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# PARALLELISMS & LIE CONNECTIONS

DAVID BLÁZQUEZ-SANZ & GUY CASALE

**Abstract:** The aim of this article is to study rational parallelisms of algebraic varieties by means of the transcendency of their symmetries. The nature of this transcendency is measured by a Galois group built from Picard-Vessiot theory of principal connections.

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## INTRODUCTION

The aim of this article is to study rational parallelisms of algebraic varieties by means of the transcendency of their symmetries. Our original motivation is to understand the obstructions to a third Lie theorem for simply transitive algebraic Lie pseudogroups. This obstruction should appear in the Galois group of certain connection associated to a Lie algebroid. However, this article is purely written in the language of regular and rational parallelisms of algebraic varieties and their symmetries. A theorem of P. Deligne says that any Lie algebra can be realized as a parallelism of an algebraic variety. This is a sort of algebraic version of third Lie theorem. The main problem is the following, given an algebraic variety with a parallelism, how far is it to be an algebraic group? It is possible to conjugate this parallelism with the canonical parallelism of invariant vector fields on an algebraic group?

In the analytic context [9], Wang proved that parallelized compact complex manifolds are bi-holomorphic to quotients of complex Lie groups by discrete cocompact subgroups. This result has

been extended by Winkelmann in [10, 11] for some open complex manifolds. Such a classification is impossible in the algebraic category but we provide a criteria to insure that an parallelized algebraic variety is isogeneous to an algebraic group. Summarizing, our main ideas are as follows. Infinitesimal symmetries of a rational parallelism are horizontal sections of a connection that we call the reciprocal Lie connection. This connection has a Galois group which is represented as a group of internal automorphisms of a Lie algebra. The obstruction to the algebraic conjugation to an algebraic group, under some assumptions, appear in the Lie algebra of this Galois group.

In Section 1 we introduce the basic definitions; several examples of parallelisms are given here. In Section 2 we study the properties of connections on the tangent bundle whose horizontal sections form a Lie algebra of vector fields. We call them Lie connections. They always come by pairs, and are characterized by having vanishing curvature and constant torsion (Proposition 1). We see that each rational parallelism has an attached pair of Lie connections, one of them having trivial Galois group. We compute the Galois groups of some parallelisms given in examples (Proposition 4), and prove that any algebraic subgroup of  $SL_2(\mathbf{C})$  appears as the differential Galois group of a  $\mathfrak{sl}_2(\mathbf{C})$ -parallelism (Theorem 1). Section 3 is devoted to the construction of the isogeny between a  $\mathfrak{g}$ -parallelized variety and an algebraic group  $G$  whose Lie algebra is  $\mathfrak{g}$ . In order to do this, we introduce the Darboux-Cartan connection, a  $G$ -connection whose horizontal sections are parallelism conjugations. We prove that if  $\mathfrak{g}$  is centerless then the Darboux-Cartan connection and the reciprocal Lie connection have isogenous Galois groups. We prove that the only centerless Lie algebras  $\mathfrak{g}$  such that a  $\mathfrak{g}$  parallelism has a trivial Galois group are algebraic Lie algebra. In particular it allows us to give a criteria for a parallelized variety to be isogeneous to an algebraic group, it is necessary and sufficient the vanishing of the Lie algebra of the Galois group of the reciprocal connection (Theorem 2). A corollary to this theorem (Corollary 3.3) says that any algebraic variety with two commuting rational  $\mathfrak{g}$ -parallelisms (with centerless  $\mathfrak{g}$ ) is isogeneous to an algebraic group. This can be seen as an algebraic version of Wang theorem. Appendix A is devoted to a geometrical presentation of Picard-Vessiot theory for linear and invariant connections. Finally, Appendix B contains a detailed proof of Deligne's theorem of the realization of a regular parallelism modeled over any finite dimensional Lie algebra. This includes also a computation of the Galois group that turns out to be, for this particular construction, an algebraic torus.

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## 1. PARALLELISMS

Let  $M$  be a smooth connected affine variety over  $\mathbf{C}$  of dimension  $r$ . We denote by  $\mathbf{C}[M]$  its ring of regular functions and by  $\mathbf{C}(M)$  its field of rational functions. Analogously, we denote by  $\mathfrak{X}[M]$  and  $\mathfrak{X}(M)$  respectively the Lie algebras of regular and rational vector fields in  $M$ , and so on.

Let  $\mathfrak{g}$  be a Lie algebra of dimension  $r$ . We fix a basis  $A_1, \dots, A_r$  of  $\mathfrak{g}$  and the following notation for the associated structure constants  $[A_i, A_j] = \sum_k \lambda_{ij}^k A_k$ .

A parallelism of type  $\mathfrak{g}$  of  $M$  is a realization of the Lie algebra  $\mathfrak{g}$  as a Lie algebra of pointwise linearly independent vector fields in  $M$ . More precisely:

**Definition.** A regular parallelism of type  $\mathfrak{g}$  in  $M$  is a Lie algebra morphism,  $\rho: \mathfrak{g} \rightarrow \mathfrak{X}[M]$  such that  $\rho A_1(x), \dots, \rho A_r(x)$  form a basis of  $T_x M$  for any point  $x$  of  $M$ .

**Example 1.** Let  $G$  be an algebraic group, and  $\mathfrak{g}$  its Lie algebra of left invariant vector fields. Then, the natural inclusion  $\mathfrak{g} \subset \mathfrak{X}[G]$  is a regular parallelism of  $G$ . The Lie algebra  $\mathfrak{g}^{rec}$  of right invariant vector fields is another regular parallelism of the same type. Left invariant and right invariant vector fields commute, hence, an algebraic group is naturally endowed with a pair of commuting parallelism of the same type.

From example 1 it is clear that any algebraic Lie algebra is realized as a parallelism of some algebraic variety. On the other hand, Theorem 7, due to P. Deligne and published in [6], ensures that any Lie algebra is realized as a regular parallelism of an algebraic variety. Analogously, we have the definitions of rational and local analytic parallelism. Note that a rational parallelism in  $M$  is a regular parallelism in a Zariski open subset  $M^* \subseteq M$ .

There is an equivalent notion of coparallelism with a dual definition. It suits better to the calculations.

**Definition.** A regular parallelism form (or coparallelism) of type  $\mathfrak{g}$  in  $M$  is a  $\mathfrak{g}$ -valued 1-form  $\omega \in \Omega^1[M] \otimes_{\mathbb{C}} \mathfrak{g}$  such that:

- (1) For any  $x \in M$ ,  $\omega_x: T_x M \rightarrow \mathfrak{g}$  is a linear isomorphism.
- (2) If  $A$  and  $B$  are in  $\mathfrak{g}$  and  $X, Y$  are vector fields such that  $\omega(X) = A$  and  $\omega(Y) = B$  then  $\omega[X, Y] = [A, B]$ .

Analogously, we define local analytic and rational coparallelism of type  $\mathfrak{g}$  in  $M$ . It is clear that each coparallelism induces a parallelism form and reciprocally, by the relation  $\omega(\rho(A)) = A$ . Thus, there is a natural equivalence between the notions of parallelism and coparallelism. From now on we fix  $\rho$  and  $\omega$  equivalent parallelism and coparallelism of type  $\mathfrak{g}$  in  $M$ .

The Lie algebra structure of  $\mathfrak{g}$  forces  $\omega$  to satisfy Maurer-Cartan structure equations,

$$d\omega + [\omega, \omega] = 0.$$

Taking components:  $\omega = \sum_i \omega_i A_i$  we have:

$$d\omega_i + \sum_{j,k=1}^r \frac{1}{2} \lambda_{jk}^i \omega_j \wedge \omega_k = 0.$$

**Example 2.** Let  $G$  be an algebraic group,  $\mathfrak{g}$  the Lie algebra of left invariant vector fields in  $G$ . Then, the structure form  $\omega$ , is a the coparallelism corresponding to the parallelism of Example 1.

**Example 3.** Let  $\mathfrak{g} = \langle A_1, A_2 \rangle$  the 2-dimensional Lie algebra with commuting relation:

$$[A_1, A_2] = A_1$$

The vector fields:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

define a regular parallelism  $\rho(A_i) = X_i$  of  $\mathbb{C}^2$ . The associated parallelism form is:

$$\omega = A_1 dx + (A_2 - xA_1) dy.$$

**Example 4** (Malgrange). Let  $\mathfrak{g} = \langle A_1, A_2, A_3 \rangle$  the 3-dimensional Lie algebra with commuting relations:

$$[A_1, A_2] = \alpha A_2 \quad [A_1, A_3] = \beta A_3.$$

With  $\alpha, \beta$ , non zero complex numbers. In particular, if  $\alpha/\beta$  is not rational, then  $\mathfrak{g}$  is a not the Lie algebra of an algebraic group. The vector fields:

$$X_1 = \frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial y} + \beta z \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z},$$

define a regular parallelism  $\rho(A_i) = X_i$  of  $\mathbb{C}^3$ . The associated parallelism form is:

$$\omega = (A_1 - A_2 \alpha y - A_3 \beta z) dx + A_2 dy + A_3 dz.$$

**Definition.** Let  $(M, \omega)$  and  $(N, \theta)$  be algebraic manifolds with coparallelisms of type  $\mathfrak{g}$ . We say that they are isogenous equivalent if there is an algebraic manifold  $(P, \eta)$  with a coparallelism of type  $\mathfrak{g}$  and dominant maps  $f: P \rightarrow M$  and  $g: P \rightarrow N$  such that  $f^*(\omega) = g^*(\theta) = \eta$ .

Clearly, the notion of isogeny of parallelized varieties extend that of isogeny of algebraic groups.

**Example 5.** Let  $f: M \dashrightarrow G$  a dominant rational map with values in an algebraic group with  $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} G$ . Then  $\theta = f^*(\omega)$  is a rational parallelism form in  $M$ .

**Example 6.** Let  $H$  be a finite subgroup of the algebraic group  $G$ ,

$$\pi: G \rightarrow M = H \backslash G = \{Hg : g \in G\}$$

the quotient by the action of  $H$  in the left side. The structure form  $\omega$  in  $G$  is left-invariant and then it is projectable by  $\pi$ . Then,  $\theta = \pi_*(\omega)$  is a regular parallelism form in  $M$ .

**Example 7.** Combining Examples 5 and 6 let  $H \subset G$  be a finite subgroup, and let  $f: M \rightarrow H \backslash G$  be a dominant rational map between manifolds of the same dimension. Then,  $\theta = f^*(\pi_*(\omega))$  is a rational parallelism of  $M$ .

**Example 8.** By application of Example 7 to the case of the multiplicative group we obtain the logarithmic forms in  $\mathbf{CP}_1$ ,  $\frac{p df}{q f}$  where  $f \in \mathbf{C}(z)$ . Thus, the logarithmic forms in  $\mathbf{CP}_1$  are the rational parallelisms isogenous to the multiplicative group.

**Example 9.** By application of Example 7 to the case of the additive group we obtain the exact forms in  $\mathbf{CP}_1$ ,  $dF$  where  $F \in \mathbf{C}(z)$ . This, the exact forms in  $\mathbf{CP}_1$  are the rational parallelisms isogenous to the additive group.

**Example 10.** Let  $H$  be a subgroup of the algebraic group  $G$ , with Lie algebra  $\mathfrak{h} \subset \mathfrak{g}$ . Let us assume that  $\mathfrak{h}$  admits a supplementary Lie algebra,  $\mathfrak{h}'$ :

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}' \quad (\text{as vector spaces})$$

We consider  $M = H \backslash G$  be the left quotient of  $G$  by the action of  $H$  and  $\pi: G \rightarrow M$  is the quotient map. It turns out that  $\mathfrak{h}'$  is a Lie algebra of vector fields in  $G$  projectable by  $\pi$ , and thus  $\pi_*|_{\mathfrak{h}'}: \mathfrak{h}' \rightarrow \mathfrak{X}[M]$  gives a parallelism of  $M$  that is regular in the open subset

$$\{Hg \in M : \text{Adj}_g(\mathfrak{h}) \cap \mathfrak{h}' = \{0\}\}.$$

It turns out to be regular in  $M$  if  $H \triangleleft G$ . Examples 3 and 4 are particular cases in which  $G$  is  $\text{Aff}(2, \mathbf{C})$  and  $\text{Aff}(3, \mathbf{C})$  respectively.

**Remark 1.** We can see also Example 10 as a coparallelism. Let us consider  $\pi': \mathfrak{G} \rightarrow \mathfrak{h}'$  be the projection given by the vector space decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ . Since  $\pi' \circ \omega$  is left invariant form in  $G$ , it is projectable by  $\pi$ . Hence, there is a form  $\omega'$  in  $M$  such that  $\pi^* \omega' = \pi' \circ \omega$ . This form  $\omega'$  is the corresponding coparallelism.

## 2. ASSOCIATED LIE CONNECTION

**2.1. Reciprocal connections.** Let  $\nabla$  be a linear connection (rational or regular) on  $TM$ . The reciprocal connection is defined as:

$$\nabla_{\vec{X}}^{rec} \vec{Y} = \nabla_{\vec{Y}} \vec{X} + [\vec{X}, \vec{Y}].$$

From this definition it is clear that the difference  $\nabla - \nabla^{rec} = \text{Tor}_{\nabla}$  is the torsion tensor,  $\text{Tor}_{\nabla} = -\text{Tor}_{\nabla^{rec}}$  and  $(\nabla^{rec})^{rec} = \nabla$ .

**2.2. Connections and parallelisms.** Let  $\omega$  be a coparallelism of type  $\mathfrak{g}$  in  $M$  and  $\rho$  its equivalent parallelism. Denote by  $\vec{X}_i$  the basis of vector fields in  $M$  such that  $\omega(\vec{X}_i) = A_i$  is a basis of  $\mathfrak{g}$ .

**Definition.** The associated connection  $\nabla$  to the parallelism  $\omega$  is the only linear connection in  $M$  for which  $\omega$  is an  $\nabla$ -horizontal form.

Clearly  $\nabla$  is a flat connection and the basis  $\{\vec{X}_i\}$  is a basis of the space of  $\nabla$ -horizontal vector fields. In this basis  $\nabla$  has vanishing Christoffel symbols.

$$\nabla_{\vec{X}_i} \vec{X}_j = 0.$$

Let us compute infinitesimal symmetries of  $\omega$ . A vector field  $\vec{Y}$  is an infinitesimal symmetry of  $\omega$  if  $\text{Lie}_{\vec{Y}} \omega = 0$ , or equivalently, it commutes with all the vector fields of the parallelism:

$$[\vec{X}_i, \vec{Y}] = 0, \quad i = 1, \dots, r.$$

**Lemma 1.** Let  $\nabla$  be the associated connection to the parallelism  $\omega$ , then for any vector field  $\vec{Y}$  and any  $j = 1, \dots, r$ ,

$$[\vec{X}_j, \vec{Y}] = \nabla_{\vec{X}_j}^{rec} \vec{Y}.$$

Thus,  $\vec{Y}$  is an infinitesimal symmetry of  $\omega$  if and only if it is an horizontal vector field for the reciprocal connection  $\nabla^{rec}$ .

*Proof.* – A direct computation yields the result. Take  $\vec{Y} = \sum_{k=1}^r f_k \vec{X}_k$ , for each  $j$  we have:

$$\nabla_{\vec{X}_j}^{rec} \vec{Y} = \sum_{k=1}^r \left( (\vec{X}_j f_k) \vec{X}_k + f_k [\vec{X}_j, \vec{X}_k] \right) = [\vec{X}_j, \vec{Y}].$$

□

The above considerations also give us the Christoffel symbols for  $\nabla^{rec}$  in the basis  $\{\vec{X}_i\}$ :

$$\nabla_{\vec{X}_i}^{rec} \vec{X}_j = [\vec{X}_i, \vec{X}_j] = \sum_{k=1}^r \lambda_{ij}^k \vec{X}_k,$$

the Christoffel symbols of  $\nabla^{rec}$  are the structure constants of the Lie algebra  $\mathfrak{g}$ .

**Lemma 2.** *Let  $\nabla$  be the associated connection to a coparallelism form in  $M$ . Then,  $\nabla^{rec}$  is flat, and the Lie bracket of two  $\nabla^{rec}$ -horizontal vector fields is  $\nabla^{rec}$ -horizontal vector field.*

*Proof.* – The flatness and the preservation of Lie bracket by  $\nabla^{rec}$  are direct consequences of Jacobi identity. Let us compute the curvature:

$$\begin{aligned} R(\vec{X}_i, \vec{X}_j, \vec{X}_k) &= \nabla_{\vec{X}_i}^{rec}(\nabla_{\vec{X}_j}^{rec} \vec{X}_k) - \nabla_{\vec{X}_j}^{rec}(\nabla_{\vec{X}_i}^{rec} \vec{X}_k) - \nabla_{[\vec{X}_i, \vec{X}_j]}^{rec} \vec{X}_k = \\ &= \rho([A_i, [A_j, A_k]] - [A_j, [A_i, A_k]] - [[A_i, A_j], A_k]) = 0. \end{aligned}$$

Let us compute the Lie bracket for  $\vec{Y}$  and  $\vec{Z}$   $\nabla^{rec}$ -horizontal vector fields:

$$\nabla_{\vec{X}_i}^{rec}[\vec{Y}, \vec{Z}] = [\vec{X}_i, [\vec{Y}, \vec{Z}]] = [[\vec{X}_i, \vec{Y}], \vec{Z}] + [\vec{Y}, [\vec{X}_i, \vec{Z}]] = [\nabla_{\vec{X}_i}^{rec} \vec{Y}, \vec{Z}] + [\vec{Y}, \nabla_{\vec{X}_i}^{rec} \vec{Z}] = 0.$$

□

**Lemma 3.** *Let  $x \in M$  be a regular point of the parallelism form  $\omega$ . The space of germs at  $x$  of horizontal vector fields for  $\nabla^{rec}$  is a Lie algebra isomorphic to  $\mathfrak{g}$ . Moreover, let  $\vec{Y}_1, \dots, \vec{Y}_r$  be horizontal vector fields with initial conditions  $\vec{Y}_i(x) = \vec{X}_i(x)$ , then  $[\vec{Y}_i, \vec{Y}_j] = -\sum_{k=1}^r \lambda_{ij}^k \vec{Y}_k$ .*

*Proof.* – We can write the vector fields  $\vec{Y}_i$  as a linear combination of the basic vector fields  $\vec{X}_i$ ,  $\vec{Y}_i = \sum_{j=1}^r a_{ji} \vec{X}_j$ . The matrix  $(a_{ij})$  satisfy the differential equation:

$$\vec{X}_k a_{ij} = -\sum_{\alpha=1}^r \lambda_{k\alpha}^i a_{\alpha j}, \quad a_{ij}(x) = \delta_{ij}.$$

On the other hand, we have  $[\vec{Y}_i, \vec{Y}_j](x) = \sum_{k=1}^r \hat{\lambda}_{ij}^k \vec{Y}_k(x)$ , for certain unknown structure constants  $\hat{\lambda}_{ij}^k$ . Let us check that  $\hat{\lambda}_{ij}^k = \lambda_{ji}^k = -\lambda_{ij}^k$ .

$$\begin{aligned} [\vec{Y}_i, \vec{Y}_j] &= \left[ \sum_{\alpha=1}^r a_{\alpha i} \vec{X}_\alpha, \sum_{\beta=1}^r a_{\beta j} \vec{X}_\beta \right] = \\ &= \sum_{\alpha, \beta, \gamma=1}^r -a_{\alpha i} \lambda_{\alpha \gamma}^\beta a_{\beta j} \vec{X}_\gamma + \sum_{\alpha, \beta, \gamma=1}^r a_{\beta j} \lambda_{\beta \gamma}^\alpha a_{\alpha i} \vec{X}_\alpha + \sum_{\alpha, \beta, \gamma=1}^r a_{\beta j} a_{\alpha i} \lambda_{\alpha \beta}^\gamma \vec{X}_\gamma \end{aligned}$$

Taking values at  $x$ , we obtain

$$[\vec{Y}_i, \vec{Y}_j](x) = \sum_{\beta=1}^r -\lambda_{ij}^\beta \vec{Y}_\beta(x) + \sum_{\alpha=1}^r \lambda_{ji}^\alpha \vec{Y}_\alpha(x) + \sum_{\gamma=1}^r \lambda_{ij}^\gamma \vec{Y}_\gamma(x) = \sum_{\alpha=1}^r \lambda_{ji}^\alpha \vec{Y}_\alpha(x).$$

□

**Example 11.** *Let  $G$  be an algebraic group with Lie algebra  $\mathfrak{g}$ . As seen in example 2 the Maurer-Cartan structure form  $\omega$  is a coparallelism in  $G$ . Let  $\nabla$  be the connection associated to this coparallelism. There is another canonical coparallelism, the right invariant Maurer-Cartan structure form  $\omega_{rec}$ , let us consider  $\mathbf{i}: G \rightarrow G$  the inversion map,*

$$\omega_{rec} = -\mathbf{i}^*(\omega).$$

*As may be expected, the connection associated to the coparallelism  $\omega_{rec}$  is  $\nabla^{rec}$ . Right invariant vector field in  $G$  are infinitesimal symmetries of left invariant vector fields and viceversa. In this case, the horizontal vector fields of  $\nabla$  and  $\nabla^{rec}$  are regular vector fields.*

In general symmetries of a rational parallelism are not rational vector fields as it is shown in the next two examples.

**Example 12.** Let us consider the Lie algebra  $\mathfrak{g}$  and the coparallelism  $\omega = A_1 dx + (A_2 - xA_1)dy$ , of example 3. Let  $\nabla$  be its associated connection. In cartesian coordinates, the only non-vanishing Christoffel symbol of the reciprocal connection is  $\Gamma_{21}^1 = -1$ . A basis of  $\nabla^{rec}$ -horizontal vector fields is:

$$\vec{Y}_1 = e^y \frac{\partial}{\partial x}, \quad \vec{Y}_2 = \frac{\partial}{\partial y}.$$

Note that they coincide with  $\vec{X}_1, \vec{X}_2$  at the origin point and  $[\vec{Y}_1, \vec{Y}_2] = -Y_1$ .

**Example 13.** Let us consider the Lie algebra  $\mathfrak{g}$  and the coparallelism  $\omega = (A_1 - \alpha x A_2 - \beta z A_3)dx + A_2 dy + A_3 dz$  of example 4. Let  $\nabla$  be its associated connection. In cartesian coordinates, the only non-vanishing Christoffel symbols of the reciprocal connection are:

$$\Gamma_{11}^2 = -\alpha, \quad \Gamma_{11}^3 = -\beta.$$

A basis of  $\nabla^{rec}$ -horizontal vector fields is:

$$\vec{Y}_1 = \frac{\partial}{\partial x}, \quad \vec{Y}_2 = e^{\alpha x} \frac{\partial}{\partial y}, \quad \vec{Y}_3 = e^{\beta x} \frac{\partial}{\partial z}.$$

Note that they coincide with  $\vec{X}_1, \vec{X}_2, \vec{X}_3$  at the origin point and:

$$[\vec{Y}_1, \vec{Y}_2] = -\alpha Y_2, \quad [\vec{Y}_1, \vec{Y}_3] = -\beta \vec{Y}_3.$$

**2.3. Lie connections.** The connections  $\nabla$  and  $\nabla^{rec}$  associated to a parallelism  $\omega$  of type  $\mathfrak{g}$  are particular cases of the following definition.

**Definition.** A Lie connection (regular or rational) in  $M$  is a flat connection  $\nabla$  in  $TM$  such that the Lie bracket of any two horizontal vector fields is an horizontal vector field.

Given a Lie connection  $\nabla$  in  $M$ , then there is a  $r$ -dimensional Lie algebra  $\mathfrak{g}$  such that the space of germs of horizontal vector fields at a regular point  $x$  is a Lie algebra isomorphic to  $\mathfrak{g}$ . We will say that  $\nabla$  is a Lie connection of type  $\mathfrak{g}$ . The following result gives several algebraic characterizations of Lie connections:

**Proposition 1.** Let  $\nabla$  be a linear connection in  $TM$ , the following statements are equivalent:

- (1)  $\nabla$  is a Lie connection.
- (2)  $\nabla^{rec}$  is a Lie connection.
- (3)  $\nabla$  is flat and has constant torsion,  $\nabla \text{Tor}_\nabla = 0$ .
- (4)  $\nabla$  and  $\nabla^{rec}$  are flat.

.

*Proof.* – Let us first see (1) $\Leftrightarrow$ (2). Let  $\nabla$  a Lie-connection, around each point of the domain of  $\nabla$  there is a parallelism, by possibly transcendental vector fields, such that  $\nabla$  is its associated connection. Then, Lemma 2 states (1) $\Rightarrow$ (2). Taking into account that  $(\nabla^{rec})^{rec} = \nabla$  we have the desired equivalence.

Let us see now that (1) $\Leftrightarrow$ (3). Let us assume that  $\nabla$  is a flat connection. For any three vector fields  $X, Y, Z$  in  $M$  we have:

$$(\nabla_X \text{Tor}_\nabla)(Y, Z) = -\text{Tor}_\nabla(\nabla_X Y, Z) - \text{Tor}_\nabla(Y, \nabla_X Z) + \nabla_X \text{Tor}_\nabla(Y, Z)$$

Let us assume that  $Y$  and  $Z$  are  $\nabla$ -horizontal vector fields. Then, we have

$$\text{Tor}_\nabla(Y, Z) = \nabla_Y Z - \nabla_Z Y - [Y, Z] = -[Y, Z]$$

and the previous equality yields:

$$(\nabla_X \text{Tor}_\nabla)(Y, Z) = -\nabla_X [Y, Z].$$

Thus, we have that  $\nabla \text{Tor}_\nabla$  vanish if and only if the Lie bracket of any two  $\nabla$ -horizontal vector fields is also  $\nabla$ -horizontal. It proves (1) $\Leftrightarrow$ (3).

Finally, let us see (1) $\Leftrightarrow$ (4). It is clear that (1) implies (4) so we only need to see (4) $\Rightarrow$ (1). Assume  $\nabla$  and  $\nabla^{rec}$  are flat. Then, locally there exist a basis of  $\nabla$ -horizontal  $\{\vec{X}_i\}$  of  $\nabla$ -horizontal vector fields and  $\{\vec{Y}_i\}$  of  $\nabla^{rec}$ -horizontal vector fields. By the definition of the reciprocal connection, we

have that a vector field  $\vec{X}$  is  $\nabla$ -horizontal if and only if it satisfies  $[\vec{X}, \vec{Y}_i] = 0$  for  $i = 1, \dots, r$ . But the, by Jacobi identity we have:

$$[[\vec{X}_i, \vec{X}_j], \vec{Y}_k] = 0,$$

the Lie brackets  $[\vec{X}_i, \vec{X}_j]$  are also  $\nabla$ -horizontal, hence  $\nabla$  is a Lie connection.  $\square$

**Lemma 4.** *Let  $\nabla$  be a Lie connection in  $M$ . Let  $x$  be a regular point and  $\vec{X}_1, \dots, \vec{X}_r$  and  $\vec{Y}_1, \dots, \vec{Y}_r$  be basis of horizontal vector field germs in  $M$  for  $\nabla$  and  $\nabla^{rec}$  respectively with same initial conditions  $\vec{X}_i(x) = \vec{Y}_i(x)$ . Then,*

$$[\vec{X}_i, \vec{X}_j](x) = -[\vec{Y}_i, \vec{Y}_j](x).$$

It follows that  $\nabla$  and  $\nabla^{rec}$  are of the same type  $\mathfrak{g}$ .

*Proof.* – By definition  $\nabla$  is the associated connection of the local analytic parallelism given by the basis  $\{\vec{X}_i\}$  of horizontal vector fields. Then, we apply Lemma 3, and conclude.  $\square$

#### 2.4. Some results on Lie connections by means of Picard-Vessiot theory.

**Proposition 2.** *Let  $\nabla$  be a rational Lie connection in  $TM$ . The  $\nabla$ -horizontal vector fields are the symmetries of a rational parallelism of  $M$  if and only if  $\text{Gal}(\nabla^{rec}) = \{1\}$ .*

*Proof.* – This is a consequence of basic properties of Galois group. It is reduced to the identity if and only if there exists a basis of rational  $\nabla^{rec}$ -horizontal sections. These sections give the wanted parallelism.  $\square$

**Proposition 3.** *For any Lie connection  $\nabla$ ,  $\text{Gal}(\nabla) \subseteq \text{Aut}(\mathfrak{g})$ .*

*Proof.* – Let us choose a point  $m \in M$  regular for  $\nabla$ . If  $\sigma$  is an isomorphism from  $\mathfrak{g}$  to  $T_m M$  then one defines  $H_{i,j}^k(\sigma)$  to be  $\frac{[X_i, X_j] \wedge X_1 \wedge \dots \wedge \widehat{X}_k \wedge \dots \wedge X_r}{X_k \wedge X_1 \wedge \dots \wedge \widehat{X}_k \wedge \dots \wedge X_r}$ . This function is a regular function of  $TM \otimes \mathfrak{g}^*$  and is  $\nabla \otimes 1$ -invariant thus  $H$  is invariant under the action of the Galois group *i.e.* The Galois group preserves the Lie bracket.  $\square$

**Proposition 4.** *Let  $\mathfrak{h}'$  be a Lie algebra in the Lie algebra of some algebraic group and let  $G$  be its algebraic envelope. Let us consider the following objects:*

- An algebraic subgroup  $H$  of  $G$  whose lie algebra  $\mathfrak{h}$  is supplementary to  $\mathfrak{h}'$  in  $\mathfrak{g}$ ,  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ .*
- The quotient map  $\pi: G \rightarrow M$  where  $M$  is the variety of cosets  $H \backslash G$ , and  $\nabla$  the Lie connection associated to the parallelism  $\pi_*: \mathfrak{h} \rightarrow \mathfrak{X}[M]$  in  $M$  (as given in Example 10).*
- Its reciprocal Lie connection  $\nabla^{rec}$  on  $M$ .*
- The Lie algebras of right invariant vector fields:*

$$\mathfrak{g}^{rec} = \mathfrak{i}_*(\mathfrak{g}), \quad \mathfrak{h}'^{rec} = \mathfrak{i}_*(\mathfrak{h}').$$

Then, the following statements are true:

- $\mathfrak{h}'$  is an ideal of  $\mathfrak{g}$  (equivalently  $\mathfrak{h}'^{rec}$  is an ideal of  $\mathfrak{g}^{rec}$ ).*
- $\mathfrak{h}$  is commutative (equivalently  $H$  is virtually abelian).*
- The adjoint action of  $G$  into  $\mathfrak{g}$  preserves  $\mathfrak{h}'^{rec}$  and thus gives, by restriction, a morphism  $\overline{\text{Adj}}: G \rightarrow \text{Aut}(\mathfrak{h}'^{rec})$ .*
- The Galois group of the connection  $\nabla^{rec}$  is  $\overline{\text{Adj}}(H) \subseteq \text{Aut}(\mathfrak{h}'^{rec})$  and thus virtually abelian.*

*Proof.* – We have that  $\mathfrak{g}$  is the algebraic hull of  $\mathfrak{h}'$ . By Lemma 9 in Appendix B we obtain  $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}'$ . Statement (i) follows straightforward. Let us consider  $A, B$  in  $\mathfrak{h}$ . Then  $[A, B]$  is in  $\mathfrak{h}$  and also in  $\mathfrak{h}'$  by the previous argument. Thus,  $[A, B] = 0$  and this finishes the proof of statement (ii). Let us denote by  $H'$  the subgroup of  $G$  image of spanned by the image of  $\mathfrak{h}'$  by the exponential map. For each element  $h \in H'$ , the adjoint action of preserves the Lie algebra  $\mathfrak{h}'$ . By continuity of the adjoint action in the Zariski topology, we have that  $\mathfrak{h}'$  is preserved by the adjoint action of all elements of  $G$ . This proves statement (iii). In order to proof the last statement in the proposition we have to construct a Picard-Vessiot extension for the connection  $\nabla^{rec}$ . Let us consider a basis  $\{A_1, \dots, A_m\}$  of  $\mathfrak{h}$  and let  $\bar{A}_i$  be the projection  $\pi_*(A_i)$ . We have an extension of differential fields:

$$(\mathbf{C}(M), \bar{\mathcal{D}}) \subseteq (\mathbf{C}(G), \mathcal{D}),$$



where  $\bar{\mathcal{D}}$  stands for the  $\mathbf{C}(M)$ -vector space of derivations spanned by  $\bar{A}_1, \dots, \bar{A}_m$  and  $\mathcal{D}$  stands for the  $\mathbf{C}(G)$ -vector space of derivations spanned by  $A_1, \dots, A_m$  (see Appendix A for our conventions on differential fields).

The projection  $\pi$  is a principal  $H$ -bundle. Any rational first integral of  $\{A_1, \dots, A_m\}$  is constant along  $H'$  and thus it is necessarily a complex number. Thus, the above extension has no new constants and it is strongly normal in the sense of Kolchin with Galois group  $H$ . Note that the differential field automorphism corresponding to an element  $h \in H$  is the pullback of functions by the left translation  $L_h^{-1}$ , that is,  $(hf)(g) = f(h^{-1}g)$ .

The horizontal sections for the connection  $\nabla^{rec}$  are characterized by the differential equations,

$$(1) \quad [\bar{A}_i, X] = 0$$

Let us consider  $\{B_1, \dots, B_m\}$  a basis of  $\mathfrak{h}^{rec}$ . From the Zariski closedness of  $H'$  in  $G$  it follows that there are regular functions  $f_{ij} \in \mathbf{C}[G]$  such that  $B_i = \sum_{j=1}^m f_{ij} A_j$ . Thus let us define  $\bar{B}_i = \sum_{j=1}^m f_{ij} \bar{A}_j$ . Those objects are vector fields in  $M$  with coefficients in  $\mathbf{C}[G]$ , and clearly satisfy equation (1). Thus, the Picard-Vessiot extension of  $\nabla^{rec}$  is spanned by the functions  $f_{ij}$  and it is embedded, as a differential field, in  $\mathbf{C}(G)$ . Let us denote such extension by  $\mathbf{L}$ . We have a chain of extensions:

$$\mathbf{C}(M) \subseteq \mathbf{L} \subseteq \mathbf{C}(G).$$

By Galois correspondence, the Galois group of  $\nabla^{rec}$  is a quotient  $H/K$  where  $K$  is the subgroup of elements of  $H$  that fix, by left translation, the functions  $f_{ij}$ . In order to prove statement (iv) we need to check that this group  $K$  is the kernel of the morphism  $\overline{\text{Adj}}$ .

Let us note that the adjoint action by  $g \in G$  of an element  $B \in \mathfrak{h}'$  is given by the left translation,  $\overline{\text{Adj}}(g)(B) = L_{g*}(B)$ . This transformation makes sense for any derivation in  $G$ , and thus we have an action of  $G$  on  $\mathfrak{X}(G)$ . Let us take  $h$  in the kernel of  $\overline{\text{Adj}}$ , thus,  $\overline{\text{Adj}}(h)(B_j) = B_j$  for any index  $j$ . Applying the transformation  $L_{h*}$  to the expression of  $B_i$  as linear combination of the left invariant vector fields  $A_j$  we obtain  $B_i = \sum_{j=1}^m L_{h*}(f_{ij} A_j) = \sum_{j=1}^m h(f_{ij}) A_j$ . The coefficients of  $B_i$  as linear combination of the  $A_j$  are unique, and thus,  $h(f_{ij}) = f_{ij}$  we conclude that  $h$  is an automorphism that fix  $\mathbf{L}$ . On the other hand, let us take  $h \in H$  fixing  $\mathbf{L}$ . Then  $L_{h*}(\sum f_{ij} A_j) = \sum f_{ij} A_j$  thus  $\overline{\text{Adj}}(h)(B_i) = B_i$  and then  $h$  is in the kernel of  $\overline{\text{Adj}}$ .  $\square$

**2.5. Some examples of  $\mathfrak{sl}_2$ -parallelisms.** We will construct some parallelized varieties as sub-varieties of the arcs space of the affine line  $\mathbb{A}_{\mathbf{C}}^1$ . This family of example will realized any subgroups of  $\text{SL}_2(\mathbf{C})$  as the Galois group of the reciprocal Lie connection.

**2.5.1. The arcs space of the affine line and its Cartan 1-form.** In our special case, the arcs space of the affine line  $\mathbb{A}_{\mathbf{C}}^1$  with affine coordinate  $z$ , is the space of all formal power series  $\hat{z} = \sum z_i \frac{x^i}{i!}$ . It will be denoted by  $\mathcal{L}$ , its ring of regular functions is  $\mathbf{C}[\mathcal{L}] = \mathbf{C}[z, z_1, z_2, \dots]$ . For a  $U \subset \mathbf{C}$  one denotes by  $\mathcal{L}U$  the set of coordinate  $\hat{z}$  with  $z \in U$ .

A biholomorphism  $f : U \rightarrow V$  between open sets of  $\mathbf{C}$  can be lift to a biholomorphism  $\mathcal{L}f : \mathcal{L}U \rightarrow \mathcal{L}V$  by composition:  $\hat{z} \rightarrow f \circ \hat{z}$ .

Let  $\hat{\chi}$  be the Lie algebra of formal vector fields  $\mathbf{C}[[x]] \frac{\partial}{\partial x}$ . One can build a rational form  $\omega : T\mathcal{L} \rightarrow \hat{\chi}$  in following way (see [3, §2]). Let  $v = \sum a_i \frac{\partial}{\partial z_i}$  be a tangent vector at the formal coordinate  $\hat{p}$ , i.e. an arc in the Zariski open subset  $\{z_1 \neq 0\}$ . The local coordinate  $\hat{p}$  can be used to have local coordinates  $p, p_1, p_2, \dots$ , on  $\mathcal{L}$  and  $v$  can be written  $v = \sum b_i \frac{\partial}{\partial p_i}$ . The form  $\omega$  is defined by  $\omega(v) = \sum b_i \frac{x^i}{i!} \frac{\partial}{\partial x}$ . This form is rational and is an isomorphism between  $T_p\mathcal{L}$  and  $\hat{\chi}$  satisfying  $d\omega = -[\omega, \omega]$  and  $(\mathcal{L}f)^*\omega = \omega$  for any biholomorphism  $f$ .

This means that  $\omega$  provides an action of  $\hat{\chi}$  commuting with the lift of biholomorphisms. This form seems to be a coparallelisms but it is not compatible with the natural structure of pro-varieties of  $\mathcal{L}$  and  $\hat{\chi} : \omega^{-1}(\frac{\partial}{\partial x}) = \sum_{i \geq 1} z_{i+1} \frac{\partial}{\partial z_i}$  is a derivation of degree +1 with respect to the pro-variety structure of  $\mathcal{L}$ . The total derivation above will be denoted by  $E$  and it gives a differential structure to the ring  $\mathbf{C}[\mathcal{L}]$

**2.5.2. The parallelized varieties.** Let  $\nu \in \mathbf{C}(z)$  be a rational function,  $f$  be a rational function on the coordinate space given by

$$2 \frac{z_3}{z_1} - 3 \left( \frac{z_2}{z_1} \right)^2 + \nu(z)(z_1)^2$$

and  $I \subset \mathbf{C}[\mathcal{L}]$  be the  $E$ -invariant ideal generated by  $p(z)z_1^2 f(z, z_1, z_2, z_3)$  where  $p$  is a minimal denominator of  $f$

**Lemma 5.** *The zero set  $V$  of  $I$  is a dimension 3 subvariety of  $\mathcal{L}$  and  $\omega(TV) = \mathfrak{sl}_2(\mathbf{C}) \subset \widehat{\chi}$ . This provides a  $\mathfrak{sl}_2$ -parallelism on  $V$ .*

*Proof.* – One can compute explicitly this parallelism using  $z, z_1$  and  $z_2$  has élate coordinates on a Zariski open subset of  $V$ . Let us first compute the  $\mathfrak{sl}_2$  action on  $\mathcal{L}$ . The standard inclusion of  $\mathfrak{sl}_2$  in  $\widehat{\chi}$  is given by  $E = \frac{\partial}{\partial x} H = x \frac{\partial}{\partial x}$  and  $F = x^2 \frac{\partial}{\partial x}$ . Their actions on  $\mathcal{L}$  are given by  $E = \sum z^{(i+1)} \frac{\partial}{\partial z^{(i)}}$ ,  $H = \sum iz^{(i)} \frac{\partial}{\partial z^{(i)}}$  and  $F = \sum i(i-1)z^{(i-1)} \frac{\partial}{\partial z^{(i)}}$ . The ideal  $I$  is generated by the functions  $E^n \cdot f$ . By definition  $E \cdot f \in I$ , a direct computation gives that  $H \cdot f = 2f \in I$   $F \cdot f = 0 \in I$ . The relations in  $\mathfrak{sl}_2$  give that  $E \cdot I \subset I$ ,  $H \cdot I \subset I, F \cdot I \subset I$  i.e. the vector fields  $E, H$  and  $F$  are tangent to  $V$ . □

Now parametrizing  $V$  by  $z, z_1$  and  $z_2$  one gets

$$\begin{aligned} E|_{\mathbf{C}^3} &= z_1 \frac{\partial}{\partial z} + z_2 \frac{\partial}{\partial z_1} + \left( \frac{-1}{2} \nu(z) (z_1)^3 + \frac{3}{2} \frac{(z_2)^2}{z_1} \right) \frac{\partial}{\partial z_2} \\ H|_{\mathbf{C}^3} &= z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} \\ F|_{\mathbf{C}^3} &= 2z_1 \frac{\partial}{\partial z_2} \end{aligned}$$

They form a rational  $\mathfrak{sl}_2$ -parallelism on  $\mathbf{C}^3$  depending on the choice of a rational function in one variable.

### 2.5.3. Symmetries and the Galois group of the reciprocal connection.

**Theorem 1.** *Any algebraic subgroup of  $\mathrm{SL}_2(\mathbf{C})$  can be realized as the Galois group of the reciprocal connection of a parallelism of  $\mathbf{C}^3$ .*

*Proof.* – A direct computation shows that  $z \mapsto \varphi(z)$  is an holomorphic function satisfying

$$2 \frac{\varphi'''}{\varphi'} - 3 \left( \frac{\varphi''}{\varphi'} \right) + \nu(\varphi) (\varphi')^2 = \nu(z)$$

if and only if its prolongation  $\mathcal{L}\varphi : \widehat{z} \mapsto \varphi(\widehat{z})$  on the space  $\mathcal{L}$  preserves  $V$  and preserves each vector fields  $E, H$  and  $F$ .

Taking infinitesimal generators of this pseudogroup, one gets for any local analytic solution of the linear equation

$$(2) \quad a''' + \nu(z)a' + \frac{\nu}{2}a = 0$$

a vector field  $X = a(z) \frac{\partial}{\partial z}$  whose prolongation on  $\mathcal{L}$  is

$$\mathcal{L}X = a(z) \frac{\partial}{\partial z} + a'(z)z_1 \frac{\partial}{\partial z_1} + (a''(z)(z_1)^2 + a'(z)z_2) \frac{\partial}{\partial z_2} + \dots$$

The equation (2) ensure that  $\mathcal{L}X$  is tangent to  $V$ . The invariance of  $\omega : (\mathcal{L}X)_* \omega = 0$  ensure that  $\mathcal{L}X$  commute with the  $\mathfrak{sl}_2$ -parallelism given above. This means that for any solution  $a$  of (2) the vector field

$$a(z) \frac{\partial}{\partial z} + a'(z)z_1 \frac{\partial}{\partial z_1} + (a''(z)(z_1)^2 + a'(z)z_2) \frac{\partial}{\partial z_2}$$

commute with  $E|_{\mathbf{C}^3}, H|_{\mathbf{C}^3}$  and  $F|_{\mathbf{C}^3}$ .

Then the linear differential system of flat section for the reciprocal connection reduces to the linear equation (2). This equation is the second symmetric power of  $y'' = \nu(z)y$ . By [4] every algebraic subgroup of  $\mathrm{SL}_2(\mathbf{C})$  can be realized as the Galois group of such an equation. □

## 3. DARBOUX-CARTAN CONNECTIONS

**3.1. Connection of parallelism conjugations.** Let  $\omega$  be a rational parallelism in  $M$  of type  $\mathfrak{g}$ , and  $G$  an algebraic group with Lie algebra of left invariant vector fields  $\mathfrak{g}$  and Maurer-Cartan form  $\theta$ . Denote by  $M^*$  the open subset of  $M$  in which  $\omega$  is regular. We will study the construction of conjugating maps between the parallelisms  $(M, \omega)$  and  $(G, \theta)$ .

Let us consider the trivial principal bundle  $\pi: P = G \times M \rightarrow M$ . In this bundle we consider the action of  $G$  by right translations  $(g, x) * g' = (gg', x)$ . Let  $\Theta$  be  $\mathfrak{g}$ -valued form  $\Theta = \theta - \omega$  in  $P$ .

**Definition.** *The kernel of  $\Theta$  is a rational flat invariant connection in the principal bundle  $\pi: P \rightarrow M$ . We call it the Darboux-Cartan connection of parallelism conjugations from  $(M, \omega)$  to  $(G, \theta)$ .*

The equation  $\Theta = 0$  defines a foliation in  $P$  transversal to the fibers of regular points of  $\omega$ . The leaves of the foliation are the graphs of analytic conjugations parallelism conjugations from  $(M, \omega)$  to  $(G, \theta)$ . By means of differential Galois theory the Darboux-Cartan connection has a Galois group  $\text{Gal}(\Theta)$  with Lie algebra  $\mathfrak{gal}(\Theta)$ . The following facts are direct consequences of the definition of the Galois group:

- (a) There is a regular covering map  $c: (M^*, \omega) \rightarrow (U, \theta)$  with  $U$  an open subset of  $G$ , and  $c^*(\theta) = \omega$  if and only if  $\text{Gal}(\Theta) = \{1\}$ .
- (b) There is a regular covering map  $c: (M^*, \omega) \rightarrow (U, q_*\theta)$  with  $U$  an open subset of  $G/H$ ,  $H$  a group of finite index, and  $c^*(q_*\theta|_U) = \omega$  if and only if  $\mathfrak{gal}(\Theta) = \{0\}$ .

In any case, the necessary and sufficient condition for  $(M, \omega)$  and  $(G, \theta)$  to be isogenous parallelized varieties is that  $\mathfrak{gal}(\Theta) = \{0\}$ .

**3.2. Darboux-Cartan connection and Picard-Vessiot.** Note that the coparallelism  $\omega$  gives a rational trivialization of  $TM$  as the trivial bundle of fiber  $\mathfrak{g}$ . In  $TM$  we have defined the connection  $\nabla^{rec}$  whose horizontal vector fields are the symmetries of the parallelism. On the other hand,  $G$  acts in  $\mathfrak{g}$  by means of the adjoint action. The Cartan-Darboux connection induces then a connection  $\nabla^{adj}$  in the associated trivial bundle  $\mathfrak{g} \times M$  of fiber  $\mathfrak{g}$ .

**Proposition 5.** *The map,*

$$\tilde{\omega}: (TM, \nabla^{rec}) \rightarrow (\mathfrak{g} \times M, \nabla^{adj}), \quad X_x \mapsto (\omega_x(X_x), x),$$

*is a birational conjugation of the linear connections  $\nabla^{rec}$  and  $\nabla^{adj}$ .*

*Proof.* – It is clear that the map  $\tilde{\omega}$  is birational. Let us consider  $\{A_1, \dots, A_m\}$  a basis of  $\mathfrak{g}$ . Let  $\rho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$  be the parallelism associated to the parallelism  $\omega$  and let us define  $X_i = \rho(A_i)$ . Then  $\{X_1, \dots, X_n\}$  is a rational frame in  $M$  and the map  $\tilde{\omega}$  conjugates the vector field  $X_i$  with the constant section  $A_i$  of the trivial bundle of fiber  $\mathfrak{g}$ . By definition of the reciprocal connection,

$$\nabla_{X_i}^{rec} X_j = [X_i, X_j],$$

and the other hand, by definition of the adjoint action, and application of the covariant derivative as in equation (3) of appendix section A.7 we obtain:

$$\nabla_{X_i}^{adj} A_j = [A_i, A_j].$$

Therefore we have that  $\tilde{\omega}$  is a rational morphism of linear connections that conjugates  $\nabla^{rec}$  with  $\nabla^{adj}$ .  $\square$

The following facts follow directly from Proposition 5, and basic properties of the Galois group (see section A.7).

**Corollary.** *Let us consider the adjoint action  $\text{Adj}: G \rightarrow \text{GL}(\mathfrak{g})$  and its derivative  $\text{adj}: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ . The following facts hold:*

- (a)  $\text{Gal}(\nabla^{rec}) \subseteq \text{Int}(\mathfrak{g})$ .
- (b)  $\text{Gal}(\nabla^{rec}) = \text{Adj}(\text{Gal}(\nabla^{adj}))$ .
- (c)  $\mathfrak{gal}(\nabla^{rec}) = \text{adj}(\mathfrak{gal}(\Theta))$ .
- (d) *If  $\mathfrak{g}$  is centerless then  $\mathfrak{gal}(\nabla^{rec})$  is isomorphic to  $\mathfrak{gal}(\Theta)$ .*
- (e) *Assume  $\mathfrak{g}$  is centerless. Then the necessary and sufficient condition for  $(M, \omega)$  and  $(G, \theta)$  to be isogenous is that  $\mathfrak{gal}(\nabla^{rec}) = \{0\}$ .*

**3.3. Algebraic Lie algebras.** Let us consider  $(M, \omega)$  a rational coparallelism of type  $\mathfrak{g}$  with  $\mathfrak{g}$  a centerless Lie algebra. We do not assume *a priori* that  $\mathfrak{g}$  is an algebraic Lie algebra. The connection  $\nabla^{rec}$  is, as said in proposition 5, conjugated to the connection in  $\mathfrak{g} \times M$  induced by the adjoint action. Note that, in order to define this connection we do not need the group operation, but just the Lie bracket in  $\mathfrak{g}$ . We have an exact sequence,

$$0 \rightarrow \mathfrak{g}' \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}^{ab} \rightarrow 0$$

where  $\mathfrak{g}'$  is the derived algebra  $[\mathfrak{g}, \mathfrak{g}]$ . Since the Galois group acts by internal automorphisms, we have that  $\mathfrak{g}' \times M$  is stabilized by the connection  $\nabla^{rec}$  and thus we have an exact sequence of connections:

$$0 \rightarrow (\mathfrak{g}' \times M, \nabla') \rightarrow (\mathfrak{g} \times M, \nabla^{rec}) \rightarrow (\mathfrak{g}^{ab} \times M, \nabla^{ab}) \rightarrow 0.$$

**Lemma 6.** *The Galois group of  $\nabla^{ab}$  is the identity and thus  $\nabla^{ab}$  has a basis of rational horizontal sections.*

*Proof.* – By definition, the action of  $\mathfrak{g}$  in  $\mathfrak{g}^{ab}$  vanishes. Thus, the constant functions  $M \rightarrow \mathfrak{g}^{ab}$  are rational horizontal sections.  $\square$

**Lemma 7.** *Let  $\omega$  be a rational coparallelism of  $M$  of type  $\mathfrak{g}$  with  $\mathfrak{g}$  a centerless Lie algebra. If  $\mathfrak{gal}(\nabla^{rec}) = \{0\}$  then  $\mathfrak{g}$  is an algebraic Lie algebra.*

*Proof.* – Let  $E$  be an algebraic envelope of  $\mathfrak{g}$  and  $\mathfrak{e}$  its Lie algebra. We may assume that  $E$  is also centerless. Let  $A_1, \dots, A_r$  be a basis of  $\mathfrak{g}$ , for  $i = 1, \dots, r$ ,  $X_i = \omega^{-1}(A_i)$ . Complete with  $B_1, \dots, B_p$  in such way that  $A_1, \dots, A_r, B_1, \dots, B_p$  is a basis of  $\mathfrak{e}$ . We consider in  $E \times M$  the distribution spanned by the vector fields  $A_i + X_i$ . This is a  $E$  principal connection.

Let  $\bar{\nabla}$  be the induced connection via the adjoint representation on  $\mathfrak{e} \times M$  then

- (1)  $\bar{\nabla}$  preserves  $\mathfrak{g}$  and  $\bar{\nabla}|_{\mathfrak{g}} = \nabla^{rec}$ . By hypothesis  $\mathfrak{gal}(\bar{\nabla}|_{\mathfrak{g}}) = \{0\}$ .
- (2) If  $\tilde{\nabla}$  is the quotient connection on  $\mathfrak{e}/\mathfrak{g}$  then  $\mathfrak{gal}(\tilde{\nabla}) = \{0\}$ .

If  $\varphi \in \mathfrak{gal}(\tilde{\nabla})$  then for any  $X \in \mathfrak{g}$ ,  $[X, Z_i] \in \mathfrak{g}$  thus  $0 = \varphi[X, Z_i] = [X, \varphi Z_i]$  and  $\varphi Z$  commute with  $\mathfrak{g}$ . From the second point above  $\varphi Z \in \mathfrak{g}$ . By hypothesis  $\varphi Z = 0$  and  $\mathfrak{gal}(\tilde{\nabla}) = \{0\}$ . The projection on  $E$  of an algebraic leaf of  $\bar{\nabla}$  containing the point gives an algebraic leaf for the foliation of  $E$  by the left translation by  $\mathfrak{g}$ . This proves the theorem.  $\square$

**Theorem 2.** *Let  $\mathfrak{g}$  be a centerless Lie algebra. An algebraic variety  $(M, \omega)$  with a rational parallelism of type  $\mathfrak{g}$  is isogenous to an algebraic group if and only if  $\mathfrak{gal}(\nabla^{rec}) = \{0\}$ .*

*Proof.* – It follows directly from Lemma 7 and cor 3.2.  $\square$

**Corollary.** *Let  $\mathfrak{g}$  be a centerless Lie algebra. Any algebraic variety endowed with a pair of commuting rational parallelisms of type  $\mathfrak{g}$  is isogenous to an algebraic group endowed with its two canonical parallelisms of left and right invariant vector fields.*

*Proof.* – Just note that to have a pair of commuting parallelism is a more restrictive condition than to have a parallelism with vanishing Lie algebra of the Galois group of its reciprocal connection.  $\square$

This result can be seen as an algebraic version of Wang result in [9]. It gives the classification of algebraic varieties endowed with pairs of commuting parallelisms. The hypothesis of centerless Lie algebra is not superfluous, note that the result clearly does not hold for abelian Lie algebras. There are rational 1-forms in  $\mathbf{CP}_1$  that are not exact (isogenous to  $(\mathbf{C}, dz)$ ) nor logarithmic (isogenous to  $(\mathbf{C}^*, d \log(z))$ ).

**Remark 2.** *In virtue of Theorem 3.3, if  $\mathfrak{g}$  is a non-algebraic centerless Lie algebra, there is no algebraic variety endowed with a pair of regular commuting parallelisms of type  $\mathfrak{g}$ . This limits the possible generalizations of Theorem 7.*

## APPENDIX A. PICARD-VESSIOT THEORY OF A PRINCIPAL CONNECTION

In this previous reasoning we have used the concept of differential Galois group of a connection. Here we present a dictionary between invariant connection and strongly normal differential field extension (in the sense of Kolchin). In our setting a differential field is going to be a pair  $(\mathcal{K}, \mathcal{D})$  where  $\mathcal{K}$  is a finitely generated field over  $\mathbf{C}$  and  $\mathcal{D}$  is a  $\mathcal{K}$ -Lie algebra of derivations of  $\mathcal{K}$ . The dimension of  $\mathcal{D}$  is called the rank of the differential field. Note that we can adapt this notion easily to that of a finite number of commuting derivations by taking a suitable basis of  $\mathcal{D}$ . However we prefer to consider the whole space of derivations. With our definition a differential field extension  $(\mathcal{K}, \mathcal{D}) \rightarrow (\mathcal{K}', \mathcal{D}')$  is a field extension  $\mathcal{K} \subset \mathcal{K}'$  such that each element of  $\mathcal{D}$  extends to a unique element of  $\mathcal{D}'$ , and such extensions span the space  $\mathcal{D}'$  as  $\mathcal{K}'$ -vector space.

**A.1. Differential field extensions and foliated varieties.** First, let us see that there is a natural dictionary between finitely generated differential fields over  $\mathbf{C}$  and irreducible foliated varieties over  $\mathbf{C}$  modulo birational equivalence. Let  $(M, \mathcal{F})$  be an irreducible foliated variety of dimension  $n$ . The distribution  $T\mathcal{F} \subset TM$  is of rank  $r \leq n$ . We denote by  $\mathfrak{X}_{\mathcal{F}}$  the space rational vector fields in  $T\mathcal{F}$ ; it is a  $\mathbf{C}(M)$ -Lie algebra of dimension  $r$ . Hence, the pair  $(\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}})$  is a differential field. The field of constants is the field  $\mathbf{C}(M)^{\mathcal{F}}$  of rational first integrals of the foliation.

Let  $(M, \mathcal{F})$  and  $(M', \mathcal{F}')$  foliated varieties. A regular (rational) map  $\phi: (M', \mathcal{F}') \dashrightarrow (M, \mathcal{F})$  is a regular (rational) morphism of foliated varieties if  $d\phi$  induces an isomorphism between  $T_x\mathcal{F}'$  and  $T_{\phi(x)}\mathcal{F}$  for (generic values of)  $x \in M'$ . It is clear that  $\mathcal{F}'$  and  $\mathcal{F}$  have the same rank.

A differential field extension, correspond here to a dominant rational map of irreducible foliated varieties  $\phi: (M', \mathcal{F}') \dashrightarrow (M, \mathcal{F})$ . It induces the extension  $\phi^*: (\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}}) \rightarrow (\mathbf{C}(M'), \mathfrak{X}_{\mathcal{F}'})$  by composition with  $\phi$ .

**Example 14.** Let  $\mathcal{F}$  the foliation of  $\mathbf{C}^2$  defined by  $\{dy - ydx = 0\}$ . It correspond to the differential field  $(\mathbf{C}(x, e^x), \langle \frac{d}{dx} \rangle)$ .

**Remark 3.** Throughout this appendix “connection” means “flat connection”.

**A.2. Invariant  $\mathcal{F}$ -connections.** Let us consider from now a foliated manifold of dimension  $n$  and rank  $r$  without rational first integrals  $(M, \mathcal{F})$ , an algebraic group  $G$  and a principal irreducible  $G$ -bundle  $\pi: P \rightarrow M$ . A  $G$ -invariant connection in the direction of  $\mathcal{F}$  is a foliation  $\mathcal{F}'$  of rank  $r$  in  $P$  such that:

- (a)  $\pi: (P, \mathcal{F}') \rightarrow (M, \mathcal{F})$  is dominant regular map of foliated varieties.
- (b) The foliation  $\mathcal{F}'$  is invariant by the action of  $G$  in  $P$ .

With this definition  $(\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}}) \rightarrow (\mathbf{C}(P), \mathfrak{X}_{\mathcal{F}'})$  is a differential field extension. Also, each element  $g \in G$  induces a differential field automorphism of  $(\mathbf{C}(P), \mathfrak{X}_{\mathcal{F}'})$  that fixes  $(\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}})$  by setting  $(g \cdot f)(x) = f(x \cdot g)$ .

Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . There is a way of defining a  $G$ -equivariant form  $\Theta_{\mathcal{F}'}$  with values in  $\mathfrak{g}$ , and defined in  $d\pi^{-1}(T\mathcal{F})$  in such way that  $T\mathcal{F}'$  is the kernel of  $\Theta_{\mathcal{F}'}$ . First, there is a canonical form  $\Theta_0$  defined in  $\ker(d\pi)$  that sends each vertical vector  $X_p \in \ker d_p\pi \subset T_pP$  to the element  $\mathfrak{g}$  that verifies,

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} p \cdot \exp \varepsilon A = X_p.$$

This form is  $G$ -equivariant in the sense that  $R_g^*(\Theta_0) = \text{Adj}_{g^{-1}} \circ \omega$ . We have a decomposition of the vector bundle  $d\pi^{-1}(T\mathcal{F}) = \ker(d\pi) \oplus T\mathcal{F}'$ . This decomposition allows to extend  $\Theta_0$  to a form  $\Theta_{\mathcal{F}'}$  defined for vectors in  $d\pi^{-1}(T\mathcal{F})$  whose kernel is precisely  $T\mathcal{F}'$ . We call *horizontal frames* to those sections  $s$  of  $\pi$  such that  $s^*(\Theta_{\mathcal{F}'}) = 0$ .

**A.3. Picard-Vessiot bundle.** We say that the principal  $G$ -bundle with invariant  $\mathcal{F}$ -connection  $\pi: (P, \mathcal{F}') \rightarrow (M, \mathcal{F})$  is a *Picard-Vessiot* bundle if there are no rational first integrals of  $\mathcal{F}'$ . The notion of Picard-Vessiot bundle correspond exactly to that of primitive extension of Kolchin. In such case  $G$  is the group of differential field automorphisms of  $(\mathbf{C}(P), \mathfrak{X}_{\mathcal{F}'})$  that that fix  $(\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}})$  and  $(\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}}) \rightarrow (\mathbf{C}(P), \mathfrak{X}_{\mathcal{F}'})$  is a strongly normal extension. Moreover, any strongly normal extensions with constant field  $\mathbf{C}$  can be constructed in this way (see [5], Ch. VI §10 Theorem 9).

One of the most remarkable properties of strongly normal extensions is the Galois correspondence (from [5], Ch. VI §4).

**Theorem 3** (Galois correspondence). *Assume that  $(\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}}) \rightarrow (\mathbf{C}(P), \mathfrak{X}_{\mathcal{F}'})$  is strongly normal with group of automorphisms  $G$ . Then, there is a bijection between the set of intermediate differential field extensions and algebraic subgroups of  $G$ . To each intermediate differential field extension, it correspond the group of automorphisms the fix such an extension point-wise. To each subgroup of automorphisms it corresponds its fixed field.*

**A.4. The Picard-Vessiot bundle of an invariant  $\mathcal{F}$ -connection.** Let us consider an irreducible principal  $G$ -bundle  $\pi: (P, \mathcal{F}') \rightarrow (M, \mathcal{F})$  endowed of a invariant  $\mathcal{F}$ -connection  $\mathcal{F}'$ . We assume that  $\mathcal{F}$  has no rational first integrals. A result of Bonnet (see [1], Theorem 1.1) ensures that for a very generic point in  $M$  the leaf passing through such point is Zariski dense in  $M$ . Let us consider such a Zariski-dense leaf  $\mathcal{L}$  of  $\mathcal{F}$  in  $M$ . Let us consider any leaf  $\mathcal{L}'$  of  $\mathcal{F}'$  in  $P$  that projects by  $\pi$  onto  $\mathcal{L}$ . Its Zariski closure is unique in the following sense:

**Theorem 4.** *Let  $\mathcal{L}'$  and  $\mathcal{L}''$  two leaves of  $\mathcal{F}'$  whose projections by  $\pi$  are Zariski dense in  $M$ . Then, there exist an element  $g \in G$  such that  $\overline{\mathcal{L}'} \cdot g = \overline{\mathcal{L}''}$ .*

*Proof.* – By construction, there is some  $x \in \pi(\mathcal{L}') \cap \pi(\overline{\mathcal{L}''})$ . Let us consider  $p \in \pi^{-1}(\{x\}) \cap \mathcal{L}'$  and  $q \in \pi^{-1}(\{x\}) \cap \overline{\mathcal{L}''}$ . Since  $p$  and  $q$  are in the same fiber, there is a unique element  $g \in G$  such that  $p \cdot g = q$ . By the  $G$ -invariance of the connection  $\mathcal{L}' \cdot g$  is the leaf of  $\mathcal{F}'$  that passes through  $q$ . The set  $\overline{\mathcal{L}''}$  is, by construction, union of leaves of  $\mathcal{F}'$  and contains the point  $q$ . Thus,  $\overline{\mathcal{L}' \cdot g} \subseteq \overline{\mathcal{L}''}$ , and  $\overline{\mathcal{L}'} \cdot g \subseteq \overline{\mathcal{L}''}$ . Now, by exchanging the roles of  $\mathcal{L}'$  and  $\mathcal{L}''$ , we prove that there is an element  $h$  such that  $\overline{\mathcal{L}''} \cdot h \subseteq \overline{\mathcal{L}'}$ . It follows  $h = g^{-1}$  and it finishes the proof.  $\square$

Let  $L$  be the Zariski closure of  $\mathcal{L}'$ . Let us consider the algebraic subgroup,

$$H = \{g \in G : L \cdot g = L\}$$

stabilizing  $L$ . The projection  $\pi$  resctricted to  $L$  is dominant, and thus there is a Zariski open subset  $M^*$  such that  $\pi^*: L^* \rightarrow M^*$  is surjective. Let us call  $\mathcal{F}^*$  the restriction of  $\mathcal{F}'$  to  $L^*$ . It follows that the bundle:  $\pi^*: (L^*, \mathcal{F}^*) \rightarrow (M^*, \mathcal{F}|_{M^*})$  is a Picard-Vessiot bundle of structure group  $H$ . The differential field extension  $(\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}}) \rightarrow (\mathbf{C}(L^*), \mathfrak{X}_{\mathcal{F}^*})$  is the so-called Picard-Vessiot extension associated to the connection. The algebraic group  $H$  is the differential Galois group of the connection.

**A.5. Split of a connection.** Let us consider a pair of morphisms of foliated varieties,

$$\phi_j: (M_j, \mathcal{F}_j) \rightarrow (M, \mathcal{F}), \quad \text{for } j = 1, 2.$$

Then, we can define in  $M_1 \times_M M_2$  a foliation  $\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$  in the following way. A vector  $X = (X_1, X_2)$  is in  $T(\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2)$  if and only if  $d\phi_1(X_1) = d\phi_2(X_2) \in T\mathcal{F}$ . Let us consider  $(P, \mathcal{F}')$  a principal  $\mathcal{F}$  connection. Note that the projection:

$$\pi_1: (M_1 \times_M P, \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}') \rightarrow (M_1, \mathcal{F}_1)$$

is a principal  $G$ -bundle endowed of a  $\mathcal{F}_1$ -connection. We call this bundle the pullback of  $(P, \mathcal{F}')$  by  $\phi_1$ .

We also may consider the trivial  $G$ -invariant connection  $\mathcal{F}_0$  in the trivial principal  $G$ -bundle,

$$\pi_0: (M \times G, \mathcal{F}_0) \rightarrow (M, \mathcal{F}),$$

for what the leafs of  $\mathcal{F}_0$  are of the form  $(\mathcal{L}, g)$  where  $\mathcal{L}$  is a leaf of  $\mathcal{F}$  and  $g$  a fixed element of  $G$ . We say that a the  $G$ -invariant connection  $(P, \mathcal{F}')$  is rationally trivial if there is a birational  $G$ -equivariant isomorphism of foliated manifolds between  $(P, \mathcal{F}')$  and  $(M \times G, \mathcal{F}_0)$ .

Invariant connections always trivialize after pullback. There is a universal  $G$ -equivariant isomorphism defined over  $P$ :

$$(P \times G, \mathcal{F}' \times_{\mathcal{F}} \mathcal{F}_0) \rightarrow (P \times_M P, \mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'), \quad (p, g) \mapsto (p, p \cdot g),$$

that trivializes any  $G$ -invariant connection. However the differential field  $(\mathbf{C}(P), \mathfrak{X}_{\mathcal{F}'})$  may have new constant elements. To avoid this, we replace the pullback to  $P$  by a pullback to the Picard-Vessiot bundle  $L^*$ :

$$(L^* \times G, \mathcal{F}^* \times_{\mathcal{F}} \mathcal{F}_0) \rightarrow (L^* \times_M P, \mathcal{F}^* \times_{\mathcal{F}} \mathcal{F}'), \quad (p, g) \mapsto (p, p \cdot g).$$

The Picard-Vessiot bundle has some minimality property. It is the smallest bundle on  $M$  that allows to trivialize the connection. We have the following result.

**Theorem 5.** *Let us consider  $\pi: (P, \mathcal{F}') \rightarrow (M, \mathcal{F})$  be as above,  $\pi^*: (L^*, \mathcal{F}^*) \rightarrow (M, \mathcal{F})$  the Picard-Vessiot bundle, and  $\phi: (\tilde{M}, \tilde{\mathcal{F}}) \rightarrow (M, \mathcal{F})$  any dominant rational map of foliated varieties such that:*

- (a)  $\tilde{\mathcal{F}}$  has no rational first integrals in  $\tilde{M}$
- (b) The pullback  $(\tilde{M} \times_M P, \tilde{\mathcal{F}} \times_{\mathcal{F}} \mathcal{F}') \rightarrow (\tilde{M}, \tilde{\mathcal{F}})$  is rationally trivial.

*There is a dominant rational map of foliated varieties  $\psi: \tilde{M} \dashrightarrow L^*$  such that  $\pi^* \circ \psi = \phi$  in their common domain.*

*Proof.* – Let us take  $\tau: \tilde{M} \times G \dashrightarrow \tilde{M} \times_M P$  a birational trivialization,  $\pi_2: \tilde{M} \times_M P \rightarrow P$  be the projection in the second factor, and  $\iota: \tilde{M} \rightarrow \tilde{M} \times G$  the inclusion  $p \mapsto (p, e)$ . Then,  $\tilde{\psi} = \pi_2 \circ \tau \circ \iota$  is a rational map from  $\tilde{M}$  to  $P$  whose differential sends  $T\tilde{\mathcal{F}}$  to  $T\mathcal{F}$ . By Bonnet theorem,  $\tilde{M}$  is the Zariski closure of a leaf of  $\tilde{\mathcal{F}}$  that projects by  $\phi$  into a Zariski dense leaf of  $\mathcal{F}$ . From this,  $\tilde{\psi}$  contains a dense leaf of  $\mathcal{F}'$  in  $P$ . By applying a suitable right translation in  $P$  and uniqueness Theorem 4, we conclude.  $\square$

**A.6. Linear connections.** Let  $(M, \mathcal{F})$  be as above, of dimension  $n$  and rank  $r$ . Let  $\xi: E \rightarrow M$  be a vector bundle of rank  $k$ . A linear integrable  $\mathcal{F}$ -connection is a foliation  $\mathcal{F}_E$  of rank  $r$  which is compatible with the structure of vector bundle in the following sense: the point-wise addition of two leaves of any dilation of a leaf is also a leaf. This can also be stated in terms of a covariant derivative operator  $\nabla$  which is defined only in the direction of  $\mathcal{F}$ . First, the kernel of  $d\xi$  is naturally projected onto  $E$  itself

$$\text{vert}_0: \ker(d\xi) \rightarrow E, \quad X_v \mapsto w,$$

where  $\frac{d}{d\varepsilon}|_{\varepsilon=0} v + \varepsilon w = X_v$ . Then, the decomposition of  $d\xi^{-1}(T\mathcal{F})$  as  $\ker(d\xi) \oplus T\mathcal{F}_E$  allows us to extend  $\text{vert}_0$  to a projection

$$\text{vert}: d\xi^{-1}(T\mathcal{F}) \rightarrow E.$$

Thus, we define for each section  $s$  its covariant derivative  $\nabla s = s^*(\text{vert} \circ ds|_{T\mathcal{F}})$ . This is a 1-form on  $M$  defined only for vectors in  $T\mathcal{F}$ . This covariant derivative has the desired properties, it is additive and satisfies the Leibniz formula:

$$\nabla(fs) = df|_{T\mathcal{F}} \otimes s + f\nabla s.$$

In general we write for  $X$  a vector in  $T\mathcal{F}$ ,  $\nabla_X s$  for the contraction of  $\nabla s$  with the vector  $X$ . It is an element of  $E$  over the same base point in  $M$  that the vector  $X$ . We call *horizontal sections* to those sections  $s$  of  $\xi$  such that  $\nabla s = 0$ .

Let  $\pi: R^1(E) \rightarrow M$  be the bundle of linear frames in  $E$ . It is a principal linear  $\text{GL}_k(\mathbf{C})$ -bundle. The foliation  $\mathcal{F}_E$  induces a foliation  $\mathcal{F}'$  in  $R^1(E)$  that is a  $G$ -invariant  $\mathcal{F}$ -connection. Let us consider the Picard-Vessiot bundle,  $(L^*, \mathcal{F}^*)$ . The uniqueness theorem 5 on the Picard-Vessiot bundle, can be rephrased algebraically in the following way. The Picard-Vessiot extension  $(\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}}) \rightarrow (\mathbf{C}(L^*), \mathfrak{X}_{\mathcal{F}^*})$  is characterized by the following facts (cf. [8] Section 1.3):

- (a) There are no new constants,  $\mathbf{C}(L^*) = \mathbf{C}$ .
- (b) It is spanned, as a field extension of  $\mathbf{C}(M)$ , by the coefficients of a fundamental matrix of solutions of the differential equation of the horizontal sections.

**A.7. Associated connections.** Let  $\pi: (P, \mathcal{F}') \rightarrow (M, \mathcal{F})$  be as a  $G$ -invariant connection, as before, where  $\mathcal{F}$  is a foliation in  $M$  without rational first integrals. Let us consider  $\phi: G \rightarrow \text{GL}(V)$  a finite dimensional linear representation of  $G$ . It is well known that the associated bundle  $\pi_P: V_P \rightarrow M$ ,

$$V_P = P \times_G V = (P \times V)/G \quad (p \cdot g, v) \sim (p, g \cdot v),$$

is a vector bundle with fiber  $V$ . Here we represent the action of  $G$  in  $V$  by the same operation symbol than before. The  $G$ -invariant connection  $\mathcal{F}'$  rises to a foliation in  $P \times G$  and then it is projected to a foliation  $\mathcal{F}_V$  in  $V_P$ . In this way, the projection

$$\pi_P: (V_P, \mathcal{F}_V) \rightarrow (M, \mathcal{F}),$$

turns out to be a linear  $\mathcal{F}$ -connection. It is called the *Lie-Vessiot* connection induced in the associated bundle. The Galois group of the principal and the associated Lie-Vessiot connection are linked in the following way.

**Theorem 6.** *Let  $H \subset G$  be the Galois group of the principal connection  $\mathcal{F}'$ . Then, the Galois group of the associated Lie-Vessiot connection  $\mathcal{F}_V$  is  $\phi(H) \subseteq \mathrm{GL}(V)$ .*

*Proof.* – Let us consider the bundle of frames  $R^1(V_P)$ , with its induced invariant connection  $\mathcal{F}''$ . Let us fix a basis  $\{v_1, \dots, v_r\}$  of  $V$ . Then, we have a map,

$$\tilde{\pi}: P \rightarrow R^1(V_P), \quad p \mapsto ([p, v_1], \dots, [p, v_r]),$$

where the pair  $[p, v]$  represents the class of the pair  $(p, v) \in P \times V$ . By construction,  $\tilde{\pi}$  sends  $T\mathcal{F}'$  to  $T\mathcal{F}''$ . It implies that, if  $\mathcal{L}^*$  is a Picard-Vessiot bundle for  $\mathcal{F}'$  then  $\tilde{\pi}(\mathcal{L}^*)$  is a Picard-Vessiot bundle for  $\mathcal{F}''$ . Second, if  $\mathcal{L}^*$  is a principal  $H$  bundle, then  $\tilde{\pi}(\mathcal{L}^*)$  is a principal  $H/K$  bundle where  $K$  is the subgroup of  $H$  that stabilizes the basis  $\{v_1, \dots, v_r\}$ .  $\square$

Let us discuss how the covariant derivative operator in  $\nabla$  is defined in terms of  $\Theta_{\mathcal{F}'}$  and the action of  $G$  in  $V$ . Let us denote by  $\phi': \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  the induced Lie algebra morphism. Let  $s$  be a local section of  $\xi$ . Let us consider the canonical projection  $\tilde{\pi}: P \times V \rightarrow V(P)$ . This turns out to be also a principal bundle, here the action on pairs is  $(p, v) \cdot g = (p \cdot g, g^{-1} \cdot v)$ . Now we can take any section  $r$  of this bundle, and define  $\tilde{s} = r \circ s$ . As  $r$  takes values in a cartesian product, we obtain  $\tilde{s} = (s_1, s_2)$  where  $s_1$  is a section of  $\pi$  and  $s_2$  is a function with values in  $V$ . Finally we obtain:

$$(3) \quad \nabla s = ds_2|_{T\mathcal{F}} - \phi'(s_1^*(\Theta_{\mathcal{F}'}))(s_2).$$

A calculation shows that it does not depend of the choice of  $r$  and it is the covariant derivative operator associated to  $\mathcal{F}_V$ . In particular if  $s_2$  is already an horizontal frame, then the covariant differential is given by the first term  $ds_s|_{T\mathcal{F}}$ .

#### APPENDIX B. DELIGNE'S REALIZATION OF LIE ALGEBRA

The proof of the existence of a regular parallelism for any complex Lie algebra  $\mathfrak{g}$  is written in a set of two letters from P. Deligne to B. Malgrange (dated from November of 2005 and February of 2010 respectively) that are published verbatim in [6]. We reproduce here the proof with some extra details.

**Theorem 7** (Deligne). *Given any complex Lie algebra  $\mathfrak{g}$  there exist an algebraic variety endowed with a regular parallelism of type  $\mathfrak{g}$ .*

**Lemma 8.** *Let  $T$  be an algebraic torus acting regularly by automorphisms in some algebraic group  $H$ , and let  $\mathfrak{t}$  be the Lie algebra of  $T$ . Let us consider the semidirect product:*

$$\mathfrak{t} \ltimes H, \quad (t, h)(t', h') = (t + t', (\exp(t') \cdot h)h')$$

*as an algebraic variety and analytic Lie group. Its left invariant vector fields form a regular parallelism of  $\mathfrak{t} \ltimes H$ . The Galois group of this parallelism is a torus.*

*Proof.* – Let us denote by  $\alpha$  the action of  $T$  in  $H$  and  $\alpha': \mathfrak{t} \rightarrow \mathfrak{X}[H]$  the Lie algebra isomorphism given by the infinitesimal generators

$$(\alpha' X)_h = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \alpha_{\exp(\varepsilon t)}(h).$$

Let  $X$  be an invariant vector field in  $\mathfrak{t}$ . Let us compute the left invariant vector field in  $\mathfrak{t} \ltimes H$  whose value at the identity is  $(X_0, 0)$ . In order to perform the computation we write the vector as an infinitesimally near point to  $(0, e)$ .

$$L_{(t, h)}(0 + \varepsilon X_0, e) = (t + \varepsilon X_t, \alpha_{\exp(\varepsilon X)}(h)) = (t + \varepsilon X_t, h + \varepsilon(\alpha' X)_h)$$

And thus  $dL_{(t, h)}(X_0, 0) = (X_t, (\alpha' X)_h)$ . We conclude that  $(X, \alpha' X) \in \mathfrak{X}[\mathfrak{t} \ltimes H]$  is the left invariant vector field whose value at  $(0, e)$ . Let us consider now  $Y$  an left invariant vector field in  $H$ . Let us compute, as before, the left invariant vector field whose value at  $(t, h)$  is  $(0, Y_h)$ .

$$L_{(t, h)}(0, e + \varepsilon Y_e) = (t, L_h(e + \varepsilon Y_e)) = (t, h + \varepsilon Y_h)$$

And thus  $(0, Y)$  is the left invariant vector field whose value at  $(0, e)$  is  $(0, Y_e)$ . These vector fields of the form  $(X, \alpha' X)$  and  $(0, Y)$  are regular and span the Lie algebra of left invariant vector fields in  $\mathfrak{t} \ltimes H$ . Hence, they form a regular parallelism.



In order to compute the Galois group of the parallelism, let us compute its reciprocal parallelism. It is formed by the right invariant vector fields in the analytic Lie group  $\mathfrak{t} \ltimes H$ . A similar computation proves that if  $X$  is an invariant vector field in  $\mathfrak{t}$  then  $(X, 0)$  is right invariant in  $\mathfrak{t} \ltimes H$ . For each element  $\tau \in T$ ,  $\alpha_\tau$  is a group automorphism of  $H$ . Thus, it induces a derived automorphism  $\alpha_{\tau*}$  of the Lie algebra of regular vector fields in  $H$ . Let  $Y$  be now a right invariant vector field in  $H$ , let us compute the right invariant vector field  $Z$  in  $\mathfrak{t} \ltimes H$  whose value at  $(0, e)$  is  $(0, Y_e)$ .

$$R_{(t,h)}(0, e + \varepsilon Y_e) = (t, \alpha_{\exp(t)}(e + \varepsilon Y_e)h) = (t, h + \varepsilon(\alpha_{\exp(t)*}Y)_h)$$

and thus,  $Z_{t,h} = (0, (\alpha_{\exp(t)*}Y)_h)$ . Those analytic vector field depend on the exponential function in a torus, and thus, we can conclude, by a standard argument of differential Galois theory, that the associated differential Galois group is a torus.  $\square$

Let us consider  $\mathfrak{g}$  an arbitrary, non algebraic, finite dimensional complex Lie algebra. We consider an embedding of  $\mathfrak{g}$  in the Lie algebra of general linear group and  $E$  the algebraic hull of the Lie algebra.  $E$  is a connected linear algebraic group, whose Lie algebra  $\mathfrak{e}$  contains  $\mathfrak{g}$ .

**Lemma 9** (Also in [2] Proposition 1). *With the above definitions and notation  $[\mathfrak{e}, \mathfrak{e}] = [\mathfrak{g}, \mathfrak{g}]$ .*

*Proof.* – Let  $H$  be the group of matrices that stabilizes  $\mathfrak{g}$  and acts trivially on  $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ . Its Lie algebra  $\mathfrak{h}$  contains  $\mathfrak{g}$  and thus  $H \supseteq E$  and  $\mathfrak{h} \supseteq \mathfrak{e}$ . By definition of  $H$  we have  $[\mathfrak{h}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}]$ , therefore  $[\mathfrak{e}, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ . Let us now consider the group  $H_1$  that stabilizes  $\mathfrak{e}$  and  $\mathfrak{g}$  and that acts trivially in  $\mathfrak{e}/[\mathfrak{g}, \mathfrak{g}]$ . This is again an algebraic group containing  $E$ , and its Lie algebra  $\mathfrak{h}_1$  satisfies  $[\mathfrak{h}_1, \mathfrak{e}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ . Taking into account  $\mathfrak{e} \subseteq \mathfrak{h}_1$  we have  $[\mathfrak{e}, \mathfrak{e}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ . The other inclusion is trivial.  $\square$

Because of Lemma 9 the abelianized Lie algebra  $\mathfrak{g}^{ab} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$  is a subspace of  $\mathfrak{e}^{ab} = \mathfrak{e}/[\mathfrak{e}, \mathfrak{e}]$ . Moreover, if we consider the quotient map,  $\pi: \mathfrak{e} \rightarrow \mathfrak{e}^{ab}$ , then  $\mathfrak{g} = \pi^{-1}(\mathfrak{g}^{ab})$ .

Let us consider an algebraic Levy decomposition  $E \simeq L \ltimes U$  (see [7], Chapter 6). Here,  $L$  is reductive and  $U$  is the unipotent radical, consisting in all the unipotent elements of  $E$ . The semidirect product structure is produced by an action of  $L$  in  $U$ , so that,  $(l_1, u_1)(l_2, u_2) = (l_1 l_2, (l_2 \cdot u_1)u_2)$ .

Since  $L$  is reductive, its commutator subgroup  $L'$  is semisimple. Let  $T$  be the center of  $L$ , which is a torus, the map:

$$\varphi: T \times L' \rightarrow L, \quad (t, l) \mapsto tl,$$

is an isogeny. The isogeny defines an action of  $T \times L'$  in  $U$  by  $(t, l) \cdot u = tl \cdot u$ . We have found an isogeny,

$$(T \times L') \ltimes U \rightarrow E$$

The Lie algebra  $\mathfrak{u}$  of  $U$  is a nilpotent Lie algebra, so that the exponential map  $\exp: \mathfrak{u} \rightarrow U$  is regular and bijective. In general, if  $V$  is an abelian quotient of  $U$  with Lie algebra  $\mathfrak{v}$  then the exponential map conjugates the addition law in  $\mathfrak{v}$  with the group law in  $V$ .

**Lemma 10.** *With the above definitions and notation, let  $\bar{u}$  be the biggest quotient of  $\mathfrak{u}^{ab}$  in which  $L$  acts by the identity. We have a Lie algebra isomorphism  $\mathfrak{e}^{ab} \simeq \mathfrak{t} \times \bar{u}$ .*

*Proof.* – Let us compute  $\mathfrak{e}^{ab}$ . We compute the commutators  $\mathfrak{e}$  by means of the isomorphism  $\mathfrak{e} \simeq (\mathfrak{t} \times L') \ltimes \mathfrak{u}$ . We obtain,

$$[(t_1, l_1, u_1), (t_2, l_2, u_2)] = [(0, [l_1, l_2], a(t_2, l_2)u_1 + [u_1, u_2]),$$

where  $a$  represents the derivative at  $(e, e)$  of the action of  $L$  in  $U$ . From this we obtain that  $[\mathfrak{e}, \mathfrak{e}]$  is spanned by  $(\{0\} \times L') \ltimes \{0\}$ ,  $\{0\} \ltimes [\mathfrak{u}, \mathfrak{u}]$  and  $\{0\} \ltimes \langle a(l)\mathfrak{u} \rangle$ . Taking into account that  $\bar{u}/(\langle a(l)\mathfrak{u} \rangle + [\mathfrak{u}, \mathfrak{u}])$  is the biggest quotient of  $\mathfrak{u}^{ab}$  in which  $L$  acts trivially, we obtain the result of the lemma.  $\square$

Let  $\mathfrak{t}$  be the Lie algebra of  $T$ . Its exponential map is a analytic group morphism, and thus we may consider the analytic action of  $\mathfrak{t} \times L'$  in  $U$  given by  $(t, l) \cdot u = (\exp(t)l) \cdot u$ . Let  $\tilde{E}$  be the algebraic variety and analytic Lie group  $(\mathfrak{t} \times L') \ltimes U$ . By application of Lemma 8 and taking into account that  $\tilde{E} \simeq \mathfrak{t} \ltimes H$  where  $H$  is the compositum group  $L' \cdot U$  we have that the left invariant vector fields in  $\tilde{E}$  are regular. Let us consider the projection,

$$\pi_1: \tilde{E} \rightarrow \mathfrak{e}^{ab} = \mathfrak{t} \times \bar{u}, \quad (t, l, u) \mapsto (t, [\log(u)]),$$

this projection is algebraic by construction, and also a morphism of Lie groups. By Lemmas 9 and 10,  $\mathfrak{g}^{ab}$  is a vector subspace of the image. Then, let us take  $\tilde{G}$  the fiber  $\pi_1^{-1}(\mathfrak{g}^{ab})$ . It is an algebraic submanifold of  $\tilde{E}$  and an analytic Lie group. The derivative at the identity of  $\pi_1$  is precisely the abelianization  $\pi$  and it follows that the Lie algebra of  $\tilde{G}$  is precisely  $\mathfrak{g}$ . Finally  $\tilde{G}$  is an algebraic variety with a regular  $\mathfrak{g}$ -parallelism. This finishes the proof of Theorem 7.

**Remark 4.** *The right invariant vector fields in  $\tilde{G}$  are constructed as in Lemma 8 by means of the exponential function in the torus. Hence, Galois groups of the parallelisms obtained via this construction are always tori.*

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