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STRUWE'S DECOMPOSITION FOR A POLYHARMONIC OPERATOR ON A COMPACT RIEMANNIAN MANIFOLD WITH OR WITHOUT BOUNDARY

SAIKAT MAZUMDAR

ABSTRACT. Given a high-order elliptic operator on a compact manifold with or without boundary, we perform the decomposition of Palais-Smale sequences for a nonlinear problem as a sum of bubbles. This is a generalization of the celebrated 1984 result of Struwe [16]. Unlike the case of second-order operators, bubbles close to the boundary might appear. Our result includes the case of a smooth bounded domain of \mathbb{R}^n .

1. INTRODUCTION

Let (M, g) be a smooth, compact Riemannian manifold of dimension n with or without boundary. In the latter case we understand that \overline{M} is a compact, oriented submanifold of (\tilde{M}, g) which is itself a smooth, compact Riemannian manifold without boundary and with the same metric g . As one checks, this includes smooth bounded domains of \mathbb{R}^n . When the boundary $\partial M \neq \emptyset$, we let ν be its outward oriented normal vector in \tilde{M} . Let k be a positive integer such that $2k < n$. We define the Sobolev space $H_{k,0}^2(M)$ as the completion of $C_c^\infty(M)$ for the norm $u \mapsto \sum_{i=0}^k \|\nabla^i u\|_2$. This norm is equivalent (see Robert [14]) to the Hilbert norm $\|u\|_{H_k^2} := \left(\sum_{l=0}^k \int_M (\Delta_g^{l/2} u)^2 dv_g \right)^{1/2}$ where $\Delta_g := -\operatorname{div}_g(\nabla)$ is the Laplace-Beltrami operator and, for α odd, $\Delta_g^\alpha u \Delta_g^\alpha v := (\nabla \Delta_g^{\frac{\alpha-1}{2}} u, \nabla \Delta_g^{\frac{\alpha-1}{2}} v)_g$ for all $u, v \in H_k^2(M)$. For details we refer to Aubin [3] and Hebey [9].

We consider the functional

$$I(u) := \frac{1}{2} \int_M (\Delta_g^{k/2} u)^2 dv_g + \frac{1}{2} \sum_{l=0}^{k-1} \int_M A_l(\nabla^l u, \nabla^l u) dv_g - \frac{1}{2_k^\#} \int_M |u|^{2_k^\#} dv_g$$

where for all $l \in \{0, \dots, k-1\}$, A_l is a smooth T_{2l}^0 -tensor field on M and A_l is symmetric (that is $A_l(X, Y) = A_l(Y, X)$ for all T_0^l -tensors X, Y on M). Here, $2_k^\# := \frac{2n}{n-2k}$ is the critical Sobolev exponent such that $H_{k,0}^2(M) \hookrightarrow L^{2_k^\#}(M)$ is continuous, which makes the definition of I consistent for all $u \in H_{k,0}^2(M)$. Critical

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points $u \in H_{k,0}^2(M)$ for I are weak solutions to the pde

$$(1) \quad \begin{cases} Pu = |u|^{2_k^\sharp - 2}u & \text{in } M \\ \partial_\nu^\alpha u = 0 & \text{on } \partial M \text{ for } |\alpha| \leq k-1 \end{cases}$$

where for any $u \in C^{2k}(M)$, we define

$$Pu := \Delta_g^k u + \sum_{l=0}^{k-1} (-1)^l \nabla^{j_l \dots j_1} ((A_l)_{i_1 \dots i_l, j_1 \dots j_l} \nabla^{i_1 \dots i_l} u)$$

and where we say that $u \in H_{k,0}^2(M)$ is a weak solution to (1) if

$$\int_M \Delta_g^{k/2} u, \Delta_g^{k/2} \varphi \, dv_g + \sum_{l=0}^{k-1} \int_M A_l (\nabla^l u \nabla^l \varphi) \, dv_g = \int_M |u|^{2_k^\sharp - 2} u \varphi \, dv_g$$

for all $\varphi \in H_{k,0}^2(M)$. As shown by the regularity theorem in Mazumdar [13], a weak solution u to (1) is indeed a strong solution, $u \in C^{2k}(\overline{M})$.

Definition 1.1. Let $(X, \|\cdot\|)$ be a Banach space and $F \in C^1(X)$. A sequence (u_α) in X is said to be a Palais-Smale sequence for F if $(F(u_\alpha))_\alpha$ has a limit in \mathbb{R} when $\alpha \rightarrow +\infty$, while $DF(u_\alpha) \rightarrow 0$ strongly in X' as $\alpha \rightarrow +\infty$.

In this paper, we describe the lack of relative compactness of Palais-Smale sequences for I , which is due to the noncompact embedding $H_{k,0}^2(M) \hookrightarrow L^{2_k^\sharp}(M)$. For Ω any open domain of \mathbb{R}^n , we let $\mathcal{D}_k^2(\Omega)$ be the completion of $C_c^\infty(\Omega)$ for the norm $u \mapsto \|\Delta^{k/2} u\|_2$. The limiting equations of (1) are

$$(2) \quad \Delta^k u = |u|^{2_k^\sharp - 2} u \text{ in } \mathbb{R}^n, \quad u \in \mathcal{D}_k^2(\mathbb{R}^n)$$

$$(3) \quad \left\{ \begin{array}{l} \Delta^k u = |u|^{2_k^\sharp - 2} u \quad \text{in } \mathbb{R}_-^n \\ \partial_\nu^\alpha u = 0 \quad \text{on } \partial \mathbb{R}_-^n \end{array} \right\}, \quad u \in \mathcal{D}_k^2(\mathbb{R}_-^n)$$

where $\Delta := \Delta_{\text{Eucl}}$ is the Laplacian on \mathbb{R}^n endowed with the Euclidean metric Eucl . Associated to the functional I is the limiting functional

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^n} (\Delta u)^2 \, dx - \frac{1}{2_k^\sharp} \int_{\mathbb{R}^n} |u|^{2_k^\sharp} \, dx \text{ for all } u \in \mathcal{D}_k^2(\mathbb{R}^n).$$

Our main theorem below shows that the lack of convergence to a solution of equation (1) is described by a sum of Bubbles:

Theorem 1.1. Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. Then there exists $d \in \mathbb{N}$ bubbles $[(x_\alpha^{(j)}), (r_\alpha^{(j)}), u^{(j)}]$, $j = 1, \dots, d$, (see Definition 2.1 below) there exists $u_\infty \in H_{k,0}^2(M)$ a solution to (1) such that, up to a subsequence,

$$u_\alpha = u_\infty + \sum_{j=1}^d B_{x_\alpha^{(j)}, r_\alpha^{(j)}}(u^{(j)}) + o(1) \text{ where } \lim_{\alpha \rightarrow +\infty} o(1) = 0 \text{ in } H_{k,0}^2(M)$$

and

$$I(u_\alpha) = I(u_\infty) + \sum_{j=1}^d E(u^{(j)}) + o(1) \text{ as } \alpha \rightarrow +\infty.$$

In Section 2, Bubbles are defined up to a term going to 0 strongly, which is relevant here. As one checks, given $u \in \mathcal{D}_k^2(\mathbb{R}^n)$ a nontrivial weak solution to (2) or (3), then multiplying the equation by u and integrating by parts yields

$$(4) \quad E(u) \geq \beta^\sharp := \frac{k}{n} K_0(n, k)^{-n/2k}$$

where $K_0(n, k)$ be the best constant of the embedding $\mathcal{D}_k^2(\mathbb{R}^n) \hookrightarrow L^{2^\sharp_k}(\mathbb{R}^n)$, that is

$$(5) \quad K_0(n, k)^{-1} = \inf_{u \in \mathcal{D}_k^2(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} (\Delta^{k/2} u)^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\sharp_k} dx \right)^{\frac{2}{2^\sharp_k}}}$$

When the Palais-Smale sequence is nonnegative, the bubbles are positive and correspond to positive solutions to (2). As shown in Lions [12], Swanson [17], Ge-Wei-Zhou [7], these solutions are exactly the extremals for (5) and are of the form

$$(6) \quad u = U_{a, \lambda} := \alpha_{n, k} \left(\frac{\lambda}{1 + \lambda^2 |\cdot - a|^2} \right)^{\frac{n-2k}{2}} \quad a \in \mathbb{R}^n, \lambda > 0$$

where $\alpha_{n, k} > 0$ is explicit. We then get the following:

Theorem 1.2. *Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. We assume that $u_\alpha \geq 0$ for all $\alpha \in \mathbb{N}$. Then there exists $u_\infty \in H_{k,0}^2(M)$ a solution to (1), there exists $d \in \mathbb{N}$, there exist $(x_\alpha^{(1)}), \dots, (x_\alpha^{(d)}) \in M$, $(r_\alpha^{(1)}), \dots, (r_\alpha^{(d)}) \in (0, +\infty)$ such that $r_\alpha^{(j)} \rightarrow 0$ and $r_\alpha^{(j)} = o(d(x_\alpha^{(j)}, \partial M))$ as $\alpha \rightarrow +\infty$ for all $j = 1, \dots, d$, such that, up to a subsequence,*

$$u_\alpha = u_\infty + \sum_{j=1}^d \eta \left((\tilde{r}_\alpha^{(j)})^{-1} \exp_{x_\alpha^{(j)}}^{-1}(\cdot) \right) \alpha_{n, k} \left(\frac{r_\alpha^{(j)}}{(r_\alpha^{(j)})^2 + d_g(\cdot, x_\alpha^{(j)})^2} \right)^{\frac{n-2k}{2}} + o(1)$$

where $\lim_{\alpha \rightarrow +\infty} o(1) = 0$ in $H_{k,0}^2(M)$, and η and $(\tilde{r}_\alpha^{(j)})$'s are as in (8). Moreover,

$$I(u_\alpha) = I(u_\infty) + d\beta^\sharp + o(1) \quad \text{as } \alpha \rightarrow +\infty$$

where β^\sharp is as in (4).

When $k = 1$ and M is a smooth bounded domain of \mathbb{R}^n , Theorem 1.1 is the pioneering result of Struwe [16]. There have been several extensions. Without being exhaustive, we refer to Hebey-Robert [11] for $k = 2$ and manifolds without boundary, Saintier [15] for the p -Laplace operator, El-Hamidi-Vétois [5] for anisotropic operators and Almaraz [1] for nonlinear boundary conditions. A general reference for description as bubbles is the monograph by Fieseler-Tintarev [18]. Another possible description is in the sense of measures as in Lions [12]: a general result of this flavour for high order elliptic operators on manifolds is in Mazumdar [13].

Palais-Smale sequence are produced via critical point techniques, like the Mountain-Pass Lemma of Ambrosetti-Rabinowitz [2] or other topological methods (see for instance the monograph Ghoussoub [8] and the references therein). Concerning higher-order problems, we refer to Bartsch-Weth-Willem [4], Ge-Wei-Zhou [7], Mazumdar [13], the general monograph Gazzola-Grunau-Sweers [6] and the references therein. Theorem 1.1 is used by the author in [13] to get Coron-type solutions to equation (1).

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2. DEFINITION OF BUBBLES

In the spirit of the exponential map, we first cook up a chart around any boundary point. We fix $x_0 \in \partial M$. Since M is a smooth submanifold of \tilde{M} , there exist Ω an open subset of \tilde{M} with $x_0 \in \Omega$, there exists $U \subset \mathbb{R}^n$ open with $0 \in U$, such that for any $x \in \Omega \cap \partial M$ there exists $\mathcal{T}_x \in C^\infty(U, \tilde{M})$ having the following properties.

$$(7) \quad \left\{ \begin{array}{l} \bullet \mathcal{T}_x(0) = x \\ \bullet \mathcal{T}_x \text{ is a smooth diffeomorphism onto its image } \mathcal{T}_x(U). \\ \bullet \mathcal{T}_x(U \cap \{x_1 < 0\}) = \mathcal{T}_x(U) \cap M \\ \bullet \mathcal{T}_x(U \cap \{x_1 = 0\}) = \mathcal{T}_x(U) \cap \partial M \\ \bullet (x, z) \mapsto \mathcal{T}_x(z) \text{ is smooth from } \Omega \times U \text{ to } \tilde{M} \\ \bullet d\mathcal{T}_x(0) : \mathbb{R}^n \rightarrow T_x M \text{ is an isometry} \\ \bullet d\mathcal{T}_x(0)[e_1] = \nu_x \text{ where } \nu_x \text{ is the outer unit normal vector to } \partial M \\ \text{at the point } x. \end{array} \right.$$

This map is defined uniformly with respect to x in a neighborhood Ω of a fixed point $x_0 \in \partial M$. By a standard abuse of notation, we will always consider $x \mapsto \mathcal{T}_x$ without any reference to Ω or x_0 : this will make sense in the sequel since the relevant points will always be in the neighborhood of a fixed point.

Definition 2.1. A “Bubble” is a triplet $[(x_\alpha), (r_\alpha), u]$ where $x_\alpha \in \overline{M}$ is a convergent sequence, $r_\alpha > 0$ for all $m \in \mathbb{N}$ with $\lim_{\alpha \rightarrow +\infty} r_\alpha = 0$ and

$$\left. \begin{array}{l} \text{either } \left\{ x_\alpha \in M, \lim_{\alpha \rightarrow +\infty} \frac{d(x_\alpha, \partial M)}{r_\alpha} = +\infty \text{ and } u \in \mathcal{D}_k^2(\mathbb{R}^n) \text{ satisfies (2)} \right\} \\ \text{or } \left\{ x_\alpha \in \partial M \text{ and } u \in \mathcal{D}_k^2(\mathbb{R}_-^n) \text{ satisfies (3)} \right\} \end{array} \right\}$$

If $x_\alpha \in M$, we let $\tilde{r}_\alpha > 0$ be such that

$$(8) \quad \lim_{\alpha \rightarrow +\infty} \tilde{r}_\alpha = \tilde{r}_\infty \in \left[0, \frac{i_g(\tilde{M})}{2} \right), \quad \lim_{\alpha \rightarrow +\infty} \frac{r_\alpha}{\tilde{r}_\alpha} = 0 \text{ and } \tilde{r}_\alpha < \frac{d_g(x_\alpha, \partial M)}{2}$$

and we define

$$B_{x_\alpha, r_\alpha}(u) := \eta \left(\frac{\exp_{x_\alpha}^{-1}(x)}{\tilde{r}_\alpha} \right) r_\alpha^{-\frac{n-2k}{2}} u \left(\frac{\exp_{x_\alpha}^{-1}(x)}{r_\alpha} \right)$$

where $\eta \in C_c^\infty(B_0(i_g(\tilde{M})))$ is identically 1 in a neighborhood of 0. Here, the exponential map is taken on the ambient manifold (\tilde{M}, g) .

If $x_\alpha \in \partial M$, we let $x_0 := \lim_{\alpha \rightarrow +\infty} x_\alpha$, and we define

$$B_{x_\alpha, r_\alpha}(u) := \eta(\mathcal{T}_{x_\alpha}^{-1}(x)) r_\alpha^{-\frac{n-2k}{2}} u \left(\frac{\mathcal{T}_{x_\alpha}^{-1}(x)}{r_\alpha} \right)$$

where \mathcal{T}_x is as in (7), Ω is a neighborhood of $x_0 \in \partial M$ and $\eta \in C_c^\infty(U)$ is identically 1 in a neighborhood of 0.

Beside $[(x_\alpha), (r_\alpha), u]$, the definition of a bubble depends on the choice of the cut-off function η , the radius \tilde{r}_α and the chart \mathcal{T}_x . However, as shown in the proposition below, after quotienting by sequences going to 0, the class of a Bubble is independent of these later parameters.

Proposition 2.1. *The definition of Bubbles depend only on $[(x_\alpha), (r_\alpha), u]$, up to a sequence going to 0 strongly in $H_{k,0}^2(M)$.*

Proof of Proposition 2.1. We first assume that $u \in \mathcal{D}_k^2(\mathbb{R}^n)$ satisfies (2) and that

$$(9) \quad \lim_{\alpha \rightarrow +\infty} \frac{d_g(x_\alpha, \partial M)}{r_\alpha} = +\infty.$$

For $i = 1, 2$, we set the bubbles $B_\alpha^i := \eta^i \left((\tilde{r}_\alpha^i)^{-1} \exp_{x_\alpha}^{-1}(\cdot) \right) r_\alpha^{-\frac{n-2k}{2}} u \left(r_\alpha^{-1} \exp_{x_\alpha}^{-1}(\cdot) \right)$, where $\eta^i \in C_c^\infty(B_0(2a_i))$, $\eta^i \equiv 1$ in $B_0(a_i)$ with $0 < 2a_i \leq \iota_g(\tilde{M})$; $\tilde{r}_\alpha^i > 0$ are as in (8). We let $r_\alpha^{max} = \max\{a_1 \tilde{r}_\alpha^1, a_2 \tilde{r}_\alpha^2\}$ and $r_\alpha^{min} = \min\{a_1 \tilde{r}_\alpha^1, a_2 \tilde{r}_\alpha^2\}$, and let $\epsilon_\alpha^{max} = r_\alpha / r_\alpha^{max}$ and $\epsilon_\alpha^{min} = r_\alpha / r_\alpha^{min}$. Then $\lim_{\alpha \rightarrow 0} \epsilon_\alpha^{max} = 0$ and $\lim_{\alpha \rightarrow 0} \epsilon_\alpha^{min} = 0$. The comparison lemma 9.1 of [13] yields $C > 0$ such that for any $R > 0$ and α large

$$\begin{aligned} \sum_{l=0}^k \|\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2)\|_2^2 &\leq \sum_{l=0}^k \int_{B_{2r_\alpha^{max}}(x_\alpha) \setminus B_{r_\alpha^{min}}(x_\alpha)} \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g \\ &\leq \sum_{i=1,2} \sum_{l=0}^k \int_{M \setminus B_{Rr_\alpha}(x_\alpha)} \left(\Delta_g^{l/2} (B_{x_\alpha, r_\alpha}^i(u)) \right)^2 dv_g. \end{aligned}$$

Therefore, using (33), we get that $B_\alpha^1 - B_\alpha^2 = o(1)$ in $H_k^2(M)$ as $\alpha \rightarrow +\infty$.

Now we consider the case of a boundary bubble, that is $x_\alpha \in \partial M$ and $u \in \mathcal{D}_k^2(\mathbb{R}^n)$ satisfies (3). For $i = 1, 2$, we set $B_\alpha^i := \eta^i \left(\mathcal{T}_{x_\alpha}^{-1}(\cdot) \right) r_\alpha^{-\frac{n-2k}{2}} u \left(r_\alpha^{-1} \mathcal{T}_{x_\alpha}^{-1}(\cdot) \right)$ where \mathcal{T}_x , $i = 1, 2$, are as in (7), U is a neighborhood of $x_0 \in \partial M$ and $\eta^1, \eta^2 \in C_c^\infty(U)$ are identically 1 in a neighborhood of 0. One has

$$\begin{aligned} \sum_{l=0}^k \int_M \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g &\leq \\ \sum_{l=0}^k \int_{D_\alpha(R) \cap M} \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g &+ \sum_{l=0}^k \int_{M \setminus D_\alpha(R)} \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g \end{aligned}$$

where $D_\alpha(R) := \mathcal{T}_{x_\alpha}^{-1}(B_0(r_\alpha R)) \cup \mathcal{T}_{x_\alpha}^2(B_0(r_\alpha R))$. It follows as in the comparison Lemma 9.1 of [13] that there exists $C > 0$ such that for α large

$$\begin{aligned} \sum_{l=0}^k \int_{D_\alpha(R) \cap M} \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g &\leq \\ C \sum_{l=0}^k \int_{(B_0(r_\alpha R) \cup \Phi_\alpha^{-1}(B_0(r_\alpha R))) \cap \mathbb{R}_-^n} \left(\Delta^{l/2} \left((B_\alpha^1 \circ \mathcal{T}_{x_\alpha}^{-1}) - (B_\alpha^2 \circ \mathcal{T}_{x_\alpha}^{-1}) \right) \right)^2 dx &\leq \\ C \sum_{l=0}^k \int_{B_0(R) \cap \mathbb{R}_-^n} \left[\Delta^{l/2} (\eta^a(r_\alpha \cdot) u) - \Delta^{l/2} (\eta^b(\Phi_\alpha(r_\alpha \cdot)) u (r_\alpha^{-1} \Phi_\alpha(r_\alpha \cdot))) \right]^2 dx &= o(1) \end{aligned}$$

where $\Phi_\alpha := \mathcal{T}_{x_\alpha}^2{}^{-1} \circ \mathcal{T}_{x_\alpha}^1$ and $d(\Phi_\alpha)_0 = Id$. Similarly to the case (9), we get that

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \sum_{l=0}^k \int_{M \setminus D_\alpha(R)} \left(\Delta_g^{l/2} (B_\alpha^1 - B_\alpha^2) \right)^2 dv_g = 0.$$

This completes the proof of Proposition 2.1. \square

3. PRELIMINARY ANALYSIS

The proof of Theorem 1.1 goes through four steps. All results are up to a subsequence. We let $(u_\alpha)_\alpha \in H_{k,0}^2(M)$ be a Palais-Smale sequence for I .

Step 1: We claim that $(u_\alpha)_\alpha$ is bounded in $H_{k,0}^2(M)$.

Proof of the claim: Since (u_α) is a Palais-Smale sequence, we have that

$$\begin{aligned} \langle DI(u_\alpha), u_\alpha \rangle &= \int_M (\Delta_g^{k/2} u_\alpha)^2 dv_g + \sum_{\alpha=0}^{k-1} \int_M A_l(\nabla^l u_\alpha, \nabla^l u_\alpha) dv_g \\ &\quad - \int_M |u_\alpha|^{2_k^\sharp} dv_g = o\left(\|u_\alpha\|_{H_k^2}\right) \end{aligned}$$

Therefore

$$(10) \quad \int_M |u_\alpha|^{2_k^\sharp} dv_g = \frac{n}{k} I(u_\alpha) + o\left(\|u_\alpha\|_{H_k^2}\right) \leq C + o\left(\|u_\alpha\|_{H_k^2}\right)$$

Since $(I(u_\alpha))_\alpha$ is bounded, then putting together these equalities yields

$$\|u_\alpha\|_{H_k^2}^2 \leq C + C \|u_\alpha\|_{H_{k-1}^2}^2 + C \int_M |u_\alpha|^{2_k^\sharp} dv_g$$

Now since the embedding of $H_{k,0}^2(M)$ in $H_{0,k-1}^2(M)$ is compact, then for any $\varepsilon > 0$ there exists a $B_\varepsilon > 0$ such that $\|u\|_{H_{k-1}^2}^2 \leq \varepsilon \|u\|_{H_k^2}^2 + B_\varepsilon \|u\|_{2_k^\sharp}^2$ for all $u \in H_k^2(M)$. Therefore, taking $\varepsilon > 0$ small enough, we get that

$$\|u_\alpha\|_{H_k^2}^2 \leq C + C \int_M |u_\alpha|^{2_k^\sharp} dv_g$$

Then using (10) we get that $\|u_\alpha\|_{H_k^2}^2 \leq C + C \|u_\alpha\|_{H_k^2}$ for all α , and therefore the sequence (u_α) is bounded in $H_{k,0}^2(M)$. This proves the claim. \square

Since (u_α) is bounded in $H_{k,0}^2(M)$, there exists $u_\infty \in H_{k,0}^2(M)$ such that

$$(11) \quad \begin{cases} u_\alpha \rightharpoonup u_\infty & \text{weakly in } H_{k,0}^2(M) \text{ and } L^{2_k^\sharp}(M), \\ u_\alpha \rightarrow u_\infty & \text{strongly in } H_{l,0}^2(M) \text{ and in } L^q(M) \text{ for } l < k, q < 2_k^\sharp, \\ u_\alpha(x) \rightarrow u_\infty(x) & \text{a.e in } M \end{cases}$$

We define $v_\alpha := u_\alpha - u_\infty$.

Step 2: We claim that

- (1) $DI(u_\infty) = 0$
- (2) (v_α) is a Palais-Smale sequence for the functional J on the space $H_{k,0}^2(M)$,
- (3) $J(v_\alpha) = I(u_\alpha) - I(u_\infty) + o(1)$ as $\alpha \rightarrow +\infty$.

where

$$J(u) := \frac{1}{2} \int_M (\Delta_g^{k/2} u)^2 dv_g - \frac{1}{2_k^\sharp} \int_M |u|^{2_k^\sharp} dv_g \text{ for } u \in H_{k,0}^2(M)$$

Proof of the claim: We fix $\varphi \in H_{k,0}^2(M)$. We have that

$$\begin{aligned} \langle DI(u_\alpha), \varphi \rangle &= \int_M \Delta_g^{k/2} u_\alpha \Delta_g^{k/2} \varphi dv_g + \sum_{\alpha=0}^{k-1} \int_M A_l(g)(\nabla^l u_\alpha, \nabla^l \varphi) \\ &\quad - \int_M |u_\alpha|^{2_k^\sharp - 2} u_\alpha \varphi dv_g = o(1) \end{aligned} \quad (12)$$

The following classical integration Lemma will be often used in the sequel (see Lemma 6.2.7 in Hebey [10] for a proof):

Lemma 3.1. *Let (M, g) be a Riemannian manifold. If (f_α) is a bounded sequence in $L^p(M)$, $1 < p < +\infty$, such that $f_\alpha \rightarrow f$ a.e in M , then $f \in L^p(M)$ and $f_\alpha \rightharpoonup f$ weakly in $L^p(M)$.*

Since $(|u_\alpha|^{2_k^\sharp - 2} u_\alpha)_\alpha$ is bounded in $L^{\frac{2_k^\sharp}{2_k^\sharp - 1}}$ and converges a.e., Lemma 3.1 yields

$$\int_M |u_\alpha|^{2_k^\sharp - 2} u_\alpha \varphi dv_g = \int_M |u_\infty|^{2_k^\sharp - 2} u_\infty \varphi dv_g + o(1) \quad (13)$$

Therefore, the weak convergence of (u_α) to u_∞ , (12) and (13) yield that u_∞ is a weak solution to (1). This proves point (1) of Step 2.

We now estimate $I(u_\alpha)$. From (11) we have

$$\begin{aligned} \int_M (\Delta_g^{k/2} u_\alpha)^2 dv_g - \int_M (\Delta_g^{k/2} u_\infty)^2 dv_g &= \int_M (\Delta_g^{k/2} v_\alpha)^2 dv_g + o(1), \\ \sum_{l=0}^{k-1} \int_M A_l(\nabla^l u_\alpha, \nabla^l u_\alpha) dv_g &= \sum_{l=0}^{k-1} \int_M A_l(\nabla^l u_\infty, \nabla^l u_\infty) dv_g + o(1) \end{aligned}$$

The following two inequalities will be of constant use in the sequel: for any $1 < p < +\infty$, there exists $C > 0$ such that

$$| |a+b|^p - |a|^p - |b|^p | \leq C (|a|^{p-1}|b| + |b|^{p-1}|a|) \quad (14)$$

$$| |a+b|^p(a+b) - |a|^p a - |b|^p b | \leq C (|a|^p|b| + |b|^p|a|) \quad (15)$$

for all $a, b \in \mathbb{R}$. It then follows from (14) that

$$\left| |u_\alpha|^{2_k^\sharp} - |u_\infty|^{2_k^\sharp} - |v_\alpha|^{2_k^\sharp} \right| \leq C \left(|v_\alpha|^{2_k^\sharp - 1} |u_\infty| + |u_\infty|^{2_k^\sharp - 1} |v_\alpha| \right),$$

and then using Lemma 3.1, we get that

$$\int_M |u_\alpha|^{2_k^\sharp} dv_g - \int_M |u_\infty|^{2_k^\sharp} dv_g = \int_M |v_\alpha|^{2_k^\sharp} dv_g + o(1)$$

Hence $I(u_\alpha) - I(u_\infty) = J(v_\alpha) + o(1)$ as $\alpha \rightarrow +\infty$, which proves point (3) of Step 2.

Next we show the sequence (v_α) is a Palais-Smale sequence for the functional J on $H_{k,0}^2(M)$. Let $\varphi \in H_{k,0}^2(M)$, we have

$$\langle DJ(v_\alpha), \varphi \rangle = \langle DI(u_\alpha), \varphi \rangle - \langle DI(u_\infty), \varphi \rangle + \int_M \Phi_\alpha \varphi dv_g + o(\|\varphi\|_{H_k^2}) \quad (16)$$

where

$$\Phi_\alpha := |v_\alpha + u_\infty|^{2_k^\# - 2} (v_\alpha + u_\infty) - |u_\infty|^{2_k^\# - 2} u_\infty - |v_\alpha|^{2_k^\# - 2} v_\alpha$$

Inequality (15) and Hölder's inequality yield

$$(17) \quad \left| \int_M \Phi_\alpha \varphi \, dv_g \right| \leq C \left(\left\| |v_\alpha|^{2_k^\# - 2} u_\infty \right\|_{\frac{2_k^\#}{2_k^\# - 1}} + \left\| |u_\infty|^{2_k^\# - 2} v_\alpha \right\|_{\frac{2_k^\#}{2_k^\# - 1}} \right) \|\varphi\|_{2_k^\#}$$

Since $v_\alpha \rightharpoonup 0$ in $L^{2_k^\#}(M)$, Lemma 3.1 yields

$$\left\| |v_\alpha|^{2_k^\# - 2} u_\infty \right\|_{\frac{2_k^\#}{2_k^\# - 1}} + \left\| |u_\infty|^{2_k^\# - 2} v_\alpha \right\|_{\frac{2_k^\#}{2_k^\# - 1}} = o(1)$$

Since (u_α) is a Palais-Smale for I , then (16), (17) and the continuous embedding $H_{k,0}^2(M) \hookrightarrow L^{2_k^\#}(M)$ yields $\langle DJ(v_\alpha), \varphi \rangle = o(\|\varphi\|_{H_k^2})$ as $\alpha \rightarrow +\infty$ uniformly wrt $\varphi \in H_{k,0}^2(M)$. This proves the claim and ends Step 2. \square

The next lemma addresses the compactness of a Palais-Smale sequence for small energy. It will be generalized to the case of small local energy in Proposition 4.1.

Step 3: Let (v_α) be a Palais-Smale sequence for J on $H_{k,0}^2(M)$. We assume that $v_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$, and that $J(v_\alpha) \rightarrow \beta$ with $\beta < \beta^\#$, where $\beta^\#$ is as in (4). We claim that $v_\alpha \rightarrow 0$ strongly in $H_{k,0}^2(M)$.

Proof of the claim: Since (v_α) is bounded and $\langle DJ(v_\alpha), v_\alpha \rangle = o(\|v_\alpha\|_{H_k^2})$, we get that

$$(18) \quad J(v_\alpha) = \frac{k}{n} \int_M (\Delta_g^{k/2} v_\alpha)^2 \, dv_g + o(1) = \frac{k}{n} \int_M |v_\alpha|^{2_k^\#} \, dv_g + o(1) = \beta + o(1).$$

As a consequence, $\beta \geq 0$. It follows from Mazumdar [13] that for any $\varepsilon > 0$ there exists $B_\varepsilon > 0$ such that

$$(19) \quad \|u\|_{2_k^\#}^2 \leq (K_0(n, k) + \varepsilon) \int_{\tilde{M}} (\Delta_g^{k/2} u)^2 \, dv_g + B_\varepsilon \|u\|_{H_{k-1}^2}^2$$

for all $u \in H_k^2(\tilde{M})$. Applying this inequality to v_α , the strong convergence to 0 in H_{k-1}^2 and (18) yield

$$\left(\frac{n}{k} \beta \right)^{2/2_k^\#} \leq (K_0(n, k) + \varepsilon) \frac{n}{k} \beta$$

Letting $\varepsilon \rightarrow 0$ and using $0 \leq \beta < \beta^\#$, we get that $\beta = 0$, and then (18) yields $v_\alpha \rightarrow 0$ strongly in $H_{k,0}^2(M)$. This proves the claim and ends Step 3. \square

Step 4: Proof of Theorem 1.1. Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. By subtracting the weak limit u_∞ , we get a Palais-Smale sequence (v_α) for the functional J with energy $J(v_\alpha) = I(u_\alpha) - I(u_\infty) + o(1)$ as $\alpha \rightarrow +\infty$. If $v_\alpha \rightarrow 0$ strongly in $H_{k,0}^2(M)$, then we end the process. If not, we apply Lemma 4.1 to subtract a bubble modeled on $v \in \mathcal{D}_k^2(\mathbb{R}^n) \setminus \{0\}$ and we get a new Palais-Smale sequence for J , but with the energy decreased by $E(v)$. If the resulting sequence goes strongly to 0, we stop the process, if not, we iterate it again. This process must stop since the energy $E(v) \geq \beta^\#$ and after finitely many steps, the energy goes below the critical threshold $\beta^\#$ and then the convergence is strong by Step 3. This proves Theorem 1.1. \square

The rest of the paper is devoted to the proof of Lemma 4.1.

4. EXTRACTION OF A BUBBLE

In the sequel, for any (M, g) as in the introduction, we let $H_k^2(M)$ be the completion of $\{u \in C^\infty(M) : \|u\|_{H_k^2} < +\infty\}$ for the norm $\|\cdot\|_{H_k^2}$. The space $H_{k,0}^2(M)$ is then a closed subspace of $H_k^2(M)$. The following lemma is the main ingredient in the proof of Theorem 1.1

Lemma 4.1. *Let (v_α) be a Palais-Smale sequence for the functional J on $H_{k,0}^2(M)$ such that $v_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$ but not strongly. Then there exists a bubble $(B_{x_\alpha, r_\alpha}(v))$ such that upto a subsequence, the following holds:*

- $w_\alpha := v_\alpha - B_{x_\alpha, r_\alpha}(v)$ is a Palais-Smale sequence for J ,
- $J(w_\alpha) = J(v_\alpha) - E(v) + o(1)$ as $\alpha \rightarrow +\infty$.

The proof of this lemma goes through 10 steps.

Step 1: We prove a strong convergence Lemma for small energies. This is a localized version of Step 3 of Section 3.

Proposition 4.1. *Let (N, g_∞) be a Riemannian manifold with positive injectivity radius.*

- Let $(g_i)_i$ be metrics on N such that $g_i \rightarrow g_\infty$ in C_{loc}^p as $i \rightarrow +\infty$ for all p .
- Let $(P_i)_i$ be a family of operators on $C^\infty(N)$ such that

$$P_i := \Delta_{g_i}^k + \sum_{l=0}^{k-1} (-1)^l \nabla^{i_1 \dots i_l} \left((A_l^i)_{i_1 \dots i_l j_1 \dots j_l} \nabla^{j_1 \dots j_l} \right)$$

with families of symmetric tensors $(A_l^i) \rightarrow A_l$ in C_{loc}^p as $i \rightarrow +\infty$ for all p .

- We fix $\Omega \subset N$ an open smooth domain, and we define

$$(20) \quad J_i(u) := \frac{1}{2} \int_{\Omega} u P_i u \, dv_{g_i} - \frac{1}{2_k^\sharp} \int_{\Omega} |u|^{2_k^\sharp} \, dv_{g_i} \text{ for } u \in H_k^2(\Omega),$$

such that J_i is C^1 . Here, the background metric is g_∞ .

- We let $(u_i) \in H_{k,0}^2(\Omega)$ and $u_\infty \in H_{k,0}^2(\Omega)$ be such that $u_i \rightharpoonup u_\infty$ weakly in $H_{k,0}^2(\Omega)$ as $i \rightarrow +\infty$.
- We assume that there exist a compact $K \subset N$ such that

$$\lim_{i \rightarrow +\infty} \sup_{u \in H_{k,0}^2(\Omega), \text{Supp } \varphi \subset K} \frac{\langle DJ_i(u_i), \varphi \rangle}{\|\varphi\|_{H_k^2(\Omega)}} = 0$$

- We assume that there exists $K_\infty > 0$ and $C \geq 0$ such that

$$(21) \quad \left(\int_N |u|^{2_k^\sharp} \, dv_{g_\infty} \right)^{\frac{2}{2_k^\sharp}} \leq K_\infty \int_N (\Delta_{g_\infty}^{k/2} u)^2 \, dv_{g_\infty} + C \|u\|_{H_{k-1}^2}^2 \text{ for all } u \in C_c^\infty(N).$$

We fix $x_0 \in \Omega$ and $\delta \in (0, i_{g_\infty}(N)/2)$. We assume that

$$(22) \quad \begin{cases} B_{x_0}(2\delta) \subset K \text{ (the ball is wrt } g_\infty), \\ \int_{B_{x_0}(2\delta) \cap \Omega} |u_i|^{2_k^\sharp} \, dv_{g_i} \leq \left(\frac{1}{2K_\infty} \right)^{\frac{2_k^\sharp}{2_k^\sharp - 2}} \text{ for all } i \in \mathbb{N}. \end{cases}$$

Then $u_i \rightarrow u_\infty$ strongly in $H_k^2(B_{x_0}(\delta) \cap \Omega)$.

Proof of Proposition 4.1: Up to extracting a subsequence, we assume that $u_i \rightarrow u_\infty$ strongly in $H_{k-1}^2(\omega)$ as $i \rightarrow +\infty$ for $\omega \subset \Omega$ relatively compact and $u_i(x) \rightarrow u_\infty(x)$ as $i \rightarrow +\infty$ for a.e. $x \in \Omega$. Let $\eta \in C^\infty(N)$ such that $\eta(x) = 1$ for $x \in B_{x_0}(\delta)$ and $\eta(x) = 0$ for $x \in N \setminus B_{x_0}(2\delta)$. Since η has compact support, we get that $\eta^2(u_i - u_\infty) \in H_{k,0}^2(\Omega)$ is uniformly bounded in $H_{k,0}^2(\Omega)$. Since $B_{x_0}(2\delta) \subset K$, it then follows from hypothesis (20) that

$$\langle DJ_i(u_i), \eta^2(u_i - u_\infty) \rangle = o(1) \text{ as } i \rightarrow +\infty.$$

Since $\eta^2(u_i - u_\infty) \rightarrow 0$ strongly in $H_{k-1}^2(\Omega)$, we then get that

$$(23) \quad \int_{\Omega} \Delta_{g_i}^{k/2} u_i \Delta_{g_i}^{k/2} (\eta^2(u_i - u_\infty)) dv_{g_i} = \int_{\Omega} |u_i|^{2_k^\sharp - 2} u_i \eta^2(u_i - u_\infty) dv_{g_i} + o(1)$$

as $i \rightarrow +\infty$. The weak convergence of u_i to u_∞ and the strong convergence of g_i to g_∞ on compact sets yields

$$(24) \quad \int_{\Omega} \Delta_{g_i}^{k/2} u_i \Delta_{g_i}^{k/2} (\eta^2(u_i - u_\infty)) dv_{g_i} = \int_{\Omega} \Delta_{g_i}^{k/2} (u_i - u_\infty) \Delta_{g_i}^{k/2} (\eta^2(u_i - u_\infty)) dv_{g_i} + o(1)$$

as $i \rightarrow +\infty$. As one checks, for any $\varphi \in H_k^2(\Omega)$, we have that $\Delta_{g_i}^{k/2} \varphi \Delta_{g_i}^{k/2} (\eta^2 \varphi) = \left(\Delta_{g_i}^{k/2} (\eta \varphi) \right)^2 + \sum_{p < k, l \leq k} \nabla^p \varphi \star \nabla^l \varphi$, where $A \star B$ denotes a linear combination of bilinear forms in A and B . Therefore, using again the strong convergence of $\eta^2(u_i - u_\infty)$ to 0 in H_{k-1}^2 , we get that

$$(25) \quad \int_{\Omega} \Delta_{g_i}^{k/2} u_i \Delta_{g_i}^{k/2} (\eta^2(u_i - u_\infty)) dv_{g_i} = \int_{\Omega} \left(\Delta_{g_i}^{k/2} (\eta(u_i - u_\infty)) \right)^2 dv_{g_i} + o(1)$$

as $i \rightarrow +\infty$. Moreover, since $|u_i|^{2_k^\sharp - 2} \eta^2(u_i - u_\infty)$ is uniformly bounded in $L^{2_k^\sharp / (2_k^\sharp - 1)}$ and goes to 0 almost everywhere as $i \rightarrow +\infty$, then it goes weakly to 0 in $L^{2_k^\sharp / (2_k^\sharp - 1)}$, and then $\int_{\Omega} |u_i|^{2_k^\sharp - 2} \eta^2(u_i - u_\infty) u_\infty dv_{g_i} \rightarrow 0$ as $i \rightarrow +\infty$. Therefore, plugging (24) and (25) into (23), we get that

$$\int_{\Omega} \left(\Delta_{g_i}^{k/2} (\eta(u_i - u_\infty)) \right)^2 dv_{g_i} = \int_{\Omega} |u_i|^{2_k^\sharp - 2} (\eta(u_i - u_\infty))^2 dv_{g_i} + o(1)$$

as $i \rightarrow +\infty$. Since $g_i \rightarrow g_\infty$ as $i \rightarrow +\infty$ in C^p locally on compact sets and $\eta(u_i - u_\infty)$ is uniformly bounded in $H_k^2(\Omega)$, we get that

$$\int_{\Omega} \left(\Delta_{g_\infty}^{k/2} (\eta(u_i - u_\infty)) \right)^2 dv_{g_\infty} = \int_{\Omega} |u_i|^{2_k^\sharp - 2} (\eta(u_i - u_\infty))^2 dv_{g_\infty} + o(1)$$

as $i \rightarrow +\infty$. Hölder's inequality, the Sobolev inequality (21), the convergence of (g_i) , the strong convergence in H_{k-1}^2 and (22) then yields

$$\begin{aligned} & \int_{\Omega} \left(\Delta_{g_{\infty}}^{k/2} (\eta(u_i - u_{\infty})) \right)^2 dv_{g_{\infty}} \\ & \leq \left(\int_{B_{x_0}(2\delta) \cap \Omega} |u_i|^{2_k^{\sharp}} dv_{g_{\infty}} \right)^{\frac{2_k^{\sharp} - 2}{2_k^{\sharp}}} \left(\int_N |\eta(u_i - u_{\infty})|^{2_k^{\sharp}} dv_{g_{\infty}} \right)^{\frac{2}{2_k^{\sharp}}} + o(1) \\ & \leq \frac{1}{2K_{\infty}} \left(K_{\infty} \int_N \left(\Delta_{g_{\infty}}^{k/2} (\eta(u_i - u_{\infty})) \right)^2 dv_{g_{\infty}} + C \|\eta(u_i - u_{\infty})\|_{H_{k-1}^2}^2 \right) + o(1) \\ & \leq \left(\int_{B_{x_0}(2\delta) \cap \Omega} |u_i|^{2_k^{\sharp}} dv_{g_i} \right)^{\frac{2_k^{\sharp} - 2}{2_k^{\sharp}}} K_{\infty} \int_{\Omega} \left(\Delta_{g_{\infty}}^{k/2} (\eta(u_i - u_{\infty})) \right)^2 dv_{g_{\infty}} + o(1) \end{aligned}$$

as $i \rightarrow +\infty$. Therefore, we get that $\|\Delta_{g_{\infty}}^{k/2} (\eta(u_i - u_{\infty}))\|_2 \rightarrow 0$ as $i \rightarrow +\infty$. Since $\eta(u_i - u_{\infty}) \rightarrow 0$ strongly in H_{k-1}^2 and η has compact support, we get that $\eta(u_i - u_{\infty}) \rightarrow 0$ strongly in $H_k^2(\Omega)$, and therefore $u_i \rightarrow u_{\infty}$ in $H_k^2(B_{x_0}(\delta) \cap \Omega)$. Note that this is up to a subsequence. Indeed, by uniqueness, the convergence holds for the initial sequence (u_i) . This proves Proposition 4.1. \square

Step 2: Since $\langle DJ(v_{\alpha}), v_{\alpha} \rangle = o(1)$, one has

$$J(v_{\alpha}) = \frac{k}{n} \int_M |v_{\alpha}|^{2_k^{\sharp}} dv_g + o(1) = \beta + o(1) \text{ as } \alpha \rightarrow +\infty$$

where $\beta := \lim_{\alpha \rightarrow +\infty} J(v_{\alpha})$. By Step 3 of Section 3, $\beta \geq \beta^{\sharp}$. Therefore, since \overline{M} is compact, for any $r_0 > 0$, there exists $y_0 \in \overline{M}$ and $\lambda_0 > 0$ such that

$$\int_{B_{y_0}(r_0) \cap M} |v_{\alpha}|^{2_k^{\sharp}} dv_g \geq \lambda_0$$

For any $r > 0$, we set

$$(26) \quad \mu_{\alpha}(r) := \max_{x \in \overline{M}} \int_{B_x(r) \cap M} |v_{\alpha}|^{2_k^{\sharp}} dv_g,$$

the Levy concentration function. In particular, $\mu_{\alpha}(r_0) \geq \lambda_0$ for all α . We fix

$$0 < \lambda < \epsilon_0 := \min \left\{ \lambda_0, \frac{1}{(2K_0(n, k))^{2_k^{\sharp}/(2_k^{\sharp}-2)}} \right\}$$

where $K_0(n, k)$ is the best constant in the Euclidean Sobolev inequality (5). Since $\mu_{\alpha}(0) = 0$, there exists $(r_{\alpha})_{\alpha} \in (0, r_0)$ and $(x_{\alpha})_{\alpha} \in \overline{M}$ such that:

$$(27) \quad \lambda = \mu_{\alpha}(r_{\alpha}) = \int_{B_{x_{\alpha}}(r_{\alpha}) \cap M} |v_{\alpha}|^{2_k^{\sharp}} dv_g$$

Step 3: We claim that $\lim_{\alpha \rightarrow +\infty} r_{\alpha} = 0$.

Proof of the claim. We argue by contradiction. If (r_{α}) does not go to 0 up to a subsequence, we get that there exists $\delta \in (0, i_g(\tilde{M})/2)$ such that for all $x \in M$, we have that $\int_{B_x(2\delta) \cap M} |v_{\alpha}|^{2_k^{\sharp}} dv_g \leq \lambda$ for all α . We apply Proposition 4.1 with $(N, g_{\infty}) = (\tilde{M}, g)$, $\Omega = M$, $P_{\alpha} = P$, $g_{\alpha} = g$, $J_{\alpha} = J$, and the Sobolev inequality

(19) of [13], and we get $v_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$ in $H_k^2(M \cap B_x(\delta))$ for all $x \in M$. With a finite covering, we get that $v_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$ strongly in $H_{k,0}^2(M)$, contradicting our initial hypothesis. This proves the claim and ends Step 3. \square

First assume that

$$(28) \quad \lim_{\alpha \rightarrow +\infty} \frac{d(x_\alpha, \partial M)}{r_\alpha} = +\infty.$$

We define

$$\tilde{v}_\alpha(x) := r_\alpha^{\frac{n-2k}{2}} u_\alpha(\exp_{x_\alpha}(r_\alpha x)) \text{ for } |x| < \frac{i_g(\tilde{M})}{r_\alpha} \text{ and } |x| < \frac{d(x_\alpha, \partial M)}{r_\alpha}$$

Step 4: Suppose that (28) holds. We claim that there exists $v \in \mathcal{D}_k^2(\mathbb{R}^n)$ such that for any $\eta \in C_c^\infty(\mathbb{R}^n)$, we have that

$$\eta \tilde{v}_\alpha \rightharpoonup \eta v \text{ weakly in } \mathcal{D}_k^2(\mathbb{R}^n) \text{ as } k \rightarrow +\infty.$$

Proof of the claim. Fix $\eta \in C_c^\infty(\mathbb{R}^n)$, and let $R_0 > 0$ be such that $\text{Supp } \eta \subset B_0(R_0)$. We define

$$\eta_\alpha(x) := \eta\left(\frac{\exp_{x_\alpha}^{-1}(x)}{r_\alpha}\right) \text{ for } x \in B_{x_\alpha}(R_0 r_\alpha), \text{ and } \eta_\alpha(x) := 0 \text{ outside.}$$

Up to a subsequence, there exists $x_0 \in \tilde{M}$ and $\tau > 0$ such that $B_{x_\alpha}(R_0 r_\alpha) \subset B_{x_0}(\tau) \subset \tilde{M}$. It then follows from the comparison Lemma 9.1 of Mazumdar [13] that there exists $C > 0$ such that

$$\int_{B_0(R_0 r_\alpha)} \left(\Delta^{k/2}[(\eta_\alpha v_\alpha) \circ \exp_{x_\alpha}]\right)^2 dx \leq C \int_{B_{x_\alpha}(R_0 r_\alpha)} \left(\Delta_g^{k/2}(\eta_\alpha v_\alpha)\right)^2 dv_g$$

for all α . With a change of variable, rough estimates of the differential terms and Hölder's inequality, we then get

$$(29) \quad \begin{aligned} & \int_{B_0(R_0)} \left(\Delta^{k/2}(\eta \tilde{v}_\alpha)\right)^2 dx \leq C \sum_{l=0}^k \int_{B_{x_\alpha}(R_0 r_\alpha)} |\nabla^l u_\alpha|_g^2 |\nabla^{k-l} \eta_\alpha|_g^2 dv_g \\ & \leq C \sum_{l=0}^k \int_{B_{x_\alpha}(R_0 r_\alpha)} r_\alpha^{2(l-k)} |\nabla^l v_\alpha|_g^2 dv_g \leq C \sum_{l=0}^k \|\nabla^l v_\alpha\|_{\frac{2n}{n-2(k-l)}}^2 \end{aligned}$$

It follows from Sobolev's embedding theorem that $H_{k-l}^2(M) \subset L^{\frac{2n}{n-2(k-l)}}(M)$ for all $l = 0, \dots, k$ and that this embedding is continuous. Since $(v_\alpha)_\alpha$ is bounded in H_k^2 , then $(\nabla^l v_\alpha)_\alpha$ is uniformly bounded in H_{k-l}^2 (with tensorial values), and then there exists $C > 0$ such that

$$(30) \quad \|\nabla^l v_\alpha\|_{\frac{2n}{n-2(k-l)}} \leq C \|v_\alpha\|_{H_k^2} \leq C'$$

for all $\alpha > 0$ and $l = 0, \dots, k$. It then follows from (29) that $(\eta \tilde{v}_\alpha)_\alpha$ is bounded in $\mathcal{D}_k^2(\mathbb{R}^n)$. Therefore, up to a subsequence, there exists $v_\eta \in \mathcal{D}_k^2(\mathbb{R}^n)$ such that $\eta \tilde{v}_\alpha \rightharpoonup v_\eta$ weakly in $\mathcal{D}_k^2(\mathbb{R}^n)$ as $\alpha \rightarrow +\infty$. A classical diagonal argument then yields the existence $v \in H_{k,loc}^2(\mathbb{R}^n)$ such that $\eta \tilde{v}_\alpha \rightharpoonup \eta v$ weakly in $\mathcal{D}_k^2(\mathbb{R}^n)$ as $\alpha \rightarrow +\infty$. We fix $R > 0$. For any $R' > R$, a change of variables and (30) yields

$$\int_{B_0(R)} |\nabla^l \eta_{R'} \tilde{v}_\alpha|_{g_\alpha}^{\frac{2n}{n-2(k-l)}} dv_{g_\alpha} \leq \int_{B_{x_\alpha}(R_0 r_\alpha)} |\nabla^l v_\alpha|_g^{\frac{2n}{n-2(k-l)}} dv_g \leq C$$

where $g_\alpha := \exp_{x_\alpha}^* g(r_\alpha \cdot)$. Using weak convergence and convexity, letting $\alpha \rightarrow +\infty$ and then $R \rightarrow +\infty$ yields $|\nabla^l v| \in L^{\frac{2n}{n-2(k-l)}}(\mathbb{R}^n)$. As one checks, we then have that the sequence $(\eta_R v)_R$ is a Cauchy sequence in $\mathcal{D}_k^2(\mathbb{R}^n)$, and then we get that $v \in \mathcal{D}_k^2(\mathbb{R}^n)$. This ends the proof of the claim, and ends Step 4. \square

Step 5: We assume that (28) holds. We let $v \in \mathcal{D}_k^2(\mathbb{R}^n)$ as in Claim 3. We claim that $v \not\equiv 0$ is a weak solution to $\Delta^k v = |v|^{2_k^* - 2} v$ in $\mathcal{D}_k^2(\mathbb{R}^n)$.

Proof of the claim. We fix $R > 0$ and we apply Proposition 4.1 with $(N, g_\infty) := (\mathbb{R}^n, \text{Eucl})$ and $\Omega := \mathbb{R}^n$. As above, we define a family of smooth metrics $(g_\alpha)_\alpha$ such that $g_\alpha(x) := \exp_{x_\alpha}^* g(r_\alpha x)$ for $x \in B_0(3R)$, $g_\alpha(x) = \text{Eucl}$ for $x \in \mathbb{R}^n \setminus B_0(4R)$, and $g_\alpha \rightarrow \text{Eucl}$ in $C_{loc}^p(\mathbb{R}^n)$ as $\alpha \rightarrow +\infty$ for all p . Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\text{Supp } \varphi \subset B_0(R)$. We define

$$\varphi_\alpha(x) := r_\alpha^{-\frac{n-2k}{2}} \varphi\left(\frac{\exp_{x_\alpha}^{-1}(x)}{r_\alpha}\right)$$

for all $x \in M$. As one checks, φ_α is well-defined and has support in $B_{x_\alpha}(Rr_\alpha)$. Moreover, using the comparison Lemma 9.1 in Mazumdar [13] and arguing as in Step 4, we get that $\|\varphi_\alpha\|_{H_{k,0}^2(M)} \leq C(R)\|\varphi\|_{H_{k,0}^2(\mathbb{R}^n)}$ for all $\alpha > 0$. Since (u_α) is a Palais-Smale sequence, we have that

$$\langle DJ(v_\alpha), \varphi_\alpha \rangle = o(\|\varphi_\alpha\|_{H_{k,0}^2}) = o(\|\varphi\|_{H_{k,0}^2(\mathbb{R}^n)})$$

as $\alpha \rightarrow +\infty$ uniformly for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\text{Supp } \varphi \subset B_0(R)$. With a change of variable, we get $\langle DJ(v_\alpha), \varphi_\alpha \rangle = \langle DJ_\alpha(\eta_R \tilde{v}_\alpha), \varphi \rangle$ where

$$J_\alpha(u) := \frac{1}{2} \int_{\mathbb{R}^n} (\Delta_{g_\alpha}^{k/2} u)^2 dv_{g_\alpha} - \frac{1}{2_k^*} \int_{\mathbb{R}^n} |u|^{2_k^*} dv_{g_\alpha}$$

for all $u \in H_k^2(\mathbb{R}^n)$. Therefore, $\langle DJ_\alpha(\eta_R \tilde{v}_\alpha), \varphi \rangle = o(\|\varphi\|_{H_{k,0}^2(\mathbb{R}^n)})$ as $\alpha \rightarrow +\infty$ uniformly for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\text{Supp } \varphi \subset B_R(0)$.

We fix $x_0 \in \mathbb{R}^n$ such that $B_{x_0}(1/2) \subset B_0(R)$. A change of variable yields

$$\int_{B_{x_0}(1/2) \cap B_0(2R)} |\eta_R \tilde{v}_\alpha|^{2_k^*} dv_{g_\alpha} = \int_{\exp_{x_\alpha}(r_\alpha B_{x_0}(1/2))} |u_\alpha|^{2_k^*} dv_g.$$

For $\alpha > 0$ large enough, we have that $\exp_{x_\alpha}(r_\alpha B_{x_0}(1/2)) \subset B_{\exp_{x_\alpha}(x_0)}(r_\alpha)$. Therefore, it follows from the definition of μ_α that

$$\int_{B_{x_0}(1/2) \cap B_0(2R)} |\eta_R \tilde{v}_\alpha|^{2_k^*} dv_{g_\alpha} \leq \mu_\alpha(r_\alpha) = \lambda < \epsilon_0$$

for all α large enough and $x_0 \in \mathbb{R}^n$ such that $1/2 + |x_0| < R$. With the Sobolev inequality (5) on \mathbb{R}^n , we apply Proposition 4.1 to $(\eta_R \tilde{v}_\alpha)_\alpha$, and we get that

$$\lim_{\alpha \rightarrow +\infty} \eta_R \tilde{v}_\alpha = \eta_R v \text{ strongly in } H_k^2(B_{x_0}(1/4)).$$

Using a finite covering, we then have $\tilde{v}_\alpha \rightarrow v$ strongly in $H_k^2(B_0(R/2))$ as $\alpha \rightarrow +\infty$. Sobolev's embedding theorem yield the convergence in $L^{2_k^*}(B_0(1))$. Since

$$\int_{B_0(1)} |\tilde{v}_\alpha|^{2_k^*} dv_{g_\alpha} = \int_{B_{x_\alpha}(r_\alpha)} |v_\alpha|^{2_k^*} dv_g = \mu_\alpha(r_\alpha) = \lambda > 0,$$

passing to the limit $\alpha \rightarrow +\infty$ yields $\int_{B_0(1)} |v|^{2_k^*} dx = \lambda \neq 0$, and therefore $v \not\equiv 0$. This proves the claim and ends Step 5. \square

Note that indeed, we have proved that

$$(31) \quad \lim_{\alpha \rightarrow +\infty} \tilde{v}_\alpha = v \text{ strongly in } H_k^2(B_0(R)) \text{ for all } R > 0.$$

We choose a sequence (\tilde{r}_α) of positive real numbers as in (8) with $\eta \in C_c^\infty(B_0(\delta))$ (with $\delta \in (0, i_g(\tilde{M}))$) identically 1 around 0. As in Definition 2.1, we set

$$V_\alpha(x) := B_{x_\alpha, r_\alpha}(v) := \eta\left(\frac{\exp_{x_\alpha}^{-1}(x)}{\tilde{r}_\alpha}\right) r_\alpha^{-\frac{n-2k}{2}} v\left(\frac{\exp_{x_\alpha}^{-1}(x)}{r_\alpha}\right)$$

We have that $V_\alpha \in H_{k,0}^2(M)$.

Step 6: We claim that

$$(32) \quad V_\alpha \rightarrow 0 \text{ in } H_{k,0}^2(M) \text{ as } \alpha \rightarrow +\infty.$$

Proof of the claim. We argue essentially as in [13]. We fix $0 \leq l \leq k$ and we define $\epsilon_\alpha := r_\alpha/\tilde{r}_\alpha$ such that $\lim_{\alpha \rightarrow +\infty} \epsilon_\alpha = 0$. We fix $R \geq 0$ (potentially 0). It follows from the comparison Lemma 9.1 of [13] that there exists $C > 0$ such that

$$\begin{aligned} \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} (\Delta_g^{l/2} V_\alpha)^2 dv_g &\leq C \int_{B_0(\delta\tilde{r}_\alpha) \setminus B_0(Rr_\alpha)} (\Delta^{l/2}(V_\alpha \circ \exp_{x_\alpha}))^2 dx \\ &\leq Cr_\alpha^{2(k-l)} \int_{B_0(\delta\epsilon_\alpha^{-1}) \setminus B_0(R)} \left(\Delta^{l/2}(\eta(\epsilon_\alpha \cdot)v)\right)^2 dx \\ &\leq Cr_\alpha^{2(k-l)} \int_{B_0(\delta\epsilon_\alpha^{-1}) \setminus B_0(R)} |\nabla^l(\eta(\epsilon_\alpha \cdot)v)|^2 dx \\ &\leq Cr_\alpha^{2(k-l)} \sum_{i=0}^l \int_{\mathbb{R}^n \setminus B_0(R)} |\nabla^{l-i}\eta(\epsilon_\alpha \cdot)| |\nabla^i v|^2 dx \\ &\leq Cr_\alpha^{2(k-l)} \sum_{i=0}^l \int_{\mathbb{R}^n \setminus B_0(R)} \epsilon_\alpha^{2(l-i)} |\nabla^i v|^2 dx \end{aligned}$$

Since $v \in \mathcal{D}_k^2(\mathbb{R}^n)$, we have that $\nabla^i v \in \mathcal{D}_{k-i}^2(\mathbb{R}^n)$, and therefore $|\nabla^i v| \in L^{2^\sharp(k-i)}(\mathbb{R}^n)$ where $2^\sharp(k-i) := \frac{2n}{n-2(k-i)}$. Therefore, Hölder's inequality yields

$$(33) \quad \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} (\Delta_g^{l/2} V_\alpha)^2 dv_g \leq C\tilde{r}_\alpha^{2(k-l)} \sum_{i=0}^l \left(\int_{\mathbb{R}^n \setminus B_0(R)} |\nabla^i v|^{2^\sharp(k-i)} dx \right)^{\frac{2}{2^\sharp(k-i)}}$$

Taking $R = 0$ and $l = 0, \dots, k$ yields the boundedness of $(V_\alpha)_\alpha$ in $H_{k,0}^2(M)$.

Arguing as in above, we get that for any $R > 0$ and any $l = 0, \dots, k$, we have that

$$(34) \quad \int_{B_{x_\alpha}(Rr_\alpha)} (\Delta_g^{l/2} V_\alpha)^2 dv_g \leq Cr_\alpha^{2(k-l)} \sum_{i=0}^l \int_{B_0(R)} \epsilon_\alpha^{2(l-i)} |\nabla^i v|^2 dx$$

Since $\nabla^i v \in L_{loc}^2(\mathbb{R}^n)$ for all $i = 0, \dots, k$, then taking $l = 0$ in (33) and (34), letting $\alpha \rightarrow +\infty$ and then $R \rightarrow +\infty$ yields $V_\alpha \rightarrow 0$ in $L^2(M)$. Then the weak compactness of bounded sequences yields (32). This proves the claim and ends Step 6. \square

Step 7: We claim that

$$(35) \quad DJ(V_\alpha) \rightarrow 0 \text{ strongly as } \alpha \rightarrow +\infty$$

Proof of the claim. We set $\varphi \in C_c^\infty(M)$. We have that

$$\langle DJ(V_\alpha), \varphi \rangle = \int_M \Delta_g^{k/2} V_\alpha \Delta_g^{k/2} \varphi \, dv_g - \int_M |V_\alpha|^{2_k^\sharp - 2} V_\alpha \varphi \, dv_g$$

We fix $R > 0$ and we define

$$I_{R,\alpha}(\varphi) := \int_{B_{x_\alpha}(Rr_\alpha)} \Delta_g^{k/2} V_\alpha \Delta_g^{k/2} \varphi \, dv_g - \int_{B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\sharp - 2} V_\alpha \varphi \, dv_g$$

and

$$II_{R,\alpha}(\varphi) := \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} \Delta_g^{k/2} V_\alpha \Delta_g^{k/2} \varphi \, dv_g - \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\sharp - 2} V_\alpha \varphi \, dv_g.$$

Step 7.1: we estimate $II_{R,\alpha}(\varphi)$. Via Hölder's and Sobolev inequality, we have that

$$(36) \quad |II_{R,\alpha}(\varphi)| \leq \left(\int_{D_\alpha(R)} (\Delta_g^{k/2} V_\alpha)^2 \, dv_g \right)^{\frac{1}{2}} \times \|\Delta_g^{k/2} \varphi\|_2 \\ + \left(\int_{D_\alpha(R)} |V_\alpha|^{2_k^\sharp} \, dv_g \right)^{\frac{2_k^\sharp - 1}{2_k^\sharp}} \times \|\varphi\|_{2_k^\sharp} \\ \leq \left(\left(\int_{D_\alpha(R)} (\Delta_g^{k/2} V_\alpha)^2 \, dv_g \right)^{\frac{1}{2}} + \left(\int_{D_\alpha(R)} |V_\alpha|^{2_k^\sharp} \, dv_g \right)^{\frac{2_k^\sharp - 1}{2_k^\sharp}} \right) \cdot \|\varphi\|_{H_k^2}$$

with $D_\alpha(R) := M \setminus B_{x_\alpha}(Rr_\alpha)$. Lemma 9.1 in [13] and $v \in L^{2_k^\sharp}(\mathbb{R}^n)$ yield

$$(37) \quad \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\sharp} \, dv_g \leq C \int_{\mathbb{R}^n \setminus B_0(Rr_\alpha)} |V_\alpha \circ \exp_{x_\alpha}|^{2_k^\sharp} \, dx \leq C \int_{\mathbb{R}^n \setminus B_0(R)} |v|^{2_k^\sharp} \, dx$$

Plugging (33) with $l = k$ and (37) into (36), letting $R \rightarrow +\infty$ and $\alpha \rightarrow +\infty$ yields

$$(38) \quad \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \frac{II_{R,\alpha}(\varphi)}{\|\varphi\|_{H_k^2}} = 0 \text{ uniformly wrt } \varphi \in H_{k,0}^2(M) \setminus \{0\}$$

Step 7.2: We now estimate $I_{R,\alpha}(\varphi)$. We define

$$\overline{\varphi}_\alpha(x) = \eta(\epsilon_\alpha x) r_\alpha^{\frac{n-2k}{2}} \varphi(\exp_{x_\alpha}(r_\alpha x))$$

where $\epsilon_\alpha := r_\alpha / \tilde{r}_\alpha$. As one checks, $\overline{\varphi}_\alpha \in C_c^\infty(\mathbb{R}^n)$. Using the comparison Lemma 9.1 in [13] and arguing as in (33)-(34), we get that

$$\|\overline{\varphi}_\alpha\|_{\mathcal{D}_k^2(\mathbb{R}^n)} \leq C \|\varphi\|_{H_k^2}$$

where $C > 0$ is independent of φ . As one checks,

$$I_{R,\alpha}(\varphi) = \int_{B_0(R)} \Delta_{g_\alpha}^{k/2} v \Delta_{g_\alpha}^{k/2} \overline{\varphi}_\alpha \, dv_{g_\alpha} - \int_{B_0(R)} |v|^{2_k^\sharp - 2} v \overline{\varphi}_\alpha \, dv_{g_\alpha}$$

Since $g_\alpha \rightarrow \text{Eucl}$ as $\alpha \rightarrow +\infty$ in $C_{loc}^p(\mathbb{R}^n)$ for all $p \geq 1$, we get

$$(39) \quad I_{R,\alpha}(\varphi) = \int_{B_0(R)} \Delta^{k/2} v \Delta^{k/2} \overline{\varphi}_\alpha \, dx - \int_{B_0(R)} |v|^{2_k^\sharp - 2} v \overline{\varphi}_\alpha \, dx + o\left(\|\overline{\varphi}_\alpha\|_{\mathcal{D}_k^2(\mathbb{R}^n)}\right)$$

where the convergence is uniform wrt $\bar{\varphi}_\alpha$. Since v is a weak solution to (1), then (39) yields

$$(40) \quad \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \frac{I_{R,\alpha}(\varphi)}{\|\varphi\|_{H_k^2}} = 0 \text{ uniformly wrt } \varphi \in H_{k,0}^2(M) \setminus \{0\}$$

The limits (38) and (40) yield $\langle DJ(V_\alpha), \varphi \rangle = o(\|\varphi\|_{H_k^2})$ as $\alpha \rightarrow +\infty$ uniformly wrt $\varphi \in C_c^\infty(M)$. The boundedness of (V_α) in $H_{k,0}^2(M)$ then yields $DJ(V_\alpha) \rightarrow 0$ strongly in $(H_{k,0}^2(M))'$ as $\alpha \rightarrow +\infty$. This proves (35) and ends Step 7. \square

We define $w_\alpha := v_\alpha - V_\alpha$. It follows from (32) that $w_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$.

Step 8: We claim that

$$(41) \quad DJ(w_\alpha) \longrightarrow 0 \text{ strongly}$$

Proof of the claim. For $\varphi \in H_{k,0}^2(M)$, we write

$$(42) \quad \langle DJ(w_\alpha), \varphi \rangle = \langle DJ(v_\alpha), \varphi \rangle - \langle DJ(V_\alpha), \varphi \rangle - \int_M \Phi_\alpha \varphi \, dv_g$$

where $\Phi_\alpha := |w_\alpha|^{2_k^\#-2} w_\alpha - |v_\alpha|^{2_k^\#-2} v_\alpha + |V_\alpha|^{2_k^\#-2} V_\alpha$. Then by applying the Hölder and Sobolev inequalities we get

$$\left| \int_M \Phi_\alpha \varphi \, dv_g \right| \leq C \|\varphi\|_{H_k^2} \|\Phi_\alpha\|_{2_k^\#/(2_k^\#-1)}$$

Step 8.1: We fix $R > 0$. Inequality (15) and Hölder's inequality yield

$$\begin{aligned} & \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2_k^\#/(2_k^\#-1)} \, dv_g \\ & \leq C \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} \left(|v_\alpha|^{2_k^\#-2} |V_\alpha| + |V_\alpha|^{2_k^\#-2} |v_\alpha| \right)^{2_k^\#/(2_k^\#-1)} \, dv_g \\ & \leq C \left(\int_M |v_\alpha|^{2_k^\#} \, dv_g \right)^{\frac{2_k^\#-2}{2_k^\#-1}} \left(\int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\#} \, dv_g \right)^{\frac{1}{2_k^\#-1}} \\ & \quad + C \left(\int_M |v_\alpha|^{2_k^\#} \, dv_g \right)^{\frac{1}{2_k^\#-1}} \left(\int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\#} \, dv_g \right)^{\frac{2_k^\#-2}{2_k^\#-1}} \end{aligned}$$

Since (v_α) is uniformly bounded in $H_k^2(M)$, then (37) yields

$$(43) \quad \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2_k^\#/(2_k^\#-1)} \, dv_g = 0.$$

This ends Step 8.1.

Step 8.2: We fix $R > 0$. A change of variable and inequality (15) yield

$$\begin{aligned} & \int_{B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2_k^\sharp/(2_k^\sharp-1)} dv_g \\ &= \int_{B_0(R)} \left| |\tilde{v}_\alpha - v|^{2_k^\sharp-2} (\tilde{v}_\alpha - v) - |\tilde{v}_\alpha|^{2_k^\sharp-2} \tilde{v}_\alpha + |v|^{2_k^\sharp-2} v \right|^{2_k^\sharp/(2_k^\sharp-1)} dv_{g_\alpha} \\ &\leq C \int_{B_0(R)} \left(|\tilde{v}_\alpha - v|^{\frac{(2_k^\sharp-2)2_k^\sharp}{2_k^\sharp-1}} |v|^{\frac{2_k^\sharp}{2_k^\sharp-1}} + |v|^{\frac{(2_k^\sharp-2)2_k^\sharp}{2_k^\sharp-1}} |\tilde{v}_\alpha - v|^{\frac{2_k^\sharp}{2_k^\sharp-1}} \right) dx \end{aligned}$$

For any $\eta \in C_c^\infty(\mathbb{R}^n)$, we have that $\eta\tilde{v}_\alpha \rightharpoonup \eta v$ weakly in $\mathcal{D}_k^2(\mathbb{R}^n)$. Therefore, up to extracting a subsequence, $(\tilde{v}_\alpha)_\alpha$ is uniformly bounded in $L^{2_k^\sharp}(B_0(R))$ and goes to v almost everywhere as $\alpha \rightarrow +\infty$. Therefore Lemma 3.1 yields that for any $R > 0$,

$$(44) \quad \lim_{\alpha \rightarrow +\infty} \int_{B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2_k^\sharp/(2_k^\sharp-1)} dv_g = 0.$$

The limits (43)-(44) yield $\|\Phi_\alpha\|_{2_k^\sharp/(2_k^\sharp-1)} \rightarrow 0$ as $\alpha \rightarrow +\infty$. Then by (42) we get $DJ(w_\alpha) \rightarrow 0$ in $(H_{k,0}^2(M))'$ as $\alpha \rightarrow +\infty$. This proves (41) and ends Step 8. \square

Step 9: We claim that we have the following decomposition of energy.

$$(45) \quad J(w_\alpha) = J(v_\alpha) - E(v) + o(1) \text{ where } o(1) \rightarrow 0 \text{ as } \alpha \rightarrow +\infty.$$

Proof of the claim. As one checks,

$$\begin{aligned} & J(v_\alpha) - J(w_\alpha) - J(V_\alpha) = \langle DJ(w_\alpha), V_\alpha \rangle \\ & - \frac{1}{2_k^\sharp} \int_M \left(|w_\alpha + V_\alpha|^{2_k^\sharp} - |w_\alpha|^{2_k^\sharp} - 2_k^\sharp |w_\alpha|^{2_k^\sharp-2} w_\alpha V_\alpha - |V_\alpha|^{2_k^\sharp} \right) dv_g \end{aligned}$$

We fix $R > 0$. Arguing as in the proof of (44), we get that

$$\lim_{\alpha \rightarrow +\infty} \int_{B_{x_\alpha}(Rr_\alpha)} \left(|w_\alpha + V_\alpha|^{2_k^\sharp} - |w_\alpha|^{2_k^\sharp} - 2_k^\sharp |w_\alpha|^{2_k^\sharp-2} w_\alpha V_\alpha - |V_\alpha|^{2_k^\sharp} \right) dv_g = 0.$$

As one checks, there exists $C > 0$ such that

$$\left| |a+b|^{2_k^\sharp} - |a|^{2_k^\sharp} - 2_k^\sharp |a|^{2_k^\sharp-2} ab - |b|^{2_k^\sharp} \right| \leq C \left(|a|^{2_k^\sharp-2} |b|^2 + |a| \cdot |b|^{2_k^\sharp-1} \right)$$

for all $a, b \in \mathbb{R}$. As in the proof of (43), we get that

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{D_\alpha(R)} \left(|w_\alpha + V_\alpha|^{2_k^\sharp} - |w_\alpha|^{2_k^\sharp} - 2_k^\sharp |w_\alpha|^{2_k^\sharp-2} w_\alpha V_\alpha - |V_\alpha|^{2_k^\sharp} \right) dv_g = 0,$$

where $D_\alpha(R) := M \setminus B_{x_\alpha}(Rr_\alpha)$. These yield $J(v_\alpha) = J(w_\alpha) + J(V_\alpha) + o(1)$.

We now estimate $J(V_\alpha)$. The estimates (33) and (37) yield

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} \left((\Delta_g^{k/2} V_\alpha)^2 + |V_\alpha|^{2_k^\sharp} \right) dv_g = 0$$

For $R > 0$, we have that

$$\int_{B_{x_\alpha}(Rr_\alpha)} \left(\frac{(\Delta_g^{k/2} V_\alpha)^2}{2} - \frac{|V_\alpha|^{2_k^\sharp}}{2_k^\sharp} \right) dv_g = \int_{B_0(R)} \left(\frac{(\Delta_{g_\alpha}^{k/2} v)^2}{2} - \frac{|v|^{2_k^\sharp}}{2_k^\sharp} \right) dv_{g_\alpha}$$

Since $g_\alpha \rightarrow \text{Eucl}$ locally uniformly in C^p for all p and $v \in \mathcal{D}_k^2(\mathbb{R}^n)$, we get that

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{B_{x_\alpha}(Rr_\alpha)} \left(\frac{(\Delta_g^{k/2} V_\alpha)^2}{2} - \frac{|V_\alpha|^{2_k^\sharp}}{2_k^\sharp} \right) dv_g = \int_{\mathbb{R}^n} \left(\frac{(\Delta^{k/2} v)^2}{2} - \frac{|v|^{2_k^\sharp}}{2_k^\sharp} \right) dx$$

All these estimates yield (45). This ends Step 9. \square

Step 10: Next we deal with the case

$$d_g(x_\alpha, \partial M) = O(r_\alpha) \quad \text{as } \alpha \rightarrow +\infty$$

Since $r_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$, then there exists $x_\infty \in \partial M$ such that $x_\alpha \rightarrow x_\infty$ as $\alpha \rightarrow +\infty$. For any $\alpha \in \mathbb{N}$, we let $z_\alpha \in \partial M$ be such that

$$d_g(x_\alpha, z_\alpha) = d_g(x_\alpha, \partial M)$$

In particular, $\lim_{\alpha \rightarrow +\infty} z_\alpha = x_\infty$. We choose a family of charts $z \mapsto \mathcal{T}_z$ for $z \in \Omega \cap \partial M$ as in (7). Since the $d(\mathcal{T}_z)_0$ is an isometry, there exists $\mathcal{C}_1, \mathcal{C}_2 > 0$, $\tau_1, \tau_2 > 0$ such that for any $z \in \Omega \cap \partial M$, $r < \tau_1$ and $y \in \mathbb{R}_-^n \cap B_0(\tau_2)$, one has

$$B_{\mathcal{T}_z(y)}(\mathcal{C}_1 r) \cap M \subset \mathcal{T}_z(B_y(r) \cap \mathbb{R}_-^n) \subset B_{\mathcal{T}_z(y)}(\mathcal{C}_2 r) \cap M$$

For $x \in r_\alpha^{-1}U \cap \{x_1 < 0\}$, we define

$$\tilde{v}_\alpha(x) := r_\alpha^{\frac{n-2k}{2}} v_\alpha \circ \mathcal{T}_{z_\alpha}(r_\alpha x) \quad \text{and} \quad \tilde{g}_\alpha(x) := \mathcal{T}_{z_\alpha}^* g(r_\alpha x)$$

As one checks, for any $\eta \in C_c^\infty(\mathbb{R}^n)$, we have that $\eta \tilde{v}_\alpha \in \mathcal{D}_k^2(\mathbb{R}_-^n)$. Arguing as Step 4, we get that there exists $v \in \mathcal{D}_k^2(\mathbb{R}_-^n)$ such that

$$\eta \tilde{v}_\alpha \rightharpoonup \eta v \quad \text{weakly in } \mathcal{D}_k^2(\mathbb{R}_-^n) \quad \text{as } \alpha \rightarrow +\infty.$$

Moreover, using Proposition 4.1 and arguing as in Step 5, we get that $v \not\equiv 0$ is a weak solution to (3) and $\tilde{v}_\alpha \rightarrow v$ as $\alpha \rightarrow +\infty$ strongly in $H_k^2(B_0(R) \cap \mathbb{R}_-^n)$ for all $R > 0$. As in Definition 2.1, for $\alpha \in \mathbb{N}$ and $x \in \overline{M}$, we set

$$V_\alpha(x) := B_{z_\alpha, r_\alpha}(v)(x) = \eta(\mathcal{T}_{z_\alpha}^{-1}(x)) r_\alpha^{-\frac{n-2k}{2}} v(r_\alpha^{-1} \mathcal{T}_{z_\alpha}^{-1}(x))$$

We define $w_\alpha := v_\alpha - V_\alpha$. Arguing as in Steps 6 to 9, we get that

- $w_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$
- $DJ(w_\alpha) \rightarrow 0$ weakly in $(H_{k,0}^2(M))'$
- $J(w_\alpha) = J(v_\alpha) - E(v) + o(1)$

as $\alpha \rightarrow +\infty$. This completes the proof of Lemma 4.1.

5. NONNEGATIVE PALAIS-SMALE SEQUENCES

To prove Theorem 1.2, we first set the following property:

Proposition 5.1. *Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. Let $d \in \mathbb{N}$ and d bubbles $[(x_\alpha^{(j)}), (r_\alpha^{(j)}), u^{(j)}]$, $j = 1, \dots, d$, be as in Theorem 1.1. Then, for any $N \in \{1, \dots, d\}$, there exists $L \geq 0$ sequences $(y_\alpha^j)_{\alpha > 0} \in \overline{M}$ and $(\lambda_\alpha^j)_{\alpha > 0} \in (0, +\infty)$, $j = 1, \dots, L$, such that for any $R > 0$*

$$\lim_{R' \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{(B_{x_\alpha^N}(Rr_\alpha^N) \setminus \bigcup_{j=1}^L B_{y_\alpha^j}(R'\lambda_\alpha^j)) \cap M} |u_\alpha - B_{x_\alpha^{(N)}, r_\alpha^{(N)}}(u^{(N)})|^{2_k^\sharp} dv_g = 0$$

where for any j , $j = 1, \dots, L$, $d_g(x_\alpha^N, y_\alpha^j) = o(r_\alpha^N)$ and $\lambda_\alpha^j = o(r_\alpha^N)$ as $\alpha \rightarrow +\infty$. Moreover, we have that

$$\lim_{\alpha \rightarrow +\infty} \frac{d_g(x_\alpha^i, x_\alpha^j)^2}{r_\alpha^i r_\alpha^j} + \frac{r_\alpha^i}{r_\alpha^j} + \frac{r_\alpha^j}{r_\alpha^i} = +\infty \text{ for all } i \neq j \in \{1, \dots, d\}.$$

We omit the proof that goes exactly as in Hebey-Robert [11], by using the boundary chart (7) for bubbles accumulating on the boundary.

We now prove Theorem 1.2. We let $(u_\alpha)_\alpha$ be as in the statement of the theorem, and we let $[(x_\alpha^{(j)}), (r_\alpha^{(j)}), u^{(j)}]$, $j = 1, \dots, d$, be the associated bubbles. We fix $N \in \{1, \dots, d\}$. For simplicity, we define $r_\alpha := r_\alpha^{(N)}$ and $x_\alpha := x_\alpha^{(N)}$. We assume that $r_\alpha^{-1}d(x_\alpha, \partial M) \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. It then follows from Proposition 5.1 that there exists a finite set $\mathcal{S} \subset \mathbb{R}^n$ such that $\lim_{\alpha \rightarrow +\infty} \tilde{v}_\alpha = u^N$ strongly in $L_{loc}^{2_k^\sharp}(\mathbb{R}^n \setminus \mathcal{S})$ where $\tilde{v}_\alpha(x) := r_\alpha^{\frac{n-2k}{2}} u_\alpha(\exp_{x_\alpha}(r_\alpha x))$ for $x \in \mathbb{R}^n$. Up to extracting, the convergence holds a.e. Since $u_\alpha \geq 0$, we then get that $u^N \geq 0$. It then follows from Lemma 4 in Ge-Wei-Zhou [7] that there exists $\lambda > 0$ and $a \in \mathbb{R}^n$ such that $u^N = U_{\lambda, a}$ is of the form (6).

We claim that $u^N = U_{\lambda, 0}$, that is $a = 0$. We prove the claim. Indeed, rescaling (26) and (27) yields

$$\int_{r_\alpha^{-1} \exp_{x_\alpha}^{-1}(B \exp_{x_\alpha}(r_\alpha x))} |\tilde{v}_\alpha|^{2_k^\sharp} dv_{g_\alpha} \leq \int_{B_0(1)} |\tilde{v}_\alpha|^{2_k^\sharp} dv_{g_\alpha}$$

for all $z \in \mathbb{R}^n$ and α large enough. Since the exponential is a normal chart and isometric at x_α , we get that for all $z \in \mathbb{R}^n$ and all $\epsilon > 0$

$$\exp_{x_\alpha}(r_\alpha B_z(1 - \epsilon)) \subset B \exp_{x_\alpha}(r_\alpha z).$$

Plugging these two inequalities together, letting $\alpha \rightarrow +\infty$, using the strong convergence (31), we get that $\int_{B_z(1-\epsilon)} |u^N|^{2_k^\sharp} dx \leq \int_{B_0(1)} |u^N|^{2_k^\sharp} dx$. Letting $\epsilon \rightarrow 0$ yields

$$\int_{B_z(1)} |u^N|^{2_k^\sharp} dx \leq \int_{B_0(1)} |u^N|^{2_k^\sharp} dx.$$

As one checks, since $u^N = U_{\lambda, a}$ is as in (6), the maximum of the left-hand-side is achieved if and only if $z = a$. Therefore $a = 0$ and $u^N = U_{\lambda, 0}$. This proves the claim.

As a consequence, as one checks, when $r_\alpha^{-1}d(x_\alpha, \partial M) \rightarrow +\infty$ as $\alpha \rightarrow +\infty$, the bubble rewrites

$$B_{x_\alpha, r_\alpha}(u^N) = B_{x_\alpha, \lambda r_\alpha}(U_{1, 0}) = \eta \left(\frac{\exp_{x_\alpha}^{-1}(\cdot)}{\tilde{r}_\alpha} \right) \alpha_{n, k} \left(\frac{\lambda r_\alpha}{\lambda^2 r_\alpha^2 + d_g(\cdot, x_\alpha)^2} \right)^{\frac{n-2k}{2}}.$$

We fix $N \in \{1, \dots, d\}$. We claim that $(r_\alpha^N)^{-1}d(x_\alpha^N, \partial M) \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. We argue by contradiction and we assume that the limit is finite. We argue as in the case above. Up to rescaling, and using the boundary chart (7), we get that u_α goes to u^N strongly as $\alpha \rightarrow +\infty$ in $L_{loc}^{2_k^\sharp}(\mathbb{R}^n \setminus \mathcal{S})$, where \mathcal{S} is finite. Therefore u^N is a nonnegative nonzero weak solution to (3), contradicting Lemma 3 in Ge-Wei-Zhou [7]. Therefore the limit is infinite and we are back to the previous case.

All these steps prove Theorem 1.2.

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