\}$, there exists $L \geq 0$ sequences $(y_\alpha^j)_{\alpha > 0} \in \overline{M}$ and $(\lambda_\alpha^j)_{\alpha > 0} \in (0, +\infty)$, $j = 1, \dots, L$, such that for any $R > 0$*

$$\lim_{R' \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{(B_{x_\alpha^N}(Rr_\alpha^N) \setminus \bigcup_{j=1}^L B_{y_\alpha^j}(R'\lambda_\alpha^j)) \cap M} |u_\alpha - B_{x_\alpha^{(N)}, r_\alpha^{(N)}}(u^{(N)})|^{2_k^\sharp} dv_g = 0$$

where for any j , $j = 1, \dots, L$, $d_g(x_\alpha^N, y_\alpha^j) = o(r_\alpha^N)$ and $\lambda_\alpha^j = o(r_\alpha^N)$ as $\alpha \rightarrow +\infty$. Moreover, we have that

$$\lim_{\alpha \rightarrow +\infty} \frac{d_g(x_\alpha^i, x_\alpha^j)^2}{r_\alpha^i r_\alpha^j} + \frac{r_\alpha^i}{r_\alpha^j} + \frac{r_\alpha^j}{r_\alpha^i} = +\infty \text{ for all } i \neq j \in \{1, \dots, d\}.$$

We omit the proof that goes exactly as in Hebey-Robert [11], by using the boundary chart (7) for bubbles accumulating on the boundary.

We now prove Theorem 1.2. We let $(u_\alpha)_\alpha$ be as in the statement of the theorem, and we let $[(x_\alpha^{(j)}), (r_\alpha^{(j)}), u^{(j)}]$, $j = 1, \dots, d$, be the associated bubbles. We fix $N \in \{1, \dots, d\}$. For simplicity, we define $r_\alpha := r_\alpha^{(N)}$ and $x_\alpha := x_\alpha^{(N)}$. We assume that $r_\alpha^{-1}d(x_\alpha, \partial M) \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. It then follows from Proposition 5.1 that there exists a finite set $\mathcal{S} \subset \mathbb{R}^n$ such that $\lim_{\alpha \rightarrow +\infty} \tilde{v}_\alpha = u^N$ strongly in $L_{loc}^{2_k^\sharp}(\mathbb{R}^n \setminus \mathcal{S})$ where $\tilde{v}_\alpha(x) := r_\alpha^{\frac{n-2k}{2}} u_\alpha(\exp_{x_\alpha}(r_\alpha x))$ for $x \in \mathbb{R}^n$. Up to extracting, the convergence holds a.e. Since $u_\alpha \geq 0$, we then get that $u^N \geq 0$. It then follows from Lemma 4 in Ge-Wei-Zhou [7] that there exists $\lambda > 0$ and $a \in \mathbb{R}^n$ such that $u^N = U_{\lambda, a}$ is of the form (6).

We claim that $u^N = U_{\lambda, 0}$, that is $a = 0$. We prove the claim. Indeed, rescaling (26) and (27) yields

$$\int_{r_\alpha^{-1} \exp_{x_\alpha}^{-1}(B \exp_{x_\alpha}(r_\alpha x))} |\tilde{v}_\alpha|^{2_k^\sharp} dv_{g_\alpha} \leq \int_{B_0(1)} |\tilde{v}_\alpha|^{2_k^\sharp} dv_{g_\alpha}$$

for all $z \in \mathbb{R}^n$ and α large enough. Since the exponential is a normal chart and isometric at x_α , we get that for all $z \in \mathbb{R}^n$ and all $\epsilon > 0$

$$\exp_{x_\alpha}(r_\alpha B_z(1 - \epsilon)) \subset B \exp_{x_\alpha}(r_\alpha z).$$

Plugging these two inequalities together, letting $\alpha \rightarrow +\infty$, using the strong convergence (31), we get that $\int_{B_z(1-\epsilon)} |u^N|^{2_k^\sharp} dx \leq \int_{B_0(1)} |u^N|^{2_k^\sharp} dx$. Letting $\epsilon \rightarrow 0$ yields

$$\int_{B_z(1)} |u^N|^{2_k^\sharp} dx \leq \int_{B_0(1)} |u^N|^{2_k^\sharp} dx.$$

As one checks, since $u^N = U_{\lambda, a}$ is as in (6), the maximum of the left-hand-side is achieved if and only if $z = a$. Therefore $a = 0$ and $u^N = U_{\lambda, 0}$. This proves the claim.

As a consequence, as one checks, when $r_\alpha^{-1}d(x_\alpha, \partial M) \rightarrow +\infty$ as $\alpha \rightarrow +\infty$, the bubble rewrites

$$B_{x_\alpha, r_\alpha}(u^N) = B_{x_\alpha, \lambda r_\alpha}(U_{1, 0}) = \eta \left(\frac{\exp_{x_\alpha}^{-1}(\cdot)}{\tilde{r}_\alpha} \right) \alpha_{n, k} \left(\frac{\lambda r_\alpha}{\lambda^2 r_\alpha^2 + d_g(\cdot, x_\alpha)^2} \right)^{\frac{n-2k}{2}}.$$

We fix $N \in \{1, \dots, d\}$. We claim that $(r_\alpha^N)^{-1}d(x_\alpha^N, \partial M) \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. We argue by contradiction and we assume that the limit is finite. We argue as in the case above. Up to rescaling, and using the boundary chart (7), we get that u_α goes to u^N strongly as $\alpha \rightarrow +\infty$ in $L_{loc}^{2_k^\sharp}(\mathbb{R}^n \setminus \mathcal{S})$, where \mathcal{S} is finite. Therefore u^N is a nonnegative nonzero weak solution to (3), contradicting Lemma 3 in Ge-Wei-Zhou [7]. Therefore the limit is infinite and we are back to the previous case.

All these steps prove Theorem 1.2.

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