Approximated Capacity of the Two-User Gaussian Interference Channel with Noisy Channel-Output Feedback

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***Abstract*—In this paper, the capacity region of the two-user Gaussian interference with noisy channel-output feedback (G-IC-NOF) is approximated to within 4.3 bits per channel use. To this end, a new achievable region and a new converse region are presented. Existing results such as: (a) the approximated capacity of the two-user Gaussian interference channel introduced by Etkin *et al.* in [1]; (b) the approximated capacity of the two-user Gaussian interference channel with perfect channel-output feedback introduced by Suh *et al.* in [2]; and (c) the approximated capacity of the symmetric two-user G-IC-NOF introduced by Le *et al.* in [3], can be obtained as special cases of the results presented here.**

***Index Terms*—Capacity, Interference Channel, Noisy Channel-Output Feedback.**

I. System Model

Consider the two-user Gaussian interference channel with noisy channel-output feedback (G-IC-NOF) in Fig. 1. Transmitter i, $i \in \{1,2\}$, communicates with receiver i subject to the interference produced by transmitter $j \in \{1,2\}\backslash\{i\}$. There are two independent and uniformly distributed messages, $W_i \in \{1,\ldots,2^{NR_i}\}$, $i \in \{1,2\}$, where N denotes the block-length in channel uses and R_i is the transmission rate in bits per channel use. At each block, transmitter i sends the codeword $\boldsymbol{X}_i^{(1:N)} = (X_{i,1},\ldots,X_{i,N})^{\mathsf{T}} \in \mathcal{X}_i^N$, where $\mathcal{X}_i^N$ is the codebook of transmitter i. The components of the channel input vector $\boldsymbol{X}_i^{(1:N)}$ are real with zero means and subject to an average power constraint

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left(X_{i,n}{}^2\right) \leq 1. \tag{1}$$

The channel coefficient from transmitter j to receiver i is denoted by h_{ij}; the channel coefficient from transmitter i to receiver i is denoted by $\overrightarrow{h}_{ii}$; and the channel coefficient from channel-output i to transmitter i is denoted by $\overleftarrow{h}_{ii}$. All channel coefficients are assumed to be non-negative real numbers.

At a given channel use n, the channel output at receiver i is denoted by $\overrightarrow{Y}_{i,n}$. During channel use $n \in \{1,\ldots,N\}$, the input-output relation of the channel model is given by the following equality:

$$\overrightarrow{Y}_{i,n} = \overrightarrow{h}_{ii}X_{i,n} + h_{ij}X_{j,n} + \overrightarrow{Z}_{i,n}, \tag{2}$$

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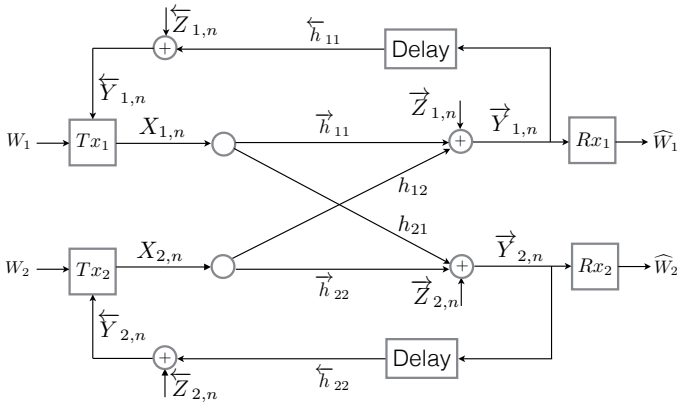

Fig. 1. Gaussian Interference Channel With Noisy Channel-Output Feedback at channel use n.

where $\overrightarrow{Z}_{i,n}$ is a real Gaussian random variable with zero mean and unit variance and it represents the noise at the input of receiver i. At a given channel use n, the feedback signal at the input of transmitter i is denoted by $\overleftarrow{Y}_{i,n}$. Let $d > 0$ be the finite feedback delay in channel uses. At the end of channel use $n > d$, transmitter i observes a scaled and noisy version of $\overrightarrow{Y}_{i,n-d}$. More specifically,

$$\forall n \in \{1,\ldots,d\}, \quad \overleftarrow{Y}_{i,n} = \overleftarrow{Z}_{i,n} \text{ and} \tag{3}$$

$$\forall n \in \{d+1,\ldots,N\}, \quad \overleftarrow{Y}_{i,n} = \overleftarrow{h}_{ii}\overrightarrow{Y}_{i,n-d} + \overleftarrow{Z}_{i,n}, \tag{4}$$

where $\overleftarrow{Z}_{i,n}$ is a real Gaussian random variable with zero mean and unit variance and it represents the noise at the input of transmitter i. The random variables $\overrightarrow{Z}_{i,n}$ and $\overleftarrow{Z}_{i,n}$ are independent and identically distributed. In the following, without loss of generality, the feedback delay is assumed to be equal to 1 channel use. The encoder of transmitter i can be modelled as a set of deterministic mappings $f_i^{(1)},\ldots,f_i^{(N)}$, with $f_i^{(1)} : \mathcal{W}_i \to \mathcal{X}_i$ and $\forall n \in \{2,\ldots,N\}$, $f_i^{(n)} : \mathcal{W}_i \times \mathbb{R}^{n-1} \to \mathcal{X}_i$, such that

$$X_{i,1} = f_i^{(1)}(W_i), \tag{5}$$

$$X_{i,n} = f_i^{(n)}\left(W_i, \overleftarrow{Y}_{i,1},\ldots,\overleftarrow{Y}_{i,n-1}\right). \tag{6}$$

The decoder of receiver i can be modelled as a deterministic function $\psi_i : \mathbb{R}_i^N \to \mathcal{W}_i$. At the end of the block, receiver i uses the sequence $\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} = \left(\overrightarrow{Y}_{i,1},\ldots,\overrightarrow{Y}_{i,N}\right)$ to obtain an estimation of the index message

$$\widehat{W}_i = \psi_i\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right). \tag{7}$$

The decoding error probability in the two-user G-IC-NOF,

denoted by p, is calculated as follows

$$p = \max\left(\Pr\left(\widehat{W_1} \neq W_1\right), \Pr\left(\widehat{W_2} \neq W_2\right)\right). \tag{8}$$

A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is said to be achievable if it satisfies the following definition

Definition 1 (Achievable Rate Pairs): A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable if there exists at least one pair of codebooks $\mathcal{X}_1^N$ and $\mathcal{X}_2^N$ with codewords of length N, and the corresponding encoding functions $f_1^{(1)}, \ldots, f_1^{(N)}$ and $f_2^{(1)}, \ldots, f_2^{(N)}$ such that the decoding error probability can be made arbitrarily small by letting the block length N grow to infinity.

The two-user G-IC-NOF in Fig. 1 can be described by six parameters: $\overrightarrow{\text{SNR}}_i$, $\overleftarrow{\text{SNR}}_i$ and INR_{ij} for all $i \in \{1,2\}$ and $j \in \{1,2\}\backslash\{i\}$ as follows

$$\overrightarrow{\text{SNR}}_i = \overrightarrow{h}_{ii}^2, \tag{9}$$

$$\text{INR}_{ij} = h_{ij}^2, \text{and} \tag{10}$$

$$\overleftarrow{\text{SNR}}_i = \overleftarrow{h}_{ii}^2 \left(\overrightarrow{\text{SNR}}_i + 2\rho\sqrt{\overrightarrow{\text{SNR}}_i \text{INR}_{ij}} + \text{INR}_{ij} + 1\right), \tag{11}$$

where ρ is the average Pearson correlation coefficient, i.e., $\rho = \frac{1}{N}\sum_{n=1}^{N}\rho_n$, with $\rho_n = \frac{\mathbb{E}[X_{i,n}X_{j,n}]}{\sqrt{\mathbb{E}[(X_{i,n})^2]\mathbb{E}[(X_{j,n})^2]}}$ for all $n \in \{1, \ldots, N\}$, and $0 \leq \rho \leq 1$.

II. Main Results

This section introduces a new achievable region, denoted by $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$, and a new converse region, denoted by $\overline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$, for the two-user G-IC-NOF with fixed parameters $\overrightarrow{\text{SNR}}_1$, $\overrightarrow{\text{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\text{SNR}}_1$ and $\overleftarrow{\text{SNR}}_2$.

The capacity region of a given multi-user channel is said to be approximated to within a constant gap if it fits the following definition [4].

Definition 2 (Approximation to within ξ units): A closed and convex region $\mathcal{T} \subset \mathbb{R}_+^n$ is approximated to within ξ units if there exist two sets $\underline{\mathcal{T}}$ and $\overline{\mathcal{T}}$ such that $\underline{\mathcal{T}} \subseteq \mathcal{T} \subseteq \overline{\mathcal{T}}$ and $\forall \boldsymbol{t} = (t_1, \ldots, t_n) \in \overline{\mathcal{T}}$ then $((t_1 - \xi)^+, \ldots, (t_n - \xi)^+) \in \underline{\mathcal{T}}$.

Denote by $\mathcal{C}_{\text{GIC}-\text{NOF}}$ the capacity region of the 2-user G-IC-NOF. The new achievable achievable region $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ and the new converse region $\overline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ approximate the capacity region $\mathcal{C}_{\text{GIC}-\text{NOF}}$ to within 4.3 bits per channel use.

A. An Achievable Region for the Two-User Gaussian Interference Channel with Noisy Channel-Output Feedback

Consider the variables $a_1, \ldots, a_7$ in (17) and $b_1, \ldots, b_7$ in (18) that are defined for a fixed 6-tuple $\left(\overrightarrow{\text{SNR}}_1, \overrightarrow{\text{SNR}}_2, \text{INR}_{12}, \text{INR}_{21}, \overleftarrow{\text{SNR}}_1, \overleftarrow{\text{SNR}}_2\right) \in \mathbb{R}_+^6$. Using this notation, the following theorem describes the new achievable region $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ for the two-user G-IC-NOF.

Theorem 1 (An achievable region for the two-user G-IC-NOF): The capacity region $\mathcal{C}_{\text{GIC}-\text{NOF}}$ contains the region $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ given by the set of non-negative rate pairs (R_1, R_2) that satisfy

$$R_1 \leqslant \min\left(a_2, a_6 + b_1, a_4 + b_1 + b_3\right), \tag{12a}$$

$$R_2 \leqslant \min\left(b_2, a_1 + b_6, a_1 + a_3 + b_4\right), \tag{12b}$$

$$R_1 + R_2 \leqslant \min(a_2 + b_4, a_4 + b_2, a_1 + a_4 + b_1 + b_7, a_1 + a_5 + b_1 + b_5, a_1 + a_7 + b_1 + b_4), \tag{12c}$$

$$2R_1 + R_2 \leqslant \min(a_2 + a_4 + b_1 + b_7, a_1 + a_4 + a_7 + 2b_1 + b_5, a_2 + a_4 + b_1 + b_5), \tag{12d}$$

$$R_1 + 2R_2 \leqslant \min(a_1 + a_5 + b_2 + b_4, a_1 + a_7 + b_2 + b_4, 2a_1 + a_5 + b_1 + b_4 + b_7), \tag{12e}$$

with $(\alpha_1, \alpha_2) \in [0,1]^2$ and $0 \leqslant \rho \leqslant \min\left(\left(1 - \frac{1}{INR_{12}}\right)^+, \left(1 - \frac{1}{INR_{21}}\right)^+\right)$.

Proof: The proof of Theorem 1 is presented in [5]. ■

The achievable region $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ in (12) is obtained using a coding scheme that combines a three-part message splitting, superposition coding and backward decoding, as first suggested in [6], [2] and [7]. This coding scheme is fully described in [5] and it is specially designed for the two-user IC-NOF. However, it can also be obtained as a special case of the more general scheme presented in [6].

B. A Converse Region for the Two-User Gaussian Interference Channel with Noisy Channel-Output Feedback

Consider the variables $\kappa_{1,i}, \ldots, \kappa_{7,i}$ in (21) for all $i \in \{1,2\}$. The variables κ_6 and $\kappa_{7,i}$ in (21f) and (21g) correspond to different expressions depending on the events $S_{1,i} - S_{5,i}$ defined in (20). Note that for all $i \in \{1,2\}$, the events $S_{1,i} - S_{5,i}$ are mutually exclusive (see Appendix A). Using this notation, the following theorem describes the new converse region $\overline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ for the two-user G-IC-NOF.

Theorem 2 (A converse region for the two-user G-IC-NOF): The capacity region $\mathcal{C}_{\text{GIC}-\text{NOF}}$ is contained into the region $\overline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ given by the set of non-negative rate pairs (R_1, R_2) that satisfy $\forall i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$:

$$R_i \leqslant \min\left(\kappa_{1,i}, \kappa_{2,i}\right), \tag{13a}$$

$$R_i \leqslant \kappa_{3,i}, \tag{13b}$$

$$R_1 + R_2 \leqslant \min\left(\kappa_4, \kappa_5\right), \tag{13c}$$

$$R_1 + R_2 \leqslant \kappa_6, \tag{13d}$$

$$2R_i + R_j \leqslant \kappa_{7,i}, \tag{13e}$$

with $0 \leqslant \rho \leqslant 1$.

Proof: The proof of Theorem 2 is presented in [5]. ■

The outer bounds (13a) and (13c) correspond to the outer bounds for the case of perfect channel-output feedback [2]. The bounds (13b), (13d) and (13e) correspond to new outer bounds.

The following corollaries highlight the fact that the achievable region $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ in (12) and the converse region $\overline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ in (13) for the two-user G-IC-NOF are a generalization of existing results such as: (a) the approximated capacity of Gaussian interference channel (G-IC) introduced by Etkin *et al.* in [1]; (b) the approximated capacity of the G-IC with perfect channel-output feedback (G-IC-POF) introduced by Suh *et al.* in [2]; and (c) the approximated capacity of the symmetric G-IC-NOF (G-SIC-NOF) introduced by Le *et al.* in [3].

Denote by $\underline{\mathcal{C}}_{\text{GIC}}$ and $\overline{\mathcal{C}}_{\text{GIC}}$ the achievable region and the converse region of the two-user G-IC introduced by Etkin *et al.* in [1]. Denote by $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}^{\dagger}$ and $\overline{\mathcal{C}}_{\text{GIC}-\text{NOF}}^{\dagger}$ the achievable region and the converse region of the two-user G-IC-NOF

obtained from Theorem 1 and Theorem 2 with $\overleftarrow{\mathrm{SNR}}_1 = 0$ and $\overleftarrow{\mathrm{SNR}}_2 = 0$. Then, the following holds

Corollary 1:

$$\underline{\mathcal{C}}^{\dagger}_{\mathrm{GIC-NOF}} \subset \underline{\mathcal{C}}_{\mathrm{GIC}} \text{ and} \tag{14a}$$

$$\overline{\mathcal{C}}_{\mathrm{GIC}} \subset \overline{\mathcal{C}}^{\dagger}_{\mathrm{GIC-NOF}}. \tag{14b}$$

Denote by $\underline{\mathcal{C}}_{\mathrm{GIC-POF}}$ and $\overline{\mathcal{C}}_{\mathrm{GIC-POF}}$ the achievable region and the converse region of the two-user G-IC-POF introduced by *Suh et al.* in [2]. Denote by $\underline{\mathcal{C}}^{\ddagger}_{\mathrm{GIC-NOF}}$ and $\overline{\mathcal{C}}^{\ddagger}_{\mathrm{GIC-NOF}}$ the achievable region and the converse region of the two-user G-IC-NOF from Theorem 1 and Theorem 2 with $\overleftarrow{\mathrm{SNR}}_1 = \infty$ and $\overleftarrow{\mathrm{SNR}}_2 = \infty$. Then, the following holds

Corollary 2:

$$\underline{\mathcal{C}}^{\ddagger}_{\mathrm{GIC-NOF}} = \underline{\mathcal{C}}_{\mathrm{GIC-POF}}, \tag{14c}$$

$$\overline{\mathcal{C}}^{\ddagger}_{\mathrm{GIC-NOF}} = \overline{\mathcal{C}}_{\mathrm{GIC-POF}}. \tag{14d}$$

Denote by $\underline{\mathcal{C}}_{\mathrm{GSIC-NOF}}$ and $\overline{\mathcal{C}}_{\mathrm{GSIC-NOF}}$ the achievable region and the converse region of the two-user G-SIC-NOF introduced by Le *et al.* in [3]. Denote by $\underline{\mathcal{C}}^{*}_{\mathrm{GIC-NOF}}$ and $\overline{\mathcal{C}}^{*}_{\mathrm{GIC-NOF}}$ the achievable region and the converse region of the two-user G-IC-NOF from Theorem 1 and Theorem 2 with $\overrightarrow{\mathrm{SNR}}_1 = \overrightarrow{\mathrm{SNR}}_2$, $\mathrm{INR}_{12} = \mathrm{INR}_{21}$ and $\overleftarrow{\mathrm{SNR}}_1 = \overleftarrow{\mathrm{SNR}}_2$. Then, the following holds

Corollary 3:

$$\underline{\mathcal{C}}_{\mathrm{GSIC-NOF}} \subset \underline{\mathcal{C}}^{*}_{\mathrm{GIC-NOF}} \text{ and} \tag{14e}$$

$$\overline{\mathcal{C}}^{*}_{\mathrm{GIC-NOF}} \subseteq \overline{\mathcal{C}}_{\mathrm{GSIC-NOF}}. \tag{14f}$$

C. A Finite Gap Between the Achievable Region and the Converse Region

Theorem 3 establishes a finite gap between the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ (Definition 2).

Theorem 3: The capacity region of the two-user G-IC-NOF is approximated to within 4.3 *bits per channel use by the achievable region* $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ *and the converse region* $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$.

Proof: The proof of Theorem 3 is presented in [5]. ■

Denote by δ_{R_1} the difference between the minimum of $\kappa_{1,1}$, $\kappa_{2,1}$ and $\kappa_{3,1}$ and the minimum of a_2, $a_6 + b_1$, $a_4 + b_1 + b_3$ and ϑ_1:

$$\delta_{R_1} = \min(\kappa_{1,1}, \kappa_{2,1}, \kappa_{3,1}) - \min(a_2, a_6 + b_1, a_4 + b_1 + b_3, \vartheta_1), \tag{15a}$$

where, ϑ_1 can be seen as an achievable rate on R_1 for a particular power allocation given in [5]. Note that δ_{R_1} represents the gap on the single rate R_1 between the achievable and the converse regions.

Denote by δ_{R_2} the difference between the minimum of $\kappa_{1,2}$, $\kappa_{2,2}$ and $\kappa_{3,2}$ and the minimum of b_2, $a_1 + b_6$, $a_1 + a_3 + b_4$ and ϑ_2:

$$\delta_{R_2} = \min(\kappa_{1,2}, \kappa_{2,2}, \kappa_{3,2}) - \min(b_2, a_1 + b_6, a_1 + a_3 + b_4, \vartheta_2). \tag{15b}$$

where, ϑ_2 can be seen as an achievable rate on R_2 for a particular power allocation given in [5]. Note that δ_{R_2} represents the gap on the single rate R_2 between the achievable and the converse regions.

Denote by δ_{2R} the difference between the minimum of κ_4, κ_5 and κ_6 and the minimum of a_2+b_4, a_4+b_2,$a_1+a_4+b_1+b_7$, $a_1 + a_5 + b_1 + b_5$, $a_1 + a_7 + b_1 + b_4$ and $\vartheta_1 + \vartheta_2$:

$$\delta_{2R} = \min(\kappa_4, \kappa_5, \kappa_6) - \min(a_2 + b_4, a_4 + b_2, a_1 + a_4 + b_1 + b_7, a_1 + a_5 + b_1 + b_5, a_1 + a_7 + b_1 + b_4, \vartheta_1 + \vartheta_2). \tag{15c}$$

Note that δ_{2R} represents the gap on the sum rate $R_1 + R_2$ between the achievable and the converse regions.

Denote by δ_{3R_1} the difference between $\kappa_{7,1}$ and the minimum of $a_2 + a_4 + b_1 + b_7$, $a_1 + a_4 + a_7 + 2b_1 + b_5$ and $a_2 + a_4 + b_1 + b_5$ and $2\vartheta_1 + \vartheta_2$:

$$\delta_{3R_1} = \kappa_{7,1} - \min(a_2 + a_4 + b_1 + b_7, a_1 + a_4 + a_7 + 2b_1 + b_5, a_2 + a_4 + b_1 + b_5, 2\vartheta_1 + \vartheta_2). \tag{15d}$$

Note that δ_{3R_1} represents the gap on the weighted sum rate $2R_1 + R_2$ between the achievable and the converse regions.

Denote by δ_{3R_2} the difference between $\kappa_{7,2}$ and the minimum of $a_1 + a_5 + b_2 + b_4$, $a_1 + a_7 + b_2 + b_4$ and $2a_1 + a_5 + b_1 + b_4 + b_7$ and $\vartheta_1 + 2\vartheta_2$:

$$\delta_{3R_2} = \kappa_{7,2} - \min(a_1 + a_5 + b_2 + b_4, a_1 + a_7 + b_2 + b_4, 2a_1 + a_5 + b_1 + b_4 + b_7, \vartheta_1 + 2\vartheta_2). \tag{15e}$$

Note that δ_{3R_2} represents the gap on the weighted sum rate $R_1 + 2R_2$ between the achievable and the converse regions.

Therefore, the finite gap between the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ of the two user G-IC-NOF can be obtained as follows

$$\delta = \max\left(\delta_R, \frac{\delta_{2R}}{2}, \frac{\delta_{3R}}{3}\right), \tag{15f}$$

where, $\delta_R = \max(\delta_{R_1}, \delta_{R_2})$ and $\delta_{3R} = \max(\delta_{3R_1}, \delta_{3R_2})$.

In most of the cases, the gap is obtained using the outer bounds of the case of perfect channel-output feedback, i.e. bounds (13a) and (13c) because it is easier to deal with them. The gap could be reduced considering the new outer bounds that are obtained for the case with noisy channel-output feedback. The capacity region of the two-user G-IC-NOF under symmetric channel conditions can be numerically approximated to within 1.1 bits per channel use as shown in Fig. 2, where $\alpha = \frac{\log \mathrm{INR}}{\log \overrightarrow{\mathrm{SNR}}}$ and $\beta = \frac{\log \overleftarrow{\mathrm{SNR}}}{\log \overrightarrow{\mathrm{SNR}}}$. The maximum gap, i.e., 1.1 bits per channel use, in Fig. 2 is obtained for $\alpha = \beta = 1.1$.

III. Conclusions

A new achievable region and a new converse region for the two-user G-IC-NOF were presented. These results are a generalization of existing results such as: (a) the approximated capacity of the two-user G-IC [1]; (b) the approximated capacity of the two-user G-IC-POF [2]; and (c) the approximated capacity of the G-SIC-NOF [3]. Finally, the capacity region of the two-user G-IC-NOF was approximated to within 4.3 bits per channel use.

Appendix A
Auxiliary Variables

This appendix introduces some auxiliary variables that are needed for describing the new achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the new converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ of the two-user G-IC-NOF.

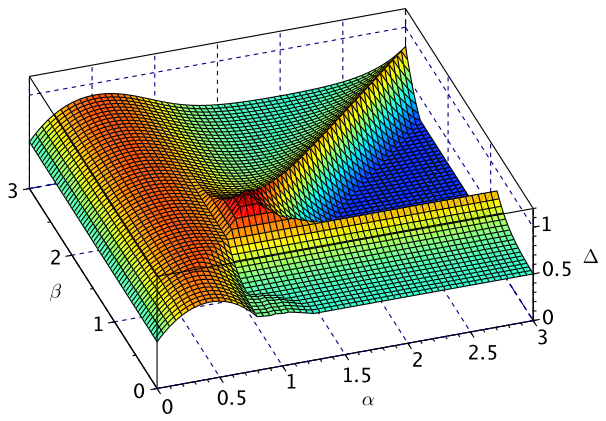

Fig. 2. A finite gap between the converse and the achievable regions for the two-user G-IC-NOF under symmetric channel conditions.

Consider the following constants for all $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$:

$$A_i = \overrightarrow{\text{SNR}}_i + 2\rho\sqrt{\overrightarrow{\text{SNR}}_i \text{INR}_{ij}} + \text{INR}_{ij}, \tag{16a}$$

$$B_i = \overrightarrow{\text{SNR}}_i - 2\sqrt{\overrightarrow{\text{SNR}}_i \text{INR}_{ji}} + \text{INR}_{ji}, \tag{16b}$$

$$C_i = \left(1-\rho^2\right)\overrightarrow{\text{SNR}}_i, \tag{16c}$$

$$D_i = \left(1-\rho^2\right)\text{INR}_{ij}, \tag{16d}$$

$$F_i = \left(\left(1-\rho\right)\text{INR}_{ij} - 1\right), \tag{16e}$$

$$G_i = \overrightarrow{\text{SNR}}_i + \text{INR}_{ij} + 2\rho\sqrt{\text{INR}_{ij}}\left(\sqrt{\overrightarrow{\text{SNR}}_i} - \sqrt{\text{INR}_{ji}}\right) + \frac{\text{INR}_{ij}\sqrt{\text{INR}_{ji}}}{\overrightarrow{\text{SNR}}_i}\left(\sqrt{\text{INR}_{ji}} - 2\sqrt{\overrightarrow{\text{SNR}}_i}\right). \tag{16f}$$

Using the equalities in (16) and for a fixed $(\alpha_1, \alpha_2) \in [0,1]^2$ and a fixed $\rho \in \left[0, \min\left(\left(1-\frac{1}{\text{INR}_{12}}\right)^+, \left(1-\frac{1}{\text{INR}_{21}}\right)^+\right)\right]$, the variables $a_1, \ldots, a_7$ that appeared in Theorem 1 are defined hereunder:

$$a_1 = \frac{1}{2}\log\left(\frac{\overleftarrow{\text{SNR}}_1(\text{INR}_{12}+1) + A_1 + 1}{\overleftarrow{\text{SNR}}_1\left(\left(1-\alpha_2\right)F_1 + 2\right) + A_1 + 1}\right), \tag{17a}$$

$$a_2 = \frac{1}{2}\log\left(A_1 + 1\right) - \frac{1}{2}, \tag{17b}$$

$$a_3 = \frac{1}{2}\log\left(\left(1-\alpha_2\right)F_1 + 2\right) - \frac{1}{2}, \tag{17c}$$

$$a_4 = \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\text{SNR}_1}}{\text{INR}_{21}}\right) - \frac{1}{2}, \tag{17d}$$

$$a_5 = \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\text{SNR}_1}}{\text{INR}_{21}} + \left(1-\alpha_2\right)F_1\right) - \frac{1}{2}, \tag{17e}$$

$$a_6 = \frac{1}{2}\log\left(\frac{\overrightarrow{\text{SNR}_1}}{\text{INR}_{21}}\left(\left(1-\alpha_1\right)F_2 + 1\right) + 2\right) - \frac{1}{2}, \tag{17f}$$

$$a_7 = \frac{1}{2}\log\left(\frac{\overrightarrow{\text{SNR}_1}}{\text{INR}_{21}}\left(\left(1-\alpha_1\right)F_2 + 1\right) + \left(1-\alpha_2\right)F_1 + 2\right) - \frac{1}{2}. \tag{17g}$$

Similarly, using the equalities in (16) and for a fixed $(\alpha_1, \alpha_2) \in [0,1]^2$ and a fixed $\rho \in \left[0, \min\left(\left(1-\frac{1}{\text{INR}_{12}}\right)^+, \left(1-\frac{1}{\text{INR}_{21}}\right)^+\right)\right]$, the variables $b_1, \ldots, b_7$ that appeared in Theorem 1 are defined hereunder:

$$b_1 = \frac{1}{2}\log\left(\frac{\overleftarrow{\text{SNR}}_2\left(\text{INR}_{21}+1\right) + A_2 + 1}{\overleftarrow{\text{SNR}}_2\left(\left(1-\alpha_1\right)F_2 + 2\right) + A_2 + 1}\right), \tag{18a}$$

$$b_2 = \frac{1}{2}\log\left(A_2 + 1\right) - \frac{1}{2}, \tag{18b}$$

$$b_3 = \frac{1}{2}\log\left(\left(1-\alpha_1\right)F_2 + 2\right) - \frac{1}{2}, \tag{18c}$$

$$b_4 = \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\text{SNR}_2}}{\text{INR}_{12}}\right) - \frac{1}{2}, \tag{18d}$$

$$b_5 = \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\text{SNR}_2}}{\text{INR}_{12}} + \left(1-\alpha_1\right)F_2\right) - \frac{1}{2}, \tag{18e}$$

$$b_6 = \frac{1}{2}\log\left(\frac{\overrightarrow{\text{SNR}_2}}{\text{INR}_{12}}\left(\left(1-\alpha_2\right)F_1 + 1\right) + 2\right) - \frac{1}{2}, \tag{18f}$$

$$b_7 = \frac{1}{2}\log\left(\frac{\overrightarrow{\text{SNR}_2}}{\text{INR}_{12}}\left(\left(1-\alpha_2\right)F_1 + 1\right) + \left(1-\alpha_1\right)F_2 + 2\right) - \frac{1}{2}, \tag{18g}$$

For all $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$, consider the following events:

$$E_{1,i}: \text{INR}_{ji} \leqslant \overrightarrow{\text{SNR}}_j, \tag{19a}$$

$$E_{2,i}: \overrightarrow{\text{SNR}}_j \leqslant \text{INR}_{ij}, \tag{19b}$$

$$E_{3,i}: \frac{\overrightarrow{\text{SNR}}_j}{\text{INR}_{ji}} \geqslant \text{INR}_{ij}. \tag{19c}$$

Using events in (19), consider the following events:

$$S_{1,i}: E_{1,i} \wedge E_{2,i}, \tag{20a}$$

$$S_{2,i}: E_{1,i} \wedge \overline{E}_{2,i} \wedge E_{3,i}, \tag{20b}$$

$$S_{3,i}: \overline{E}_{1,i} \wedge E_{2,i}, \tag{20c}$$

$$S_{4,i}: E_{1,i} \wedge \overline{E}_{2,i} \wedge \overline{E}_{3,i}, \tag{20d}$$

$$S_{5,i}: \overline{E}_{1,i} \wedge \overline{E}_{2,i}. \tag{20e}$$

Note that for all $i \in \{1,2\}$ the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ exhibit the property stated by the following lemma.

Lemma 1: For all $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ and for all $(\overrightarrow{\text{SNR}}_1, \overrightarrow{\text{SNR}}_2, \text{INR}_{12}, \text{INR}_{21}) \in \mathbb{R}_+^4$, only one of the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ is true.

Proof: The proof follows from verifying that for $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ are mutually exclusive.

For instance consider that the event $S_{1,i}$ is true. Then $E_{1,i}$ is true and $E_{2,i}$ is true, and therefore $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ are false. This proves that if $S_{1,i}$ is true then $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ are simultaneously false. The same verification can be done for $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$. Finally following the same reasoning it can be verified that if any 4-tuple of the events $\{S_{1,i}, S_{2,i}, S_{3,i}, S_{4,i}, S_{5,i}\}$ is false, the remaining event is necessarily true. This completes the proof. ■

Events (19) and (20) and equalities (21), (22) and (23) are used to define the converse region $\overline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ of the two-user

G-IC-NOF in Theorem 2, and these equalities are defined as follows

$$\kappa_{1,i}=\frac{1}{2}\log\left(A_i+1\right), \tag{21a}$$

$$\kappa_{2,i}=\frac{1}{2}\log\left(1+D_j\right)+\frac{1}{2}\log\left(1+\frac{C_i}{1+D_j}\right), \tag{21b}$$

$$\kappa_{3,i}=\frac{1}{2}\log\left(C_i+1\right)+\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_j\left(C_i+D_j+1\right)}{\left(A_j+1\right)\left(C_i+(1-\rho^2)\right)}+1\right), \tag{21c}$$

$$\kappa_4=\frac{1}{2}\log\left(1+\frac{C_1}{1+D_2}\right)+\frac{1}{2}\log\left(A_2+1\right), \tag{21d}$$

$$\kappa_5=\frac{1}{2}\log\left(1+\frac{C_2}{1+D_1}\right)+\frac{1}{2}\log\left(A_1+1\right), \tag{21e}$$

$$\kappa_6=\begin{cases}\kappa_{6,1} & \text{if } (S_{1,2}\vee S_{2,2}\vee S_{3,2})\wedge\\ & (S_{1,1}\vee S_{2,1}\vee S_{3,1})=\text{True}\\ \kappa_{6,2} & \text{if } (S_{1,2}\vee S_{2,2}\vee S_{3,2})\wedge(S_{4,1}\vee S_{5,1})=\text{True}\\ \kappa_{6,3} & \text{if } (S_{4,2}\vee S_{5,2})\wedge(S_{1,1}\vee S_{2,1}\vee S_{3,1})=\text{True}\\ \kappa_{6,4} & \text{if } (S_{4,2}\vee S_{5,2})\wedge(S_{4,1}\vee S_{5,1})=\text{True}\end{cases}, \tag{21f}$$

$$\kappa_{7,i}=\begin{cases}\kappa_{7,i,1} & \text{if } (S_{1,i}\vee S_{2,i}\vee S_{3,i})=\text{True}\\ \kappa_{7,i,2} & \text{if } (S_{4,i}\vee S_{5,i})=\text{True}\end{cases}, \tag{21g}$$

where,

$$\begin{aligned}\kappa_{6,1}=&\frac{1}{2}\log\left(A_1+D_1\mathrm{INR}_{21}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)\\&+\frac{1}{2}\log\left(1+\frac{D_2\overleftarrow{\mathrm{SNR}}_2}{A_2+1}\right)+\frac{1}{2}\log\left(A_2+D_1\mathrm{INR}_{21}\right)\\&-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)+\frac{1}{2}\log\left(1+\frac{D_1\overleftarrow{\mathrm{SNR}}_1}{A_1+1}\right),\end{aligned} \tag{22a}$$

$$\begin{aligned}\kappa_{6,2}=&\frac{1}{2}\log\left(A_1+D_1\mathrm{INR}_{21}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)\\&+\frac{1}{2}\log\left(1+\frac{D_2}{\overrightarrow{\mathrm{SNR}}_2}\left(\mathrm{INR}_{12}+\frac{\overleftarrow{\mathrm{SNR}}_2\left(B_2\right)}{A_2+1}\right)\right)\\&-\frac{1}{2}\log\left(1+\frac{D_1\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\right)+\frac{1}{2}\log\left(1+\frac{D_1\overleftarrow{\mathrm{SNR}}_1}{A_1+1}\right)\\&+\frac{1}{2}\log\left(G_2+\frac{D_1\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\left(\overrightarrow{\mathrm{SNR}}_2+B_2\right)\right)\\&-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right),\end{aligned} \tag{22b}$$

$$\begin{aligned}\kappa_{6,3}=&\frac{1}{2}\log\left(G_1+\mathrm{INR}_{12}+\frac{D_1\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\left(\overrightarrow{\mathrm{SNR}}_1+B_1\right)\right)\\&-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)+\frac{1}{2}\log\left(1+\frac{D_2\overleftarrow{\mathrm{SNR}}_2}{A_2+1}\right)\\&+\frac{1}{2}\log\left(A_2+D_1\mathrm{INR}_{21}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)\\&+\frac{1}{2}\log\left(1+\frac{D_1}{\overrightarrow{\mathrm{SNR}}_1}\left(\mathrm{INR}_{21}+\frac{\overleftarrow{\mathrm{SNR}}_1\left(B_1\right)}{A_1+1}\right)\right)\\&-\frac{1}{2}\log\left(1+\frac{D_1\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\right),\end{aligned} \tag{22c}$$

$$\begin{aligned}\kappa_{6,4}=&\frac{1}{2}\log\left(G_1+\frac{D_1\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\left(\overrightarrow{\mathrm{SNR}}_1+B_1\right)\right)\\&-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)\\&+\frac{1}{2}\log\left(1+\frac{D_2}{\overrightarrow{\mathrm{SNR}}_2}\left(\mathrm{INR}_{12}+\frac{\overleftarrow{\mathrm{SNR}}_2\left(B_2\right)}{A_2+1}\right)\right)\\&-\frac{1}{2}\log\left(1+\frac{D_1\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\right)-\frac{1}{2}\log\left(1+\frac{D_1\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\right)\\&+\frac{1}{2}\log\left(G_2+\frac{D_1\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\left(\overrightarrow{\mathrm{SNR}}_2+B_2\right)\right)\\&+\frac{1}{2}\log\left(1+\frac{D_1}{\overrightarrow{\mathrm{SNR}}_1}\left(\mathrm{INR}_{21}+\frac{\overleftarrow{\mathrm{SNR}}_1\left(B_1\right)}{A_1+1}\right)\right),\end{aligned} \tag{22d}$$

and

$$\begin{aligned}\kappa_{7,i,1}=&\frac{1}{2}\log\left(A_i+1\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{ij}\right)\\&+\frac{1}{2}\log\left(1+\frac{D_j\overleftarrow{\mathrm{SNR}}_j}{A_j+1}\right)+\frac{1}{2}\log\left(A_j+D_i\mathrm{INR}_{ji}\right)\\&+\frac{1}{2}\log\left(1+C_i+D_j\right)-\frac{1}{2}\log\left(1+D_j\right),\end{aligned} \tag{23a}$$

$$\begin{aligned}\kappa_{7,i,2}=&\frac{1}{2}\log\left(A_i+1\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{ij}\right)-\frac{1}{2}\log\left(1+D_j\right)\\&+\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\frac{\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\left(\mathrm{INR}_{ij}+\right.\right.\\&\left.\left.\frac{\overleftarrow{\mathrm{SNR}}_j\left(B_j\right)}{A_j+1}\right)\right)-\frac{1}{2}\log\left(1+\frac{D_i\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\right)\\&+\frac{1}{2}\log\left(G_j+\frac{D_i\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\left(\overrightarrow{\mathrm{SNR}}_j+B_j\right)\right)\\&+\frac{1}{2}\log\left(1+C_i+D_j\right).\end{aligned} \tag{23b}$$

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