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# Approximate Capacity of the Gaussian Interference Channel with Noisy Channel-Output Feedback

Victor Quintero, Samir M. Perlaza, Iñaki Esnaola and Jean-Marie Gorce

**Abstract**—In this paper, an achievability region and a converse region for the two-user Gaussian interference channel with noisy channel-output feedback (G-IC-NOF) are presented. The achievability region is obtained using a random coding argument and three well-known techniques: rate splitting, superposition coding and backward decoding. The converse region is obtained using some of the existing perfect-output feedback outer-bounds as well as a set of new outer-bounds that are obtained by using genie-aided models of the original G-IC-NOF. Finally, it is shown that the achievability region and the converse region approximate the capacity region of the G-IC-NOF to within a constant gap in bits per channel use.

**Index Terms**—Capacity, Interference Channel, Noisy Channel-Output Feedback.

## I. NOTATION

Throughout this paper,  $(\cdot)^+$  denotes the positive part operator, i.e.,  $(\cdot)^+ = \max(\cdot, 0)$  and  $\mathbb{E}_X[\cdot]$  denotes the expectation with respect to the distribution of the random variable  $X$ . The logarithm function  $\log$  is assumed to be base 2.

## II. SYSTEM MODEL

Consider the two-user G-IC-NOF in Fig. 1. Transmitter  $i$ , with  $i \in \{1, 2\}$ , communicates with receiver  $i$  subject to the interference produced by transmitter  $j$ , with  $j \in \{1, 2\} \setminus \{i\}$ . There are two independent and uniformly distributed messages,  $W_i \in \mathcal{W}_i$ , with  $\mathcal{W}_i = \{1, 2, \dots, 2^{NR_i}\}$ , where  $N$  denotes the block-length in channel uses and  $R_i$  is the transmission rate in bits per channel use. At each block, transmitter  $i$  sends the codeword  $\mathbf{X}_i = (X_{i,1}, X_{i,2}, \dots, X_{i,N})^T \in \mathcal{X}_i^N$ , where  $\mathcal{X}_i$  and  $\mathcal{X}_i^N$  are respectively the channel-input alphabet and the codebook of transmitter  $i$ .

The channel coefficient from transmitter  $j$  to receiver  $i$  is denoted by  $h_{ij}$ ; the channel coefficient from transmitter  $i$  to receiver  $i$  is denoted by  $\bar{h}_{ii}$ ; and the channel coefficient from channel-output  $i$  to transmitter  $i$  is denoted by  $\bar{h}_{ii}$ . All channel coefficients are assumed to be non-negative real numbers. At a given channel use  $n \in \{1, 2, \dots, N\}$ , the channel output at receiver  $i$  is denoted by  $\bar{Y}_{i,n}$ . During channel use  $n$ , the input-output relation of the channel model is given by:

$$\bar{Y}_{i,n} = \bar{h}_{ii} X_{i,n} + h_{ij} X_{j,n} + \bar{Z}_{i,n}, \quad (1)$$

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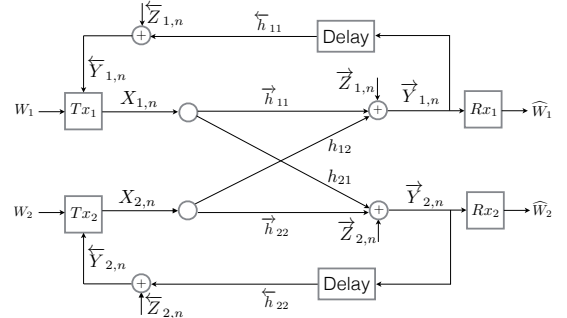


Fig. 1. Gaussian interference channel with noisy channel-output feedback at channel use  $n$ .

where  $\bar{Z}_{i,n}$  is a real Gaussian random variable with zero mean and unit variance that represents the noise at the input of receiver  $i$ . Let  $d > 0$  be the finite feedback delay in channel uses. At the end of channel use  $n$ , transmitter  $i$  observes  $\bar{Y}_{i,n}$ , which consists of a scaled and noisy version of  $\bar{Y}_{i,n-d}$ . More specifically,

$$\bar{Y}_{i,n} = \begin{cases} \bar{Z}_{i,n}, & \text{for } n \in \{1, 2, \dots, d\} \\ \bar{h}_{ii} \bar{Y}_{i,n-d} + \bar{Z}_{i,n}, & \text{for } n \in \{d+1, d+2, \dots, N\}, \end{cases} \quad (2)$$

where  $\bar{Z}_{i,n}$  is a real Gaussian random variable with zero mean and unit variance that represents the noise in the feedback link of transmitter-receiver pair  $i$ . The random variables  $\bar{Z}_{i,n}$  and  $\bar{Z}_{i,n}$  are independent and identically distributed. In the following, without loss of generality, the feedback delay is assumed to be one channel use ( $d = 1$ ). The encoder of transmitter  $i$  is defined by a set of deterministic functions  $f_i^{(1)}, \dots, f_i^{(N)}$ , with  $f_i^{(1)} : \mathcal{W}_i \rightarrow \mathcal{X}_i$  and for all  $n \in \{2, \dots, N\}$ ,  $f_i^{(n)} : \mathcal{W}_i \times \mathbb{R}^{n-1} \rightarrow \mathcal{X}_i$ , such that

$$X_{i,1} = f_i^{(1)}(W_i), \quad \text{and} \quad (3a)$$

$$X_{i,n} = f_i^{(n)}(W_i, \bar{Y}_{i,1}, \dots, \bar{Y}_{i,n-1}). \quad (3b)$$

The components of the input vector  $\mathbf{X}_i$  are real numbers subject to an average power constraint:

$$\frac{1}{N} \sum_{n=1}^N \mathbb{E}(X_{i,n}^2) \leq 1, \quad (4)$$

where the expectation is taken over the joint distribution of the message indexes  $W_1, W_2$ , and the noise terms, i.e.,  $\bar{Z}_{1,1}, \bar{Z}_{2,1}, \bar{Z}_{1,2},$  and  $\bar{Z}_{2,2}$ . The dependence of  $X_{i,n}$  on  $W_1, W_2$ , and the previously observed noise realizations is due to the effect of feedback as shown in (2) and (3).

Assume that during a given communication,  $T$  blocks are transmitted. Hence, the decoder of receiver  $i$  is defined by a deterministic function  $\psi_i : \mathbb{R}_i^{NT} \rightarrow \mathcal{W}_i^T$ . At the end of the communication, receiver  $i$  uses the sequence  $(\vec{Y}_{i,1}, \vec{Y}_{i,2}, \dots, \vec{Y}_{i,NT})$  to obtain an estimation of the message indices

$$(\widehat{W}_i^{(1)}, \widehat{W}_i^{(2)}, \dots, \widehat{W}_i^{(T)}) = \psi_i(\vec{Y}_{i,1}, \vec{Y}_{i,2}, \dots, \vec{Y}_{i,NT}), \quad (5)$$

where  $\widehat{W}_i^{(t)}$  is an estimation of the message index sent during block  $t \in \{1, 2, \dots, T\}$ . The decoding error probability in the two-user G-IC-NOF during block  $t$ , denoted by  $P_e^{(t)}(N)$ , is given by

$$P_e^{(t)}(N) = \max\left(\Pr\left[\widehat{W}_1^{(t)} \neq W_1^{(t)}\right], \Pr\left[\widehat{W}_2^{(t)} \neq W_2^{(t)}\right]\right). \quad (6)$$

The definition of an achievable rate pair  $(R_1, R_2) \in \mathbb{R}_+^2$  is given below.

*Definition 1 (Achievable Rate Pairs):* A rate pair  $(R_1, R_2) \in \mathbb{R}_+^2$  is achievable if there exists at least one pair of codebooks  $\mathcal{X}_1^N$  and  $\mathcal{X}_2^N$  with codewords of length  $N$ , and the corresponding encoding functions  $f_1^{(1)}, \dots, f_1^{(N)}$  and  $f_2^{(1)}, \dots, f_2^{(N)}$  such that the decoding error probability  $P_e^{(t)}(N)$  can be made arbitrarily small by letting the block-length  $N$  grow to infinity, for all blocks  $t \in \{1, \dots, T\}$ .

The two-user G-IC-NOF in Fig. 1 can be fully described by six parameters:  $\overrightarrow{\text{SNR}}_i$ ,  $\overleftarrow{\text{SNR}}_i$ , and  $\text{INR}_{ij}$ , with  $i \in \{1, 2\}$  and  $j \in \{1, 2\} \setminus \{i\}$ , which are defined as follows

$$\overrightarrow{\text{SNR}}_i = \vec{h}_{ii}^2, \quad (7)$$

$$\text{INR}_{ij} = h_{ij}^2 \text{ and} \quad (8)$$

$$\overleftarrow{\text{SNR}}_i = \overleftarrow{h}_{ii}^2 (\vec{h}_{ii}^2 + 2\vec{h}_{ii}h_{ij} + h_{ij}^2 + 1). \quad (9)$$

### III. MAIN RESULTS

This section introduces an achievable region (Theorem 1) and a converse region (Theorem 2), denoted by  $\underline{\mathcal{C}}_{\text{G-IC-NOF}}$  and  $\overline{\mathcal{C}}_{\text{G-IC-NOF}}$  respectively, for the two-user G-IC-NOF with fixed parameters  $\overrightarrow{\text{SNR}}_1, \overrightarrow{\text{SNR}}_2, \text{INR}_{12}, \text{INR}_{21}, \overleftarrow{\text{SNR}}_1$ , and  $\overleftarrow{\text{SNR}}_2$ . In general, the capacity region of a given multi-user channel is said to be approximated to within a constant gap according to the following definition.

*Definition 2 (Approximation to within  $\xi$  units):* A closed and convex set  $\mathcal{T} \subset \mathbb{R}_+^n$  is approximated to within  $\xi$  units by the sets  $\underline{\mathcal{T}}$  and  $\overline{\mathcal{T}}$  if  $\underline{\mathcal{T}} \subseteq \mathcal{T} \subseteq \overline{\mathcal{T}}$  and for all  $\mathbf{t} = (t_1, \dots, t_m) \in \overline{\mathcal{T}}$  then  $((t_1 - \xi)^+, \dots, (t_m - \xi)^+) \in \underline{\mathcal{T}}$ .

Denote by  $\mathcal{C}_{\text{GIC-NOF}}$  the capacity region of the 2-user G-IC-NOF. The achievable region  $\underline{\mathcal{C}}_{\text{G-IC-NOF}}$  and the converse region  $\overline{\mathcal{C}}_{\text{G-IC-NOF}}$  approximate the capacity region  $\mathcal{C}_{\text{GIC-NOF}}$  to within 4.4 bits per channel use (Theorem 3).

#### A. An Achievable Region for the Two-User G-IC-NOF

The description of the achievable region  $\underline{\mathcal{C}}_{\text{G-IC-NOF}}$  is presented using the constants  $a_{1,i}$ ; the functions  $a_{2,i} : [0, 1] \rightarrow \mathbb{R}_+$ ,  $a_{7,i} : [0, 1]^3 \rightarrow \mathbb{R}_+$ , which are defined as follows, for all  $i \in \{1, 2\}$ , with  $j \in \{1, 2\} \setminus \{i\}$ :

$$a_{1,i} = \frac{1}{2} \log \left( 2 + \frac{\overrightarrow{\text{SNR}}_i}{\text{INR}_{ji}} \right) - \frac{1}{2}, \quad (10a)$$

$$a_{2,i}(\rho) = \frac{1}{2} \log \left( b_{1,i}(\rho) + 1 \right) - \frac{1}{2}, \quad (10b)$$

$$a_{3,i}(\rho, \mu) = \frac{1}{2} \log \left( \frac{\overleftarrow{\text{SNR}}_i (\text{INR}_{ij} + 1) + b_{1,i}(\rho) + 1}{\overleftarrow{\text{SNR}}_i ((1-\mu)b_{2,i}(\rho) + 2) + b_{1,i}(\rho) + 1} \right), \quad (10c)$$

$$a_{4,i}(\rho, \mu) = \frac{1}{2} \log \left( (1-\mu)b_{2,i}(\rho) + 2 \right) - \frac{1}{2}, \quad (10d)$$

$$a_{5,i}(\rho, \mu) = \frac{1}{2} \log \left( 2 + \frac{\overrightarrow{\text{SNR}}_i}{\text{INR}_{ji}} + (1-\mu)b_{2,i}(\rho) \right) - \frac{1}{2}, \quad (10e)$$

$$a_{6,i}(\rho, \mu) = \frac{1}{2} \log \left( \frac{\overrightarrow{\text{SNR}}_i}{\text{INR}_{ji}} \left( (1-\mu)b_{2,j}(\rho) + 1 \right) + 2 \right) - \frac{1}{2}, \quad (10f)$$

$$a_{7,i}(\rho, \mu_1, \mu_2) = \frac{1}{2} \log \left( \frac{\overrightarrow{\text{SNR}}_i}{\text{INR}_{ji}} \left( (1-\mu_i)b_{2,j}(\rho) + 1 \right) + (1-\mu_j)b_{2,i}(\rho) + 2 \right) - \frac{1}{2}. \quad (10g)$$

where the functions  $b_{l,i} : [0, 1] \rightarrow \mathbb{R}_+$ , with  $(l, i) \in \{1, 2\}^2$  are defined as follows

$$b_{1,i}(\rho) = \overrightarrow{\text{SNR}}_i + 2\rho \sqrt{\overrightarrow{\text{SNR}}_i \text{INR}_{ij} + \text{INR}_{ij}} \text{ and} \quad (11a)$$

$$b_{2,i}(\rho) = (1-\rho)\text{INR}_{ij} - 1, \quad (11b)$$

with  $j \in \{1, 2\} \setminus \{i\}$ .

Note that the functions in (10) and (11) depend on  $\overrightarrow{\text{SNR}}_1, \overrightarrow{\text{SNR}}_2, \text{INR}_{12}, \text{INR}_{21}, \overleftarrow{\text{SNR}}_1$ , and  $\overleftarrow{\text{SNR}}_2$ , however as these parameters are fixed in this analysis, this dependence is not emphasized in the definition of these functions. Finally, using this notation, Theorem 1 is presented on the next page.

*Proof:* The proof of Theorem 1 is presented in [1]. ■

#### B. Comments on the Achievability

The achievable region is obtained using a random coding argument and combining three classical tools: rate splitting, superposition coding, and backward decoding. This coding scheme is described in [1] and it is specially designed for the two-user IC-NOF. Consequently, only the strictly needed number of superposition code-layers is used. Other achievable schemes, as reported in [2], can also be obtained as special cases of the more general scheme presented in [3]. However, in this more general case, the resulting code for the IC-NOF contains a handful of unnecessary superposing code-layers, which complicates the error probability analysis.

#### C. A Converse Region for the Two-User G-IC-NOF

The description of the converse region  $\overline{\mathcal{C}}_{\text{G-IC-NOF}}$  is determined by the ratios  $\frac{\text{INR}_{ij}}{\overrightarrow{\text{SNR}}_j}$ , and  $\frac{\text{INR}_{ji}}{\overrightarrow{\text{SNR}}_j}$ , for all  $i \in \{1, 2\}$ , with  $j \in \{1, 2\} \setminus \{i\}$ . All relevant scenarios regarding these

*Theorem 1: The capacity region  $\mathcal{C}_{\text{GIC-NOF}}$  contains the region  $\underline{\mathcal{C}}_{\text{G-IC-NOF}}$  given by the closure of the set of all possible non-negative achievable rate pairs  $(R_1, R_2)$  that satisfy*

$$R_1 \leq \min \left( a_{2,1}(\rho), a_{6,1}(\rho, \mu_1) + a_{3,2}(\rho, \mu_1), a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{4,2}(\rho, \mu_1) \right), \quad (12a)$$

$$R_2 \leq \min \left( a_{2,2}(\rho), a_{3,1}(\rho, \mu_2) + a_{6,2}(\rho, \mu_2), a_{3,1}(\rho, \mu_2) + a_{4,1}(\rho, \mu_2) + a_{1,2} \right), \quad (12b)$$

$$R_1 + R_2 \leq \min \left( a_{2,1}(\rho) + a_{1,2}, a_{1,1} + a_{2,2}(\rho), a_{3,1}(\rho, \mu_2) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{7,2}(\rho, \mu_1, \mu_2), \right. \\ \left. a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{5,2}(\rho, \mu_1), a_{3,1}(\rho, \mu_2) + a_{7,1}(\rho, \mu_1, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{1,2} \right), \quad (12c)$$

$$2R_1 + R_2 \leq \min \left( a_{2,1}(\rho) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{7,2}(\rho, \mu_1, \mu_2), \right. \\ \left. a_{3,1}(\rho, \mu_2) + a_{1,1} + a_{7,1}(\rho, \mu_1, \mu_2) + 2a_{3,2}(\rho, \mu_1) + a_{5,2}(\rho, \mu_1), a_{2,1}(\rho) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{5,2}(\rho, \mu_1) \right), \quad (12d)$$

$$R_1 + 2R_2 \leq \min \left( a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{2,2}(\rho) + a_{1,2}, a_{3,1}(\rho, \mu_2) + a_{7,1}(\rho, \mu_1, \mu_2) + a_{2,2}(\rho) + a_{1,2}, \right. \\ \left. 2a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{1,2} + a_{7,2}(\rho, \mu_1, \mu_2) \right), \quad (12e)$$

with  $(\rho, \mu_1, \mu_2) \in \left[0, \left(1 - \max\left(\frac{1}{\text{INR}_{12}}, \frac{1}{\text{INR}_{21}}\right)\right)^+\right] \times [0, 1] \times [0, 1]^2$ .

ratios are described by two events denoted by  $S_{l_1,1}$  and  $S_{l_2,2}$ , where  $(l_1, l_2) \in \{1, \dots, 5\}^2$ :

$$S_{1,i}: \overrightarrow{\text{SNR}}_j < \min(\text{INR}_{ij}, \text{INR}_{ji}), \quad (13a)$$

$$S_{2,i}: \text{INR}_{ji} \leq \overrightarrow{\text{SNR}}_j < \text{INR}_{ij}, \quad (13b)$$

$$S_{3,i}: \text{INR}_{ij} \leq \overrightarrow{\text{SNR}}_j < \text{INR}_{ji}, \quad (13c)$$

$$S_{4,i}: \max(\text{INR}_{ij}, \text{INR}_{ji}) \leq \overrightarrow{\text{SNR}}_j < \text{INR}_{ij} \text{INR}_{ji}, \quad (13d)$$

$$S_{5,i}: \overrightarrow{\text{SNR}}_j \geq \max(\text{INR}_{ij}, \text{INR}_{ji}, \text{INR}_{ij} \text{INR}_{ji}). \quad (13e)$$

Note that for all  $i \in \{1, 2\}$ , the events  $S_{1,i}$ ,  $S_{2,i}$ ,  $S_{3,i}$ ,  $S_{4,i}$ , and  $S_{5,i}$  are mutually exclusive. This observation shows that given any 4-tuple  $(\overrightarrow{\text{SNR}}_1, \overrightarrow{\text{SNR}}_2, \text{INR}_{12}, \text{INR}_{21})$ , there always exists one and only one pair of events  $(S_{l_1,1}, S_{l_2,2})$ , with  $(l_1, l_2) \in \{1, \dots, 5\}^2$ , that identifies a unique scenario. Note also that the pairs of events  $(S_{2,1}, S_{2,2})$  and  $(S_{3,1}, S_{3,2})$  are not feasible. In view of this, twenty-three different scenarios can be identified using the events in (13). Once the exact scenario is identified, the converse region is described using the functions  $\kappa_{l,i} : [0, 1] \rightarrow \mathbb{R}_+$ , with  $(l, i) \in \{1, \dots, 3\} \times \{1, 2\}$ ;  $\kappa_l : [0, 1] \rightarrow \mathbb{R}_+$ , with  $l \in \{4, 5\}$ ;  $\kappa_{6,l} : [0, 1] \rightarrow \mathbb{R}_+$ , with  $l \in \{1, \dots, 4\}$ ; and  $\kappa_{7,i,l} : [0, 1] \rightarrow \mathbb{R}_+$ , with  $(i, l) \in \{1, 2\}^2$ . These functions are defined as follows for all  $i \in \{1, 2\}$ , with  $j \in \{1, 2\} \setminus \{i\}$

$$\kappa_{1,i}(\rho) = \frac{1}{2} \log(b_{1,i}(\rho) + 1), \quad (14a)$$

$$\kappa_{2,i}(\rho) = \frac{1}{2} \log\left(1 + b_{5,j}(\rho)\right) + \frac{1}{2} \log\left(1 + \frac{b_{4,i}(\rho)}{1 + b_{5,j}(\rho)}\right), \quad (14b)$$

$$\kappa_{3,i}(\rho) = \frac{1}{2} \log\left(\frac{\overleftarrow{\text{SNR}}_j (b_{4,i}(\rho) + b_{5,j}(\rho) + 1)}{(b_{1,j}(\rho) + 1)(b_{4,i}(\rho) + (1 - \rho^2))} + 1\right) \\ + \frac{1}{2} \log(b_{4,i}(\rho) + 1), \quad (14c)$$

$$\kappa_4(\rho) = \frac{1}{2} \log\left(1 + \frac{b_{4,1}(\rho)}{1 + b_{5,2}(\rho)}\right) + \frac{1}{2} \log(b_{1,2}(\rho) + 1), \quad (14d)$$

$$\kappa_5(\rho) = \frac{1}{2} \log\left(1 + \frac{b_{4,2}(\rho)}{1 + b_{5,1}(\rho)}\right) + \frac{1}{2} \log(b_{1,1}(\rho) + 1), \quad (14e)$$

$$\kappa_6(\rho) = \begin{cases} \kappa_{6,1}(\rho) & \text{if } (S_{1,2} \vee S_{2,2} \vee S_{5,2}) \\ & \wedge (S_{1,1} \vee S_{2,1} \vee S_{5,1}) \\ \kappa_{6,2}(\rho) & \text{if } (S_{1,2} \vee S_{2,2} \vee S_{5,2}) \\ & \wedge (S_{3,1} \vee S_{4,1}) \\ \kappa_{6,3}(\rho) & \text{if } (S_{3,2} \vee S_{4,2}) \\ & \wedge (S_{1,1} \vee S_{2,1} \vee S_{5,1}) \\ \kappa_{6,4}(\rho) & \text{if } (S_{3,2} \vee S_{4,2}) \wedge (S_{3,1} \vee S_{4,1}) \end{cases} \quad (14f)$$

$$\kappa_{7,i}(\rho) = \begin{cases} \kappa_{7,i,1}(\rho) & \text{if } (S_{1,i} \vee S_{2,i} \vee S_{5,i}) \\ \kappa_{7,i,2}(\rho) & \text{if } (S_{3,i} \vee S_{4,i}) \end{cases} \quad (14g)$$

where

$$\kappa_{6,1}(\rho) = \frac{1}{2} \log(b_{1,1}(\rho) + b_{5,1}(\rho) \text{INR}_{21}) - \frac{1}{2} \log(1 + \text{INR}_{12}) \\ + \frac{1}{2} \log\left(1 + \frac{b_{5,2}(\rho) \overleftarrow{\text{SNR}}_2}{b_{1,2}(\rho) + 1}\right) \\ + \frac{1}{2} \log(b_{1,2}(\rho) + b_{5,1}(\rho) \text{INR}_{21}) - \frac{1}{2} \log(1 + \text{INR}_{21}) \\ + \frac{1}{2} \log\left(1 + \frac{b_{5,1}(\rho) \overleftarrow{\text{SNR}}_1}{b_{1,1}(\rho) + 1}\right) + \log(2\pi e), \quad (15a)$$

$$\kappa_{6,2}(\rho) = \frac{1}{2} \log(b_{1,1}(\rho) + b_{5,1}(\rho) \text{INR}_{21}) - \frac{1}{2} \log(1 + \text{INR}_{12}) \\ + \frac{1}{2} \log\left(1 + \frac{b_{5,2}(\rho)}{\overrightarrow{\text{SNR}}_2} \left(\text{INR}_{12} + \frac{\overleftarrow{\text{SNR}}_2 (b_{3,2})}{b_{1,2}(\rho) + 1}\right)\right) \\ - \frac{1}{2} \log\left(1 + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\overrightarrow{\text{SNR}}_2}\right) \\ + \frac{1}{2} \log\left(1 + \frac{b_{5,1}(\rho) \overleftarrow{\text{SNR}}_1}{b_{1,1}(\rho) + 1}\right) \\ + \frac{1}{2} \log\left(b_{6,2}(\rho) + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\overrightarrow{\text{SNR}}_2} (\overrightarrow{\text{SNR}}_2 + b_{3,2})\right) \\ - \frac{1}{2} \log(1 + \text{INR}_{21}) + \log(2\pi e), \quad (15b)$$

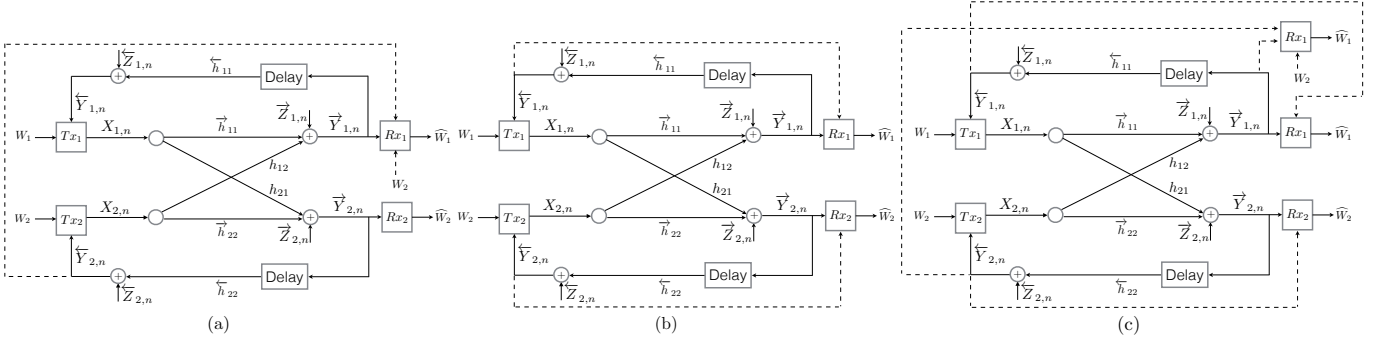


Fig. 2. Genie-Aided G-IC-NOF models for channel use  $n$ . (a) Model used to calculate the outer-bound on  $R_1$ ; (b) Model used to calculate the outer-bound on  $R_1 + R_2$ ; and (c) Model used to calculate the outer-bound on  $2R_1 + R_2$

$$\begin{aligned}
\kappa_{6,3}(\rho) &= \frac{1}{2} \log \left( b_{6,1}(\rho) + \text{INR}_{12} \right. \\
&\quad \left. + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\overrightarrow{\text{SNR}}_1} \left( \overrightarrow{\text{SNR}}_1 + b_{3,1} \right) \right) \\
&\quad - \frac{1}{2} \log \left( 1 + \text{INR}_{12} \right) + \frac{1}{2} \log \left( 1 + \frac{b_{5,2}(\rho) \overleftarrow{\text{SNR}}_2}{b_{1,2}(\rho) + 1} \right) \\
&\quad + \frac{1}{2} \log \left( b_{1,2}(\rho) + b_{5,1}(\rho) \text{INR}_{21} \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{21} \right) \\
&\quad + \frac{1}{2} \log \left( 1 + \frac{b_{5,1}(\rho)}{\overrightarrow{\text{SNR}}_1} \left( \text{INR}_{21} + \frac{\overleftarrow{\text{SNR}}_1(b_{3,1})}{b_{1,1}(\rho) + 1} \right) \right) \\
&\quad - \frac{1}{2} \log \left( 1 + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\overrightarrow{\text{SNR}}_1} \right) + \log(2\pi e), \quad (15c)
\end{aligned}$$

$$\begin{aligned}
\kappa_{6,4}(\rho) &= \frac{1}{2} \log \left( b_{6,1}(\rho) + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\overrightarrow{\text{SNR}}_1} \left( \overrightarrow{\text{SNR}}_1 + b_{3,1} \right) \right) \\
&\quad - \frac{1}{2} \log \left( 1 + \text{INR}_{12} \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{21} \right) \\
&\quad + \frac{1}{2} \log \left( 1 + \frac{b_{5,2}(\rho)}{\overrightarrow{\text{SNR}}_2} \left( \text{INR}_{12} + \frac{\overleftarrow{\text{SNR}}_2(b_{3,2})}{b_{1,2}(\rho) + 1} \right) \right) \\
&\quad - \frac{1}{2} \log \left( 1 + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\overrightarrow{\text{SNR}}_2} \right) \\
&\quad - \frac{1}{2} \log \left( 1 + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\overrightarrow{\text{SNR}}_1} \right) \\
&\quad + \frac{1}{2} \log \left( b_{6,2}(\rho) + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\overrightarrow{\text{SNR}}_2} \left( \overrightarrow{\text{SNR}}_2 + b_{3,2} \right) \right) \\
&\quad + \frac{1}{2} \log \left( 1 + \frac{b_{5,1}(\rho)}{\overrightarrow{\text{SNR}}_1} \left( \text{INR}_{21} + \frac{\overleftarrow{\text{SNR}}_1(b_{3,1})}{b_{1,1}(\rho) + 1} \right) \right) \\
&\quad + \log(2\pi e), \quad (15d)
\end{aligned}$$

and

$$\begin{aligned}
\kappa_{7,i,1}(\rho) &= \frac{1}{2} \log \left( b_{1,i}(\rho) + 1 \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{ij} \right) \\
&\quad + \frac{1}{2} \log \left( 1 + \frac{b_{5,j}(\rho) \overleftarrow{\text{SNR}}_j}{b_{1,j}(\rho) + 1} \right) \\
&\quad + \frac{1}{2} \log \left( b_{1,j}(\rho) + b_{5,i}(\rho) \text{INR}_{ji} \right) \\
&\quad + \frac{1}{2} \log \left( 1 + b_{4,i}(\rho) + b_{5,j}(\rho) \right) - \frac{1}{2} \log \left( 1 + b_{5,j}(\rho) \right) \\
&\quad + 2 \log(2\pi e), \quad (16a)
\end{aligned}$$

$$\begin{aligned}
\kappa_{7,i,2}(\rho) &= \frac{1}{2} \log \left( b_{1,i}(\rho) + 1 \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{ij} \right) \\
&\quad - \frac{1}{2} \log \left( 1 + b_{5,j}(\rho) \right) + \frac{1}{2} \log \left( 1 + b_{4,i}(\rho) + b_{5,j}(\rho) \right) \\
&\quad + \frac{1}{2} \log \left( 1 + (1 - \rho^2) \frac{\text{INR}_{ji}}{\overrightarrow{\text{SNR}}_j} \left( \text{INR}_{ij} + \right. \right. \\
&\quad \left. \left. \frac{\overleftarrow{\text{SNR}}_j(b_{3,j})}{b_{1,j}(\rho) + 1} \right) \right) - \frac{1}{2} \log \left( 1 + \frac{b_{5,i}(\rho) \text{INR}_{ji}}{\overrightarrow{\text{SNR}}_j} \right) \\
&\quad + \frac{1}{2} \log \left( b_{6,j}(\rho) + \frac{b_{5,i}(\rho) \text{INR}_{ji}}{\overrightarrow{\text{SNR}}_j} \left( \overrightarrow{\text{SNR}}_j + b_{3,j} \right) \right) \\
&\quad + 2 \log(2\pi e), \quad (16b)
\end{aligned}$$

where the functions  $b_{l,i}$ , with  $(l,i) \in \{1,2\}^2$  are defined in (11);  $b_{3,i}$  are constants; and the functions  $b_{l,i} : [0,1] \rightarrow \mathbb{R}_+$ , with  $(l,i) \in \{4,5,6\} \times \{1,2\}$  are defined as follows, with  $j \in \{1,2\} \setminus \{i\}$ :

$$b_{3,i} = \overrightarrow{\text{SNR}}_i - 2\sqrt{\overrightarrow{\text{SNR}}_i \text{INR}_{ji}} + \text{INR}_{ji}, \quad (17a)$$

$$b_{4,i}(\rho) = (1 - \rho^2) \overrightarrow{\text{SNR}}_i, \quad (17b)$$

$$b_{5,i}(\rho) = (1 - \rho^2) \text{INR}_{ij}, \quad (17c)$$

$$\begin{aligned}
b_{6,i}(\rho) &= \overrightarrow{\text{SNR}}_i + \text{INR}_{ij} + 2\rho\sqrt{\text{INR}_{ij}} \left( \sqrt{\overrightarrow{\text{SNR}}_i} - \sqrt{\text{INR}_{ji}} \right) \\
&\quad + \frac{\text{INR}_{ij} \sqrt{\text{INR}_{ji}}}{\overrightarrow{\text{SNR}}_i} \left( \sqrt{\text{INR}_{ji}} - 2\sqrt{\overrightarrow{\text{SNR}}_i} \right). \quad (17d)
\end{aligned}$$

Note that the functions in (14), (15), (16) and (17) depend on  $\overrightarrow{\text{SNR}}_1$ ,  $\overrightarrow{\text{SNR}}_2$ ,  $\text{INR}_{12}$ ,  $\text{INR}_{21}$ ,  $\overleftarrow{\text{SNR}}_1$ , and  $\overleftarrow{\text{SNR}}_2$ . However, these parameters are fixed in this analysis, and therefore, this dependence is not emphasized in the definition of these functions. Finally, using this notation, Theorem 2 is presented below.

**Theorem 2:** *The capacity region  $\mathcal{C}_{\text{GIC-NOF}}$  is contained within the region  $\overline{\mathcal{C}}_{\text{G-IC-NOF}}$  given by the closure of the set of non-negative rate pairs  $(R_1, R_2)$  that satisfy for all  $i \in \{1,2\}$ , with  $j \in \{1,2\} \setminus \{i\}$ :*

$$R_i \leq \min(\kappa_{1,i}(\rho), \kappa_{2,i}(\rho)), \quad (18a)$$

$$R_i \leq \kappa_{3,i}(\rho), \quad (18b)$$

$$R_1 + R_2 \leq \min(\kappa_4(\rho), \kappa_5(\rho)), \quad (18c)$$

$$R_1 + R_2 \leq \kappa_6(\rho), \quad (18d)$$

$$2R_i + R_j \leq \kappa_{7,i}(\rho), \quad (18e)$$

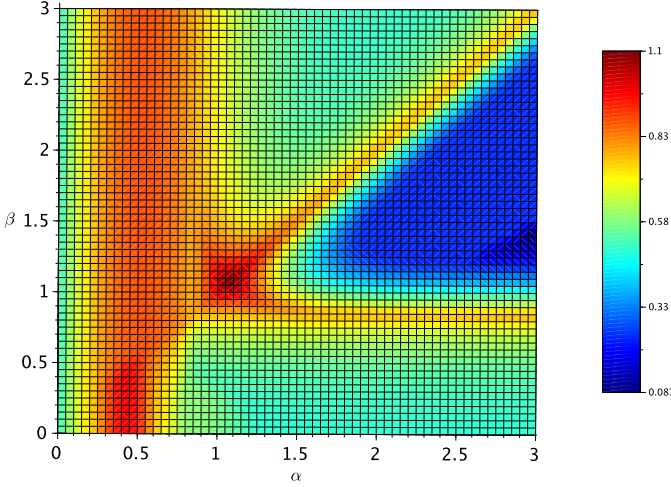


Fig. 3. Gap between the converse region  $\bar{\mathcal{C}}_{\text{G-IC-NOF}}$  and the achievable region  $\underline{\mathcal{C}}_{\text{G-IC-NOF}}$  of the two-user G-IC-NOF, under symmetric channel conditions, i.e.,  $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$ ,  $\text{INR}_{12} = \text{INR}_{21} = \text{INR}$ , and  $\check{\text{SNR}}_1 = \check{\text{SNR}}_2 = \check{\text{SNR}}$ , as a function of  $\alpha = \frac{\log \text{INR}}{\log \text{SNR}}$  and  $\beta = \frac{\log \check{\text{SNR}}}{\log \text{SNR}}$ .

with  $\rho \in [0, 1]$ .

*Proof:* The proof of Theorem 2 is presented in [1]. ■

#### D. Comments on the Converse Region

The outer bounds (18a) and (18c) correspond to the outer bounds for the case of perfect channel-output feedback [4]. The bounds (18b), (18d) and (18e) correspond to new outer bounds that generalize those presented in [2] for the two-user symmetric G-IC-NOF. These new outer-bounds were obtained using the genie-aided models shown in Fig. 2.

#### E. A Gap Between the Achievable Region and the Converse Region

Theorem 3 describes the gap between the achievable region  $\underline{\mathcal{C}}_{\text{G-IC-NOF}}$  and the converse region  $\bar{\mathcal{C}}_{\text{G-IC-NOF}}$  using the approximation notion described in Definition 2.

*Theorem 3:* The capacity region of the two-user G-IC-NOF is approximated to within 4.4 bits per channel use by the achievable region  $\underline{\mathcal{C}}_{\text{G-IC-NOF}}$  and the converse region  $\bar{\mathcal{C}}_{\text{G-IC-NOF}}$ .

*Proof:* The proof of Theorem 3 is presented in [1]. ■

The gap, denoted by  $\delta$ , between the sets  $\bar{\mathcal{C}}_{\text{G-IC-NOF}}$  and  $\underline{\mathcal{C}}_{\text{G-IC-NOF}}$  can be approximated (Definition 2) as follows

$$\delta \leq \max \left( \delta_{R_1}, \delta_{R_2}, \frac{\delta_{2R}}{2}, \frac{\delta_{3R_1}}{3}, \frac{\delta_{3R_2}}{3} \right), \quad (19)$$

where

$$\begin{aligned} \delta_{R_1} \triangleq & \min \left( \kappa_{1,1}(\rho), \kappa_{2,1}(\rho), \kappa_{3,1}(\rho) \right) - \min \left( a_{2,1}(\rho), \right. \\ & \left. a_{6,1}(\rho, \mu_1) + a_{3,2}(\rho, \mu_1), a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{4,2}(\rho, \mu_1) \right), \end{aligned} \quad (20a)$$

$$\begin{aligned} \delta_{R_2} \triangleq & \min \left( \kappa_{1,2}(\rho), \kappa_{2,2}(\rho), \kappa_{3,2}(\rho) \right) - \min \left( a_{2,2}(\rho), \right. \\ & \left. a_{3,1}(\rho, \mu_2) + a_{6,2}(\rho, \mu_2), a_{3,1}(\rho, \mu_2) + a_{4,1}(\rho, \mu_2) + a_{1,2} \right), \end{aligned} \quad (20b)$$

$$\begin{aligned} \delta_{2R} \triangleq & \min \left( \kappa_4(\rho), \kappa_5(\rho), \kappa_6(\rho) \right) - \min \left( a_{2,1}(\rho) + a_{1,2}, \right. \\ & a_{1,1} + a_{2,2}(\rho), \\ & a_{3,1}(\rho, \mu_2) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{7,2}(\rho, \mu_1, \mu_2), \\ & a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{5,2}(\rho, \mu_1), \\ & \left. a_{3,1}(\rho, \mu_2) + a_{7,1}(\rho, \mu_1, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{1,2} \right), \end{aligned} \quad (20c)$$

$$\begin{aligned} \delta_{3R_1} \triangleq & \kappa_{7,1}(\rho) - \min \left( a_{2,1}(\rho) + a_{1,1} + a_{3,2}(\rho, \mu_1) \right. \\ & \left. + a_{7,2}(\rho, \mu_1, \mu_2), a_{3,1}(\rho, \mu_2) + a_{1,1} + a_{7,1}(\rho, \mu_1, \mu_2) \right. \\ & \left. + 2a_{3,2}(\rho, \mu_1) + a_{5,2}(\rho, \mu_1), a_{2,1}(\rho) + a_{1,1} + a_{3,2}(\rho, \mu_1) \right. \\ & \left. + a_{5,2}(\rho, \mu_1) \right), \end{aligned} \quad (20d)$$

$$\begin{aligned} \delta_{3R_2} \triangleq & \kappa_{7,2}(\rho) - \min \left( a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{2,2}(\rho) \right. \\ & \left. + a_{1,2}, a_{3,1}(\rho, \mu_2) + a_{7,1}(\rho, \mu_1, \mu_2) + a_{2,2}(\rho) + a_{1,2}, \right. \\ & \left. 2a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{1,2} \right. \\ & \left. + a_{7,2}(\rho, \mu_1, \mu_2) \right). \end{aligned} \quad (20e)$$

Note that  $\delta_{R_1}$  and  $\delta_{R_2}$  represent the gap between the active achievable single-rate bound and the active converse single-rate bound;  $\delta_{2R}$  represents the gap between the active achievable sum-rate bound and the active converse sum-rate bound; and,  $\delta_{3R_1}$  and  $\delta_{3R_2}$  represent the gap between the active achievable weighted sum-rate bound and the active converse weighted sum-rate bound.

Finally, it is important to highlight that, as suggested in [2], [4], and [5], the gap between  $\underline{\mathcal{C}}_{\text{G-IC-NOF}}$  and  $\bar{\mathcal{C}}_{\text{G-IC-NOF}}$  can be calculated more precisely. However, the choice in (19) eases the calculations at the expense of less precision.

Fig. 3 presents the exact gap existing between the achievable region  $\underline{\mathcal{C}}_{\text{G-IC-NOF}}$  and the converse region  $\bar{\mathcal{C}}_{\text{G-IC-NOF}}$  for the case in which  $\overline{\text{SNR}}_1 = \overline{\text{SNR}}_2 = \overline{\text{SNR}}$ ,  $\text{INR}_{12} = \text{INR}_{21} = \text{INR}$ , and  $\check{\text{SNR}}_1 = \check{\text{SNR}}_2 = \check{\text{SNR}}$  as a function of  $\alpha = \frac{\log \text{INR}}{\log \text{SNR}}$  and  $\beta = \frac{\log \check{\text{SNR}}}{\log \text{SNR}}$ . Note that in this case, the maximum gap is 1.1 bits per channel use and occurs when  $\alpha = \beta = 1.1$ .

#### IV. CONCLUSIONS

An achievable region and a converse region for the two-user G-IC-NOF have been introduced. It has been shown that these regions approximate the capacity region of the two-user G-IC-NOF to within 4.4 bits per channel use.

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