



HAL
open science

Integro-partial differential equations with singular terminal condition

A Popier

► **To cite this version:**

A Popier. Integro-partial differential equations with singular terminal condition. 2017. hal-01293775v2

HAL Id: hal-01293775

<https://hal.science/hal-01293775v2>

Preprint submitted on 31 Jan 2017 (v2), last revised 31 Jan 2017 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Integro-partial differential equations with singular terminal condition.

A. Popier

LUNAM Université, Université du Maine,
Laboratoire Manceau de Mathématiques,
Avenue O. Messiaen, 72085 Le Mans cedex 9
France

January 31, 2017

Abstract

In this paper, we show that the minimal solution of a backward differential equation gives a probabilistic representation of the minimal viscosity solution of an integro-partial differential equation both with a singular terminal condition. Singularity means that at the final time, the value of the solution can be equal to infinity. Different types of regularity of this viscosity solution are investigated: Sobolev, Hölder or strong regularity.

Keywords. Integro-partial differential equations, viscosity solution, backward stochastic differential equation, singular condition.

AMS 2010 classification. 35R09, 35D40, 60G99, 60H30, 60J75.

Introduction

The notion of backward stochastic differential equations (BSDEs) was first introduced in [12] in the linear setting and in [38] for non linear equation. One particular interest for the study of BSDE is the application to partial differential equations (PDEs). Indeed as proved in the paper [37], BSDEs can be seen as generalization of the Feynman-Kac formula for non linear PDEs. Since then a large literature has been developed on this topic (see in particular the books [19], [47], [17], [39] and the references therein).

In this context the considered PDEs are semi-linear, i.e. of the following form:

$$(1) \quad \frac{\partial u}{\partial t}(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), \nabla u(t, x)\sigma(t, x)) = 0$$

where $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the unknown function, b and σ are given functions¹ defined on \mathbb{R}^d and with values respectively in \mathbb{R}^d and $\mathbb{R}^{d \times d}$. \mathcal{L} is a second-order differential operator defined by

$$(2) \quad \mathcal{L}\phi = (\nabla u)b + \frac{1}{2}\text{Trace}((D^2u)\sigma\sigma^*).$$

The notations ∇ and D^2 are respectively the gradient and the Hessian matrix w.r.t. x . Moreover the solution u should satisfy the terminal condition $u(T, x) = g(x)$. When f is equal to zero, the Feynman-Kac formula states that the solution u is given by:

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad u(t, x) = \mathbb{E}(g(X_T^{t,x})),$$

where $X^{t,x}$ is the solution of a (forward) stochastic differential equation (SDE): for any $0 \leq t \leq s \leq T$ and $x \in \mathbb{R}^d$

$$(3) \quad X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r.$$

W is a d -dimensional Brownian motion. In general, the probabilistic representation of the solution u is given by a couple of two SDEs, the forward SDE (3) and a backward equation: for any $0 \leq t \leq s \leq T$ and $x \in \mathbb{R}^d$

$$(4) \quad Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r.$$

The main result of [37] can be summarized by the equality:

$$u(t, x) = Y_t^{t,x}.$$

Roughly speaking, if we can solve the system (3) and (4), then $Y_t^{t,x}$ is a deterministic function and is a (weak) solution of (1). This is a method of characteristics to solve the parabolic PDE. The converse assertion can be proved provided we can apply Itô's formula. If the solution u is enough regular, then Itô's formula gives: $Y_s^{t,x} = u(s, X_s^{t,x})$. See for example [9] for more details. The extension to quasi-linear PDEs (when the functions b and σ depend on u and ∇u) or to fully non linear PDEs has been already developed (see among other [32] and [48]).

Here we are interesting in another development of the theory: the case of Poisson random noise. In [5], Barles et al. show that we can add in the system (3)–(4) a Poisson random measure μ :

$$(5) \quad \begin{cases} X_s^{t,x} &= x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r + \int_t^s \int_E \beta(X_{r-}^{t,x}, e)\tilde{\mu}(de, dr) \\ Y_s^{t,x} &= g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, U_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r \\ &\quad - \int_s^T \int_E U_r^{t,x}(e)\tilde{\mu}(de, dr). \end{cases}$$

¹ b and σ can also depend on time with straightforward modifications in the proofs.

Here $\tilde{\mu}(de, ds) = \mu(de, ds) - \lambda(de)ds$ is the martingale measure related to μ , λ being the characteristic measure of μ . And if we can find a solution $(X^{t,x}, Y^{t,x}, Z^{t,x}, U^{t,x})$ to this problem, then again $u(t, x) = Y_t^{t,x}$ will be a weak solution of a integro-partial differential equation (IPDE):

$$(6) \quad \frac{\partial}{\partial t}u(t, x) + \mathcal{L}u(t, x) + \mathcal{I}(t, x, u) + f(t, x, u, (\nabla u)\sigma, \mathcal{B}(t, x, u)) = 0$$

where \mathcal{L} is given by (2) and \mathcal{I} and \mathcal{B} are two integro-differential operators²:

$$\begin{aligned} \mathcal{I}(t, x, \phi) &= \int_E [\phi(x + \beta(x, e)) - \phi(x) - (\nabla\phi)(x)\beta(x, e)]\lambda(de) \\ \mathcal{B}(t, x, \phi) &= \int_E [\phi(x + \beta(t, x, e)) - \phi(x)]\gamma(x, e)\lambda(de). \end{aligned}$$

The term γ is related to the generator f which is supposed to have the special form:

$$f(t, x, y, z, u) = f\left(t, x, y, z, \int_E u(e)\gamma(x, e)\lambda(de)\right).$$

In [5], weak solution means viscosity solution. Since this paper, several authors have weakened the assumptions of [5]. The books [47] (Section 8.17) and [17] (Chapter 4) give a nice review of these results (and several references on this topic). Let us mention that in [34] the authors obtain a similar result, but for Sobolev-type solution of the IPDE (6).

Among all semi-linear PDEs like (1), a particular form has been widely studied:

$$(7) \quad \frac{\partial u}{\partial t}(t, x) + \mathcal{L}u(t, x) - u(t, x)|u(t, x)|^q = 0.$$

Baras & Pierre [3], Marcus & Veron [33] (and many other papers) have given existence and uniqueness results for this PDE. In [33] it is shown that every positive solution of (7) possesses a uniquely determined final trace g which can be represented by a couple (\mathcal{S}, μ) where \mathcal{S} is a closed subset of \mathbb{R}^d and μ a non negative Radon measure on $\mathcal{R} = \mathbb{R}^d \setminus \mathcal{S}$. The final trace can also be represented by a positive, outer regular Borel measure ν , and ν is not necessary locally bounded. The two representations are related by:

$$\forall A \subset \mathbb{R}^m, A \text{ Borel}, \begin{cases} \nu(A) = \infty & \text{if } A \cap \mathcal{S} \neq \emptyset \\ \nu(A) = \mu(A) & \text{if } A \subset \mathcal{R}. \end{cases}$$

The set \mathcal{S} is the set of singular final points of u and it corresponds to a “blow-up” set of u . From the probabilistic point of view Dynkin & Kuznetsov [18] and Le Gall [31] have proved similiary results for the PDE (7) in the case $0 < q \leq 1$: they use the theory of superprocesses.

Now if we want to represent u using a FBSDE, we have to deal with a *singular* terminal condition ξ in (5), which means that $\mathbb{P}(\xi = +\infty) > 0$. This singular case has been studied first in [41] and recently it was used to solve a stochastic control problem for portfolio

²Again β and γ can be time-depend, the results would not be changed.

liquidation (see [2] or [23]). In [28] we enlarge the known results on this subject in several directions: more general generator f (than $f(y) = -y|y|^q$) and almost no assumption on the filtration \mathbb{F} (instead of the Brownian-Poisson filtration). Finally in [41] we established a link between the solution Y of the BSDE with singular terminal condition and the viscosity solution u of the PDE (7).

In this paper we generalize the results of [41] and using our recent papers [28] we want to study of the related IPDE (6) when the terminal condition $u(T, \cdot) = g$ is *singular* in the sense that g takes values in $\mathbb{R}_+ \cup \{+\infty\}$ and the set

$$\mathcal{S} = \{x \in \mathbb{R}^d, \quad g(x) = +\infty\}$$

is a non empty closed subset of \mathbb{R}^d . Again in the non singular case, if the terminal function g is of linear growth, the relation $u(t, x) = Y_t^{t,x}$ is also obtained in [5]. Moreover several papers have studied the existence and the uniqueness of the solution of such IPDE (see among others [1], [8], [10] or [25]). To our best knowledges **the study of (6) with a singularity at time T** is completely new. There is no probabilistic representation of such IPDE using superprocesses and no deterministic works on this topic. In the PhD thesis of Piozin, we have studied the case when $f(t, y, z, u) = f(y) = -y|y|^q$. Hence the aim of the paper is to prove that this minimal solution Y of the singular BSDE is the probabilistic representation of the minimal positive viscosity solution u of the IPDE for general function f with a singular terminal condition.

One applied motivation for this study is developed in [24] and [28]. Indeed the optimal solution of a stochastic control problem with constraint is the minimal solution of the FBSDE (5). The value function v and the optimal state can be computed directly with Y (Z and U are not involved here). In other words v will be the minimal viscosity solution of a IPDE with singular terminal condition.

The paper is organized as follows. In the first part we describe the mathematical setting. Since we are interesting in singular terminal condition, the generator f of the BSDE has to satisfy special conditions, in particular **C8**. When y becomes large, the function $y \mapsto f(t, x, y, z, u) - f(t, x, y, 0, 0)$ decreases at least like $-y|y|^q$. We recall the precise result concerning the link between (5) and (6) when the terminal condition is non singular. From the system (5) we get a continuous viscosity solution of the equation (6).

In the second section, we show that the minimal solution Y of the BSDE provides the minimal viscosity solution u for the IPDE (see Theorem 3 in Section 2). In details we show that $Y_t^{t,x} = u(t, x)$ is a (discontinuous) viscosity solution of (6) on any interval $[0, T - \varepsilon]$ for $\varepsilon > 0$ with $\liminf_{t \rightarrow T} u(t, x) \geq g(x)$. Here we mainly use a stability result on viscosity solutions: roughly speaking an increasing sequence of viscosity solutions is itself a viscosity solution. Extra assumptions (denoted by **(D)**) are supposed to prove that $u(t, x)$ converges to $g(x)$ as t goes to T . Then we prove minimality of this solution, which requires a comparison result for viscosity solution for IPDE adapted for our setting.

The last part is devoted to study the regularity of this minimal solution on $[0, T - \varepsilon] \times \mathbb{R}^d$ for $\varepsilon > 0$. Indeed the minimal viscosity solution constructed before is the increasing limit of continuous functions. Hence it is lower semicontinuous, but the continuity is an open

question. Therefore we give several conditions on the coefficients of the forward SDE and on the Lévy measure λ in order to obtain:

- Sobolev-type regularity: u and ∇u are in some \mathbb{L}^2 weighted space (Proposition 3). Only the coefficients of the forward SDE are supposed to be regular.
- Hölder regularity of u (Proposition 4). We will impose some conditions on λ , but no additional regularity condition on the parameters.
- Classical regularity: u is of class $C^{1,2}$ on $[0, T] \times \mathbb{R}^d$ (Proposition 5). The matrix diffusion σ is supposed to be uniformly elliptic and λ is not too singular on 0.

Of course we do not claim that we are exhaustive. Our conditions are quite classical and widely used. But different sets of assumptions could be also used to obtain similar results.

1 Setting and known results

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$. We assume that this set supports a d -dimensional Brownian motion W and a Poisson random measure μ with intensity $\lambda(de)dt$ on the space $E \subset \mathbb{R}^{d'} \setminus \{0\}$. The filtration \mathbb{F} is generated by W and μ . We will denote \mathcal{E} the Borelian σ -field of E and $\tilde{\mu}$ is the compensated measure: for any $A \in \mathcal{E}$ such that $\lambda(A) < +\infty$, then $\tilde{\mu}([0, t] \times A) = \mu([0, t] \times A) - t\lambda(A)$ is a martingale. The measure λ is σ -finite on (E, \mathcal{E}) satisfying

$$\int_E (1 \wedge |e|^2) \lambda(de) < +\infty.$$

In this paper for a given $T \geq 0$, we denote:

- \mathcal{P} : the predictable σ -field on $\Omega \times [0, T]$ and

$$\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{E}.$$

- On $\tilde{\Omega} = \Omega \times [0, T] \times E$, a function that is $\tilde{\mathcal{P}}$ -measurable, is called predictable. $G_{loc}(\mu)$ is the set of $\tilde{\mathcal{P}}$ -measurable functions ψ on $\tilde{\Omega}$ such that for any $t \geq 0$ a.s.

$$\int_0^t \int_E (|\psi_s(e)|^2 \wedge |\psi_s(e)|) \lambda(de) < +\infty.$$

- \mathcal{D} (resp. $\mathcal{D}(0, T)$): the set of all predictable processes on \mathbb{R}_+ (resp. on $[0, T]$). $L_{loc}^2(W)$ is the subspace of \mathcal{D} such that for any $t \geq 0$ a.s.

$$\int_0^t |Z_s|^2 ds < +\infty.$$

We refer to [26] for details on random measures and stochastic integrals. On \mathbb{R}^d , $|\cdot|$ denotes the Euclidean norm and $\mathbb{R}^{d \times d'}$ is identified with the space of real matrices with d rows and d' columns. If $z \in \mathbb{R}^{d \times d'}$, we have $|z|^2 = \text{trace}(zz^*)$.

Now to define the solution of our BSDE, let us introduce the following spaces for $p \geq 1$.

- $\mathbb{D}^p(0, T)$ is the space of all adapted càdlàg processes X such that

$$\mathbb{E} \left(\sup_{t \in [0, T]} |X_t|^p \right) < +\infty.$$

For simplicity, $X_* = \sup_{t \in [0, T]} |X_t|$.

- $\mathbb{H}^p(0, T)$ is the subspace of all processes $X \in \mathcal{D}(0, T)$ such that

$$\mathbb{E} \left[\left(\int_0^T |X_t|^2 dt \right)^{p/2} \right] < +\infty.$$

- $\mathbb{L}_\mu^p(0, T) = \mathbb{L}_\mu^p(\Omega \times (0, T) \times E)$: the set of processes $\psi \in G_{loc}(\mu)$ such that

$$\mathbb{E} \left[\left(\int_0^T \int_E |\psi_s(e)|^2 \lambda(de) ds \right)^{p/2} \right] < +\infty.$$

- $\mathbb{L}_\lambda^p(E) = \mathbb{L}^p(E, \lambda; \mathbb{R}^m)$: the set of measurable functions $\psi : E \rightarrow \mathbb{R}^m$ such that

$$\|\psi\|_{\mathbb{L}_\lambda^p}^p = \int_E |\psi(e)|^p \lambda(de) < +\infty.$$

- $\mathbb{S}^p(0, T) = \mathbb{D}^p(0, T) \times \mathbb{H}^p(0, T) \times \mathbb{L}_\mu^p(0, T)$.

Concerning function spaces, in the sequel $\Pi_{pg}(0, T)$ will denote the space of functions $\phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$ of polynomial growth, i.e. for some non negative constants δ and C

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |\phi(t, x)| \leq C(1 + |x|^\delta).$$

For a continuous function $\phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $\alpha \in [0, 1)$, we define

$$\begin{aligned} \|\phi\|_\infty &= \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} |\phi(t, x)|, \\ \|\phi\|_\alpha &= \sup_{(t, x) \neq (s, y), |x-y| \leq 1} \frac{|\phi(t, x) - \phi(s, y)|}{|t-s|^{\alpha/2} + |x-y|^\alpha}. \end{aligned}$$

For $k \in \mathbb{N}$, $C^{k, 2k} = C^{k, 2k}([0, T] \times \mathbb{R}^d)$ is the subset of continuous functions $\phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ whose partial derivatives of order less than or equal to k w.r.t. t and $2k$ w.r.t. x are continuous on $[0, T] \times \mathbb{R}^d$. For $\alpha \in [0, 1)$, the set $H^{k+\alpha/2, 2k+\alpha}$ is the subset of $C^{k, 2k}$ such that $\|\partial_t^k \phi\|_\alpha + \|\partial_x^{2k} \phi\|_\alpha < +\infty$. We denote $C_{l, b}^k(\mathbb{R}^d)$ the set of C^k -functions which grow at most linearly at infinity and whose partial derivatives of order less than or equal to k are bounded.

1.1 Our forward backward SDE, assumptions on the coefficients

First of all we consider the forward SDE: for any $0 \leq t \leq s \leq T$ and any $x \in \mathbb{R}^d$

$$(8) \quad X_s^{t,x} = x + \int_t^s b(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dW_r + \int_t^s \int_E \beta(X_{r-}^{t,x}, e) \tilde{\mu}(de, dr).$$

Moreover for $0 \leq s < t$, $X_s^{t,x} = x$. The coefficients $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times k}$ and $\beta : \mathbb{R}^d \times E \rightarrow \mathbb{R}^d$ satisfy the following conditions.

A1. b and σ are Lipschitz continuous w.r.t. x , i.e. there exists a constant $K_{b,\sigma}$ such that for any x and y in \mathbb{R}^d :

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq K_{b,\sigma} |x - y|$$

A2. β is Lipschitz continuous w.r.t. x uniformly in e , i.e. there exists a constant K_β such that for all $e \in E$, for any x and y in \mathbb{R}^d :

$$|\beta(x, e) - \beta(y, e)| \leq K_\beta |x - y| (1 \wedge |e|).$$

A3. b and σ growth at most linearly:

$$|b(x)| + |\sigma(x)| \leq C_{b,\sigma} (1 + |x|).$$

A4. β is bounded w.r.t. x and there exists a constant C_β such that

$$|\beta(x, e)| \leq C_\beta (1 \wedge |e|).$$

Under these assumptions, the forward SDE (8) has a unique strong solution $X^{t,x} = \{X_s^{t,x}, t \leq s \leq T\}$ (see [35] or [43]). Moreover for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $p \geq 2$

$$(9) \quad \mathbb{E} \left[\sup_{t \leq s \leq T} |X_s^{t,x} - x|^p \right] \leq C(1 + |x|^p)(T - t).$$

In Section 3, we will also assume that the next conditions hold.

A5. σ and b are bounded: there exists a constant C s.t.

$$\forall x \in \mathbb{R}^d, \quad |b(x)| + |\sigma(x)| \leq C;$$

A6. $\sigma\sigma^*$ is uniformly elliptic, i.e. there exists $\Lambda_0 > 0$ s.t. for all $x \in \mathbb{R}^d$:

$$\forall y \in \mathbb{R}^d, \quad \sigma\sigma^*(x)y \cdot y \geq \Lambda_0 |y|^2.$$

The use of these two hypotheses will be precised when necessary.

Definition 1 (Assumptions (A) and (A+)) *When Conditions A1, A2, A3 and A4 hold, f verifies Conditions (A). When the coefficients satisfy (A) with A5 and A6, f satisfies Assumptions (A+).*

The terminal condition ξ of the BSDE will satisfy several assumptions, denoted by **Conditions (B)**.

B1. There exists a function g defined on \mathbb{R}^d with values in $\mathbb{R}_+ \cup \{+\infty\}$ such that

$$\xi = g(X_T^{t,x}).$$

We denote

$$\mathcal{S} := \{x \in \mathbb{R}^d \text{ s.t. } g(x) = \infty\}$$

the set of singularity points for the terminal condition induced by g . This set \mathcal{S} is supposed to be non empty and closed. We also denote by $\partial\mathcal{S}$ the boundary of \mathcal{S} .

B2. Integrability condition:

$$g(X_T^{t,x})\mathbf{1}_{\mathbb{R}^d \setminus \mathcal{S}}(X_T^{t,x}) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}).$$

B3. Continuity condition: g is continuous from \mathbb{R}^d to $\mathbb{R}_+ \cup \{+\infty\}$.

This last assertion implies that for any $n \in \mathbb{N}^*$, $x \mapsto g_n(x) = g(x) \wedge n$ is a continuous function on \mathbb{R}^d .

Now we consider the BSDE: for any $t \leq s \leq T$

$$(10) \quad Y_s^{t,x} = \xi + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, U_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r - \int_s^T \int_E U_r^{t,x}(e)\tilde{\mu}(de, dr).$$

The generator f of the BSDE (10) is a deterministic function $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{L}_\lambda^2 \rightarrow \mathbb{R}$. The unknowns are $(Y^{t,x}, Z^{t,x}, U^{t,x})$ such that

- $Y^{t,x}$ is progressively measurable and càdlàg with values in \mathbb{R} ;
- $Z^{t,x} \in L_{loc}^2(W)$, with values in \mathbb{R}^d ;
- $U^{t,x} \in G_{loc}(\mu)$ with values in \mathbb{R} .

The BSDE is called *singular* since the probability $\mathbb{P}(\xi = +\infty)$ can be positive.

First the function f has the special structure for u in \mathbb{L}_λ^2 .

C1. There exists a function γ from $\mathbb{R}^d \times E$ to \mathbb{R} such that

$$f(t, x, y, z, u) = f\left(t, x, y, z, \int_E u(e)\gamma(x, e)\lambda(de)\right).$$

For simplicity we denote with the same function f the right and the left hand side. For notational convenience we will denote $f_r^0 = f_r^{0,t,x} = f(r, X_r^{t,x}, 0, 0, 0)$.

C2. The process $f^{0,t,x}$ is non negative for any $(t, x) \in [0, T] \times \mathbb{R}^d$.

C3. The function $y \mapsto f(t, x, y, z, u)$ is monotone: there exists $\chi \in \mathbb{R}$ such that for any $t \in [0, T]$, $x \in \mathbb{R}^d$, $z \in \mathbb{R}^k$ and $u \in \mathbb{R}$

$$(f(t, x, y, z, u) - f(t, x, y', z, u))(y - y') \leq \chi(y - y')^2.$$

C4. f is locally Lipschitz continuous w.r.t. y : for all $R > 0$, there exists L_R such that for any y and y' and any (t, x, z, u)

$$|y| \leq R, |y'| \leq R \implies |f(t, x, y, z, u) - f(t, x, y', z, u)| \leq L_R |y - y'|.$$

C5. f is Lipschitz in z , uniformly w.r.t. all parameters: there exists $L > 0$ such that for any (t, x, y, u) , z and z' :

$$|f(t, x, y, z, u) - f(t, x, y, z', u)| \leq L|z - z'|.$$

C6. The function $u \in \mathbb{R} \mapsto f(t, x, y, z, u)$ is Lipschitz and non decreasing for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^k$:

$$\forall u \leq u', \quad 0 \leq f(t, x, y, z, u') - f(t, x, y, z, u) \leq L(u' - u).$$

C7. There exists a function $\vartheta \in \mathbb{L}_\lambda^2$ such that for all $(x, e) \in \mathbb{R}^d \times E$

$$0 \leq \gamma(x, e) \leq \vartheta(e).$$

Since the terminal condition may be singular, to ensure that the solution component Y attains the value ∞ on \mathcal{S} at time T but is finite a.s. before time T , we suppose that

C8. There exists a constant $q > 0$ and a positive function $a : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that for any $y \geq 0$

$$f(r, X_r^{t,x}, y, z, q) \leq -a(r, X_r^{t,x})y|y|^q + f(r, X_r^{t,x}, 0, z, q).$$

$p = 1 + \frac{1}{q}$ is the Hölder conjugate of $1 + q$. Moreover, in order to derive the a priori estimate, the following assumptions will hold.

C9. The function

$$(t, x) \mapsto \frac{1}{a(t, x)^{1/q}} + |f(t, x, 0, 0, 0)|$$

belongs to $\Pi_{pg}(0, T)$.

C10. There exists $\ell > 1$ such that the function ϑ in **C5** belongs to $\mathbb{L}_\lambda^{\tilde{\ell}}$ with $\tilde{\ell} = \ell/(\ell - 1)$.

Again to lighten the notations, $a(r, X_r^{t,x})$ will be denoted a_r or $a_r^{t,x}$ if we do not need to precise the variables t and x .

Since we want to work on the link with IPDE, in order to use the work of Barles et al. [5], we need extra assumptions on the regularity of f w.r.t. t and x .

C11. The function $t \mapsto f(t, x, y, z, u)$ is continuous on $[0, T]$.

C12. For all $R > 0$, $t \in [0, T]$, $|x| \leq R$, $|x'| \leq R$, $|y| \leq R$, $z \in \mathbb{R}^k$, $u \in \mathbb{R}$,

$$|f(t, x, y, z, u) - f(t, x', y, z, u)| \leq \varpi_R(|x - x'|)(1 + |z|),$$

where $\varpi_R(s) \rightarrow 0$ when $s \searrow 0$.

C13. There exists $C_\gamma > 0$ such that for all $(x, x') \in (\mathbb{R}^d)^2$, $e \in E$,

$$|\gamma(x, e) - \gamma(x', e)| \leq C_\gamma |x - x'| (1 \wedge |e|^2).$$

Definition 2 (Conditions (C)) *If f satisfies all conditions C1 to C13, we say that f verifies Conditions (C).*

1.2 Comments on the hypotheses (C) and examples

The previous list is rather long. It is the union of the conditions of [5] and [28]. Let us clarify several points. The condition **C1** is classical (see [5], [17], [25], etc.)

The conditions **C2** to **C7** are assumed in [27] to ensure that if ξ and f_r^0 are in L^p for some $p > 1$, the BSDE (10) has a unique solution in $\mathbb{S}^p(0, T)$. Indeed by **C4**, for every $n > 0$ the function

$$\sup_{|y| \leq n} |f(r, X_r^{t,x}, y, 0, 0) - f_r^0| \leq nL_n$$

is bounded on $[0, T]$ and thus in $L^1(0, T)$.

Lemma 1 *Under Hypotheses C6 and C7, for all $(t, x, y, z, u, v) \in [0, T] \times \mathbb{R}^{d+1+k} \times (\mathbb{L}_\lambda^2)^2$, there exists a progressively measurable process $\kappa = \kappa^{t,x,y,z,u,v} : \Omega \times \mathbb{R}_+ \times E \rightarrow \mathbb{R}$ such that*

$$(11) \quad f(r, X_r^{t,x}, y, z, u) - f(r, X_r^{t,x}, y, z, v) \leq \int_E (u(e) - v(e)) \kappa_r^{t,x,y,z,u,v}(e) \lambda(de)$$

with $\mathbb{P} \otimes \text{Leb} \otimes \lambda$ -a.e. for any (t, x, y, z, u, v) , $0 \leq \kappa_t^{t,x,y,z,u,v}(e)$ and $|\kappa_t^{t,x,y,z,u,v}(e)| \leq \vartheta(e)$.

Proof. From Hypotheses **C1** and **C6**, we have

$$\begin{aligned} & f(r, X_r^{t,x}, y, z, u) - f(r, X_r^{t,x}, y, z, v) \\ &= f\left(r, X_r^{t,x}, y, z, \int_E u(e) \gamma(x, e) \lambda(de)\right) - f\left(r, X_r^{t,x}, y, z, \int_E v(e) \gamma(x, e) \lambda(de)\right) \\ &= \int_E (u(e) - v(e)) F_r^{t,x,y,z,u,v} \gamma(x, e) \lambda(de) \\ &= \int_E (u(e) - v(e)) \kappa_r^{t,x,y,z,u,v}(e) \lambda(de), \end{aligned}$$

with

$$F_r^{t,x,y,z,u,v} = \frac{f\left(r, X_r^{t,x}, y, z, \int_E u(e) \gamma(x, e) \lambda(de)\right) - f\left(r, X_r^{t,x}, y, z, \int_E v(e) \gamma(x, e) \lambda(de)\right)}{\int_E (u(e) - v(e)) \gamma(x, e) \lambda(de)}$$

and

$$\kappa_t^{x,y,z,u,v}(e) = F_t^{t,x,y,z,u,v} \gamma(x, e).$$

Since f is non decreasing and from **C7**, $\kappa_r^{t,x,y,z,u,v}(e) \geq 0$ and from the Lipschitz condition,

$$|\kappa_r^{t,x,y,z,u,v}(e)| \leq L\vartheta(e).$$

This achieves the proof. \square

The previous lemma implies that f is Lipschitz continuous w.r.t. u uniformly in ω , t , y and z :

$$|f(t, x, y, z, u) - f(t, x, y, z, v)| \leq L\|\vartheta\|_{L_\lambda^2}\|u - v\|_{L_\lambda^2}.$$

Hence we can apply Theorems 1 and 2, together with Proposition 2 in [27] and deduce the existence and the uniqueness of the solution of the BSDE under suitable integrability conditions on ξ and f^0 . Moreover we can compare two solutions of the BSDE (10) with different terminal conditions (see Theorem 4.1 and Assumption 4.1 in [44] or Proposition 4 in [27]).

Remark 1 *By very classical arguments we can suppose w.l.o.g. that $\chi = 0$ in **C3**.*

The assumptions **C8**, **C9** and **C10** are used to deal with singular terminal condition ξ , that is when $\mathbb{P}(\xi = +\infty) > 0$.

Lemma 2 *For any $\eta > 0$ and $\ell > 0$*

$$(12) \quad \mathbb{E} \int_0^T (T-s)^{-1+\eta} \left[\left(\frac{1}{qa_s} \right)^{1/q} + (T-s)^{1+1/q} f_s^0 \right]^\ell ds < +\infty.$$

Proof. From integrability property (9) of X , **C9** implies that $a^{-1/q}$ and f^0 belongs to any $L^\delta((0, T) \times \Omega)$ for any $\delta > 1$. Hence:

$$\begin{aligned} & \mathbb{E} \int_0^T (T-s)^{-1+\eta} \left[\left(\frac{1}{qa_s^{t,x}} \right)^{1/q} + (T-s)^{1+1/q} f_s^{0,t,x} \right]^\ell ds \\ & \leq C(1 + |x|^{\delta\ell}) \mathbb{E} \int_0^T (T-s)^{-1+\eta} ds < +\infty \end{aligned}$$

for any $0 < \eta$. \square

This lemma implies in particular that there exists $\ell > 1$ such that

$$\mathbb{E} \int_0^T \left[\left(\frac{1}{qa(r, X_r^{t,x})} \right)^{1/q} + (T-r)^{1+1/q} f_r^{0,t,x} \right]^\ell dr < +\infty$$

and from **C10**, ϑ is in $\mathbb{L}_\lambda^{\tilde{\ell}}$, where $\tilde{\ell}$ is the Hölder conjugate of ℓ .

Remark 2 *Assumptions **C8** and **C9** imply that the function a must be bounded.*

Finally with Condition **C11**, we will deduce existence of a viscosity solution for the IPDE as in [5], whereas **C12** and **C13** are assumed to ensure uniqueness of the viscosity solution.

Now let us give two examples of generators f satisfying Conditions **(C)**.

- Assume that $y \mapsto f(y)$ is a non increasing function of class C^1 , with $f(0) \geq 0$ and such that for some constant $a > 0$ and any $y \geq 0$: $f(y) - f(0) \leq -ay|y|^q$. Then **(C)** holds. In particular, $f(y) = -y|y|^q$ (for some $q > 0$) is a classical example.
- In [28] the generator related to the optimal closure portfolio strategy is given by:

$$f(t, x, y, u) = -\frac{y|y|^q}{q\eta(t, x)^q} + f^0(t, x)$$

The parameter $\eta > 0$ is the price impact parameter and $f^0 \geq 0$ is the risk measure of the open position. Here $a(t, x) = -\frac{1}{q\eta(t, x)^q}$ and η and f^0 are continuous functions of polynomial growth.

1.3 A first link with viscosity solution of a IPDE

First assume that $(t, x) \in [0, T] \times \mathbb{R}^d$ is fixed. Under Conditions **C1** to **C7**, from Theorems 1 or 2 in [27], there exists a unique solution for the truncated version of BSDE (10), where the terminal condition ξ is replaced by $\xi \wedge n$ and where the generator f is replaced by f_n for some $n > 0$:

$$f_n(r, y, z, u) = (f(r, X_r^{t,x}, y, z, u) - f_r^0) + (f_r^0 \wedge n).$$

The solution of this truncated BSDE will be denoted by $(Y^{n,t,x}, Z^{n,t,x}, U^{n,t,x})$: for any $t \leq s \leq T$

$$(13) \quad \begin{aligned} Y_s^{n,t,x} &= \xi \wedge n + \int_s^T f_n(r, Y_r^{n,t,x}, Z_r^{n,t,x}, U_r^{n,t,x}) dr - \int_s^T Z_r^{n,t,x} dW_r \\ &\quad - \int_s^T \int_E U_r^{n,t,x}(e) \tilde{\mu}(dr, de). \end{aligned}$$

Moreover $(Y^{n,t,x}, Z^{n,t,x}, U^{n,t,x}) \in \mathbb{S}^\delta(0, T)$ for any $\delta > 1$.

If **(C)** holds, we work with almost the same setting as in [5]. The only difference is that f is not Lipschitz continuous w.r.t. y . But for a fixed n , a straightforward consequence of the comparison principle for BSDE implies that $Y^{n,t,x}$ is bounded by $n(T + 1)$ (see Proposition 4 in [27]). We can replace in the BSDE (13) our generator f_n by \widehat{f}_n with $\widehat{f}_n(t, x, y, z, q) = f_n(t, x, \mathcal{T}_n(y), z, q)$ with $\mathcal{T}_n(y) = (n(T + 1)y)/(|y| \vee n(T + 1))$. From Condition **C4**, \widehat{f}_n is Lipschitz w.r.t. y .

We will use the notion of viscosity solution of the IPDE (6). The reason will be clearer later. For a locally bounded function v in $[0, T] \times \mathbb{R}^d$, we define its upper (resp. lower) semicontinuous envelope v^* (resp. v_*) by:

$$v^*(t, x) = \limsup_{(s,y) \rightarrow (t,x)} v(s, y) \quad (\text{resp. } v_*(t, x) = \liminf_{(s,y) \rightarrow (t,x)} v(s, y)).$$

For such equation (6) we introduce the notion of viscosity solution as in [1] (see also Definition 3.1 in [5] or Definitions 1 and 2 in [8]). Since we do not assume the continuity of the involved function u , we adapt the definition of discontinuous viscosity solution (see Definition 4.1 and 5.1 in [25]).

Definition 3 A locally bounded function v is

1. a **viscosity subsolution** of (6) if it is upper semicontinuous on $[0, T] \times \mathbb{R}^d$ and if

$$v(T, x) \leq h(x), \quad x \in \mathbb{R}^d,$$

and if for any $\phi \in C^2([0, T] \times \mathbb{R}^d)$ wherever $(t, x) \in [0, T[\times \mathbb{R}^d$ is a global maximum point of $v - \phi$,

$$-\frac{\partial}{\partial t}\phi(t, x) - \mathcal{L}\phi(t, x) - \mathcal{I}(t, x, \phi) - f(t, x, v, (\nabla\phi)\sigma, \mathcal{B}(t, x, \phi)) \leq 0.$$

2. a **viscosity supersolution** of (6) if it is lower semicontinuous on $[0, T] \times \mathbb{R}^d$ and if

$$v(T, x) \geq h(x), \quad x \in \mathbb{R}^d,$$

and if for any $\phi \in C^2([0, T] \times \mathbb{R}^d)$ wherever $(t, x) \in [0, T[\times \mathbb{R}^d$ is a global minimum point of $v - \phi$,

$$-\frac{\partial}{\partial t}\phi(t, x) - \mathcal{L}\phi(t, x) - \mathcal{I}(t, x, \phi) - f(t, x, v, (\nabla\phi)\sigma, \mathcal{B}(t, x, \phi)) \geq 0.$$

3. a **viscosity solution** of (6) if its upper envelope v^* is a subsolution and if its lower envelope v_* is a supersolution of (6).

This definition is equivalent to Definition 4.1 in [25]. Note that if a comparison principle holds for (6), then $v_* = v^*$ and thus a viscosity solution is a continuous function. We can also give another definition like Definition 5.1 in [25]. For any $\delta > 0$, the operators \mathcal{I} and \mathcal{B} will be split in two parts:

$$\begin{aligned} \mathcal{I}^{1,\delta}(t, x, \phi) &= \int_{|e| \leq \delta} [\phi(t, x + \beta(x, e)) - \phi(t, x) - (\nabla\phi)(t, x)\beta(x, e)]\lambda(de) \\ \mathcal{I}^{2,\delta}(t, x, p, \phi) &= \int_{|e| > \delta} [\phi(t, x + \beta(x, e)) - \phi(t, x) - p\beta(x, e)]\lambda(de), \\ \mathcal{B}^\delta(t, x, \phi, v) &= \int_{|e| \leq \delta} [\phi(t, x + \beta(x, e)) - \phi(t, x)]\gamma(x, e)\lambda(de) \\ &\quad + \int_{|e| > \delta} [v(t, x + \beta(x, e)) - v(t, x)]\gamma(x, e)\lambda(de). \end{aligned}$$

Definition 4 A locally bounded and upper (resp. lower) semicontinuous function v is a **viscosity sub (resp. super) solution** of (6) if

$$v(T, x) \leq h(x) \quad (\text{resp. } v(T, x) \geq h(x)), \quad x \in \mathbb{R}^d,$$

and if for any $\delta > 0$, for any $\phi \in C^2([0, T] \times \mathbb{R}^d)$ wherever $(t, x) \in [0, T[\times \mathbb{R}^d$ is a global maximum (resp. minimum) point of $v - \phi$ on $[0, T] \times B(x, R_\delta)$,

$$\begin{aligned} -\frac{\partial}{\partial t}\phi(t, x) - \mathcal{L}\phi(t, x) - \mathcal{I}^{1,\delta}(t, x, \phi) - \mathcal{I}^{2,\delta}(t, x, \nabla\phi, v) \\ - f(t, x, v, (\nabla\phi)\sigma, \mathcal{B}^\delta(t, x, \phi, v)) \leq 0 \quad (\text{resp. } \geq 0). \end{aligned}$$

We refer to Remark 3.2 and Lemma 3.3 in [5], to condition (NLT), Proposition 1 and Section 2.2 in [8] and to Appendix in [25] for the discussion (and the proof) on the equivalence between Definitions 3 and 4.

From Theorem 3.4 and Theorem 3.5 in [5], we have directly the next result.

Proposition 1 *Under conditions (A) on the coefficients of the SDE (8) and assumptions (B) and (C) on the terminal condition and on the generator of the BSDE (10), the function $u_n(t, x) := Y_t^{n,t,x}$, $(t, x) \in [0, T] \times \mathbb{R}^d$, is the unique bounded (by $n(T+1)$) continuous viscosity solution of (6) with terminal condition g_n and generator f_n .*

1.4 Known results on singular BSDE

In [28], we extend the result of [41] and [2] concerning BSDE with a singular terminal condition, i.e. when $\mathbb{P}(\xi = +\infty) > 0$. Note that the special structure **C1** of the generator is useless here.

Theorem 1 (of [28]) *Under Conditions C2 to C10, the sequence $(Y^{n,t,x}, Z^{n,t,x}, U^{n,t,x})$ converges to $(Y^{t,x}, Z^{t,x}, U^{t,x})$ on $\mathbb{S}^\ell(t, r)$ for any $t \leq r < T$ and*

- $Y_r^{t,x} \geq 0$ a.s. for any $t \leq r \leq T$.
- $(Y^{t,x}, Z^{t,x}, U^{t,x})$ belongs to $\mathbb{S}^\ell(t, r)$ for any $t \leq r < T$.
- For all $t \leq s \leq s' < T$:

$$Y_s^{t,x} = Y_{s'}^{t,x} + \int_s^{s'} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, U_r^{t,x}) dr - \int_s^{s'} Z_r^{t,x} dW_r - \int_s^{s'} \int_E U_r^{t,x}(e) \tilde{\mu}(dr, de).$$

- $(Y^{t,x}, Z^{t,x}, U^{t,x})$ is a super-solution in the sense that: a.s.

$$(14) \quad \liminf_{r \rightarrow T} Y_r^{t,x} \geq \xi = g(X_T^{t,x}).$$

Definition 5 *Any process $(\tilde{Y}, \tilde{Z}, \tilde{U})$ satisfying the previous four items is called **super-solution** of the BSDE (10) with singular terminal condition ξ .*

A key point of the proof is the a priori estimate: for any n , a.s. for any $t \leq s \leq T$

$$(15) \quad Y_s^{n,t,x} \leq Y_s^{t,x} \leq \frac{K_{\ell,L,\vartheta}}{(T-s)^{1+1/q}} \left\{ \mathbb{E} \left(\int_s^T \left[\left(\frac{1}{qa_r} \right)^{1/q} + (T-r)^{1+1/q} f_r^0 \right]^\ell dr \middle| \mathcal{F}_s \right) \right\}^{1/\ell}$$

where $K_{\ell,L,\vartheta}$ is a non negative constant depending only on ℓ , L and ϑ and this constant is a non decreasing function of L and ϑ and a non increasing function of ℓ .

In [28], we have also proved minimality of the constructed solution in the sense that if $(\tilde{Y}, \tilde{Z}, \tilde{U})$ is another non negative super-solution, then for all $r \in [t, T]$, \mathbb{P} -a.s. $\tilde{Y}_r \geq Y_r^{t,x}$.

In [42], we studied the behaviour of this minimal solution $Y^{t,x}$ at time T . In particular we gave sufficient conditions in order to obtain an equality in (14):

$$\liminf_{r \rightarrow T} Y_r^{t,x} = \xi = g(X_T^{t,x}).$$

From Lemma 2 and Proposition 2 in [42], there exist two constants C and δ independent of (n, t, x) such that the process $(Z^{n,t,x}, U^{n,t,x})$ satisfies:

$$(16) \quad \mathbb{E} \left[\int_t^T (T-s)^\rho \left(|Z_s^{n,t,x}|^2 + \|U_s^{n,t,x}\|_{\mathbb{L}_\lambda^2}^2 \right) ds \right]^{\ell/2} \leq C(1 + |x|^\delta).$$

The constant ρ is given by:

$$(17) \quad \rho = \frac{2}{q} + 2 \left(1 - \frac{1}{\ell} \right) + \frac{2\eta}{\ell}.$$

In order to prove that $\liminf_{t \rightarrow T} Y_t = \xi$, we needed to control a term due to the covariance between the jumps of the SDE (8) and the jumps of the BSDE (10). Thus we made a link between the singularity set \mathcal{S} and the jumps of the forward process X . More precisely we assume

Conditions (D).

D1. The boundary $\partial\mathcal{S}$ is compact and of class C^2 .

D2. For any $x \in \mathcal{S}$, any $s \in [0, T]$ and λ -a.s.

$$x + \beta(s, x, e) \in \mathcal{S}.$$

Furthermore there exists a constant $\nu > 0$ such that if $x \in \partial\mathcal{S}$, then for any $s \in [0, T]$, $d(x + \beta(s, x, e), \Gamma) \geq \nu$, λ -a.s.

These assumptions mean in particular that if $X_{s-} \in \mathcal{S}$, then $X_s \in \mathcal{S}$ a.s. Moreover if X_{s-} belongs to the boundary of \mathcal{S} , and if there is a jump at time s , then X_s is in the interior of \mathcal{S} .

Theorem 2 (Theorem 3 in [42]) *Under Conditions (A)-(B)-(C)-(D), if $\rho < 1$ and if $f^0 \in L^1((0, T) \times \Omega)$, the minimal supersolution $Y^{t,x}$ satisfies a.s.*

$$\liminf_{r \rightarrow T} Y_r^{t,x} = \xi.$$

Again **C1** is not involved in this result. The condition $\rho < 1$ is a balance between the non linearity q and the singularity of the generator f .

Remark 3 (On the coefficient ρ) *From Lemma 2, Estimate (12) holds for any $0 < \eta < 1$. Then $\rho < 1$ for $\ell < 2$ and $q > \frac{2\ell}{2-\ell}$. In other words if $q > 2$, we can take $\ell \in (1, 2)$ such that $q > 2\ell/(2-\ell)$ and thus $\rho < 1$. The counterpart is that ϑ should be in $\mathbb{L}_\lambda^2 \cap \mathbb{L}_\lambda^{\tilde{\ell}}$ with $\tilde{\ell} = \ell/(\ell-1)$. If the generator is $f(y) = -y|y|^q$, then $\rho < 1$ if $q > 2$, which was supposed in [41].*

Therefore the minimal solution Y of the singular BSDE (10) is obtained as the increasing limit of Y^n . And it is well known that viscosity solutions are stable by monotone limit. That is the reason why we use this notion of weak solutions.

2 Existence of a minimal viscosity solution with singular data

In [28] we have proved that for any $t \leq s \leq T$

$$\lim_{n \rightarrow +\infty} Y_s^{n,t,x} = Y_s^{t,x}$$

and $Y^{t,x}$ also satisfies (15). As before we define the function u by:

$$u(t, x) = Y_t^{t,x}.$$

Therefore the sequence $u_n(t, x)$ converges to $u(t, x)$. Since a and f^0 depend only on $X^{t,x}$, using Condition **C9** and Property (9), the a priori estimate (15) becomes: there exists two constants $K > 0$ and $\delta > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$:

$$(18) \quad 0 \leq u_n(t, x) \leq u(t, x) \leq \frac{K}{(T-t)^{1/q}}(1 + |x|^\delta).$$

Since u_n is a continuous function, the function u is lower semi-continuous on $[0, T] \times \mathbb{R}^d$ and satisfies for all $x_0 \in \mathbb{R}^d$:

$$(19) \quad \liminf_{(t,x) \rightarrow (T,x_0)} u(t, x) \geq g(x_0).$$

Definition 6 (Viscosity solution with singular data) *A function u is a viscosity solution of (6) with terminal data g if u is a viscosity solution on $[0, T[\times \mathbb{R}^d$ and satisfies:*

$$\lim_{(t,x) \rightarrow (T,x_0)} u(t, x) = g(x_0).$$

The aim of this section is to prove the following result.

Theorem 3 *Under conditions (A)-(B)-(C), $u(t, x) = Y_t^{t,x}$ is a viscosity solution of the IPDE (6) on $[0, T[\times \mathbb{R}^d$. Moreover u is the minimal viscosity solution among all non negative solutions satisfying (19).*

Finally if we add Conditions (D) and if $\rho < 1$, then

$$\lim_{(t,x) \rightarrow (T,x_0)} u(t, x) = g(x_0).$$

Note that we do not prove the continuity of u because of the lack of uniform convergence of the approximating sequence u_n . But we are also not able to show that u is discontinuous.

2.1 Viscosity solution

In order to prove that u is a viscosity solution, the main tool is the half-relaxed upper- and lower-limit of the sequence of functions $\{u_n\}$, i.e.

$$\bar{u}(t, x) = \limsup_{\substack{n \rightarrow +\infty \\ (t',x') \rightarrow (t,x)}} u_n(t', x') \quad \text{and} \quad \underline{u}(t, x) = \liminf_{\substack{n \rightarrow +\infty \\ (t',x') \rightarrow (t,x)}} u_n(t', x').$$

In our case, $\underline{u} = u \leq \bar{u} = u^*$ because the sequence $\{u_n\}$ is non decreasing and u_n is continuous for all $n \in \mathbb{N}^*$. Note that u^* also satisfies estimate (18).

Lemma 3 *The function u is a viscosity solution of (6) on $[0, T] \times \mathbb{R}^d$.*

Proof. First $u = u_* = \underline{u}$ is lower semi-continuous on $[0, T] \times \mathbb{R}^d$. From the estimate (18), for all $\varepsilon > 0$, there exists a constant K_ε such that for every $n \in \mathbb{N}^*$ and all $(t, x) \in [0, T - \varepsilon] \times \mathbb{R}^d$,

$$(20) \quad 0 \leq u_n(t, x) \leq u(t, x) \leq K_\varepsilon(1 + |x|^\delta).$$

In other words u_n and u belong to $\Pi_{pg}(0, T - \varepsilon)$.

Since u_n is a supersolution of the IPDE (6), passing to the limit with a stability result (see the proof of Theorem 4.1 in [25] or the results in [1], [8] or [10]), we can obtain that u is a supersolution of (6) on $[0, T] \times \mathbb{R}^d$.

For convenience of the reader, let us give the main ideas (for details see the proof of Theorem 4.1 in [25]). Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and let ϕ be a function which belongs to $C^{1,2}([0, T] \times \mathbb{R}^d) \cap \Pi_{pg}$ such that $u - \phi$ has a strict global minimum in (t, x) on $[0, T] \times \mathbb{R}^d$ and we assume w.l.o.g. that $u(t, x) = \phi(t, x)$. Now let $\delta > 0$ and (t_n, x_n) be the global minimum of $u_n - \phi$ on $[0, T] \times \overline{B(x, R_\delta)}$. R_δ is a positive number such that R_δ tends to zero when $\delta \rightarrow 0$. As in [25], one can prove that

$$\lim_n (t_n, x_n) = (t, x), \quad \lim_n u_n(t_n, x_n) = u(t, x).$$

The bound (20) is crucial here. Now since u_n is a viscosity supersolution, by Definition 4,

$$(21) \quad -\frac{\partial}{\partial t} \phi(t_n, x_n) - \mathcal{L}\phi(t_n, x_n) - \mathcal{I}^{1,\delta}(t_n, x_n, \phi) - \mathcal{I}^{2,\delta}(t_n, x_n, \nabla \phi, u_n) \\ - f_n(t_n, x_n, u_n, (\nabla \phi)\sigma(t_n, x_n), \mathcal{B}^\delta(t_n, x_n, \phi, u_n)) \geq 0.$$

By continuity of ϕ and of the coefficients of the SDE (8), we can pass to the limit as n goes to ∞ :

$$\lim_{n \rightarrow +\infty} \left[-\frac{\partial}{\partial t} \phi(t_n, x_n) - \mathcal{L}\phi(t_n, x_n) \right] = -\frac{\partial}{\partial t} \phi(t, x) - \mathcal{L}\phi(t, x)$$

and

$$\lim_{n \rightarrow +\infty} \mathcal{I}^{1,\delta}(t_n, x_n, \phi) = \mathcal{I}^{1,\delta}(t, x, \phi).$$

Since $u_n - \phi \geq 0$ attains his minimum at (t_n, x_n)

$$\mathcal{I}^{2,\delta}(t_n, x_n, \nabla \phi, u_n) \\ = \int_{|e| > \delta} [u_n(t_n, x_n + \beta(x_n, e)) - u_n(t_n, x_n) - \nabla \phi(t_n, x_n)\beta(x_n, e)] \lambda(de) \\ \geq \int_{|e| > \delta} [\phi(t_n, x_n + \beta(x_n, e)) - \phi(t_n, x_n) - \nabla \phi(t_n, x_n)\beta(x_n, e)] \lambda(de),$$

and

$$\begin{aligned}
& \mathcal{B}^\delta(t_n, x_n, \phi, u_n) \\
&= \int_{|e| \leq \delta} [\phi(t_n, x_n + \beta(x_n, e)) - \phi(t_n, x_n)] \gamma(x_n, e) \lambda(de) \\
&\quad + \int_{|e| > \delta} [u_n(t_n, x_n + \beta(x_n, e)) - u_n(t_n, x_n)] \gamma(x_n, e) \lambda(de) \\
&\geq \int_{|e| \leq \delta} [\phi(t_n, x_n + \beta(x, e)) - \phi(t_n, x_n)] \gamma(x_n, e) \lambda(de) \\
&\quad + \int_{|e| > \delta} [\phi(t_n, x_n + \beta(x, e)) - \phi(t_n, x_n)] \gamma(x, e) \lambda(de)
\end{aligned}$$

and by Fatou's lemma

$$\liminf_{n \rightarrow +\infty} \mathcal{I}^{2,\delta}(t_n, x_n, \nabla \phi, u_n) \geq \mathcal{I}^{2,\delta}(t, x, \nabla \phi, \phi)$$

and

$$\liminf_{n \rightarrow +\infty} \mathcal{B}^\delta(t_n, x_n, \phi, u_n) \geq \mathcal{B}(t, x, \phi).$$

The function $f(t, x, y, z, u)$ is non decreasing w.r.t. u and continuous w.r.t. all variables:

$$\liminf_{n \rightarrow +\infty} f_n(t_n, x_n, u_n, (\nabla \phi)\sigma(t_n, x_n), \mathcal{B}^\delta(t_n, x_n, \phi, u_n)) \geq f(t, x, u, (\nabla \phi)\sigma(t, x), \mathcal{B}(t, x, \phi)).$$

Passing to the limit in (21) we obtain:

$$\begin{aligned}
& -\frac{\partial}{\partial t} \phi(t, x) - \mathcal{L}\phi(t, x) - \mathcal{I}^{1,\delta}(t, x, \phi) \\
& \geq \mathcal{I}^{2,\delta}(t, x, \nabla \phi, \phi) + f(t, x, u, (\nabla \phi)\sigma(t, x), \mathcal{B}(t, x, \phi)).
\end{aligned}$$

Thus u is a supersolution of (6) on $[0, T] \times \mathbb{R}^d$.

By the same argument we can show that u^* is a subsolution on $[0, T] \times \mathbb{R}^d$. Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\phi \in C^{1,2}([0, T] \times \mathbb{R}^d) \cap \Pi_g$ such that $u^* - \phi$ has a strict global maximum at (t, x) on $[0, T] \times \mathbb{R}^d$ with $u^*(t, x) = \phi(t, x)$. As in [25] there exists a subsequence n_k such that

- (t_{n_k}, x_{n_k}) is the global maximum of $u_{n_k} - \phi$ on $[0, T] \times \overline{B(x, R_\delta)}$;
- as k goes to ∞ , $(t_{n_k}, x_{n_k}) \rightarrow (t, x)$ and $u_{n_k}(t_{n_k}, x_{n_k}) \rightarrow u^*(t, x)$.

Now for k large, since u_{n_k} is a subsolution, we have again by Definition 4,

$$\begin{aligned}
(22) \quad & -\frac{\partial}{\partial t} \phi(t_{n_k}, x_{n_k}) - \mathcal{L}\phi(t_{n_k}, x_{n_k}) - \mathcal{I}^{1,\delta}(t_{n_k}, x_{n_k}, \phi) - \mathcal{I}^{2,\delta}(t_{n_k}, x_{n_k}, \nabla \phi, u_{n_k}) \\
& - f^{n_k}(t_{n_k}, x_{n_k}, u_{n_k}, (\nabla \phi)\sigma(t_{n_k}, x_{n_k}), \mathcal{B}^\delta(t_{n_k}, x_{n_k}, \phi, u_{n_k})) \leq 0.
\end{aligned}$$

Again since $u_{n_k} - \phi$ has a maximum at the point (t_{n_k}, x_{n_k}) , we have

$$\begin{aligned} & \mathcal{I}^{2,\delta}(t_{n_k}, x_{n_k}, \nabla\phi, u_{n_k}) \\ &= \int_{|e|>\delta} [u_{n_k}(t_{n_k}, x_{n_k} + \beta(x_{n_k}, e)) - u_{n_k}(t_{n_k}, x_{n_k}) - \nabla\phi(t_{n_k}, x_{n_k})\beta(x_{n_k}, e)]\lambda(de) \\ &\leq \int_{|e|>\delta} [\phi(t_{n_k}, x_{n_k} + \beta(x_{n_k}, e)) - u_{n_k}(t_{n_k}, x_{n_k}) - \nabla\phi(t_{n_k}, x_{n_k})\beta(x_{n_k}, e)]\lambda(de) \end{aligned}$$

and

$$\mathcal{B}^\delta(t_{n_k}, x_{n_k}, \phi, u_{n_k}) \leq \mathcal{B}(t_{n_k}, x_{n_k}, \phi).$$

By continuity and monotonicity w.r.t. q of f , Lebesgue's theorem and since $u(t, x) = \phi(t, x)$

$$\begin{aligned} & -\frac{\partial}{\partial t}\phi(t, x) - \mathcal{L}\phi(t, x) - \mathcal{I}^{1,\delta}(t, x, \phi) \\ & \leq \mathcal{I}^{2,\delta}(t, x, \nabla\phi, \phi) + f(t, x, u, (\nabla\phi)\sigma(t, x), \mathcal{B}(t, x, \phi)). \end{aligned}$$

Thus u^* is a subsolution on $[0, T] \times \mathbb{R}^d$. □

As for the singular BSDE (10), the main difficulty is to show that

$$\limsup_{(t,x) \rightarrow (T,x_0)} u(t, x) \leq g(x_0) = u(T, x_0).$$

We will prove that u^* is locally bounded on a neighbourhood of T on the open set $\mathcal{R} = \{g < +\infty\}$. Then, we deduce u^* is a subsolution with relaxed terminal condition and we apply this to demonstrate that $u^*(T, x) \leq g(x)$ if $x \in \{g < +\infty\}$, which shows the wanted inequality on u .

Lemma 4 *Assumptions (A)-(B)-(C)-(D) hold. For any $\varepsilon > 0$, if we define the closed subset of \mathcal{R}*

$$\Gamma(\varepsilon) := \{x \in \mathcal{R} : d(x, \partial\mathcal{S}) \geq \varepsilon\}$$

u^* is a subsolution with relaxed terminal condition:

$$\begin{cases} -\frac{\partial u^*}{\partial t} - \mathcal{L}u^* - \mathcal{I}u^* - f(t, x, u^*, \nabla u^* \sigma, \mathcal{B}(t, x, u^*)) = 0, & \text{in } [0, T) \times \Gamma(\varepsilon); \\ \min \left[-\frac{\partial u^*}{\partial t} - \mathcal{L}u^* - \mathcal{I}u^* - f(t, x, u^*, \nabla u^* \sigma, \mathcal{B}(t, x, u^*)); u^* - g \right] \leq 0, & \text{in } \{T\} \times \Gamma(\varepsilon). \end{cases}$$

Proof. We make the same calculation as in [42], Section 3.4. Hence we only give here the main steps.

Remember that \mathcal{S} is the singular set of g , $\mathcal{R} = \mathcal{S}^c$ is open and for any $\varepsilon > 0$ we define

$$\Gamma(\varepsilon) := \{x \in \mathcal{R} : d(x, \partial\mathcal{S}) \geq \varepsilon\}.$$

$d(\cdot, \partial\mathcal{S})$ is the distance to the boundary $\partial\mathcal{S}$. By the C^∞ Urysohn lemma, there exists a C^∞ function ψ such that $\psi \in [0, 1]$, $\psi \equiv 1$ on $\Gamma(\varepsilon)$ and $\psi \equiv 0$ on $\Gamma(\varepsilon/2)^c$. In particular

the support of ψ is included in \mathcal{R} and since $\partial\mathcal{S}$ is compact, ψ belongs to $C_b^\infty(\mathbb{R}^d)$. We take $\gamma > 2(q+1)/q$ and we define

$$(23) \quad \phi = \psi^\gamma.$$

Note that ϕ also takes its values in $[0, 1]$, $\phi \equiv 1$ on $\Gamma(\varepsilon)$ and $\phi \equiv 0$ on $\Gamma(\varepsilon/2)^c$. Moreover again with **(D)**, we can choose ε_0 small enough such that for any $0 < \varepsilon < \varepsilon_0$:

$$(24) \quad \psi(X_{s-}) = 0 \Rightarrow \psi(X_s) = 0, \quad \frac{\psi(X_s)}{\psi(X_{s-})} = \psi(X_s) \mathbf{1}_{\Gamma(\varepsilon)}(X_{s-}).$$

(see Lemma 4 in [42] for the details).

Lemma 5 *There exists a constant C independent of n and t such that*

$$u_n(t, x)\phi(x) \leq C(1 + |x|^\delta).$$

The proof of this lemma is postponed at the end of this section.

For any $0 < \varepsilon < \varepsilon_0$, u_n is bounded on $[0, T] \times \Gamma(\varepsilon)$ by $C(1 + |x|^\delta)$ uniformly w.r.t. to n . Therefore, u^* is bounded on $[0, T] \times \Gamma(\varepsilon)$ by $C(1 + |x|^\delta)$. We know that u_n is a subsolution of the IPDE (6) restricted to $[0, T] \times \Gamma(\varepsilon)$, i.e. for $(t, x) \in [0, T] \times \Gamma(\varepsilon)$

$$-\frac{\partial u_n}{\partial t}(t, x) - \mathcal{L}u_n(t, x) - \mathcal{I}(t, x, u_n) - f_n(t, x, u_n, (\nabla u_n)\sigma(t, x), \mathcal{B}(t, x, u_n)) = 0$$

with the terminal condition

$$u_n(T, x) = (g \wedge n)(x), \quad x \in \Gamma(\varepsilon).$$

From Lemma 3, u^* is a subsolution of the IPDE (6) on $[0, T] \times \Gamma(\varepsilon)$.

The behaviour at time T is an adaptation of Theorem 4.1 in [4] (see also section 4.4.5 in [4]). Since g is continuous (Hypothesis **B3**),

$$g(x) = \bar{g}(x) = \limsup_{\substack{n \rightarrow +\infty \\ x' \rightarrow x}} (g \wedge n)(x').$$

Now assume that for $\phi \in C^{1,2}([0, T] \times \mathbb{R}^d) \cap \Pi_{pg}$ such that $u^* - \phi$ has a strict global maximum on $[0, T] \times \Gamma(\varepsilon)$ at (T, x) and suppose that $u^*(T, x) > g(x)$. There exists a subsequence n_k such that (t_{n_k}, x_{n_k}) is the global maximum of $u_{n_k} - \phi$ on $[0, T] \times \overline{B(x, R_\delta)}$ and as k goes to ∞ , $(t_{n_k}, x_{n_k}) \rightarrow (T, x)$ and $u_{n_k}(t_{n_k}, x_{n_k}) \rightarrow u^*(T, x)$. This implies in particular that $t_{n_k} < T$ for any k large enough. If not, then up to a subsequence (still denoted n_k),

$$u^*(t, x) = \limsup_k u_{n_k}(t_{n_k}, x_{n_k}) = \limsup_k u_{n_k}(T, x_{n_k}) = \limsup_k (g \wedge n_k)(x_{n_k}) \leq g(x).$$

Since u_{n_k} is a subsolution, we still have (22) and passing through the limit we obtain

$$\begin{aligned} & -\frac{\partial}{\partial t}\phi(T, x) - \mathcal{L}\phi(T, x) - \mathcal{I}^{1,\delta}(T, x, \phi) \\ & \leq \mathcal{I}^{2,\delta}(T, x, \nabla\phi, \phi) + f(T, x, u^*, (\nabla\phi)\sigma(T, x), \mathcal{B}(T, x, \phi)). \end{aligned}$$

Thus u^* is a subsolution on $[0, T] \times \Gamma(\varepsilon)$. \square

Now Theorem 4.7 in [4] (with straightforward modifications) shows that $u^* \leq g$ in $\{T\} \times \Gamma(\varepsilon)$. In other words for any $x_0 \in \mathcal{R}$,

$$\limsup_{(t,x) \rightarrow (T,x_0)} u(t,x) \leq g(x_0).$$

With Inequality (19), we obtain the desired behaviour of u near terminal time T . This achieves the proof of Theorem 3 (except minimality).

Now we prove Lemma 5.

Proof. From assumptions **(D)**, Γ is compact and of class C^1 , then there exists a constant $\varepsilon_0 > 0$ such that for every $y \in \mathcal{R} \cap \Gamma(\varepsilon_0)^c$, there exists a unique $z \in \partial\mathcal{S}$ such that $d(y, \partial\mathcal{S}) = \|y - z\|$.

We use Itô's formula to the process $Y^{n,t,x}\phi(X^{t,x})$ between t and T and we take the expectation since (Y^n, Z^n, U^n, M^n) belongs to $\mathbb{S}^2(0, T)$, X is in $\mathbb{H}^2(0, T)$, and ϕ and the derivatives of ϕ are supposed to be bounded. Thus we obtain for $x \in \mathbb{R}^d$ and $t \in [0, T]$:

$$(25) \quad \begin{aligned} u_n(t,x)\phi(x) &= \mathbb{E}[Y_T^{n,t,x}\phi(X_T^{t,x})] - \mathbb{E} \int_t^T Y_{s-}^{n,t,x} [\mathcal{L}\phi(s, X_s^{t,x}) + \mathcal{I}(s, X_{s-}^{t,x}, \phi)] ds \\ &\quad + \mathbb{E} \left[\int_t^T \phi(X_{s-}^{t,x}) f_n(s, Y_s^{n,t,x}, Z_s^{n,t,x}, U_s^{n,t,x}) ds \right] \\ &\quad - \mathbb{E} \left[\int_t^T \nabla\phi(X_s^{t,x}) \sigma(X_s^{t,x}) Z_s^{n,t,x} ds \right] \\ &\quad - \mathbb{E} \left[\int_t^T \int_E (\phi(X_s^{t,x}) - \phi(X_{s-}^{t,x})) U_s^{n,t,x}(e) \lambda(de) ds \right]. \end{aligned}$$

From the Assumptions **B1** and **B2** on $\xi = g(X_T^{t,x})$, we have for any n :

$$\mathbb{E}(Y_T^{n,t,x} \phi(X_T^{t,x})) \leq \mathbb{E}(g(X_T^{t,x}) \phi(X_T^{t,x})) < +\infty.$$

Now we decompose the quantity with the generator f_n as follows:

$$(26) \quad \begin{aligned} &\mathbb{E} \left[\int_t^T \phi(X_{s-}^{t,x}) f_n(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x}, U_s^{n,t,x}) ds \right] \\ &= \mathbb{E} \left[\int_t^T \phi(X_{s-}^{t,x}) (f(s, X_s^{t,x}, Y_s^{n,t,x}, 0, 0) - f_s^{0,t,x}) ds \right] \\ &\quad + \mathbb{E} \left[\int_t^T \phi(X_{s-}^{t,x}) (f_s^{0,t,x} \wedge n) ds \right] \\ &\quad + \mathbb{E} \left[\int_t^T \phi(X_{s-}^{t,x}) \zeta_s^n Z_s^{n,t,x} ds \right] + \mathbb{E} \left[\int_t^T \phi(X_{s-}^{t,x}) \mathcal{U}_s^{n,t,x} ds \right] \end{aligned}$$

where ζ_s^n is a k -dimensional random vector defined by:

$$\zeta_s^{i,n} = \frac{(f(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x}, 0) - f(s, X_s^{t,x}, Y_s^{n,t,x}, 0, 0))}{Z_s^{i,n}} \mathbf{1}_{Z_s^{i,n,t,x} \neq 0}$$

and

$$\mathcal{U}_s^n = f(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x}, U_s^{n,t,x}) - f(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x}, 0).$$

Now from Condition **C5**, $|\zeta_s^n| \leq K$. Using Estimate (16), where ρ is given by (17), by Hölder inequality and since $\rho < 1$, there exists a constant C such that for any n

$$(27) \quad \mathbb{E} \int_t^T |(\nabla \phi(X_s^{t,x}) \sigma(X_s^{t,x}) + \phi(X_s^{t,x}) \zeta_s^n) Z_s^{n,t,x}| ds \leq C(1 + |x|^\delta).$$

The same estimate holds for $U^{n,t,x}$. Indeed from Lemma 1:

$$|\mathcal{U}_s^n| \leq \int_E \vartheta(e) |U_s^{n,t,x}(e)| \lambda(de),$$

and again by Hölder inequality and since $\rho < 1$,

$$(28) \quad \mathbb{E} \int_t^T \left(\int_E |\phi(X_s^{t,x}) - \phi(X_{s-}^{t,x})| |U_s^{n,t,x}(e)| \lambda(de) + |\phi(X_{s-}^{t,x})| |\mathcal{U}_s^n| \right) ds \leq C(1 + |x|^\delta).$$

In (26), by Condition **C9** and Property (9), since ϕ is bounded

$$(29) \quad \mathbb{E} \left[\int_t^T \phi(X_{s-}^{t,x}) (f_s^{0,t,x} \wedge n) ds \right] \leq C(1 + |x|^\delta).$$

Now we treat the two terms with the operators \mathcal{L} and \mathcal{I} . First

$$(30) \quad \mathbb{E} \left[\int_t^T |Y_{s-}^{n,t,x} \mathcal{L} \phi(s, X_s^{t,x})| ds \right] \leq C \left[\mathbb{E} \int_t^T a(s, X_s^{t,x}) \phi(X_s^{t,x}) (Y_s^{n,t,x})^{q+1} ds \right]^{1/(q+1)}.$$

To obtain this inequality, we use that $\gamma > 2(q+1)/q$ in (23), Hölder's inequality, and the existence of a constant C such that

$$\mathcal{L} \phi = \mathcal{L}(\psi^\gamma) \leq C \psi^{\gamma-2}$$

which can be found in [33]. The same inequality holds for \mathcal{I} :

$$(31) \quad \mathbb{E} \left[\int_t^T Y_{s-}^{n,t,x} |\mathcal{I}(s, X_{s-}^{t,x}, \phi)| ds \right] \leq C \left[\mathbb{E} \int_t^T a(s, X_s^{t,x}) \phi(X_{s-}^{t,x}) (Y_s^{n,t,x})^{q+1} ds \right]^{\frac{1}{q+1}}.$$

To prove this estimate, we use Assumptions **(D)** and the properties (24).

Now we come to the conclusion. By Condition **C8**

$$(32) \quad \begin{aligned} & -\mathbb{E} \left[\int_t^T \phi(X_{s-}^{t,x}) (f(s, X_s^{t,x}, Y_s^{n,t,x}, 0, 0) - f_s^{0,t,x}) ds \right] \\ & \geq \mathbb{E} \left[\int_0^t \phi(X_{s-}^{t,x}) a(s, X_s^{t,x}) (Y_s^{n,t,x})^{1+q} ds \right]. \end{aligned}$$

The relations (27), (28), (29), (30), (31) and (32) hold. Thus, we have:

$$\begin{aligned} & -\mathbb{E} \int_t^T \phi(X_s^{t,x}) f_n(s, Y_s^{n,t,x}, Z_s^{n,t,x}, U_s^{n,t,x}) ds \\ & + \mathbb{E} \int_t^T Y_{s-}^{t,x,n} [\mathcal{L} \phi(X_s^{t,x}) + \mathcal{I}(s, X_{s-}^{t,x}, \phi)] ds \leq C(1 + |x|^\delta). \end{aligned}$$

The constant C does not depend on n and t . In the left hand side, the second term is controlled by the first one raised to a power strictly smaller than 1 using Hölder's inequality (see (30) and (31)). Therefore, there exists a constant C :

$$\mathbb{E} \int_t^T \phi(X_s^{t,x}) |f_n(s, Y_s^{n,t,x}, Z_s^{n,t,x}, U_s^{n,t,x})| ds \leq C(1 + |x|^\delta).$$

From (25) we deduce that there exists a constant C independent of n and t such that

$$u_n(t, x)\phi(x) \leq C(1 + |x|^\delta).$$

This achieves the proof of the lemma. \square

2.2 Minimality of the solution

The aim here is to prove minimality of the viscosity solution obtained by approximation among all non negative viscosity solutions (Theorem 3). We compare a viscosity solution v (in the sense of Definition 6) with u_n , for all integer n : for all $(t, x) \in [0, T] \times \mathbb{R}^d$, $u_n(t, x) \leq v_*(t, x)$. We deduce that $u \leq v_* \leq v$. Recall that $g : \mathbb{R}^d \rightarrow \overline{\mathbb{R}^+}$ is continuous from **B3**, which implies that $g \wedge n : \mathbb{R}^d \rightarrow \mathbb{R}^+$ is continuous.

To simplify the notation, we will denote F the following function on $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \times \mathbb{R}^2$:

$$(33) \quad F(t, x, u, p, X, I, B) = -pb(x) - \frac{1}{2}\text{Trace}(X(\sigma\sigma^*)(x)) - I - f(t, x, u, p\sigma(x), B).$$

\mathbb{S}_d is the set of symmetric matrices of size $d \times d$.

Proposition 2 $u_n \leq v_*$, where v is a non negative viscosity solution of the PDE (6).

Proof. This result seems to be a direct consequence of a well-known maximum principle for viscosity solutions (see [4] or [16] when $\mathcal{I} = 0$, [5], [8] or [25] in general). But to the best of our knowledge, this principle was not proved for solutions which can take the value $+\infty$. Recall that the terminal condition and the generator can be singular at time T . Thus, following the proof of Proposition 4.1 in [25] or Theorem 3 in [8], we just give here the main points.

The beginning of the proof is exactly the same as the proof of Proposition 23 in [41]. We fix $\varepsilon > 0$ and $n \geq 1$ and we define $u_{n,\varepsilon}(t, x) = u_n(t, x) - \frac{\varepsilon}{t}$. We will prove that $u_{n,\varepsilon} \leq v_*$ for every ε , hence we deduce $u_n \leq v_*$.

We suppose that there exists $(s, z) \in [0, T] \times \mathbb{R}^d$ such that $u_{n,\varepsilon}(s, z) - v_*(s, z) \geq \nu > 0$ and we will find a contradiction. First of all, it is clear that s is not equal to 0 or T , because $u_{n,\varepsilon}(0, z) = -\infty$ and $v_*(T, z) \geq g(z)$ (by definition of a supersolution).

$u_{n,\varepsilon}$ and $-v_*$ are bounded from above on $[0, T] \times \mathbb{R}^d$ respectively by $n(T + 1)$ and 0. Thus, for $(\eta, \chi) \in (\mathbb{R}^*)^2$, if we define:

$$m(t, x, y) = u_{n,\varepsilon}(t, x) - v_*(t, y) - \frac{\eta}{2}|x - y|^2 - \chi(|x|^2 + |y|^2),$$

m has a supremum $M_{\eta,\chi}$ on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ and the penalization terms assure that the supremum is attained at a point $(\hat{t}, \hat{x}, \hat{y}) = (t_{\eta,\chi}, x_{\eta,\chi}, y_{\eta,\chi})$. By classical arguments we prove that if χ is sufficiently small

$$(34) \quad \nu/2 \leq M_{\eta,\chi}, \quad |\hat{x}|^2 + |\hat{y}|^2 \leq \frac{n(T+1)}{\chi} \quad \text{and} \quad |\hat{x} - \hat{y}|^2 \leq \frac{2n(T+1)}{\eta}.$$

Moreover for η large enough, the time \hat{t} satisfies $0 < \hat{t} < T$ (see [41] for the details).

For η large enough, we can apply Jensen-Ishii's Lemma for non local operator established by Barles and Imbert (Lemma 1 and Corollary 2 in [8]) with $u_{n,\varepsilon}$ subsolution, v_* supersolution and $\phi(x, y) = \frac{\eta}{2}|x - y|^2 + \chi(|x|^2 + |y|^2)$ at the point $(\hat{t}, \hat{x}, \hat{y})$. For any $\delta > 0$ there exists $\zeta > 0$ and $(a, p, X), (b, q, Y)$ such that

- $a = b, p = \nabla_x \phi(\hat{x}, \hat{y}) = \eta(\hat{x} - \hat{y}) + 2\chi\hat{x}, q = -\nabla_y \phi(\hat{x}, \hat{y}) = -\eta(\hat{y} - \hat{x}) - 2\chi\hat{y}$
- X and Y are symmetric matrices of size $d \times d$ such that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \eta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\chi \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + o_\zeta(1)$$

- the non local operators become

$$\begin{aligned} I^{n,\varepsilon} &= \mathcal{I}^{1,\delta}(\hat{t}, \hat{x}, \phi_\zeta(\cdot, \hat{y})) + \mathcal{I}^{2,\delta}(\hat{t}, \hat{x}, p, u_{n,\varepsilon}(\hat{t}, \hat{x})) \\ I^* &= \mathcal{I}^{1,\delta}(\hat{t}, \hat{y}, -\phi_\zeta(\hat{x}, \cdot)) + \mathcal{I}^{2,\delta}(\hat{t}, \hat{y}, q, v_*(\hat{t}, \hat{y})) \\ B^{n,\varepsilon} &= \mathcal{B}^\delta(\hat{t}, \hat{x}, \phi_\zeta(\cdot, \hat{y}), u_{n,\varepsilon}(\hat{t}, \hat{x})) \\ B^* &= \mathcal{B}^\delta(\hat{t}, \hat{y}, -\phi_\zeta(\hat{x}, \cdot), v_*(\hat{t}, \hat{y})) \end{aligned}$$

- and finally

$$\begin{aligned} -a + F(\hat{t}, \hat{x}, u_{n,\varepsilon}(\hat{t}, \hat{x}), p, X, I^{n,\varepsilon}, B^{n,\varepsilon}) &\leq -\frac{\varepsilon}{T^2} \\ -b + F(\hat{t}, \hat{y}, v_*(\hat{t}, \hat{y}), q, Y, I^*, B^*) &\geq 0. \end{aligned}$$

The result holds for any $0 < \zeta < \bar{\zeta}$ and the value $\bar{\zeta} > 0$ depends on the coefficients of the IPDE. The function ϕ_ζ is defined in the same way as in [8]. Proposition 3 in [8] shows that we can replace ϕ_ζ in $\mathcal{I}^{1,\delta}$ by ϕ up to some $o(\zeta)$. We subtract the two previous inequalities:

$$(35) \quad \frac{\varepsilon}{T^2} + o(\zeta) \leq -F(\hat{t}, \hat{x}, u_{n,\varepsilon}(\hat{t}, \hat{x}), p, X, I^{n,\varepsilon}, B^{n,\varepsilon}) + F(\hat{t}, \hat{y}, v_*(\hat{t}, \hat{y}), q, Y, I^*, B^*).$$

Let us separate the local terms with the non local ones. For the first ones we have:

$$\begin{aligned} &\frac{1}{2}\text{Trace}(\sigma\sigma^*(\hat{x})X) - \frac{1}{2}\text{Trace}(\sigma\sigma^*(\hat{y})Y) \\ &+ (b(\hat{x}) - b(\hat{y})) \cdot \eta(\hat{x} - \hat{y}) + 2\chi(b(\hat{x}) \cdot \hat{x} + b(\hat{y}) \cdot \hat{y}) \end{aligned}$$

As in [41], using **A1** and **A3**, we prove that there exists a constant K independent of η and χ such that:

$$(36) \quad \begin{aligned} & (b(\hat{x}) - b(\hat{y})) \cdot \eta(\hat{x} - \hat{y}) + 2\chi (b(\hat{x}) \cdot \hat{x} + b(\hat{y}) \cdot \hat{y}) \\ & \leq \eta K |\hat{x} - \hat{y}|^2 + 2\chi K (1 + |\hat{x}|^2 + |\hat{y}|^2), \end{aligned}$$

and

$$(37) \quad \text{Trace}(\sigma\sigma^*(\hat{x})X) - \text{Trace}(\sigma\sigma^*(\hat{y})Y) \leq K\eta|\hat{x} - \hat{y}|^2 + K\chi(1 + |\hat{x}|^2 + |\hat{y}|^2).$$

Now we deal with the non local terms. First we control

$$\begin{aligned} & \mathcal{I}^{2,\delta}(\hat{t}, \hat{x}, p, u_{n,\varepsilon}(\hat{t}, \hat{x})) - \mathcal{I}^{2,\delta}(\hat{t}, \hat{y}, q, v_*(\hat{t}, \hat{y})) \\ & = \int_{|e|>\delta} [u_{n,\varepsilon}(\hat{t}, \hat{x} + \beta(\hat{x}, e)) - u_{n,\varepsilon}(\hat{t}, \hat{x}) - p\beta(\hat{x}, e)]\lambda(de) \\ & \quad - \int_{|e|>\delta} [v_*(\hat{t}, \hat{y} + \beta(\hat{y}, e)) - v_*(\hat{t}, \hat{y}) - q\beta(\hat{y}, e)]\lambda(de) \end{aligned}$$

We use the following inequality:

$$(38) \quad \begin{aligned} & u_{n,\varepsilon}(\hat{t}, \hat{x} + \beta(\hat{x}, e)) - v_*(\hat{t}, \hat{y} + \beta(\hat{y}, e)) \leq m(\hat{t}, \hat{x}, \hat{y}) \\ & \quad + \frac{\eta}{2} |\hat{x} + \beta(\hat{x}, e) - \hat{y} - \beta(\hat{y}, e)|^2 + \chi (|\hat{x} + \beta(\hat{x}, e)|^2 + |\hat{y} + \beta(\hat{y}, e)|^2) \\ & \leq m(\hat{t}, \hat{x}, \hat{y}) + \frac{\eta}{2} |\hat{x} - \hat{y}|^2 + \chi (|\hat{x}|^2 + |\hat{y}|^2) \\ & \quad + \frac{\eta}{2} |\beta(\hat{x}, e) - \beta(\hat{y}, e)|^2 + \chi (|\beta(\hat{x}, e)|^2 + |\beta(\hat{y}, e)|^2) \\ & \quad + \eta(\hat{x} - \hat{y})(\beta(\hat{x}, e) - \beta(\hat{y}, e)) + 2\chi (\hat{x}\beta(\hat{x}, e) + \hat{y}\beta(\hat{y}, e)) \\ & = u_{n,\varepsilon}(\hat{t}, \hat{x}) - v_*(\hat{t}, \hat{y}) + p\beta(\hat{x}, e) - q\beta(\hat{y}, e) \\ & \quad + \frac{\eta}{2} |\beta(\hat{x}, e) - \beta(\hat{y}, e)|^2 + \chi (|\beta(\hat{x}, e)|^2 + |\beta(\hat{y}, e)|^2). \end{aligned}$$

By construction, $v_*(\hat{t}, \hat{y}) \leq u_{n,\varepsilon}(\hat{t}, \hat{x})$ and therefore by assumptions **A2** and **A4** on β , there exists K independent of η and χ such that:

$$(39) \quad \mathcal{I}^{2,\delta}(\hat{t}, \hat{x}, p, u_{n,\varepsilon}(\hat{t}, \hat{x})) - \mathcal{I}^{2,\delta}(\hat{t}, \hat{y}, q, v_*(\hat{t}, \hat{y})) \leq K \left(\frac{\eta}{2} |\hat{x} - \hat{y}|^2 + \chi(1 + |\hat{x}| + |\hat{y}|) \right).$$

Now

$$(40) \quad \begin{aligned} & \mathcal{I}^{1,\delta}(\hat{t}, \hat{x}, \phi(\cdot, \hat{y})) - \mathcal{I}^{1,\delta}(\hat{t}, \hat{x}, -\phi(\hat{x}, \cdot)) \\ & = \int_{|e|\leq\delta} [\phi(\hat{x} + \beta(\hat{x}, e), \hat{y}) - \phi(\hat{x}, \hat{y}) - (\nabla_x \phi)(\hat{x}, \hat{y})\beta(\hat{x}, e)]\lambda(de) \\ & \quad - \int_{|e|\leq\delta} [-\phi(\hat{x}, \hat{y} + \beta(\hat{y}, e)) + \phi(\hat{x}, \hat{y}) + (\nabla_y \phi)(\hat{x}, \hat{y})\beta(\hat{y}, e)]\lambda(de) \\ & = \left(\frac{\eta}{2} + \chi \right) \int_{|e|\leq\delta} (|\beta(\hat{x}, e)|^2 + |\beta(\hat{y}, e)|^2) \lambda(de) \\ & \leq 2K^2 \left(\frac{\eta}{2} + \chi \right) \int_{|e|\leq\delta} (1 \wedge |e|^2) \lambda(de) = 2K^2 \left(\frac{\eta}{2} + \chi \right) O(\delta), \end{aligned}$$

by assumption **A4**. To finish we need to control

$$f_n(\hat{t}, \hat{x}, u_{n,\varepsilon}(\hat{t}, \hat{x}), p\sigma(\hat{x}), B^{n,\varepsilon}) - f(\hat{t}, \hat{y}, v_*(\hat{t}, \hat{y}), q\sigma(\hat{y}), B^*).$$

Recall that we can replace ϕ_ζ by ϕ (up to some $o_\zeta(1)$). We begin with

$$\begin{aligned} B^{n,\varepsilon} - B^* &= \mathcal{B}^\delta(\hat{t}, \hat{x}, \phi_\zeta(\cdot, \hat{y}), u_{n,\varepsilon}(\hat{t}, \hat{x})) - \mathcal{B}^\delta(\hat{t}, \hat{y}, -\phi_\zeta(\hat{x}, \cdot), v_*(\hat{t}, \hat{y})) \\ &= \int_{|e| \leq \delta} [\phi_\zeta(\hat{x} + \beta(\hat{x}, e), \hat{y}) - \phi_\zeta(\hat{x}, \hat{y})] \gamma(\hat{x}, e) \lambda(de) \\ &\quad + \int_{|e| > \delta} [u_{n,\varepsilon}(\hat{t}, \hat{x} + \beta(\hat{x}, e)) - u_{n,\varepsilon}(\hat{t}, \hat{x})] \gamma(\hat{x}, e) \lambda(de) \\ &\quad + \int_{|e| \leq \delta} [\phi_\zeta(\hat{x}, \hat{y} + \beta(\hat{y}, e)) - \phi_\zeta(\hat{x}, \hat{y})] \gamma(\hat{y}, e) \lambda(de) \\ &\quad - \int_{|e| > \delta} [v^*(\hat{t}, \hat{y} + \beta(\hat{y}, e)) - v^*(\hat{t}, \hat{y})] \gamma(\hat{y}, e) \lambda(de) \\ &\leq \int_{|e| > \delta} [u_{n,\varepsilon}(\hat{t}, \hat{x} + \beta(\hat{x}, e)) \gamma(\hat{x}, e) - v^*(\hat{t}, \hat{y} + \beta(\hat{y}, e)) \gamma(\hat{y}, e)] \lambda(de) \\ &\quad + \int_{|e| > \delta} u_{n,\varepsilon}(\hat{t}, \hat{x}) [\gamma(\hat{y}, e) - \gamma(\hat{x}, e)] \lambda(de) \\ &\quad + \int_{|e| \leq \delta} [\phi_\zeta(\hat{x} + \beta(\hat{x}, e), \hat{y}) - \phi_\zeta(\hat{x}, \hat{y})] \gamma(\hat{x}, e) \lambda(de) \\ &\quad + \int_{|e| \leq \delta} [\phi_\zeta(\hat{x}, \hat{y} + \beta(\hat{y}, e)) - \phi_\zeta(\hat{x}, \hat{y})] \gamma(\hat{y}, e) \lambda(de). \end{aligned}$$

The last two integrals are almost the same:

$$\begin{aligned} &\int_{|e| \leq \delta} [\phi_\zeta(\hat{x} + \beta(\hat{x}, e), \hat{y}) - \phi_\zeta(\hat{x}, \hat{y})] \gamma(\hat{x}, e) \lambda(de) \\ &= (2\chi\hat{x} + \eta|\hat{x} - \hat{y}|) \int_{|e| \leq \delta} \beta(\hat{x}, e) \gamma(\hat{x}, e) \lambda(de) + \left(\chi + \frac{\eta}{2}\right) \int_{|e| \leq \delta} |\beta(\hat{x}, e)|^2 \gamma(\hat{x}, e) \lambda(de) \\ &\leq K \left(2\chi|\hat{x}| + \eta|\hat{x} - \hat{y}| + \chi + \frac{\eta}{2}\right) O(\delta) \end{aligned}$$

for some constant K depending only on β and γ . Then from Condition **C13**

$$(41) \quad \begin{aligned} &\int_{|e| > \delta} u_{n,\varepsilon}(\hat{t}, \hat{x}) [\gamma(\hat{y}, e) - \gamma(\hat{x}, e)] \lambda(de) \leq u_{n,\varepsilon}(\hat{t}, \hat{x}) C_\gamma |\hat{x} - \hat{y}| \int_{|e| > \delta} (1 \wedge |e|^2) \lambda(de) \\ &\leq K u_{n,\varepsilon}(\hat{t}, \hat{x}) |\hat{x} - \hat{y}| \leq Kn(T+1) |\hat{x} - \hat{y}|. \end{aligned}$$

The remaining term can be controlled as follows:

$$\begin{aligned} &\int_{|e| > \delta} [u_{n,\varepsilon}(\hat{t}, \hat{x} + \beta(\hat{x}, e)) \gamma(\hat{x}, e) - v^*(\hat{t}, \hat{y} + \beta(\hat{y}, e)) \gamma(\hat{y}, e)] \lambda(de) \\ &= \int_{|e| > \delta} u_{n,\varepsilon}(\hat{t}, \hat{x} + \beta(\hat{x}, e)) [\gamma(\hat{x}, e) - \gamma(\hat{y}, e)] \lambda(de) \\ &\quad + \int_{|e| > \delta} [u_{n,\varepsilon}(\hat{t}, \hat{x} + \beta(\hat{x}, e)) - v^*(\hat{t}, \hat{y} + \beta(\hat{y}, e))] \gamma(\hat{y}, e) \lambda(de) \end{aligned}$$

The first term on the right-hand side can be bounded as (41). For the second one, we use inequality (38):

$$\begin{aligned}
& \int_{|e|>\delta} [u_{n,\varepsilon}(\hat{t}, \hat{x} + \beta(\hat{x}, e)) - v^*(\hat{t}, \hat{y} + \beta(\hat{y}, e))] \gamma(\hat{y}, e) \lambda(de) \\
& \leq \int_{|e|>\delta} \left[p\beta(\hat{x}, e) - q\beta(\hat{y}, e) + \frac{\eta}{2} |\beta(\hat{x}, e) - \beta(\hat{y}, e)|^2 + \chi (|\beta(\hat{x}, e)|^2 + |\beta(\hat{y}, e)|^2) \right] \gamma(\hat{y}, e) \lambda(de) \\
& \leq K \left(\frac{\eta}{2} |\hat{x} - \hat{y}|^2 + \chi(1 + |\hat{x}| + |\hat{y}|) \right).
\end{aligned}$$

Hence all these bounds give

$$\begin{aligned}
B^{n,\varepsilon} - B^* & \leq 2n(T+1)K|\hat{x} - \hat{y}| + K \left(\frac{\eta}{2} |\hat{x} - \hat{y}|^2 + \chi(1 + |\hat{x}| + |\hat{y}|) \right) \\
& \quad + 2K \left(\chi(|\hat{x}| + |\hat{y}|) + \eta|\hat{x} - \hat{y}| + \chi + \frac{\eta}{2} \right) O(\delta) = \mathfrak{h}(\hat{x}, \hat{y}).
\end{aligned}$$

We can now control the term

$$\begin{aligned}
(42) \quad & f_n(\hat{t}, \hat{x}, u_{n,\varepsilon}(\hat{t}, \hat{x}), p\sigma(\hat{x}), B^{n,\varepsilon}) - f(\hat{t}, \hat{y}, v_*(\hat{t}, \hat{y}), q\sigma(\hat{y}), B^*) \\
& \leq [(f^0 \wedge n) - f^0] + \varpi_{n/\chi}(|\hat{x} - \hat{y}|(1 + |p\hat{x}|)) + L|p\sigma(\hat{x}) - q\sigma(\hat{y})| + L\mathfrak{h}(\hat{x}, \hat{y}) \\
& \leq \varpi_{n/\chi}(|\hat{x} - \hat{y}|(1 + |p\hat{x}|)) + LK(\eta|\hat{x} - \hat{y}|^2 + \chi(1 + |\hat{x}|^2 + |\hat{y}|^2)) + L\mathfrak{h}(\hat{x}, \hat{y})
\end{aligned}$$

We have used that the function $f = f(t, x, y, z, u)$ is Lipschitz continuous w.r.t. z (Condition **C5**), locally Lipschitz continuous w.r.t. x (Condition **C12**), is non increasing w.r.t. y (Condition **C3**) and non decreasing w.r.t. u (Condition **C6**).

Finally plugging (36), (37), (39) and (40), (42) in (35) we obtain:

$$(43) \quad \frac{\varepsilon}{T^2} + o(\zeta) \leq \omega_1(\eta, \chi, \hat{x}, \hat{y}) O(\delta) + \omega_2(\eta|\hat{x} - \hat{y}|^2, \chi(1 + |\hat{x}|^2 + |\hat{y}|^2), |\hat{x} - \hat{y}|)$$

where we have gathered in the ω_1 terms, all terms multiplied by $O(\delta)$. The ω_2 term contains all terms of the form $\eta|\hat{x} - \hat{y}|^2$, $\chi(1 + |\hat{x}|^2 + |\hat{y}|^2)$ or $|\hat{x} - \hat{y}|$. We let ζ and δ go to zero and since

$$\lim_{\eta \rightarrow +\infty} \lim_{\chi \rightarrow 0} \left(\frac{\eta}{2} |\hat{x} - \hat{y}|^2 + \chi(|\hat{x}|^2 + |\hat{y}|^2) \right) = 0,$$

the inequality (43) leads to a contradiction taking χ sufficiently small and η sufficiently large. Hence $u_{n,\varepsilon} \leq v_*$ and it is true for every $\varepsilon > 0$, so the result is proved. \square

3 Regularity of the minimal solution

The function u is the minimal non negative viscosity solution of the PDE (6). From (18) we know that u is finite on $[0, T] \times \mathbb{R}^d$ and For $\varepsilon > 0$, u is bounded on $[0, T - \varepsilon] \times \mathbb{R}^d$ by $K(1 + |x|^\delta)\varepsilon^{-1/q}$. We cannot expect regularity on $[0, T] \times \mathbb{R}^d$, but only on $[0, T - \varepsilon] \times \mathbb{R}^d$ for any $\varepsilon > 0$. In order to obtain a smoother solution u , some assumptions are imposed on the coefficients. We distinguish three different conditions.

- *Sobolev regularity*: the viscosity solution is a weak solution in the Sobolev sense if the coefficients on the forward SDE (8) are smooth and if the *linkage operator* $x \mapsto x + \beta(x, e)$ is a C^2 -diffeomorphism. Under these assumptions, we can control the stochastic flow of $X^{t,x}$ and then deduce the Sobolev regularity of u .
- *Hölder regularity*: if the Lévy measure λ verifies **(E)**, then the viscosity solution is locally Hölder continuous. Remark that under the stronger condition **A6**, **(E)** is unnecessary.
- *Strong regularity*. u can be a classical solution provided **(A+)** holds, if the function in **C9** is bounded, and under different settings.
 - If the measure λ is finite and if f only depend on (t, x, y) , then using Veretenikov's result, u is in $C^{1,2}([0, T - \varepsilon] \times \mathbb{R}^d)$ for any $\varepsilon > 0$. Note that here no regularity on b , σ or f is required.
 - Again if λ is finite and if $f(t, x, 0, z, u)$ is bounded with $(t, x) \mapsto f(t, x, y, z, u) \in H^{\alpha/2, \alpha}$ uniformly w.r.t. (t, x) , then we can use the technic developped in Ma et al. [32].
 - In the setting of Garroni and Menaldi [22], i.e. for some $\gamma < 2$

$$(44) \quad \int_E (1 \wedge |e|^\gamma) \lambda(de) < +\infty$$

and the linkage operator satisfies

$$(45) \quad \det(\text{Id}_d + \nabla_x \beta(x, e)) \geq c_1 > 0,$$

the existence of a Green function G with suitable properties will ensure a regularizing effect of the operator $\mathcal{L} + \mathcal{I}$.

Of course, none of these settings gives necessary conditions. For sure other sufficient assumptions could be exhibited.

3.1 Sobolev regularity of the solution

The solution u is the increasing limit of u_n . And on u_n we can apply Theorem 1 of [34]. Indeed let us fix a continuous positive and integrable weight function ρ such that $1/\rho$ is locally integrable. We define $\mathbb{L}_\rho^2([0, T] \times \mathbb{R}^d)$ the Hilbert space of function $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\int_0^T \int_{\mathbb{R}^d} |v(t, x)|^2 \rho(x) dx dt < +\infty.$$

We assume that

- The functions b , σ , $\beta(\cdot, e)$ are in $C_{t,b}^3(\mathbb{R}^d)$ for any $e \in E$. Condition **A4** holds also for all derivatives of β of order less than or equal to 3.

- For each $e \in E$ the linkage operator $x \mapsto x + \beta(x, e)$ is a C^2 -diffeomorphism.

These extra assumptions are used to control the stochastic flow generated by $X^{t,x}$ (see Proposition 2 in [34]).

Recall that we can replace in the BSDE (13) our generator f_n by \widehat{f}_n where \widehat{f}_n is Lipschitz continuous w.r.t. y . Hence all assumptions of Theorem 1 in [34] are fulfilled: $u_n(t, x) = Y_t^{n,t,x}$ is the unique Sobolev solution of IPDE (6) in the space

$$\mathcal{H}_T = \{v \in L^2_\rho([0, T] \times \mathbb{R}^d), \quad \sigma^* \nabla v \in L^2_\rho([0, T] \times \mathbb{R}^d)\}.$$

The definition of Sobolev solution is given in Definition 1 in [34]. Moreover $(\sigma^* \nabla u)(t, x) = Z_t^{n,t,x}$. In particular for any $\varepsilon > 0$, and each function $\phi \in C^\infty([0, T] \times C_c^\infty(\mathbb{R}^d))$ (i.e. the space of infinite differentiable functions in a neighborhood on $[0, T]$ with compact support in \mathbb{R}^d), for any $t \leq T - \varepsilon$

$$\begin{aligned} & \int_t^{T-\varepsilon} (u_n(s, x), \partial_s \phi(s, x)) ds + (u_n(t, x), \phi(t, x)) - (u_n(T - \varepsilon, x), \phi(T - \varepsilon, x)) \\ & \quad - \int_t^{T-\varepsilon} (u_n(s, x), \mathcal{A}^* \phi(s, x)) ds \\ & = \int_t^{T-\varepsilon} (f(s, x, u_n(s, x)), \sigma^* \nabla u_n(s, x), \mathcal{B}(s, x, u_n)), \phi(s, x)) ds \end{aligned}$$

where $(v, w) = \int_{\mathbb{R}^d} u(x)v(x)dx$ is the scalar product on $L^2(\mathbb{R}^d)$ and \mathcal{A}^* is the adjoint operator of the operator $\mathcal{L} + \mathcal{I}$.

Moreover for any $\varepsilon > 0$, on $[0, T - \varepsilon]$, by the estimate (18), and from Inequality (16), we deduce that u_n and $\sigma^* \nabla u_n$ are bounded from above by $C(1 + |x|^\delta)$ for some $C > 0$ and $\delta > 0$. Hence if we choose the suitable weight ρ , u_n and $\sigma^* \nabla u_n$ are bounded in $\mathcal{H}(0, T - \varepsilon)$. Therefore the next result is proved.

Proposition 3 *Under conditions (A)-(B)-(C), if the coefficients b , σ and β satisfy the above conditions, then $u \in \mathcal{H}_{T-\varepsilon}$ and is a Sobolev solution of the IPDE (6) on $[0, T - \varepsilon]$ for any $\varepsilon > 0$.*

Note that in the case $f(y) = -y|y|^q$, the only hypotheses in order to have a Sobolev solution are on the coefficients of the forward diffusion.

3.2 Lipschitz/Hölder regularity of the solution

Recently there have been several papers [6, 7, 13, 14, 15, 45, 46] (among many others) dealing with C^α estimates and regularity of the solution of the IPDE (6). Here we will mainly use the paper written by Barles et al. [6, 7].

In our setting we defined F by (33) and from Conditions (A) and (C) we can easily check that F is continuous and *degenerate elliptic* and (H0) and (H2) of [6] hold:

- If $X \geq Y$, $I \geq I'$, $B \geq B'$, $F(t, x, u, p, X, I, B) \leq F(t, x, u, p, Y, I', B')$.

- For any $t \in [0, T]$, $x \in \mathbb{R}^d$, u, v in \mathbb{R} , $p \in \mathbb{R}^d$, $X \in \mathbb{S}_d$ and $(I, B) \in \mathbb{R}^2$,

$$F(t, x, u, p, X, l, B) - F(t, x, v, p, X, l, B) \geq 0 = 0 \times (u - v), \text{ when } u \geq v.$$

- $(I, B) \mapsto F(\cdot, I, B)$ is Lipschitz continuous, uniformly with respect to all the other variables.

Since we are just interesting in a global regularity property, we add the *strict ellipticity condition* (H) of [6]: there exist two non negative functions Λ_1 and Λ_2 defined on \mathbb{R}^d , a positive constant Λ_0 such that $\Lambda_1(x) + \Lambda_2(x) \geq \Lambda_0 > 0$, and a modulus of continuity ϖ_F such that for any $\varepsilon > 0$, $t \in [0, T]$, $u \in \mathbb{R}$, x, y in \mathbb{R}^d , $B \in \mathbb{R}$,

- if X, Y in \mathbb{S}_d satisfy the matrix inequality:

$$-\frac{1}{\varepsilon} \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{1}{\varepsilon} \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix}$$

with $Z = Id - \omega \hat{z} \otimes \hat{z}$ for some unit vector $\hat{z} \in \mathbb{R}^d$ and $\omega \geq 1$,

- and if $I' \geq I$,

then

$$\begin{aligned} F(t, y, u, p, Y, I', B) - F(t, x, u, p, X, I, B) &\leq \Lambda_1(x)(I - I') \\ &+ \Lambda_2(x)\text{Trace}(X - Y) + \varpi_F \left(\frac{|x - y|^2}{\varepsilon} + |x - y|(1 + |p|) \right) \end{aligned}$$

As explained in the introduction of [6], the diffusion term gives the ellipticity in certain directions whereas it is given by the non local term in the complementary directions.

In our setting, the existence of ϖ_F follows from our conditions **(A)** and **C12** if we assume that ϖ_R in **C12** does not depend on R (see Section 4.1 in [6]). Moreover F is linear w.r.t. I . Hence with $\Lambda_1(x) = 1$

$$F(t, x, u, p, X, I', B) - F(t, x, u, p, X, I, B) = \Lambda_1(x)(I - I').$$

Moreover $2\Lambda_2(x)$ is the minimal eigenvalue of the matrix $\sigma\sigma^*(x)$. Thus $\Lambda_2(x) \geq 0$. Therefore the strict ellipticity condition (H) is satisfied with $\Lambda_0 = 1$.

Now when the strict ellipticity is involved by the non local terms, we need some extra conditions on the Lévy measure λ and on the coefficient β in the SDE (8). These assumptions are denoted by (J1) to (J5) in [6]. In the following, B is the unit ball in E and B_ε is the ball centered at zero with radius $\varepsilon > 0$. Remember that λ is a Lévy measure on \mathbb{R}^d :

$$\int_E (1 \wedge |e|^2) \lambda(de) < +\infty.$$

From **(A)**, the next conditions hold already.

- From Condition **A4**, there exists a constant $C_{\lambda, \beta}$ such that for all $x \in \mathbb{R}^d$:

$$\int_B |\beta(x, e)|^2 \lambda(de) + \int_{\mathbb{R}^d \setminus B} \lambda(de) \leq C_{\lambda, \beta}.$$

- (partially). From **A2** there exists a constant K_β such that for all $e \in B$, and x and y in \mathbb{R}^d :

$$|\beta(x, e) - \beta(y, e)| \leq K_\beta |e| |x - y|.$$

Moreover from **A4** for all $(e, x) \in E \times \mathbb{R}^d$

$$|\beta(x, e)| \leq C_\beta |e|.$$

- Assumption **A2** implies that for all $e \in E$, $|e| \geq 1$, and x and y in \mathbb{R}^d :

$$|\beta(x, e) - \beta(y, e)| \leq K_\beta |x - y|.$$

To verify all conditions of [6] we add these additional hypotheses.

E1. There exists $c_\beta > 0$ such that for all $(e, x) \in E \times \mathbb{R}^d$, $c_\beta |e| \leq |\beta(x, e)|$.

E2. There exists $\tau \in (0, 2)$ such that for every $a \in \mathbb{R}^d$, there exists $0 < \eta < 1$ and a constant $\tilde{C}_\lambda > 0$ such that the following holds for any $x \in \mathbb{R}^d$

$$\forall \varepsilon > 0, \quad \int_{C_{\eta, \varepsilon}(a)} |\beta(x, e)|^2 \lambda(de) \geq \tilde{C}_\lambda \eta^{\frac{d-1}{2}} \varepsilon^{2-\tau}$$

with $C_{\eta, \varepsilon}(a) = \{e; |\beta(x, e)| \leq \varepsilon, (1 - \eta)|\beta(x, e)||a| \leq |a \cdot \beta(x, e)|\}$.

E3. There exists $\tau \in (0, 2)$ such that for $\varepsilon > 0$ small enough

$$\int_{B \setminus B_\varepsilon} |e| \lambda(de) \leq \begin{cases} \hat{C}_\lambda \varepsilon^{1-\tau} & \text{for } \tau \neq 1, \\ \hat{C}_\lambda |\ln(\varepsilon)| & \text{for } \tau = 1. \end{cases}$$

We denote by **(E)** the three conditions **E1**, **E2**, **E3**. If **(E)** holds then Conditions (J1) to (J5) of [6] are satisfied. Recall that our terminal condition g is continuous from \mathbb{R}^d to $\mathbb{R} \cup \{+\infty\}$ (Hypothesis **B3**). Now we state the main result of this part. This Proposition is a modification of Corollary 7 of [6].

Proposition 4 *Assume that Conditions (A)-(B)-(C)-(E) are satisfied. Moreover the modulus of continuity in **C12** does not depend on R .*

- Assume that $\tau > 1$ and that for all $M \geq 0$, g is a Lipschitz continuous function on the set $\mathcal{O}_M = \{|g| \leq M\}$. Then for all $\varepsilon > 0$, u is locally Lipschitz continuous on \mathbb{R}^d , uniformly w.r.t. $t \in [0, T - \varepsilon]$: for all M , there exists a constant $C_{M, \varepsilon}$ such that

$$\forall |x| \leq M, \forall |y| \leq M, |u(t, x) - u(t, y)| \leq C_{M, \varepsilon} |x - y|.$$

The constant $C_{M, \varepsilon}$ depends only on ε , on M on the dimension d , and on the constants in Assumption **(E)**.

- If $\tau \leq 1$, and if for some $\alpha < \tau$, g is α -Hölder continuous function on the set $\mathcal{O}_M = \{|g| \leq M\}$ for all $M \geq 0$, then u is locally α -Hölder continuous on \mathbb{R}^d , uniformly w.r.t. $t \in [0, T - \varepsilon]$.

Proof. For any $n \in \mathbb{N}$, u_n is a continuous viscosity solution of (6) with terminal condition g_n . We know that this function is bounded on $[0, T] \times \mathbb{R}^d$ by $n(T + 1)$ and has a uniform w.r.t. n upper bound given by (18). We can apply to u_n the results of Corollary 7 in [6].

- Assume that $\tau > 1$. Our condition on g implies that g_n is Lipschitz on \mathbb{R}^d . From [6], u_n is Lipschitz continuous w.r.t. x on $[0, T]$, i.e. there exists a constant C_n such that for all $t \in [0, T]$, all $(x, y) \in (\mathbb{R}^d)^2$,

$$|u_n(t, x) - u_n(t, y)| \leq C_n |x - y|.$$

The key point here is that the constant C_n depends only on $\|u_n\|_\infty$, on the dimension d , and on the constants in Assumption **(E)**. From the upper bound (20), we deduce that for any $\varepsilon > 0$, on $[0, T - \varepsilon]$, u_n is locally Lipschitz continuous w.r.t. x and the Lipschitz constant $C_{M, \varepsilon}$ does not depend on n . Thus by Arzelà-Ascoli theorem, the limit u is locally Lipschitz continuous with the same constants.

- If $\tau \leq 1$, then u_n is α -Hölder continuous w.r.t. x on $[0, T]$, i.e. there exists a constant C such that for all $t \in [0, T]$, all $(x, y) \in (\mathbb{R}^d)^2$,

$$|u_n(t, x) - u_n(t, y)| \leq C |x - y|^\alpha.$$

And C depends on the norm of u_n , which does not depend on n if we take t in $[0, T - \varepsilon]$. The conclusion follows immediately. □

Remark 4 (On Condition **(E))** *If the matrix $\sigma\sigma^*$ is uniformly elliptic (see Condition **A6**), then the conclusion of the previous proposition still holds without Assumption **(E)** and the regularity of u depends on the regularity of g (no more on τ).*

Indeed Condition **(E)** is crucial to have regularity estimates when the local second order differential operator \mathcal{L} becomes degenerated. But if \mathcal{L} remains strictly elliptic, then regularity of the solution can be derived directly without the help of the non local operator \mathcal{I} .

3.3 Strong regularity of the solution

Here we briefly explain how we can derive that this minimal viscosity solution is a regular function on $[0, T - \varepsilon] \times \mathbb{R}^d$ under additional conditions. For results concerning classical solutions of the IPDE (6), the book of Garroni and Menaldi [22] is an important reference (see also references therein).

To have more regularity on the solution u we will assume that Conditions **(A+)** hold, that is σ and b are bounded and $\sigma\sigma^*$ is uniformly elliptic:

$$(A6) \quad \forall y \in \mathbb{R}^d, \quad \sigma\sigma^*(x)y \cdot y \geq \Lambda_0 |y|^2.$$

Recall that **A6** implies Condition **H** of the previous section.

In this first lemma, we obtain a regularity result with a transformation of the non local operator \mathcal{I} . This idea is used in [32] and [40]. This transformation requires the finiteness of the measure λ . But after we can use all results concerning (linear) PDE, especially the ones contained in the reference book [30].

Lemma 6 Under **(A+)** and **(C)**, if the measure λ is finite and if f depends only on (t, x, v) with $f(t, x, 0)$ bounded uniformly w.r.t. (t, x) , then for every bounded and continuous function ϕ , the Cauchy problem

$$(46) \quad \partial_t v + \mathcal{L}v + \mathcal{I}(t, x, v) + f(t, x, v) = 0$$

with terminal condition $v(T, \cdot) = \phi$, has a unique bounded classical solution v in the sense that $v \in C^{1,2}([0, T] \times \mathbb{R}^d) \cap C([0, T] \times \mathbb{R}^d)$.

Here in fact no regularity assumption on the coefficients b , σ and $f(\cdot, \cdot, 0)$ is required. Only boundedness and **A6** are important.

Proof. We will use the scheme done in [41] (Proposition 24), in Ma et al. [32] (Theorem 1) and in Pham [40] (Proposition 5.3). For a given continuous and bounded function ϕ , the problem (46) has a unique bounded and continuous viscosity solution v (just apply Theorem 3.4 and Theorem 3.5 in [5]). All regularity estimates come from the book of Ladyzhenskaya et al. [30].

We consider the following Cauchy problem: on $[0, T] \times \mathbb{R}^d$

$$(47) \quad \partial_t w + \tilde{\mathcal{L}}w + f_v = 0,$$

with

- the differential operator $\tilde{\mathcal{L}}$:

$$\tilde{\mathcal{L}}\phi = \frac{1}{2} \text{Trace}(D^2 \phi \sigma \sigma^*(x)) + \tilde{b}(x) \nabla \phi,$$

- the drift term \tilde{b} :

$$\tilde{b}(x) = b(x) - \int_E \beta(x, e) \lambda(de).$$

- the generator f_v :

$$f_v(t, x) = f(t, x, v(t, x)) + \int_E [v(x + \beta(x, e)) - v(x)] \lambda(de).$$

First note that v is also the unique bounded viscosity solution of (47).

Then the assumptions **A4**, **A5** and **C4** imply that the drift term \tilde{b} and the function f_v are also bounded. Using the result of Veretennikov [49], Theorem 3.1, the problem (47) has a unique solution w in the class $C([0, T] \times \mathbb{R}^d; \mathbb{R}^+) \cap \bigcap_{p>1} W_{p,loc}^{1,2}([0, T] \times \mathbb{R}^d)$. Now we

define three processes for all $s \geq t$:

$$\bar{X}_s^{t,x} = x + \int_t^s \tilde{b}(u, \bar{X}_u^{t,x}) du + \int_t^s \sigma(u, \bar{X}_u^{t,x}) dB_u,$$

and

$$\bar{Y}_s^{t,x} = w(s, \bar{X}_s^{t,x}), \text{ and } \bar{Z}_s^{t,x} = \nabla w(s, \bar{X}_s^{t,x}) \sigma(s, \bar{X}_s^{t,x}).$$

We can apply the Itô formula to the function w (see [29], section 2.10). We have for all $s \geq t$:

$$\overline{Y}_s^{t,x} = \overline{Y}_{T-\delta}^{t,x} - \int_s^{T-\delta} f_v(r, \overline{X}_r^{t,x}) dr - \int_s^{T-\delta} \overline{Z}_r^{t,x} dB_r.$$

Since f_v is bounded, it is well known (see [36]) that the function $(t, x) \mapsto \overline{Y}_t^{t,x}$ is a continuous and bounded viscosity solution of (47). Therefore $w = v$ and thus the viscosity solution v belongs to $\bigcap_{p>1} W_{p,loc}^{1,2}([0, T] \times \mathbb{R}^d)$.

Hence for all $\alpha < 1$, v belongs to the space $H^{\beta/2, 1+\beta}$ (the set of functions which are $\beta/2$ -Hölder-continuous in time and $1 + \beta$ -Hölder-continuous in space), and the Hölder norm of v depends just on the L^∞ bound of v . Thus f_v is also Hölder continuous, and from the existence result of [30] (see section IV, theorems 5.1 and 10.1), v is a classical solution of (47) and of (46). \square

Lemma 7 *Under (A+) and (C), if the measure λ is finite, if $f(t, x, 0, z, u)$ is bounded, and if $(t, x) \mapsto f(t, x, y, z, u)$ is in $H^{\alpha/2, \alpha}$ uniformly w.r.t. (t, x) , then for every bounded and continuous function ϕ , the Cauchy problem*

$$(48) \quad \partial_t v + \mathcal{L}v + \mathcal{I}(t, x, v) + f(t, x, v, \nabla v \sigma(t, x), \mathcal{B}(t, x, v)) = 0$$

with terminal condition $v(T, \cdot) = \phi$, has a unique bounded classical solution v in the sense that $v \in C^{1,2}([0, T] \times \mathbb{R}^d) \cap C([0, T] \times \mathbb{R}^d)$.

Proof. If the functions σ , b , β , f and γ were of class C^1 with bounded derivatives w.r.t. all parameters (x, y, z, u) , then the conditions of [32] would be satisfied. Thus from Theorem 1 in [32], if $\phi \in H^{2+\alpha}(\mathbb{R}^d)$ for some $\alpha > 0$, then $w \in H^{1+\alpha/2, 2+\alpha}([0, T] \times \mathbb{R}^d)$. But we claim that we do not need these conditions if we relax the regularity condition on the solution w .

Indeed the first key point in [32] is the transformation of (48):

$$(49) \quad \partial_t w + \tilde{\mathcal{L}}w + \tilde{f}(t, x, w, \nabla w \sigma, \mathcal{B}(t, x, w)) = 0,$$

where $\tilde{\mathcal{L}}$ is the same as in (47) and

$$\begin{aligned} \tilde{f}(t, x, w, \nabla w \sigma, \mathcal{B}(t, x, w)) = \\ f(t, x, w(t, x), \nabla w(t, x) \sigma(t, x), \mathcal{B}(t, x, w)) + \int_E [w(x + \beta(x, e)) - w(x)] \lambda(de). \end{aligned}$$

The bounded viscosity solution v of (48) is also the unique viscosity solution of (49). Moreover the bound on v depends on the Lipschitz constants of f and the bound on ϕ . A classical solution being a viscosity solution, by uniqueness of the viscosity solution, a classical solution is bounded with the same bound. As already denoted, we can truncate f w.r.t. y and assume that f is in fact Lipschitz w.r.t. y (from Condition **C4**).

Then we follow the same steps as in [32]. For a classical solution w the first estimate is:

$$\|\nabla w\|_\infty + \|w\|_{H^{\alpha/2, \alpha}([0, T] \times \mathbb{R}^d)} \leq M$$

where M and $\alpha \in (0, 1)$ depend only on the bounds in **A5**, **A6** and the Lipschitz constants of the coefficients. Then (49) can be seen as a linear equation with Hölder continuous leading coefficients, all other coefficients being bounded. Hence L^p estimates give that

$$\|w, \partial_t w, \nabla w, D^2 w\|_{L^p_{loc}} \leq C_p$$

where C_p depends only on M and the bound on w . Thereby for all $\alpha < 1$, w belongs to the space $H^{\beta/2, 1+\beta}$, and the Hölder norm of w depends just on M and on the L^∞ bound of w . Again after this step, we can deduce the existence of a classical solution w by Schauder estimate. \square

Now if f does not depend on z (or on ∇v), a straightforward modification of the proof shows the next result.

Lemma 8 *The conclusion of the lemma 7 holds under the assumptions:*

- f does not depend on z ;
- ϕ is bounded Lipschitz continuous function;
- there exists a constant $C_{\lambda, \beta}$ such that for all $x \in \mathbb{R}^d$,

$$\int_E |\beta(x, e)| \lambda(de) \leq C_{\lambda, \beta}.$$

As an example if the measure λ satisfies (44) with $\gamma = 1$, then from Assumption **A4** the above condition holds.

Proof. Indeed the previous section 3.2 shows that v is also Lipschitz continuous. By Condition **A4**, β is bounded uniformly w.r.t. $x \in \mathbb{R}^d$. Hence there exists a constant C such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\int_E |v(t, x + \beta(x, e)) - v(t, x)| \lambda(de) \leq C_M \int_E |\beta(x, e)| \lambda(de) \leq C_M C_{\lambda, \beta}.$$

Moreover with **C7**

$$\int_E |v(t, x + \beta(x, e)) - v(t, x)| \gamma(x, e) \lambda(de) \leq C_M \int_E |\beta(x, e)| \gamma(x, e) \lambda(de) \leq C_M C_\vartheta C_\beta,$$

in other words $\mathcal{B}(t, x, v)$ is a bounded function on $[0, T] \times \mathbb{R}^d$. Thus \tilde{b} and f_v are still bounded functions. The rest of the proof does not change. \square

The main drawback of the previous lemmas is the finiteness of the measure λ . To avoid this condition, we must use regularity results on IPDE. Some results can be found in [11] (see Chapter 3, theorems 3.2 and 3.3), but with Condition (44) with $\gamma = 1$. The main reference (from our best knowledge) is the book of Garroni and Menaldi [22] (with a short version [21] and the book [20] on Green functions for IPDE).

Lemma 9 *Assume that*

- The measure λ satisfies (44) for some $\gamma < 2$.
- The function β is differentiable w.r.t. x and (45) holds.
- f is Hölder-continuous w.r.t. (t, x) uniformly w.r.t. the other parameters, that is there exists $\alpha \in (0, 1)$ such that there exists a constant C such that for all (y, z, u)

$$|f(t, x, y, z, u) - f(t', x', y, z, u)| \leq C(|t - t'|^{\alpha/2} + |x - x'|^\alpha).$$

Then for any bounded and continuous function ϕ , the Cauchy problem (48) has a unique solution v in the set $C([0, T] \times \mathbb{R}^d) \cap H^{1+\delta/2, 2+\delta}([0, T] \times \mathbb{R}^d)$ where $\delta = \alpha$ if $\gamma < 2 - \alpha$ and $\delta \in (0, 2 - \alpha)$ if $\gamma \in [2 - \alpha, 2)$. Moreover v is bounded on $[0, T] \times \mathbb{R}^d$.

Proof. We follow the ideas developed in the proof of Theorem 4.2, Chapter VI (see also Theorem 6.1, Chapter V) in [30]. For $0 \leq \rho \leq 1$, we consider the family of linear problems: on $[0, T] \times \mathbb{R}^d$

$$(50) \quad \frac{\partial}{\partial t} v(t, x) + \mathcal{L}v(t, x) + \mathcal{I}(t, x, v) + \rho f(t, x, w, (\nabla w)\sigma, \mathcal{B}(t, x, w)) = 0$$

with terminal condition $v(T, \cdot) = \phi$. This defines an operator Ψ which associates each function $w \in H^{1+\delta/2, 2+\delta}([0, T] \times \mathbb{R}^d)$ with a solution v of the linear problem (50): $v = \Psi(w, \rho)$. Its fixed points for $\rho = 1$ are solutions of our initial problem.

Now let us recall some facts contained in [20, 21, 22]. First if $w \in H^{1+\delta/2, 2+\delta}$, then for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\|\mathcal{B}(t, x, w)\|_\delta = \varepsilon \|\nabla w\|_\delta + C(\varepsilon) \|\phi\|_\delta.$$

The next point concerns the existence and the uniqueness of the Green function. Under **(A+)**, there exists a unique Green function G associated with the parabolic second-order integro-differential operator

$$\partial_t u - \mathcal{L}u - \mathcal{I}u,$$

and G has the representation

$$G(x, t, y, s) = G_{\mathcal{L}}(x, t, y, s) + \int_s^t du \int_{\mathbb{R}^d} G_{\mathcal{L}}(x, t, \xi, u) Q(\xi, u, y, s) d\xi,$$

where $G_{\mathcal{L}}$ is the Green function associated with the differential operator \mathcal{L} . G and Q belong to some Green Function Spaces (see Chapter VII, Definition 1.1 in [20]). The key properties are inherited from the properties of $G_{\mathcal{L}}$. The Green function $G_{\mathcal{L}}$ is studied in Chapter IV, Sections 12 to 14 of [30]. The properties of the Green functions $G_{\mathcal{L}}$ and G imply regularizing property for the parabolic operator $\partial_t u - \mathcal{L}u - \mathcal{I}u$. Namely, from Theorem VIII.2.1 of [20] and Theorem 3.2 of [21], if ϕ is a continuous and bounded function, and $f \in H^{\alpha/2, \alpha}$, then the function v defined by

$$v(t, x) = \int_{\mathbb{R}^d} G(x, t, y, 0) \phi(y) dy + \int_0^t \int_{\mathbb{R}^d} G(x, t, y, s) f(y, s) dy ds = v_1(t, x) + v_2(t, x)$$

on $(0, +\infty) \times \mathbb{R}^d$ is in $H^{1+\delta/2, 2+2\delta}((0, \tau] \times \mathbb{R}^d)$ for any $\tau > 0$, and solves the IPDE (50) with $f(t, x)$ in the right-hand side and the initial condition $v(0, \cdot) = \phi$. Moreover the estimates on v and its derivatives on $[\varepsilon, \tau] \times \mathbb{R}^d$ only depend on ε , on the bound on ϕ and on f . More precisely, there exists a constant C independent of ϕ such that:

$$\varepsilon^{1+\delta/2} \|v_1\|_{H^{1+\delta/2, 2+2\delta}([\varepsilon, \tau] \times \mathbb{R}^d)} \leq C \|\phi\|_\infty,$$

and

$$\|v_2\|_{H^{1+\delta/2, 2+2\delta}([\varepsilon, \tau] \times \mathbb{R}^d)} \leq C \|f\|_\delta.$$

These inequalities can be found in Theorem 3.2 of [21] and Section IV.14 of [30]. The Green function G is involved in the constant C . The time reversion will give the same inequality on $[0, T - \varepsilon] \times \mathbb{R}^d$, if ϕ is a terminal condition at time T .

In searching for the fixed points of Ψ we apply the Leray-Schauder principle (see [30] for the details).

- Ψ is equicontinuous in w and ρ and uniformly compact. This comes immediately from the previous inequality since $w \in H^{1+\delta/2, 2+2\delta}([0, T] \times \mathbb{R}^d)$ implies that $f(t, x, w, (\nabla w)\sigma, \mathcal{B}(t, x, w))$ is in $H^{\delta/2, \delta}$.
- The operator $\Psi(\cdot, 0)$ has a unique fixed point. Existence is implied by the existence of the Green function G , and uniqueness can be deduced from the uniqueness of the solution of the BSDE.
- Since $\Psi(w, 0)$ takes all w into a single element, $w \mapsto w - \Psi(w, 0)$ invertible in a neighborhood of the unique fixed point of $\Psi(\cdot, 0)$.

Hence for each $\rho \in [0, 1]$, there exists at least one fixed point v^ρ for Ψ . Using again the regularizing result of [20], $v^\rho \in H^{1+\delta/2, 2+2\delta}([0, T] \times \mathbb{R}^d)$. We just have to take $\rho = 1$ to obtain v .

The continuity and the boundedness on the solution v comes from classical a priori estimate on the BSDE (see [5]). \square

Proposition 5 *Under (A+), (B) and (C), we assume that the conditions of Lemmas 6, 7, 8 or 9 hold. Moreover the function in C9 is bounded. Then for all $\varepsilon > 0$:*

$$(51) \quad u \in C^{1,2}([0, T - \varepsilon] \times \mathbb{R}^d; \mathbb{R}^+).$$

Proof. The proof of Proposition 2 shows that there is a unique continuous viscosity solution in Π_{pg} of the Cauchy problem (48) (or (46)). Moreover, the Cauchy problem (48) (or (46)) has a classical solution for every continuous bounded function ϕ .

Recall that u_n is jointly continuous in (t, x) and from (20), on $[0, T - \varepsilon] \times \mathbb{R}^d$, u_n is bounded by:

$$0 \leq u_n(t, x) \leq K_\varepsilon(1 + |x|^\delta).$$

But if the function involved in C9 is bounded, then we can take $\delta = 0$. Thus, the problem (46) with condition $\phi = u_n(T - \varepsilon, \cdot)$ has a bounded classical solution. Since every classical

solution is a viscosity solution and since u_n is the unique bounded and continuous viscosity solution of (46), we deduce that:

$$\forall \delta > 0, u_n \in C^{1,2}([0, T - \delta] \times \mathbb{R}^d; \mathbb{R}^+).$$

From the construction of the classical solution u_n , we also know that the sequence $\{u_n\}$ is locally bounded in $H^{\alpha, 1+\alpha}([0, T - \delta/2] \times \mathbb{R}^d)$. The bound is given by the L^∞ norm of u_n which is smaller than $(T - \delta/4)^{-1/q}$. Therefore u is continuous on $[T - \delta/2] \times \mathbb{R}^d$ and if we consider the problem (46) with continuous terminal data $u(T - \delta, \cdot)$, with the same argument as for u_n , we obtain that u is a classical solution, i.e. $u \in C^{1,2}([0, T - \delta] \times \mathbb{R}^d; \mathbb{R}^+)$. This achieves the proof. \square

If (D) is also satisfied, then u is continuous on $[0, T] \times \mathbb{R}^d$.

References

- [1] O. Alvarez and A. Tourin. Viscosity solutions of nonlinear integro-differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(3):293–317, 1996.
- [2] S. Ankirchner, M. Jeanblanc, and T. Kruse. BSDEs with Singular Terminal Condition and a Control Problem with Constraints. *SIAM J. Control Optim.*, 52(2):893–913, 2014.
- [3] P. Baras and M. Pierre. Problèmes paraboliques semi-linéaires avec données mesures. *Applicable Anal.*, 18(1-2):111–149, 1984.
- [4] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*, volume 17 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Paris, 1994.
- [5] G. Barles, R. Buckdahn, and É. Pardoux. Backward stochastic differential equations and integral-partial differential equations. *Stochastics Stochastics Rep.*, 60(1-2):57–83, 1997.
- [6] G. Barles, E. Chasseigne, A. Ciomaga, and C. Imbert. Lipschitz regularity of solutions for mixed integro-differential equations. *J. Differential Equations*, 252(11):6012–6060, 2012.
- [7] G. Barles, E. Chasseigne, and C. Imbert. Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations. *J. Eur. Math. Soc. (JEMS)*, 13(1):1–26, 2011.
- [8] G. Barles and C. Imbert. Second-order elliptic integro-differential equations: viscosity solutions’ theory revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(3):567–585, 2008.
- [9] G. Barles and E. Lesigne. SDE, BSDE and PDE. In *Backward stochastic differential equations (Paris, 1995–1996)*, volume 364 of *Pitman Res. Notes Math. Ser.*, pages 47–80. Longman, Harlow, 1997.

- [10] I. Bensaoud and A. Sayah. Stability results for Hamilton-Jacobi equations with integro-differential terms and discontinuous Hamiltonians. *Arch. Math. (Basel)*, 79(5):392–395, 2002.
- [11] A. Bensoussan and J.-L. Lions. *Contrôle impulsif et inéquations quasi variationnelles*, volume 11 of *Méthodes Mathématiques de l'Informatique [Mathematical Methods of Information Science]*. Gauthier-Villars, Paris, 1982.
- [12] J.-M. Bismut. Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.*, 44:384–404, 1973.
- [13] L. Caffarelli and L. Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.
- [14] H. Chang-Lara and G. Dávila. Regularity for solutions of non local parabolic equations. *Calc. Var. Partial Differential Equations*, 49(1-2):139–172, 2014.
- [15] H. Chang-Lara and G. Dávila. Regularity for solutions of nonlocal parabolic equations II. *J. Differential Equations*, 256(1):130–156, 2014.
- [16] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.
- [17] L. Delong. *Backward stochastic differential equations with jumps and their actuarial and financial applications*. European Actuarial Academy (EAA) Series. Springer, London, 2013. BSDEs with jumps.
- [18] E. B. Dynkin and S. E. Kuznetsov. Nonlinear parabolic P.D.E. and additive functionals of superdiffusions. *Ann. Probab.*, 25(2):662–701, 1997.
- [19] Nicole El Karoui and Laurent Mazliak, editors. *Backward stochastic differential equations*, volume 364 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, 1997. Papers from the study group held at the University of Paris VI, Paris, 1995–1996.
- [20] M. G. Garroni and J.-L. Menaldi. *Green functions for second order parabolic integro-differential problems*, volume 275 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1992.
- [21] M.G. Garroni and J.L. Menaldi. Regularizing effect for integro-differential parabolic equations. *Comm. Partial Differential Equations*, 18(12):2023–2050, 1993.
- [22] M.G. Garroni and J.L. Menaldi. *Second order elliptic integro-differential problems*, volume 430 of *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [23] P. Graewe, U. Horst, and J. Qiu. A Non-Markovian Liquidation Problem and Backward SPDEs with Singular Terminal Conditions. *ArXiv e-prints*, 2013.

- [24] P. Graewe, U. Horst, and E. Séré. Smooth solutions to portfolio liquidation problems under price-sensitive market impact. *ArXiv e-prints*, 2013.
- [25] S. Hamadène and X. Zhao. Systems of Integro-PDEs with Interconnected Obstacles and Multi-Modes Switching Problem Driven by Lévy Process. *ArXiv e-prints*, August 2014.
- [26] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.
- [27] T. Kruse and A. Popier. BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration. *Stochastics An International Journal of Probability and Stochastic Processes*, 2015. <http://dx.doi.org/10.1080/17442508.2015.1090990>.
- [28] T. Kruse and A. Popier. Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position. *Stochastic Processes and their Applications*, pages –, 2016.
- [29] N. V. Krylov. *Controlled diffusion processes*, volume 14 of *Applications of Mathematics*. Springer-Verlag, New York-Berlin, 1980. Translated from the Russian by A. B. Aries.
- [30] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.
- [31] J.-F. Le Gall. A probabilistic approach to the trace at the boundary for solutions of a semilinear parabolic partial differential equation. *J. Appl. Math. Stochastic Anal.*, 9(4):399–414, 1996.
- [32] J. Ma, J. Yong, and Y. Zhao. Four step scheme for general Markovian forward-backward SDEs. *J. Syst. Sci. Complex.*, 23(3):546–571, 2010.
- [33] M. Marcus and L. Véron. Initial trace of positive solutions of some nonlinear parabolic equations. *Comm. Partial Differential Equations*, 24(7-8):1445–1499, 1999.
- [34] A. Matoussi, W. Sabbagh, and C. Zhou. The obstacle problem for semilinear parabolic partial integro-differential equations. *Stoch. Dyn.*, 15(1):1550007, 38, 2015.
- [35] B. Øksendal and A. Sulem. *Applied stochastic control of jump diffusions*. Universitext. Springer, Berlin, second edition, 2007.
- [36] É. Pardoux. BSDEs, weak convergence and homogenization of semilinear PDEs. In *Nonlinear analysis, differential equations and control (Montreal, QC, 1998)*, volume 528 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 503–549. Kluwer Acad. Publ., Dordrecht, 1999.

- [37] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, volume 176 of *Lecture Notes in Control and Inform. Sci.*, pages 200–217. Springer, Berlin, 1992.
- [38] É. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.
- [39] E. Pardoux and A. Rascanu. *Stochastic Differential Equations, Backward SDEs, Partial Differential Equations*, volume 69 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, 2014.
- [40] H. Pham. Optimal stopping of controlled jump diffusion processes: a viscosity solution approach. *J. Math. Systems Estim. Control*, 8(1):27 pp. (electronic), 1998.
- [41] A. Popier. Backward stochastic differential equations with singular terminal condition. *Stochastic Process. Appl.*, 116(12):2014–2056, 2006.
- [42] A. Popier. Limit behaviour of the minimal solution of a bsde with jumps and with singular terminal condition. *ArXiv e-prints*, 2016.
- [43] Philip E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. *Stochastic Modelling and Applied Probability*.
- [44] M.-C. Quenez and A. Sulem. BSDEs with jumps, optimization and applications to dynamic risk measures. *Stochastic Process. Appl.*, 123(8):3328–3357, 2013.
- [45] L. Silvestre. Hölder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana Univ. Math. J.*, 55(3):1155–1174, 2006.
- [46] L. Silvestre. Hölder continuity for integro-differential parabolic equations with polynomial growth respect to the gradient. *Discrete Contin. Dyn. Syst.*, 28(3):1069–1081, 2010.
- [47] Rong Situ. *Theory of stochastic differential equations with jumps and applications*. Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005. Mathematical and analytical techniques with applications to engineering.
- [48] N. Touzi. Second order backward SDEs, fully nonlinear PDEs, and applications in finance. In *Proceedings of the International Congress of Mathematicians. Volume IV*, pages 3132–3150. Hindustan Book Agency, New Delhi, 2010.
- [49] A Yu Veretennikov. Parabolic equations and Itô’s stochastic equations with coefficients discontinuous in the time variable. *Mathematical Notes*, 31(4):278–283, 1982.