A note on certain representations in characteristic p and associated functions

Federico Pellarin

A NOTE ON CERTAIN REPRESENTATIONS IN CHARACTERISTIC p AND ASSOCIATED FUNCTIONS

F. PELLARIN

1. Introduction

In [6], the author has introduced some special functions related to the arithmetic of function fields of positive characteristic (and more precisely, to the arithmetic of the ring $A = \mathbb{F}_q[\theta]$ with θ an indeterminate), namely, L-values and vector valued modular forms (the vectors having entries in certain ultrametric Banach algebras). The purpose of [6] was to produce a new class of functional identities for L-values, and only very particular examples of these new special functions were required, in order to obtain the results in that paper. The theory was later developed along several axes (see for example [1], which also contains quite a detailed bibliography).

The aim of this note is to highlight the connection that these special functions have with representations of algebras, groups etc. associated to A, and to present families of examples which can be the object of further studies. In particular, we are interested in certain irreducible representations of $\mathrm{SL}_2(\mathbb{F}_q[\theta])$ or $\mathrm{GL}_2(\mathbb{F}_q[\theta])$. We also provide a few explicit examples and properties of such representations.

The plan of the note is the following. In §2, we shall discuss of algebra representations of A and we will consider their associated ω-values and L-values. In §3, we present a class of irreducible representations ρ^I sitting inside symmetric powers in the case $q = p$. In §4, We apply the results of §3 to show that certain tensor products ρ^{II} are irreducible. In §5 series we use these results to show that the entries of certain vectorial Poincaré series generalizing those introduced in [6] are linearly independent and we present a conjecture on the rank of a certain module of vectorial modular forms.

In all the following, $q = p^e$ with p a prime number and $e > 0$ we set

$$\Gamma = \mathrm{GL}_2(\mathbb{F}_q[\theta]),$$

where θ is an indeterminate over $\mathbb{F}_q$ and $q = p^e$ for some prime number p and an integer $e > 0$. We shall write $A = \mathbb{F}_q[\theta]$ (so that $\Gamma = \mathrm{GL}_2(A)$). All along the paper, if $a = a_0 + a_1\theta + \cdots + a_r\theta^r$ is an element of A with $a_0, \ldots, a_r \in \mathbb{F}_q$ and if t is an element of an $\mathbb{F}_q$-algebra B, then $a(t)$ denotes the element $a_0 + a_1 t + \cdots + a_r t^r \in B$. Also, we set $K = \mathbb{F}_q(\theta)$.

2. Algebra representations

In this section, we consider an integral, commutative $\mathbb{F}_q$-algebra $\boldsymbol{A}$ and we denote by $\boldsymbol{K}$ its fraction field. We choose an injective algebra representation

$$A \xrightarrow{\sigma} \mathrm{Mat}_{d\times d}(\boldsymbol{K}), \tag{1}$$

which is completely determined by the choice of the image $\vartheta := \sigma(\theta)$. Note that σ is not injective if and only if ϑ has all its eigenvalues in $\mathbb{F}_q^{ac}$, algebraic closure of $\mathbb{F}_q$ (In all the following, if L is a field, L^{ac} denotes an algebraic closure of L). We have:

Lemma 1. *The representation σ is irreducible over $\boldsymbol{K}$ if and only if ϑ has d distinct eigenvalues in an algebraic closure $\boldsymbol{K}^{ac}$ of $\boldsymbol{K}$, which are pairwise conjugated.*

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Proof. From the Jordan canonical form of ϑ over $\boldsymbol{K}^{ac}$, there is a decomposition $(\boldsymbol{K}^{ac})^d = \oplus_i V_i$ where V_i is a vector space over $\boldsymbol{K}^{ac}$ such that the left multiplication by ϑ is equal to the left multiplication by a Jordan block associated to an eigenvalue λ_i of this left multiplication by ϑ. The set $\{V_i\}$ is $\mathrm{Aut}(\boldsymbol{K}^{ac}/\boldsymbol{K})$-invariant. If there exists i such that $\dim_{\boldsymbol{K}^{ac}}(V_i) > 1$ then, there exists a proper sub-vector space over $\boldsymbol{K}$ which is left invariant so that the induced representation is a direct sum of conjugate right lower corner sub-representations, and σ cannot be irreducible over $\boldsymbol{K}$. Hence, if σ is irreducible (over $\boldsymbol{K}$), the spaces V_i are one-dimensional over $\boldsymbol{K}^{ac}$ and pairwise conjugate; in this case, the eigenvalues are distinct and pairwise conjugated. Reciprocally, if the eigenvalues of ϑ are distinct and pairwise conjugated, then σ is irreducible over $\boldsymbol{K}$. Indeed, assuming the converse, there would exist a proper sub-vector space $V \subset (\boldsymbol{K}^{ac})^d$ defined over $\boldsymbol{K}$ invariant under the left multiplication by ϑ. Then, with V_i one-dimensional, we would have $V = \oplus_{i\in I} V_i$ with $\emptyset \neq I \subsetneq \{1,\dots,d\}$. But then, V would not be $\mathrm{Aut}(\boldsymbol{K}^{ac}/\boldsymbol{K})$-invariant. □

2.1. **An example.** We consider here θ as an indeterminate over $\boldsymbol{K}$. More generally, we can suppose that $\boldsymbol{K}$ is the fraction field of a commutative integral $\mathbb{F}_q$-algebra $\boldsymbol{A}$. We denote by $\boldsymbol{A}[\theta]^+$ the set of polynomials which are monic in θ. Let P be a polynomial in $\boldsymbol{A}[\theta]^+$, let d be the degree of P in θ. The euclidean division in $\boldsymbol{A}[\theta]$ by P defines for all $a \in \boldsymbol{A}[\theta]$, in an unique way, a matrix $\sigma_P(a) \in \mathrm{Mat}_{d\times d}(\boldsymbol{A})$ such that

$$aw \equiv \sigma_P(a)w \pmod{P\boldsymbol{A}[\theta]},$$

where w is the column vector with entries $1, \theta, \dots, \theta^{d-1}$. Explicitly, if $P = \theta^d + P_{d-1}\theta^{d-1} + \cdots + P_0$ with $P_i \in \boldsymbol{A}$, then

$$\sigma_P(\theta) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -P_0 & -P_1 & \cdots & -P_{d-1} \end{pmatrix}.$$

Hence, the map σ_P defines an algebra homomorphism

$$A \to \mathrm{End}(\boldsymbol{A}^d)$$

and provides a representation of the monoid $A^* = A\setminus\{0\}$ in $\mathrm{Mat}_{d\times d}(\boldsymbol{A})^*$. The representation σ_P is faithful if P has no roots in $\mathbb{F}_q^{ac}$ and is irreducible if and only if P is irreducible over $\boldsymbol{K}$.

2.2. **L-values and ω-values of algebra representations and semi-characters.** We continue with the settings of the beginning of this section. Here, we shall give a few elementary properties of certain basic objects that can be associated to representations such as in (1). Since the proofs are in fact obvious generalizations of the arguments of [6, 1], we will only sketch our proofs.

For a ring R, we denote by R^* the set R with the underlying multiplicative monoid. We recall that A^+ denotes the subset of A of monic polynomials; it is a multiplicative monoid. Let $\boldsymbol{M}$ be an $\mathbb{F}_q$-algebra (e.g. $\boldsymbol{M} = \mathrm{Mat}_{d\times d}(\boldsymbol{K}_s)$).

Definition 2. A monoid homomorphism

$$\sigma : A^+ \to \boldsymbol{M}^*$$

is a *semi-character* if there exist pairwise commuting $\mathbb{F}_q$-algebra homomorphisms

$$\sigma_1, \dots, \sigma_s : A \to \boldsymbol{M}$$

such that, for $a \in A^+$,

$$\sigma(a) = \sigma_1(a)\cdots\sigma_s(a).$$

The trivial map $\sigma(a) = 1$ for all a is a semi-character, according to the convention that an empty product is equal to one. If, for all $i = 1, \dots, s$, $\vartheta_i = \sigma_i(\theta)$ has a well defined minimal polynomial over $\boldsymbol{K}_s$, we say that the semi-character σ is of *Dirichlet type*. For example, if $\boldsymbol{M} = \mathrm{Mat}_{d\times d}(\boldsymbol{K}_s)$, then every element $\vartheta \in \boldsymbol{M}$ has a minimal polynomial over $\boldsymbol{K}_s$. The *conductor* of a semi-character of Dirichlet type is the product of all the pairwise distinct minimal polynomials of the elements $\vartheta_1, \dots, \vartheta_s$.

Example. If we choose $\boldsymbol{M} = \mathbb{F}_q^{ac}$, then a semi-character $\sigma : A^+ \to \mathbb{F}_q^{ac}$ is always of Dirichlet type, and our definition coincides in fact with the usual notion of a Dirichlet-Goss character $A^+ \to \mathbb{F}_q^{ac}$. There are non-pairwise conjugated elements $\zeta_1, \dots, \zeta_s \in \mathbb{F}_q^{ac}$, of minimal polynomials $P_1, \dots, P_s \in A$, such that $\sigma(a) = a(\zeta_1)^{n_1} \cdots a(\zeta_s)^{n_s}$ for all $a \in A$, with $0 < n_i < q^{d_i} - 1$ for all i, with d_i the degree of P_i. The conductor is the product $P_1 \cdots P_s$.

Non-example. We set $\boldsymbol{M} = \mathbb{F}_q[x]$ and we consider the map $\sigma : A^+ \to \boldsymbol{M}$ defined by $a \mapsto x^{\deg_\theta(a)}$. Then, σ is a monoid homomorphism which is not a semi-character. Indeed, assuming the converse, then $\sigma = \sigma_1 \cdots \sigma_s$ for algebra homomorphisms $\sigma_i : A \to \boldsymbol{M}$. But since $\sigma(\theta) = \sigma_1(\theta) \cdots \sigma_s(\theta) = t$, we get $s = 1$ and σ would be an algebra homomorphism, which is certainly false.

From now on, we suppose, for commodity, that $\boldsymbol{M} = \mathrm{Mat}_{d\times d}(\boldsymbol{K}_s)$ (but some of the arguments also hold with $\boldsymbol{M}$ any $\mathbb{F}_q$-algebra).

2.2.1. *ω-values of an algebra representation.* We consider a d-dimensional representation as in (1), in $\boldsymbol{M} = \mathrm{Mat}_{d\times d}(\boldsymbol{K}_s)$. Let λ_θ be a root $(-\theta)^{\frac{1}{q-1}} \in K^{ac}$ of $-\theta$. We consider the following element of $\lambda_\theta K_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M} \subset \mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M}$, the topological product $\widehat{\otimes}_{\mathbb{F}_q}$ being considered with respect to the trivial norm over $\boldsymbol{M}$, and where K_∞ is the local field $\mathbb{F}_q((1/\theta))$. We can quite easily view the ring $K_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M}$: it is the ring of the formal series $f = \sum_{i \geq i_0} c_i \theta^{-i}$, with $c_i \in \boldsymbol{M}$; in other words, it is equal to the ring $\mathrm{Mat}_{d\times d}(\boldsymbol{K}_s((\theta^{-1})))$. We denote by Π_σ the convergent product

$$\Pi_\sigma = \prod_{i \geq 0} (I_d - \sigma(\theta)\theta^{-q^i})^{-1} \in \mathrm{GL}(K_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M})$$

(where I_d denotes the identity matrix). Let $\mathbb{C}_\infty$ be the completion $\widehat{K_\infty^{ac}}$, where K_∞^{ac} denotes an algebraic closure of K_∞. The *ω-value* associated to σ is the product $\omega_\sigma = \lambda_\theta \Pi_\sigma \in \lambda_\theta K_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M}$. We have, on the other hand, an algebra endomorphism τ of $\mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M}$ uniquely defined by setting $\tau(\lambda \otimes \mu) = \lambda^q \otimes \mu$.

Lemma 3. *The element ω_σ is a generator of the free $\boldsymbol{M}$-submodule of rank one of $\mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M}$ of the solutions of the τ-difference equation*

$$\tau(X) = (\sigma(\theta) - \theta I_d)X.$$

Proof. (Sketch.) It is easy to verify that ω_σ is a solution of the above equation. If ω' is another solution, then $Y = \omega' \omega_\sigma^{-1}$ is solution of $\tau(Y) = Y$; it is then straightforward to check that the solutions of the latter difference equation are precisely the elements of $\boldsymbol{M}$, hence, $Y \in \boldsymbol{M}$. □

Remark 4. It is also easy to prove that, for σ as in (1), $\det(\omega_\sigma) = \omega_\alpha$, where $\alpha \in \boldsymbol{K}_s[\theta]$ is -1 times the characteristic polynomial of $\sigma(\theta)$ and ω_α is the function defined in [1, §6].

Let T be another indeterminate. The ring $\mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M}$ is endowed with a structure of $A[T]$-module in two ways. The first structure is that in which the multiplication by θ is given by the usual diagonal multiplication, and the multiplication by T is given by the left multiplication by $\sigma(\theta)$; this defines indeed, uniquely, a module structure. The second structure, called *Carlitz module structure*, denoted by $C(\mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M})$, has the same multiplication by T and has the multiplication C_θ by θ independent of the choice of σ, and defined as follows. If $m \in C(\mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M})$, then $C_\theta(m) = \theta m + \tau(m)$. Note that both structures depend in fact on the choice of σ, and that the dependence is limited in the multiplication by T.

We have the exponential map

$$\exp_C : \mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M} \to C(\mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M})$$

defined by $\exp_C(f) = \sum_{i \geq 0} D_i^{-1} \tau^i(f)$, where D_i is the product of the monic polynomials of A of degree i. It is quite standard to check that this is a continuous, open, surjective $A[T]$-module homomorphism, of kernel $\widetilde{\pi} A \otimes_{\mathbb{F}_q} \boldsymbol{M}$.

Lemma 5. *We have $\omega_\sigma = \exp_C\left(\widetilde{\pi}(\theta I_d - \rho(\theta))^{-1}\right)$.*

Proof. We set $f = \exp_C\left(\widetilde{\pi}(\theta I_d - \rho(\theta))^{-1}\right)$. Since $(C_\theta - \rho(\theta))(f) = 0$ in $C(\mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M})$, the lemma follows from Lemma 3. □

2.2.2. *L-values associated to a semi-character.* Let σ be a semi-character $A^+ \to \boldsymbol{M}$, with $\boldsymbol{M}$ an $\mathbb{F}_q$-algebra. To fix our ideas, we again suppose that $\boldsymbol{M} = \mathrm{Mat}_{d\times d}(\boldsymbol{K}_s)$, with $\boldsymbol{K}_s$ as in the previous sections. Let n be a positive integer. The *n-th L-value* associated to σ is the following element of $\mathrm{GL}(K_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M})$:

$$L_\sigma(n) = \prod_P \left(I_d - \sigma(P)P^{-n}\right)^{-1} = \sum_{a\in A^+} \sigma(a)a^{-n}.$$

Remark 6. We write $\sigma = \sigma_1 \cdots \sigma_s$ for $\mathbb{F}_q$-algebra homomorphisms $\sigma_i : A \to \boldsymbol{M}$. The elements $L_\sigma(n)$ and $\omega_{\sigma_1}\cdots\omega_{\sigma_s}$ commute each other. Further, if $\lambda_{i,1}, \ldots, \lambda_{i,d} \in \boldsymbol{K}_s^{ac}$ are the eigenvalues of $\sigma_i(\theta)$ for $i = 1, \ldots, s$ (considered with multiplicities), then we have the formula

$$\det(L_\sigma(n)) = \prod_{j=1}^{s}\prod_{i=1}^{d} L(\chi_t, n)|_{t=\lambda_{i,j}} \in K_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{K}_s,$$

where $L(\chi_t, n) = \sum_{a\in A^+} a^{-n}a(t)$.

We study in some more detail the case of $n = 1$. We write $\sigma = \sigma_1 \cdots \sigma_s$ as above. Since for all $a \in A^+$, $C_a(\omega_{\sigma_i}) = \sigma_i(a) \in \boldsymbol{M}$, we have the convergent series identity

$$\sum_{a\in A^+} a^{-1} C_a(\omega_{\sigma_1}) \cdots C_a(\omega_{\sigma_s}) = L_\sigma(1)\omega_{\sigma_1}\cdots\omega_{\sigma_s} \in \mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \boldsymbol{M}.$$

2.2.3. *A Simple application of a result of Anglès et al.* We now invoke the result of B. Anglès, F. Tavares Ribeiro and the author [1, Theorem 8.2] (note that in the statement, we can set $Z = 1$). For completeness, we mention the following result, which is a very easy consequence of ibid.:

Proposition 7. *For every semi-character $\sigma : A^+ \to \boldsymbol{M}$ with $\sigma = \sigma_1 \cdots \sigma_s$ as above,*

$$(\omega_{\sigma_1}\cdots\omega_{\sigma_s})^{-1} \exp_C(\omega_{\sigma_1}\cdots\omega_{\sigma_s} L_\sigma(1)) = S_\sigma \in A \otimes_{\mathbb{F}_q} \boldsymbol{M}.$$

Further, if $s \equiv 1 \pmod{q-1}$ the matrix with polynomial entries S_σ is zero. In particular, in this case, if $s > 1$ and $s \equiv 1 \pmod{q-1}$,

$$L_\sigma(1) = \widetilde{\pi}(\omega_{\sigma_1}\cdots\omega_{\sigma_s})^{-1}\mathbb{B}_\sigma,$$

where $\mathbb{B}_\sigma$ is a matrix with polynomial entries of $A \otimes_{\mathbb{F}_q} \boldsymbol{M}$.

Hence, $L_\sigma(1)$ is some kind of Taelman unit, in the sense of [9]. If $\sigma : A \to \boldsymbol{M}$ is an algebra homomorphism, we have the simple explicit formula

$$L_\sigma(1) = \omega_\sigma^{-1}(I_d\theta - \sigma(\theta))^{-1}\widetilde{\pi}$$

which can be proved in a way very similar to that of [6, §4]. We are going to see, in §5.2.1 how $L_\sigma(n)$ is related to certain vectorial Eisenstein series, when $n \equiv s \pmod{q-1}$.

2.3. **Representations of Γ associated to an algebra representation.** Let σ be a d-dimensional representation as in (1). We associate to it, canonically, a representation of $\Gamma = \mathrm{GL}_2(A)$ in $\mathrm{GL}_{2d}(\boldsymbol{K})$.

We consider the map $\Gamma \xrightarrow{\rho_\sigma} \mathrm{Mat}_{2d\times 2d}(\boldsymbol{K})$, defined by

$$\rho_\sigma\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d)\end{pmatrix}.$$

Then, ρ_σ determines a representation $\Gamma \to \mathrm{GL}_{2d}(\boldsymbol{K})$. Indeed, $\sigma(a), \sigma(b)$ commute each other, for $a, b \in A$. Furthermore, we have:

Lemma 8. *ρ_σ is irreducible if and only if σ is irreducible.*

Proof. It is easy to see that if σ is reducible, then ρ_σ is reducible. We suppose now that σ is irreducible. Then, by Lemma 1, $\vartheta = \sigma(\theta)$ has distinct eigenvalues $\lambda_1, \dots, \lambda_d \in \boldsymbol{K}^{ac}$ which are conjugate over $\boldsymbol{K}$, and, over $\boldsymbol{K}^{ac}$, σ splits as a direct sum of one-dimensional representations $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_d$ with $(\boldsymbol{K}^{ac})^d = V_1 \oplus \cdots \oplus V_d$ and $\sigma_i : A \to \mathrm{End}_{\boldsymbol{K}^{ac}}(V_i)$ for all i. This shows that ρ_σ splits as a direct sum $\rho_1 \oplus \cdots \oplus \rho_d$ with $\rho_i : \Gamma \to \mathrm{GL}(W_i)$ irreducible, $(\boldsymbol{K}^{ac})^{2d} = W_1 \oplus \cdots \oplus W_d$, and $W_i \cong V_i \oplus V_i$. Let us suppose by contradiction that there exists a proper, non-trivial subvector space $V \subset \boldsymbol{K}^{2d}$ such that $\rho_\sigma(\gamma)(V) \subset V$ for all $\gamma \in \Gamma$. Then, for all i, $\rho_\sigma(\gamma)((V \otimes \boldsymbol{K}^{ac}) \cap W_i) \subset (V \otimes \boldsymbol{K}^{ac}) \cap W_i$. Now, reasoning on the dimension, we see that there exists i such that $(V \otimes \boldsymbol{K}^{ac}) \cap W_i \neq \{0\}$. This vector space cannot have dimension 1 because the representation ρ_i is irreducible. Hence, $V \otimes \boldsymbol{K}^{ac}$ contains W_i but the W_i's are all Galois-conjugate because the V_i's are so, due to the fact that σ is irreducible. Hence, $V \otimes \boldsymbol{K}^{ac}$ contains W_i for all i and $V \otimes \boldsymbol{K}^{ac} = (\boldsymbol{K}^{ac})^{2d}$ which is again impossible. □

Example. We can construct, in particular, the representation $\rho_P = \rho_{\sigma_P} : \Gamma \to \mathrm{GL}_{2d}(\boldsymbol{A})$ which is irreducible if and only if P is irreducible.

3. Symmetric powers

In the first part of this section, we suppose that q is a prime number; $p = q$. Let B be an $\mathbb{F}_p$-algebra. We denote by ρ the tautological representation $\mathrm{GL}_2(B) \to \mathrm{GL}_2(B)$. We consider the representation

$$\rho_r = \mathrm{Sym}^r(\rho) : \mathrm{GL}_2(B) \to \mathrm{GL}_{r+1}(B),$$

where Sym^r denotes the r-th symmetric power realized in the space of polynomials homogeneous of degree $r+1$ with coefficients in B. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(B)$, then

$$\rho_r(\gamma)(X^{r-i}Y^i) = (aX + cY)^{r-i}(bX + dY)^i, \quad i = 0, \dots, r.$$

Associated to an integer $l \geq 0$ with p-expansion $l = l_0 + l_1 p + \cdots + l_s p^s$ $(0 \leq l_i \leq p-1)$, we also consider the representations

$$\rho_l^I = \rho_{l_0} \otimes \rho_{l_1}^{(1)} \otimes \cdots \otimes \rho_{l_s}^{(s)},$$

where, for a matrix M with entries in B, $M^{(i)}$ denotes the the matrix obtained from M raising all its entries to the power p^i. The dimension of ρ_l^I is equal to

$$\phi_p(l) = \prod_i (l_i + 1).$$

Remark 9. We notice the following algorithm to construct the sequence of dimensions $(\phi_p(l))_{l \geq 1}$. Define $a_0 = (1)$, $a_1 = (1, 2, \dots, p)$ (equal to the concatenation $[a_0, 2a_0, \dots, pa_0]$) and then, inductively,

$$a_n = [a_{n-1}, 2a_{n-1}, \dots, pa_{n-1}].$$

Since it is clear that for all n, a_{n-1} is a prefix of a_n, there is a well defined inductive limit a_∞ of the sequence $(a_n)_{n \geq 0}$ which is easily seen to be equal to the sequence $(\phi_p(l))_{l \geq 1}$.

Lemma 10. *The representation ρ_l^I is isomorphic to a sub-representation of ρ_l.*

Proof. We actually construct the sub-representation explicitly; the Lemma will follow easily. We consider, for $\gamma \in \mathrm{GL}_2(B)$, the matrix $\rho_l^\star(\gamma)$ which is the square matrix extracted from

$$\rho_l(\gamma) = (\rho_{i,j})_{1 \leq i,j \leq l+1}$$

in the following way. If $0 \leq r \leq l$ is such that $\binom{l}{r} \equiv 0 \pmod{p}$, we drop the $(r+1)$-th row and the $(r+1)$-th column. In other words, one uses the row matrix

$$\mathcal{D}_l = \left(\binom{l}{l}, \dots, \binom{l}{r}, \dots, \binom{l}{0} \right)$$

and discards rows and columns of ρ_l according with the vanishing of the corresponding entry of $\mathcal{D}_l$ and what is left precisely defines the matrix $\rho_l^\star$. By Lucas formula, $\rho_l^\star$ has dimension $\phi_p(l)$ and it is easy to see, by induction on the number of digits of the p-expansion of l, that $\rho_l^\star \cong \rho_l^I$. □

Example. If $l = 1 + p$, we have

$$\mathcal{D}_l^I = (1,1,1,1) = \left(\binom{l}{0}, \binom{l}{1}, \binom{l}{p}, \binom{l}{l}\right).$$

In this case, we find, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

$$\rho_l^\star(\gamma) = \begin{pmatrix} a^{p+1} & a^p b & ab^p & b^{p+1} \\ a^p c & a^p d & b^p c & b^p d \\ ac^p & bc^p & ad^p & bd^p \\ c^{p+1} & c^p d & cd^p & d^{p+1} \end{pmatrix}.$$

3.1. **Representations of $\mathrm{SL}_2(\mathbb{F}_{q'})$.** Let us set $q' = p^f$ with $f > 0$. Then, $B = \mathbb{F}_{q'}$ is an $\mathbb{F}_p$-algebra and we can construct the representations ρ_l and $\rho_l^\star \cong \rho_L^I$ of the beginning of this section. We denote by $\overline{\rho}_l$ the representation ρ_l^I with $B = \mathbb{F}_{q'}$ restricted to $\mathrm{SL}_2(B)$. By using the fact that any non-zero stable subspace in a representation of a p-group over a vector space has a non-zero fixed vector, it is easy to show and in fact well known that, for all $l \geq 0$, $\overline{\rho}_l$ is an irreducible representation if and only if $l < q'$. By Schur's theory, one shows that the representations $\overline{\rho}_l$ with $l < q'$ exhaust all the conjugacy classes of irreducible representations of $\mathrm{SL}_2(\mathbb{F}_{q'})$ over $\mathbb{F}_p^{ac}$. Indeed, counting conjugacy classes of $\mathrm{SL}_2(\mathbb{F}_{q'})$ is an easy task and we know that their number coincides with the number of conjugacy classes of irreducible representations so it suffices to check that the representations above are mutually inequivalent which is an elementary task. This explicit description first appears in the paper [2] of Brauer and Nesbitt.

3.2. **Some representations of $\mathrm{GL}_2(A)$.** We now set $q = p^e$ with $e > 0$. We also set $\boldsymbol{K} := \mathbb{F}_p(t)$ for a variable t all along this subsection. We consider the algebra homomorphism $\chi_t : A \to \mathbb{F}_q[t]$ defined by $\chi_t(a) = a(t) = a_0 + a_1 t + \cdots + a_d t^d$ for $a = a_0 + a_1\theta + \cdots + a_d\theta^d \in A$, with coefficients $a_i \in \mathbb{F}_q$. We extend our notations by setting, for a matrix M with entries in A, $\chi_t(M)$ the matrix obtained applying χ_t entry-wise. We denote by $\rho_t, \rho_{t,l}, \rho_{t,l}^I$ the representations $\chi_t \circ \rho, \chi_t \circ \rho_l, \chi_t \circ \rho^I$ over $\boldsymbol{K}$-vector spaces with the appropriate dimension, of the group $\Gamma = \mathrm{GL}_2(A)$ when $B = A$.

Lemma 11. *For all l as above, the representation $\rho_{t,l}^I$ is irreducible.*

Proof. It suffices to show that the restriction to $\mathrm{SL}_2(\mathbb{F}_p[\theta]) \subset \Gamma$ is irreducible. Let us consider an element $\zeta \in \mathbb{F}_q^{ac}$ of degree f and let us denote by $\mathbb{F}_{q'}$ with $q' = p^f$ the subfield $\mathbb{F}_p(\zeta)$ of $\mathbb{F}_p^{ac}$. The group homomorphism

$$\mathrm{ev}_\zeta : \mathrm{SL}_2(\mathbb{F}_p[\theta]) \to \mathrm{GL}_2(\mathbb{F}_{q'})$$

defined by the entry-wise evaluation ev_ζ of θ by ζ has image $\mathrm{SL}_2(\mathbb{F}_{q'})$. Indeed, the evaluation map $\mathrm{ev}_\zeta : \mathbb{F}_p[\theta] \to \mathbb{F}_{q'}$ is surjective, the image of $\mathrm{SL}_2(\mathbb{F}_p[\theta])$ by ev_ζ clearly contains the subgroup of unipotent upper and lower matrices with coefficients in $\mathbb{F}_{q'}$, which are known to generate $\mathrm{SL}_2(\mathbb{F}_{q'})$ ([1]).

We set $N = \phi_p(l)$. Let V be a non-zero $\boldsymbol{K}$-subvector space of $\boldsymbol{K}^N$ which is stable under the action of the restriction of $\rho_{t,l}^I$ to $\mathrm{SL}_2(\mathbb{F}_p[\theta])$. Let us fix a basis b of V. We choose f big enough so that $q' > l, q$ and the image b' of b in $\mathbb{F}_{q'}^N$ by the evaluation at $t = \zeta$ is well defined and non-zero. Then, the $\mathbb{F}_{q'}$-span of b' is a non-trivial sub-vector space of $\mathbb{F}_{q'}^N$ which is left invariant under the action of $\overline{\rho}_l$, which is impossible. □

Remark 12. Let m be a class of $\mathbb{Z}/(q-1)\mathbb{Z}$ and let us consider the representation

$$\rho_{t,l,m}^I : \Gamma \to \mathrm{GL}_{\phi_p(l)}(\boldsymbol{K})$$

defined by

$$\rho_{t,l,m}^I := \rho_{t,l}^I \otimes \det^{-m}.$$

By Lemma 11, it is irreducible. However, the representations $\rho_{t,l,m}^I$ do not cover all the irreducible representations of Γ in $\mathrm{GL}_N(\boldsymbol{K})$ for some N. Due to the fact that we evaluate the functor GL_N on a ring which

[1] If p is odd and $q' \neq 9$, to generate $\mathrm{SL}_2(\mathbb{F}_{q'})$ one only needs two unipotent elements, thanks to Dickson's Theorem.

is not a field (here, the ring A), we may well have irreducible representations which, after specialization at roots of unity, do not deliver irreducible representations of $\mathrm{SL}_2(\mathbb{F}_{q'})$.

Remark 13. The group $S_{(p)}$ of p-adic digit permutations of $\mathbb{Z}_p$ discussed by Goss in [3] acts on the positive integers l by means of their expansions in base p. This defines an action of the group $S_{(p)}$ on the set of representations ρ_l^I by $\nu(\rho_l^I) = \rho_{\nu(l)}^I$, for $\nu \in S_{(p)}$. It is easy to show that $\nu(\rho_l^I) \cong \rho_{l'}^I$ if and only if $\nu(l) = l'$.

Remark 14. We are thankful to Pietro Corvaja for having noticed the following property: *Let k be a perfect field. Then, for all $\gamma \in \mathrm{SL}_2(k^{ac})$ there exists a morphism $\phi : \mathbb{A}^1 \to \mathrm{SL}_2$ defined over k and $\alpha \in k^{ac}$, such that $\phi(\alpha) = \gamma$.*

4. Products of representations

Let $t_1, \dots, t_s$ be independent variables. We denote by $\underline{t}_s$ the set of variables $(t_1, \dots, t_s)$ and we set $\boldsymbol{K}_s = \mathbb{F}_q(\underline{t}_s)$. If $s = 1$, we write $t = t_1$ and we have $\boldsymbol{K}_s = \boldsymbol{K}$, the field of §3.2. We also consider $\underline{l} = (l_1, \dots, l_s)$ an s-tuple with entries in $\mathbb{Z}$ which are ≥ 1.

Theorem 15. *the representation*

$$\rho_{\underline{t},\underline{l}}^{II} := \rho_{t_1,l_1}^I \otimes \cdots \otimes \rho_{t_s,l_s}^I : \Gamma \to \mathrm{GL}_{\phi_p(l_1)\cdots\phi_p(l_s)}(\boldsymbol{K}_s)$$

is irreducible.

Proof. We set $N = \phi_p(l_1)\cdots\phi_p(l_s)$. Let us suppose that the statement is false. Then, there exists a $\boldsymbol{K}_s$-sub-vector space $V \neq \{0\} \subset \boldsymbol{K}_s^N$ such that for all $\gamma \in \Gamma$, $\rho_{\underline{t},\underline{l}}^{II}(\gamma)(V) \subset V$. Let us fix a basis $v = (v_1, \dots, v_r)$ of V. For integers $0 \leq k_1 \leq \cdots \leq k_s$, we denote by ev the map $\mathbb{F}_q[\underline{t}_s] \to \mathbb{F}_q[t]$ which sends $a(t_1, \dots, t_s) \in \mathbb{F}_q[\underline{t}_s]$ to $a(t^{k_1}, \dots, t^{k_s}) \in \mathbb{F}_q[t]$. This map is a ring homomorphism whose kernel is the prime ideal $\mathcal{P}$ generated by the polynomials $t_j - t_{j-1}^{q^{k_j - k_{j-1}}}$, $j = 2, \dots, s$. We consider the associated multiplicative set $S = \mathbb{F}_q[\underline{t}_s] \setminus \mathcal{P}$. Then, the evaluation map ev extends to $S^{-1}\mathbb{F}_q[\underline{t}_s]$ which is Zariski dense in $\boldsymbol{K}_s = \mathbb{F}_q(\underline{t}_s)$. We now extend ev coefficient-wise on every matrix, vector, etc. with entries in $S^{-1}\mathbb{F}_q[\underline{t}_s]$. In particular, if k_1 is big enough, $\mathrm{ev}(v)$ is well defined and non-zero.

We can in fact choose $k_1, \dots, k_s$ so that we also have $\mathrm{ev}(\rho_{\underline{t},\underline{l}}^{II}) = \rho_{t,l}^I$ for some $l \geq 0$. Indeed, if we write the p-expansions $l_i = l_{i,0} + l_{i,1}p + \cdots + l_{i,r}p^r$ $(i = 1, \dots, s)$ for some $r \geq 0$, then we can choose $k_1, \dots, k_s$ so that there is no carry over in the p-expansion of the sum $l = l_1 q^{k_1} + l_2 q^{k_2} + \cdots + l_s q^{k_s}$; for such a choice of $k_1, \dots, k_s$, $\mathrm{ev}(\rho_{\underline{t},\underline{l}}^{II})$ is thus irreducible.

We now set W to be the $\boldsymbol{K}$-span of $\mathrm{ev}(v)$ (well defined, non-zero, and proper in $\boldsymbol{K}^N$), where we write $\boldsymbol{K} = \mathbb{F}_q(t)$. Let w be in W. We can write $w = a_1 \,\mathrm{ev}(v_1) + \cdots + a_r \,\mathrm{ev}(v_r)$ for elements $a_i \in \boldsymbol{K}$. Then,

$$\rho_{t,l}^I(\gamma)(w) = a_1 \rho_{t,l}^I(\gamma)(\mathrm{ev}(v_1)) + \cdots + a_r \rho_{t,l}^I(\gamma)(\mathrm{ev}(v_r)) = a_1 \,\mathrm{ev}(\rho_{\underline{t},\underline{l}}^{II}(\gamma)(v_1)) + \cdots + a_r \,\mathrm{ev}(\rho_{\underline{t},\underline{l}}^{II}(\gamma)(v_r))$$

is a vector of W, hence contradicting the irreducibility of $\rho_{t,l}^I$. □

5. Applications to Poincaré series

Definition 16. We say that a representation $\Gamma \xrightarrow{\rho} \mathrm{GL}_N(\boldsymbol{K})$ is *normal to the depth* $L \in \{1, \dots, N\}$ if for all $\gamma \in H = \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\} \subset \Gamma$, we have that $\rho(\gamma) = \left(\begin{smallmatrix} * & * \\ 0 & I_L \end{smallmatrix}\right)$, where I_L denotes the identity matrix of size L.

A representation as above which is normal to the depth N has finite image. If, for some ring R, we have that $M \in \mathrm{Mat}_{N\times N}(R) = \left(\begin{smallmatrix} * & * \\ X & Y \end{smallmatrix}\right)$ with $Y \in \mathrm{Mat}_{L\times L}$, we set $M_L = (X, Y) \in \mathrm{Mat}_{L\times N}(R)$. In other words, M_L is the matrix constituted by the last L lines of M.

We denote by $\Omega = \mathbb{C}_\infty \setminus K_\infty$ the Drinfeld "upper-half plane" of $\mathbb{C}_\infty$. We recall the following notation from [6]. we also choose m a non-negative integer, an integer $w \in \mathbb{Z}_{>0}$, and, for $\delta \in \Gamma$ and $z \in \Omega$, we set $\mu_{w,m}(\delta, z) = \det(\delta)^{-m} J_\gamma^w(z)$, where $J_\gamma(z)$ is the usual "Drinfeldian" factor of automorphy defined, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ by $J_\gamma(z) = cz + d$. We also denote by $u(z)$ the uniformizer at infinity of Ω, that is, the function $u(z) = \frac{1}{e_C(z)}$ with e_C the exponential function $\mathbb{C}_\infty \to \mathbb{C}_\infty$ with lattice period $A \subset \mathbb{C}_\infty$.

We consider a representation $\Gamma \xrightarrow{\rho} \mathrm{GL}_N(\boldsymbol{K})$, normal to the depth L. Following [6, §2.4], we set, for $\delta \in \Gamma$,

$$f_\delta = \mu_{w,m}(\delta, z)^{-1} u^m(\delta(z))\rho(\delta)_L : \Omega \to \mathrm{Mat}_{L\times N}(\boldsymbol{C}),$$

where $\boldsymbol{C} := \mathrm{Frac}(\widehat{\mathbb{C}_\infty \otimes_{\mathbb{F}_q}} \boldsymbol{K})$.

Endowing $\boldsymbol{K}$ with the trivial norm, it is easy to show that the series

$$\mathcal{E}_{w,m,\rho}(z) = \sum_{\delta \in H\backslash\Gamma} f_\delta,$$

the sum being over the representatives of the cosets of Γ modulo the left action of $H = \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\}$, converges to a holomorphic function

$$\mathcal{E}_{w,m,\rho} : \Omega \to \mathrm{Mat}_{L\times N}(\boldsymbol{C}),$$

in the sense of [7].

Remark 17. For convenience of the reader, we recall here the definition of an holomorphic function $\Omega \to \boldsymbol{C}$. For $z \in \Omega$, we set $|z|_\infty := \inf_{\lambda\in K_\infty} |z-\lambda|$, which is non-zero. We also define, on Ω, a Stein-like structure by considering the affinoids $U_n = \{z \in \Omega; |z| \le q^n \text{ and } |z|_\infty \ge q^{-n}\}$, so that $\Omega = \cup_{n\in\mathbb{N}} U_n$. For n fixed, a function $f : U_n \to \boldsymbol{C}$ is *holomorphic* if it is a uniform limit of a converging sequence of rational functions $U_n \to \boldsymbol{C}$, without poles in U_n. A function $f : U_n \to \boldsymbol{C}$ is *holomorphic* if, for all $n \ge 0$, the restriction of f to U_n is holomorphic.

Following and readapting the proof of [6, Proposition 22], we obtain:

Proposition 18. *The following properties hold, for $w \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}_{\ge 0}$ and ρ a representation $\Gamma \to \mathrm{GL}_N(\boldsymbol{K})$:*

(1) For all $\gamma \in \Gamma$, we have

$$\mathcal{E}_{w,m,\rho}(\gamma(z)) = \det(\gamma)^{-m} J_\gamma(z)^w \mathcal{E}_{w,m,\rho}(z)\cdot\rho(\gamma)^{-1},$$

(2) $\mathcal{E}_{w,m,\rho}(z)$ is tempered ([2]); there exists $h \in \mathbb{Z}$ such that

$$u(z)^h \mathcal{E}_{w,m,\rho}(z) \to 0 \in \mathrm{Mat}_{L\times N}(\boldsymbol{C})$$

as $u(z) \to 0$.

The proposition means that the L columns of the transposed of $\mathcal{E}_{w,m,\rho}$ are vectorial modular forms of weight w and type m with respect to the contragredient representation of ρ. In the next two subsections, we analyze two particular classes of representations.

5.1. **Vectorial Poincaré series associated to representations ρ_σ.** Let us consider an irreducible, faithful algebra representation $A \xrightarrow{\sigma} \mathrm{Mat}_{d\times d}(\boldsymbol{K})$, and the associated representation $\Gamma \xrightarrow{\rho_\sigma} \mathrm{GL}_N(\boldsymbol{K})$ with $N = 2d$, which is easily seen to be normal to the depth $L = d$.

Proposition 19. *The following properties hold, with $\rho = \rho_\sigma$.*

(1) If $w - 1 \not\equiv 2m \pmod{q-1}$, then $\mathcal{E}_{w,m,\rho} = 0$, identically.

(2) If $w - 1 \equiv 2m \pmod{q-1}$ and $w \ge (q+1)m + 1$, then, the rank of the matrix function $\mathcal{E}_{w,m,\rho} \ne 0$ is d.

(3) In the second case, each row of the matrix function $\mathcal{E}_{w,m,\rho}$ has the entries which are $\boldsymbol{C}$-linearly independent.

Proof. Since σ is irreducible and faithful, the matrix $\vartheta = \sigma(\theta)$ has distinct conjugate eigenvalues $\lambda_1, \dots, \lambda_d \in \boldsymbol{K}^{ac}$ none of which lies in $\mathbb{F}_q^{ac}$. We consider a corresponding basis $v_1, \dots, v_d \in (\boldsymbol{K}^{ac})^d$ of eigenvectors (considered as column matrices) which are then common eigenvectors for all the elements of the image of σ. In $(\boldsymbol{K}^{ac})^{2d} = (\boldsymbol{K}^{ac})^N$ we consider the basis $w_1, \dots, w_{2d}$ defined by $w_i = v_i \oplus 0$ and $w_{d+i} = 0 \oplus v_i$ for

[2] In the sense of [6].

$i = 1, \dots, d$. We also denote by $M \in \mathrm{GL}_N(\boldsymbol{K}^{ac})$ the matrix whose columns are the w_i's for $i = 1, \dots, 2d$. then, for $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have

$$M \cdot \rho(\delta)_L = (v_1, \dots, v_d) * \delta(t) := (c(\lambda_1)v_1, \dots, c(\lambda_d)v_d, d(\lambda_1)v_1, \dots, d(\lambda_d)v_d).$$

Hence, we have, with the same significance of the product $*$, that

$$M \cdot \mathcal{E}_{w,m,\rho} = (v_1, \dots, v_d) * \mathcal{E}_{w,m,\chi_t},$$

where $\mathcal{E}_{w,m,\chi_t} : \Omega \to \mathrm{Mat}_{1\times 2}(\boldsymbol{C})$ (where $\boldsymbol{C} = \mathrm{Frac}(\widehat{\mathbb{C}_\infty \otimes_{\mathbb{F}_q}} \mathbb{F}_q(t))$) is the function defined by

$$\mathcal{E}_{w,m,\chi_t}(z) = \sum_{\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H\backslash\Gamma} \mu_{w,m}(\delta, z)^{-1} u^m(\delta(z))(c(t), d(t)).$$

This matrix function is the deformation of vectorial Poincaré series $\mathcal{E}_{w,m}(z,t)$ considered in [6, Proposition 22] and we know the following:

- If $w - 1 \not\equiv 2m \pmod{q-1}$, then $\mathcal{E}_{w,m,\chi_t}$ is identically zero.
- If $w - 1 \equiv 2m \pmod{q-1}$ and $w \geq (q+1)m + 1$, then all the entries of $\mathcal{E}_{w,m,\chi_t}$ are non-zero.

We now observe that if C is a complete field containing A and if $f(t) = \sum_{i\geq 0} f_i t^i \in C[[t]]$ is a non-zero formal power series, with $f_i \to 0$ for $i \to \infty$ (an element of the Tate algebra of formal series with coefficients in C in the variable t) then, for $\lambda \in \boldsymbol{K}^{ac} \setminus \mathbb{F}_q^{ac}$, we have that $f(\lambda) = \sum_{i\geq 0} f_i \lambda^i$ converges in the complete field $\boldsymbol{C}^\sharp := \mathrm{Frac}(\widehat{C \otimes_{\mathbb{F}_q}} \boldsymbol{K}^{ac})$ (with $\boldsymbol{K}^{ac}$ carrying the trivial norm) to a non-zero element.

Since M is invertible, $\mathcal{E}_{w,m,\rho}$ is identically zero if and only if $\mathcal{E}_{w,m,\chi_t}$ is identically zero (we have supposed that ϑ has no eigenvalues in $\mathbb{F}_q^{ac}$) and in the case of non-vanishing, the rank is maximal equal to d. Properties 1) and 2) of our proposition hence follow from Proposition 22 of loc. cit.

It remains to show the part 3), but this is a standard statement which follows from Lemma 8. Indeed, assuming that $w - 1 \equiv 2m \pmod{q-1}$ and $w \geq (q+1)m + 1$, let i be an index between 1 and d; we know that it is non-zero. Let us assume, by contradiction, that the entries of $\mathcal{E}$ are linearly dependent; then, the vector space V whose elements are the vectors $v \in (\boldsymbol{C}^\sharp)^N$ such that $\mathcal{E}(z) \cdot v = 0$ for all $z \in \Omega$, is non-trivial.

Let v be in V. For all $\gamma \in \Gamma$, we have

$$0 = \mathcal{E}(\gamma(z)) \cdot v = \det(\gamma)^{-m} J_\gamma(z)^w \mathcal{E}(z) \cdot \rho(\gamma)^{-1} \cdot v.$$

This means that $\rho(\gamma)^{-1} \cdot v \in V$ so that $\rho(\gamma)(V) \subset V$ for all $\gamma \in \Gamma$. Since ρ is irreducible, we thus have that $V = (\boldsymbol{C}^\sharp)^N$ but $\mathcal{E}$ is non-zero, whence a contradiction. □

5.2. **Vectorial Poincaré series associated to the representations $\rho^{II}_{\underline{t},\underline{l}}$.** We now consider the settings of §4 and we return to independent variables $\underline{t}_s$ and to the field $\boldsymbol{K}_s = \mathbb{F}_q(\underline{t}_s)$. We also consider $\underline{l} = (l_1, \dots, l_s)$ an s-tuple with entries in $\mathbb{Z}$ which are ≥ 1 and we note that the representation $\rho = \rho^{II}_{\underline{t},\underline{l}}$ of Theorem 15 is normal to the depth $L = 1$.

Proposition 20. *We have the following properties, for $w > 0$ and $m \geq 0$, and with ρ the above considered representation.*

(1) The function

$$\mathcal{E}_{w,m,\rho} : \Omega \to \mathrm{Mat}_{1\times N}(\boldsymbol{C}),$$

with $N = \phi_p(l_1) \cdots \phi_p(l_s)$ is well defined, holomorphic, tempered, and satisfies

$$\mathcal{E}_{w,m,\rho}(\gamma(z)) = \det(\gamma)^{-m} J_\gamma(z)^w \mathcal{E}_{w,m,\rho}(z) \cdot \rho(\gamma)^{-1}, \quad \gamma \in \Gamma.$$

(2) If $w' := w - l_1 - \cdots - l_s \not\equiv 2m + 1 \pmod{q-1}$, then $\mathcal{E}_{w,m,\rho} \equiv 0$.
(3) With w' defined as above, if $w' \equiv 2m + 1$ and $w' \geq (q+1)m + 1$, then $\mathcal{E}_{w,m,\rho} \neq 0$.
(4) If $m = 0$ and $w' \equiv 1 \pmod{q-1}$, then $\mathcal{E}_{w,m,\rho} \neq 0$.
(5) In all cases in which $\mathcal{E}_{w,m,\rho} \neq 0$, its entries are linearly independent over $\boldsymbol{C}$.

Proof. The first two properties and the last one are simple variants of the corresponding parts of Proposition 19 so we skip on the details of the proofs. For the third property, we consider the matrix function $F_{\underline{l}} : \Omega \to \mathrm{Mat}_{N\times 1}(\mathbb{C}_\infty)$ with $N = \phi_p(l_1)\cdots\phi_p(l_s)$, defined by

$$F_{\underline{l}}(z) = \mathrm{Sym}^{l_{1,0}}(F) \otimes \cdots \otimes \mathrm{Sym}^{l_{1,r_1}}(F^{(r_1)}) \otimes \cdots \otimes \mathrm{Sym}^{l_{s,0}}(F) \otimes \cdots \otimes \mathrm{Sym}^{l_{s,r_s}}(F^{(r_s)}),$$

with $F(z) = \binom{z}{1}$ and where we have used the expansions in base p of $l_1, \ldots, l_s$: $l_i = l_{i,0} + l_{i,1}p + \cdots + l_{i,r_i}p^{r_i}$ with $r_i \neq 0$ for $i = 1, \ldots, s$. Then, as in [6], we note that

$$(\mathcal{E}_{w,m,\rho} \cdot F_{\underline{l}})_{t_i=\theta} = P_{w',m}$$

the Poincaré series of weight w' and type m so that we can conclude with [4, Proposition 10.5.2]. The property 3) is not enough to show the property 4), but we can proceed more directly by noticing that in this case,

$$\mathcal{E}_{w,0,\rho} = \sum_{\delta\in H\backslash\Gamma} J_\gamma^{-w}\rho(\gamma)_1.$$

Hence, if we suppose that $u(z) \to \infty$, then $\mathcal{E}_{w,0,\rho} \to (0, \ldots, 0, 1)$. □

The transposed of the matrix functions $\mathcal{E}_{w,m,\rho}$ are thus vectorial modular forms of weight w, type m associated to the representations ${}^t\rho$ in the sense of [6].

5.2.1. *Eisenstein series.* We consider $\mathbb{F}_q$-algebra representations $\sigma_1, \ldots, \sigma_s : A \to \mathrm{Mat}_{d\times d}(\boldsymbol{K})$. Let σ be the semi-character $\sigma_1 \cdots \sigma_s$. We set:

$$\mathcal{G}_{w,\sigma}(z) = \sum_{(a,b)\in A^2}{}' (az+b)^{-w}(\sigma_1(a), \sigma_1(b)) \otimes \cdots \otimes (\sigma_s(a), \sigma_s(b))$$

(the dash $'$ denotes a sum which avoids the couple $(0, 0)$). This defines a holomorphic function

$$\mathcal{G}_{w,\sigma} : \Omega \to \mathrm{Mat}_{d^s\times 2d^s}(\boldsymbol{C}).$$

Let, on the other side, ρ be the representation $\rho : \Gamma \to \mathrm{Mat}_{2d^s\times 2d^s}(\boldsymbol{K})$ defined by $\rho = \rho_{\sigma_1} \otimes \cdots \otimes \rho_{\sigma_s}$. The following lemma is easy to verify.

Lemma 21. *We have the identity $\mathcal{G}_{w,\sigma} = L_\sigma(w)\mathcal{E}_{w,0,\rho}$.*

The matrix $L_\sigma(w)$ is the L-value associated to the semi-character σ as defined in §2.2.2. This and Proposition 7 suggest that the Eisenstein series $\mathcal{G}_{w,\sigma}$ should be some kind of Taelman units. Of course, this is quite speculative, because at the moment, we do not have at our disposal any kind of metric over the spaces of vectorial modular forms that we consider, allowing us to define an appropriate notion of unit group of Taelman in this setting. However, this seems to suggest the following conjecture.

Conjecture 22. *The $\boldsymbol{C}$-module of the vectorial modular forms of weight one and type 0 associated to a representation $\rho_{\sigma_1} \otimes \cdots \otimes \rho_{\sigma_s}$ with $\sigma_1, \ldots, \sigma_s$ algebra representations $A \to \mathrm{Mat}_{d\times d}(\boldsymbol{K})$ is of rank one, generated by the Eisenstein series $\mathcal{G}_{w,\sigma}$, where $\sigma = \sigma_1 \cdots \sigma_s$.*

6. Other representations of Γ

Let $\boldsymbol{K}$ be a field containing a fixed base field k of positive characteristic (e.g. $k = \mathbb{F}_q$) and let us consider a group representation $\rho : G \to \mathrm{GL}_N(\boldsymbol{K})$. The *essential dimension* (over k) of ρ is the transcendence degree over k of the field generated by the entries of the image of ρ. If G is finite, then the essential dimension of ρ is zero. In this paper, we have studied several examples in the case $k = \mathbb{F}_q$ and $G = \Gamma$. For instance, the essential dimension of the tautological representation $\Gamma \to \mathrm{GL}_2(A)$ is one, and the essential dimension of a representation $\rho^{II}_{\underline{t},\underline{l}}$ as in Theorem 15 is s, the number of variables in $\underline{t}_s$.

As a conclusion of the present note, we would like to point out that there are irreducible, finite dimensional representations of $\mathrm{GL}_2(k[t])$ with infinite essential dimension. Indeed, for any field k, a Theorem of Nagao (see [5, Theorem 2] and Serre, [8, II.1.6]) asserts that

$$\mathrm{GL}_2(k[t]) \cong \mathrm{GL}_2(k) *_{B(k)} B(k[t]), \tag{2}$$

where, for a commutative ring R, $B(R)$ denotes the group of upper triangular matrices with entries in R with invertible determinant and where $*_{B(k)}$ stands for the amalgamated product along $B(k)$. Therefore, we have the following:

Proposition 23. *Any automorphism ϕ of $\mathrm{GL}_2(k)$ extends to a group isomorphism between $\mathrm{GL}_2(k[t])$ and the subgroup Φ^∞ of $\mathrm{GL}_2(k[x_1, x_2, \ldots])$ generated by $\mathrm{GL}_2(k)$ and the matrices $\left(\begin{smallmatrix}\lambda & x_i\\ 0 & \mu\end{smallmatrix}\right)$, where $x_1, x_2, \ldots$ are independent indeterminates over k and $\lambda, \mu \in k^\times$.*

Proof. By (2), we see that the association $\left(\begin{smallmatrix}\lambda & t^i\\ 0 & \mu\end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix}\lambda & x_i\\ 0 & \mu\end{smallmatrix}\right)$ extends to give the above group isomorphism. □

The above proposition exhibits representations $\mathrm{GL}_2(k[t]) \to \mathrm{GL}_2(\boldsymbol{K})$ which have infinite essential dimension over $\mathbb{F}_q$.

Remark 24. The group $\mathrm{SL}_2(\mathbb{Z})$ has uncountably many conjugacy classes of irreducible complex representations and their explicit classification is not yet understood. A similar question arises with the group $\mathrm{GL}_2(A)$ and its representations in a complete algebraically closed field of characteristic p. The complete classification for $\mathrm{SL}_2(\mathbb{Z})$ is however accessible if we impose a bound on the dimension. In [10], Tuba and Wenzl obtained a complete classification of irreducible representations of the braid group B_3 of dimension ≤ 5 yelding a similar result for irreducible complex representations of $\mathrm{SL}_2(\mathbb{Z})$; it turns out that these families algebraically depend on finitely many parameters (eigenvalues, characters etc.). It would be nice to have a similar result at hand for $\mathrm{GL}_2(A)$.

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