

## Soft and hard anisotropic interface in composite materials

F. Lebon<sup>1</sup>, S. Dumont<sup>1,2</sup>, R. Rizzoni<sup>3</sup>, J. C. López-Realpozo<sup>4</sup>, R. Guinovart-Díaz<sup>4</sup>,  
R. Rodríguez-Ramos<sup>4</sup>, J. Bravo-Castillero<sup>4</sup>, F. J. Sabina<sup>5</sup>

<sup>1</sup>Laboratoire de Mécanique et d'Acoustique, Université Aix-Marseille, CNRS, Centrale  
Marseille, 4 Impasse Nikola Tesla, CS 40006, 13453 Marseille Cedex 13, France

[lebon@lma.cnrs-mrs.fr](mailto:lebon@lma.cnrs-mrs.fr)

<sup>2</sup>Université de Nîmes, Site des Carmes, Place Gabriel Péri, 30000 Nîmes, France

[serge.dumont@unimes.fr](mailto:serge.dumont@unimes.fr)

<sup>3</sup>ENDIF, Università di Ferrara, Italy [raffaella.rizzoni@unife.it](mailto:raffaella.rizzoni@unife.it)

<sup>4</sup>Facultad de Matemática y Computación. Universidad de La Habana, San Lázaro y L,  
Vedado, Habana 4. CP-10400, Cuba

[jclrealpozo@matcom.uh.cu](mailto:jclrealpozo@matcom.uh.cu), [guino@matcom.uh.cu](mailto:guino@matcom.uh.cu), [reinaldo@matcom.uh.cu](mailto:reinaldo@matcom.uh.cu),

[jbravo@matcom.uh.cu](mailto:jbravo@matcom.uh.cu)

<sup>5</sup>Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas. Universidad  
Nacional Autónoma de México, Apartado Postal 20-726. Delegación de Álvaro Obregón,  
01000 México, DF., México [fjs@mym.iimas.unam.mx](mailto:fjs@mym.iimas.unam.mx)

### Abstract

For a large class of composites, the adhesion at the fiber–matrix interface is imperfect i.e. the continuity conditions for displacements and often for stresses is not satisfied. In the present contribution, effective elastic moduli for this kind of composites are obtained by means of the Asymptotic Homogenization Method (AHM). Interaction between fiber and matrix is considered for linear elastic fibrous composites with parallelogram periodic cell. In this case, the contrast or jump in the displacements on the boundary of each phase is proportional to the corresponding component of the tension on the interface. A general anisotropic behavior of the interphase is assumed and the interface stiffnesses are explicitly given in terms of the elastic constants of the interphase. The constituents of the composites exhibit transversely isotropic properties. A doubly periodic parallelogram array of cylindrical inclusions is considered. Comparisons with theoretical and experimental results verified that the present model is efficient for the analysis of composites with presence of imperfect interface and parallelogram cell. The present method can provide benchmark results for other numerical and approximate methods.

**Keywords:** Fibres; Interface/interphase; Parallelogram cell; Fibrous composites; imperfect contact; asymptotic homogenization method

## **Introduction**

In this work, micromechanical analysis method is applied to a periodic composite with unidirectional fibers and parallelogram cells. The analytical expressions of the homogenized elastic properties are calculated for two phase composite with hard and soft interfaces. The Asymptotic Homogenization Method (AHM), for two-phase fibrous periodic composites with imperfect adhesion and oblique cell is used for the calculation of the plane elastic effective coefficients. This contribution is an extension of previous works by the authors (Rodriguez-Ramos et al. 2011 [1], Guinovart-Diaz et al. 2011 [2]), where only the perfect contact was considered for the antiplane problem. Besides, the present investigation is different of those of Lopez-Realpozo et al. 2011 [3] and Rodriguez-Ramos et al. 2013 [4] since the plane problem is solved for the calculation of the effective coefficients for composites with parallelogram cell. The novelty of the present work is that the imperfection of the interface in the composite with parallelogram cell is taken into account introducing two spring-type stiffnesses ( $K_n, K_t$ ) for plane problems.

Using a classical approach [5], the spring parameters can be identified from a three phase problem where the interphase coating the fiber is very thin. The paper is organized as follows. In the first part of the paper the derivation of the contact law mechanically equivalent to the interphase coating the fiber is reviewed on the basis of an energy method [6]. The method allows obtaining the spring-type interface law for a general anisotropic behavior of the interphase and the interface parameters ( $K_n, K_t$ ) are explicitly given in terms of the elastic constants of the interphase.

The results of the micromechanical analysis presented in the second part of this paper are mainly focused on the impact of the arrangement of the fibers and the mechanic imperfection at the interface on the plane properties in the composites. Moreover, the theoretical approach is validated with some theoretical models.

## **Modeling of imperfect contact**

The interphase coating the fiber is represented as a thin layer  $B^\varepsilon$  with uniform small thickness  $\varepsilon \ll 1$  and cross-section  $A$ . The interphase joins the fiber and the matrix, assumed to occupy the reference configurations  $S_1^\varepsilon$  and  $S_2^\varepsilon$ , respectively.

Let  $\Gamma_1^\varepsilon, \Gamma_2^\varepsilon$  be the interfaces between the adhesive and the adherents and let  $S^\varepsilon = S_1^\varepsilon \cup S_2^\varepsilon \cup B^\varepsilon \cup \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon$  denote the composite made of the adhesive and the two adherents.

Adhesive and adherents are assumed to be perfectly bonded in order to ensure the continuity of the displacement and stress vector fields across  $\Gamma_1^\varepsilon, \Gamma_2^\varepsilon$ .

Let  $(O, i_1, i_2, i_3)$  be an orthonormal Cartesian basis and let  $(O, x_1, x_2, x_3)$  be the coordinates a particle. The origin is taken at the center of the interphase midplane and the  $x_3$  – axis is perpendicular to the interphase midplane.

The materials are homogeneous and linear elastic with  $C_{ijkl}^1, C_{ijkl}^2$  and  $C_{ijkl}^\varepsilon$  the elasticity tensors of the adherents and of the interphase, respectively. The elasticity tensors are assumed to be symmetric, with the minor and major symmetries, and positive definite.

The adhesive is assumed to be *soft*, i.e.  $C_{ijkl}^\varepsilon = \varepsilon C_{ijkl}$  with  $C_{ijkl}$  independent of  $\varepsilon$ .

The adherents are subjected to a body force density  $f : S_1^\varepsilon \cup S_2^\varepsilon \rightarrow \mathbb{R}^3$  and to a surface force density  $g : \Gamma_\varepsilon \rightarrow \mathbb{R}^3$  on  $\Gamma_\varepsilon \subset (\partial S_1^\varepsilon \setminus \Gamma_1^\varepsilon) \cup (\partial S_2^\varepsilon \setminus \Gamma_2^\varepsilon)$ . Body forces are negligible in the adhesive.

On the complementary part  $\Gamma_u^\varepsilon \subset (\partial S_1^\varepsilon \setminus \Gamma_1^\varepsilon) \cup (\partial S_2^\varepsilon \setminus \Gamma_2^\varepsilon) \setminus \Gamma_\varepsilon$  homogeneous boundary conditions are assigned:  $u^\varepsilon = 0$  on  $\Gamma_u^\varepsilon$ , where  $u^\varepsilon : S^\varepsilon \rightarrow \mathbb{R}^3$  is the displacement field defined from  $S^\varepsilon$ . The sets  $\Gamma_g^\varepsilon, \Gamma_u^\varepsilon$  are assumed to be located far from the interphase and the fields of the external forces are sufficient regularity to ensure the existence of equilibrium configuration.

The approach used in [6] to obtain the contact law is based on the fact that stable equilibrium configurations of the composite assemblage minimize the total energy:

$$E^\varepsilon(\mathbf{u}) = \int_{S_1^\varepsilon} \left( \frac{1}{2} C_{ijkl}^1 u_{i,j} u_{k,l} - f_i u_i \right) dV_x + \int_{S_2^\varepsilon} \left( \frac{1}{2} C_{ijkl}^2 u_{i,j} u_{k,l} - f_i u_i \right) dV_x - \int_{\Gamma_g^\varepsilon} \mathbf{g}_i u_i dA_x \\ + \int_{B^\varepsilon} \left( \frac{1}{2} \varepsilon C_{ijkl} u_{i,j} u_{k,l} \right) dV_x,$$

in the space of kinematically admissible displacements:

$$V^\varepsilon = \left\{ \mathbf{u} \in H(S^\varepsilon; \mathbb{R}^3) : \mathbf{u} = 0 \text{ on } \Gamma_u^\varepsilon \right\},$$

Where  $H(S^\varepsilon; \mathbb{R}^3)$  is the space of the vector-valued functions on the set  $S^\varepsilon$  which are continuous and differentiable as many times as necessary. Under suitable regularity assumptions, the existence of a unique minimizer  $\mathbf{u} \in V^\varepsilon$  is ensured [7].

For the asymptotic analysis, it is convenient to introduce the following change of variables  $\hat{\mathbf{p}} : (x_1, x_2, x_3) \rightarrow (z_1, z_2, z_3)$  in the adhesive:

$$z_1 = x_1, \quad z_2 = x_2, \quad z_3 = x_3/\varepsilon,$$

which gives

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial z_2} = \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial z_3} = \varepsilon \frac{\partial}{\partial x_3}.$$

A change of variable  $\bar{\mathbf{p}} : (x_1, x_2, x_3) \rightarrow (z_1, z_2, z_3)$  is also introduced in the adherents:

$$z_1 = x_1, \quad z_2 = x_2, \quad z_3 = x_3 \pm 1/2(1 - \varepsilon),$$

where the plus (minus) sign applies whenever  $x \in S_1^\varepsilon$  ( $x \in S_2^\varepsilon$ ) and one has

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial z_2} = \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial z_3} = \frac{\partial}{\partial x_3}.$$

After the change of variables  $\hat{\mathbf{p}}$ , the interphase occupies the domain

$$B = \left\{ (z_1, z_2, z_3) \in \mathbb{R}^3 : (z_1, z_2) \in A, |z_3| < 1/2 \right\},$$

and the adherents occupy the domains  $S_{1,2} = S_{1,2}^\varepsilon \pm 1/2(1 - \varepsilon)\mathbf{i}_3$ . The sets

$\Gamma_{1,2} = \left\{ (z_1, z_2, z_3) \in \mathbb{R}^3 : (z_1, z_2) \in A, z_3 = \pm 1/2 \right\}$  are taken to denote the interfaces between

$B$ , and  $S_{1,2}$  and  $S = S_1 \cup S_2 \cup B \cup \Gamma_1 \cup \Gamma_2$  is called the rescaled configuration of the

composite body. Lastly,  $\Gamma_u$  and  $\Gamma_g$  indicate the images of  $\Gamma_u^\varepsilon$  and  $\Gamma_g^\varepsilon$  under the change of

variables,  $\bar{\mathbf{f}} := \mathbf{f} \circ \bar{\mathbf{p}}^{-1}$  and  $\bar{\mathbf{g}} = \mathbf{g} \circ \bar{\mathbf{p}}^{-1}$  the rescaled external forces.

Using the changes of variables given by  $\bar{\mathbf{p}}$ ,  $\hat{\mathbf{p}}$  and denoting  $\bar{\mathbf{u}} = \mathbf{u}^\varepsilon \circ \bar{\mathbf{p}}^{-1}$  and  $\hat{\mathbf{u}} = \mathbf{u}^\varepsilon \circ \hat{\mathbf{p}}^{-1}$  the displacement fields from the rescaled adhesive and adherents, respectively, the total energy takes the rescaled form:

$$\begin{aligned} E^\varepsilon(\hat{\mathbf{u}}^\varepsilon, \bar{\mathbf{u}}^\varepsilon) = & \int_{S_1} \left( \frac{1}{2} C_{ijkl}^1 \bar{u}_{i,j}^\varepsilon \bar{u}_{k,l}^\varepsilon - \bar{f}_i \bar{u}_{i,j}^\varepsilon \right) dV_z + \int_{S_2} \left( \frac{1}{2} C_{ijkl}^2 \bar{u}_{i,j}^\varepsilon \bar{u}_{k,l}^\varepsilon - \bar{f}_i \bar{u}_{i,j}^\varepsilon \right) dV_z - \int_{\Gamma_\varepsilon} \bar{g}_i \bar{u}_i^\varepsilon dA_z \\ & + \int_{B^\varepsilon} \frac{1}{2} \left( \varepsilon^{-1} K_{ki}^{33} \hat{u}_{k,3}^\varepsilon \hat{u}_{i,3}^\varepsilon + 2K_{ki}^{\alpha 3} \hat{u}_{i,3}^\varepsilon \hat{u}_{k,\alpha}^\varepsilon + \varepsilon K_{ki}^{\alpha\beta 3} \hat{u}_{i,\alpha}^\varepsilon \hat{u}_{k,\beta}^\varepsilon \right) dV_z, \end{aligned}$$

where the matrices  $K^{jl}$  (with  $j, l = 1, 2, 3$ ) are defined by the relation:

$$K_{ki}^{jl} := C_{ijkl}.$$

In view of the symmetry properties of the elasticity tensor  $C_{ijkl}$  it results that  $(K^{jl})^T = K^{lj}$ ,  $j, l = 1, 2, 3$ .

Next, the existence of asymptotic expansions of the displacement fields with respect to the small parameter  $\varepsilon$  is assumed:

$$\begin{aligned} \mathbf{u}^\varepsilon &= \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \varepsilon^2 \mathbf{u}^2 + o(\varepsilon^2), \\ \hat{\mathbf{u}}^\varepsilon &= \hat{\mathbf{u}}^0 + \varepsilon \hat{\mathbf{u}}^1 + \varepsilon^2 \hat{\mathbf{u}}^2 + o(\varepsilon^2), \\ \bar{\mathbf{u}}^\varepsilon &= \bar{\mathbf{u}}^0 + \varepsilon \bar{\mathbf{u}}^1 + \varepsilon^2 \bar{\mathbf{u}}^2 + o(\varepsilon^2). \end{aligned}$$

Substituting these expansions into the rescaled energy, one obtains

$$E^\varepsilon(\hat{\mathbf{u}}^\varepsilon, \bar{\mathbf{u}}^\varepsilon) = E^0(\hat{\mathbf{u}}^0, \bar{\mathbf{u}}^0) + o(1),$$

with

$$\begin{aligned} E^0(\hat{\mathbf{u}}^0, \bar{\mathbf{u}}^0) = & \int_{S_1} \left( \frac{1}{2} C_{ijkl}^1 \bar{u}_{i,j}^0 \bar{u}_{k,l}^0 - \bar{f}_i \bar{u}_{i,j}^0 \right) dV_z + \int_{S_2} \left( \frac{1}{2} C_{ijkl}^2 \bar{u}_{i,j}^0 \bar{u}_{k,l}^0 - \bar{f}_i \bar{u}_{i,j}^0 \right) dV_z - \int_{\Gamma_\varepsilon} \bar{g}_i \bar{u}_i^0 dA_z \\ & + \int_{B^\varepsilon} \frac{1}{2} \left( \varepsilon^{-1} K_{ki}^{33} \hat{u}_{k,3}^0 \hat{u}_{i,3}^0 \right) dV_z. \end{aligned}$$

The energy  $E^0(\hat{\mathbf{u}}^0, \bar{\mathbf{u}}^0)$  is now minimized with respect to couples  $(\hat{\mathbf{u}}^0, \bar{\mathbf{u}}^0)$  in the set of the rescaled admissible displacements:

$$V^0 = \left\{ (\hat{\mathbf{u}}, \bar{\mathbf{u}}) \in H(S_{1,2}; \mathbb{R}^3) \times (B; \mathbb{R}^3) : \bar{\mathbf{u}} = 0 \text{ on } \Gamma_u, \bar{\mathbf{u}} = \hat{\mathbf{u}} \text{ on } \Gamma_{1,2} \right\}.$$

Using standard arguments, the following equilibrium equations are obtained:

$$\begin{aligned} \left( C_{ijkl}^{1,2} \bar{u}_{k,l}^0 \right)_{,j} + \bar{f}_i &= 0 \text{ in } S_{1,2}, \\ \left( C_{ijkl}^{1,2} \bar{u}_{k,l}^0 \right) n_j &= \bar{g}_i \text{ on } \Gamma_g, \end{aligned}$$

$$\begin{aligned} \left( C_{ijkl}^{1,2} \bar{u}_{k,l}^0 \right) n_j &= 0 \text{ on } \partial S_{1,2} \setminus (\Gamma_g \cup \Gamma_u \cup \Gamma_{1,2}), \\ \left( K_{jk}^{33} \hat{u}_{k,l}^0 \right)_{,3} &= 0 \text{ in } B, \\ \left( C_{ijkl}^{1,2} \bar{u}_{k,l}^0 \right) &= K_{ir}^{33} \hat{u}_{r,3}^0 \text{ on } \Gamma_{1,2}, \end{aligned}$$

where  $n$  is taken to denote the outward normal. The first three equations are the equilibrium equations of the adherents at the order zero, with the suitable boundary conditions.

The remaining two equations imply the continuity of the traction vector, say  $\bar{\sigma}^0 i_3 := (C^{1,2} \bar{u}^0) i_3$  at the order zero across the rescaled interphase, i.e.

$$\|\bar{\sigma}^0\| i_3 = 0,$$

where  $\|\psi\| = \psi^{(1)} - \psi^{(2)}$  indicates the jump in the quantity  $\psi$  across the rescaled interphase  $B$ . After integration of equation  $(K_{jk}^{33} \hat{u}_{k,3}^0)_{,3} = 0$  with respect to  $z_3$ , using the natural condition  $C_{i3kl}^{1,2} \bar{u}_{k,l}^0 = K_{ir}^{33} \hat{u}_{r,3}^0$  on  $\Gamma_{1,2}$  and the continuity of the displacement vector fields at the surfaces  $\Gamma_{1,2}$ , one obtains

$$\|\bar{\sigma}^0\| i_3 = K^{33} \|\bar{u}^0\|,$$

which represents the classical law for a soft interface. In [6] it is shown that this law can be rephrased in the composite's limit configuration obtained as  $\varepsilon \rightarrow 0$ . In this case one obtains:

$$\sigma^0 i_3 = K^{33} \llbracket u^0 \rrbracket,$$

where  $\llbracket \psi \rrbracket$  indicates the jump of  $\psi$  across the surface  $\Gamma$  defined on the adherents' limit configuration as  $\varepsilon \rightarrow 0$  and  $\sigma^0 i_3$  is the traction vector field defined on the same limit configuration.

### Spring-like interface law for a plane anisotropic adhesive

The elastic coefficient of the adhesive's material enter the interface law through the matrix  $K^{33}$ .

For the general case of anisotropic (triclinic) material, the matrix  $K^{33}$  has components

$$\mathbf{K}^{33} = \begin{pmatrix} C_{1313} & C_{1323} & C_{1333} \\ C_{1323} & C_{2323} & C_{2333} \\ C_{1333} & C_{2333} & C_{3333} \end{pmatrix},$$

in the original coordinate system  $(O, x_1, x_2, x_3)$ . If the adhesive is monoclinic with symmetry plane orthogonal to  $i_2$ , then  $\mathbf{K}^{33}$  has four non vanishing components

$$\mathbf{K}_{\text{monoclinic}, i_2}^{33} = \begin{pmatrix} C_{1313} & 0 & C_{1333} \\ 0 & C_{2323} & 0 \\ C_{1333} & 0 & C_{3333} \end{pmatrix}.$$

If the adhesive is monoclinic with symmetry plane orthogonal to  $i_3$ , then  $\mathbf{K}^{33}$  is diagonal:

$$\mathbf{K}_{\text{monoclinic}, i_3}^{33} = \begin{pmatrix} C_{1313} & 0 & 0 \\ 0 & C_{2323} & 0 \\ 0 & 0 & C_{3333} \end{pmatrix}.$$

The same holds if the adhesive is orthotropic and the coordinated planes are the symmetry planes of the material.

For an orthotropic adhesive, the non-vanishing coefficient in  $\mathbf{K}_{\text{monoclinic}, i_3}^{33}$  can be expressed in terms of the engineering constants as follows [8]:

$$\begin{aligned} C_{1313} &= G_{31}, \\ C_{2323} &= G_{23}, \\ C_{3333} &= \frac{1 - \nu_{12}\nu_{21}}{E_1 E_2 \Delta}, \end{aligned}$$

where  $G_{31}$ ,  $G_{23}$  are the shear moduli in the (3,1) and (2,3) planes, respectively;  $E_i$ ,  $i = 1, 2, 3$  is the Young's modulus in the  $i$ -direction;  $\nu_{ij}$ ,  $i, j = 1, 2, 3$  are the Poisson's ration, and they satisfy the three reciprocal relations

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}, \quad i, j = 1, 2, 3, \quad i \neq j,$$

and

$$\Delta = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}}{E_1 E_2 E_3}.$$

If the material is transversely isotropic and  $i_3$  is the direction of the axis of transverse symmetry, then  $\mathbf{K}^{33}$  is diagonal and  $C_{1313} = C_{2323}$ . Finally, for an isotropic material, one has

$$\mathbf{K}_{\text{isotropic}}^{33} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 2\mu + \lambda \end{pmatrix},$$

with  $\mu, \lambda$  the Lamé constants. In terms of the engineering constants,  $E, \nu, G$  one has

$$\mu = G, \quad 2\mu + \lambda = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}.$$

### Contact law modeling a curved anisotropic thin adhesive

The spring-like interface laws obtained in the previous sections are local and thus they can be locally applied to curved thin adhesives. In particular, it is introduced a local reference system with axes parallel to the tangential  $t$  and the normal  $n$  direction of the adhesive midline let  $s$  denote the third direction, orthogonal to the adhesive's plane. Based on the previous analysis, one could propose a contact law of the form

$$\begin{pmatrix} \mathbf{T}_n \\ \mathbf{T}_t \\ \mathbf{T}_s \end{pmatrix} = \mathbf{K} \begin{pmatrix} \llbracket \mathbf{u}_n \rrbracket \\ \llbracket \mathbf{u}_t \rrbracket \\ \llbracket \mathbf{u}_s \rrbracket \end{pmatrix},$$

with  $\mathbf{u}$  the displacement,  $\mathbf{T} = \sigma^0 i_3$  the traction vector and  $\mathbf{K} \equiv \mathbf{K}^{33}$  the spring stiffness, given as shown in the previous section. In particular, the case of  $\mathbf{K}$  with constants coefficients would correspond to an adhesive material whose symmetry properties are independent of the directions, as represented in Figure 1.a.

However, one could also imagine an adhesive whose properties are “globally” constant, i.e. they are fixed with respect to a global frame, a case schematized in Figure 1.b. In this second case, the matrix  $\mathbf{K}$  entering the above interface law, which is written in the local frame, can be obtained by applying a transformation from the global reference frame



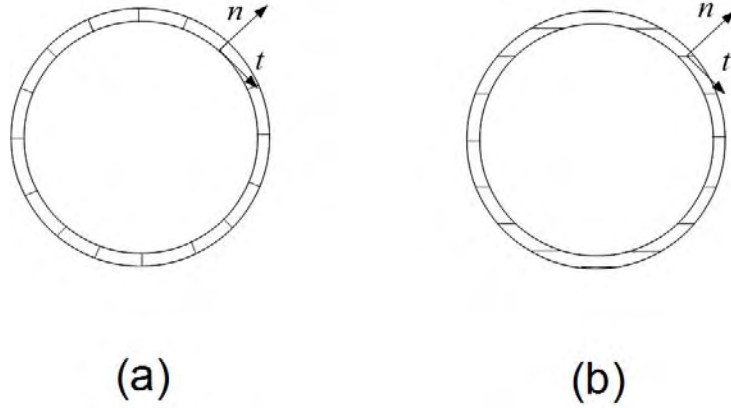


Fig. 1. (a) Adhesive properties independent of the directions  $n$  and  $t$ . (b) Adhesive properties “globally” constant.

### General considerations for heterogeneous structures with imperfect contact

A two phase uniaxial reinforced material is considered here in which fibers and matrix have transversely isotropic elastic properties; the axis of transverse symmetry coincides with the fiber direction, which is taken as the  $Ox_3$  axis. The fibers cross-section is circular. Moreover, the fibers are periodically distributed without overlapping in directions parallel to  $Ox_1$  and  $Ox_2$  axis,  $w_1$  and  $w_2$  are two complex numbers, which define the parallelogram periodic cell of the two phase composite. Therefore, the composite  $\Omega$  consists of a parallelogram array of identical circular cylinders embedded in a homogeneous medium (Fig. 2).

The overall properties of the above periodic medium are sought using the well-known AHM [9, 10], and the following considerations are assumed. Two variables are introduced, i.e.  $\mathbf{x}$  and  $\mathbf{y} = \mathbf{x}/\alpha$ . They are referred as the slow or macroscopic and fast or microscopic variables, respectively, where  $\alpha = 1/L$  is a small dimensionless parameter,  $L$  is a linear dimension of the body and  $l$  is the diameter of the unit cell.

Then, it follows that in terms of the fast variable  $\mathbf{y}$ , the appropriate periodic unit cell  $S$  is taken as a regular parallelogram cell in the  $y_1y_2$ -plane so that  $S = S_1 \cup S_2$  with  $S_1 \cap S_2 = \emptyset$ , where the domain  $S_2$  is occupied by the matrix and its complement  $S_1$  (fiber) is considered by a circle of radius  $R$  and center at the origin  $O$  (Fig. 2). The common interface between the fiber and the matrix is denoted by  $\Gamma$ .

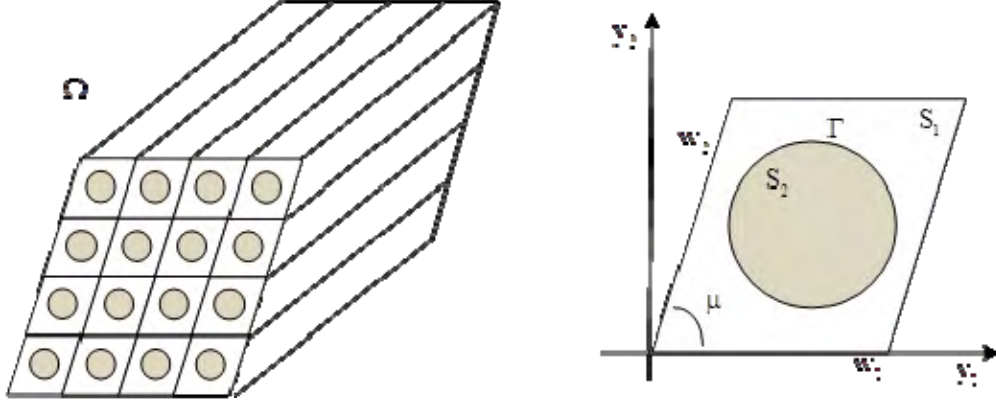


Fig. 2. Fibrous composite material and regular parallelogram cell with circular cell

Using the Einstein summation convention in which repeated subscripts are summed over the range of  $i, j, k, l = 1, 2, 3$ , the constitutive equation is

$$\sigma_{ij}(\mathbf{x}, \mathbf{y}) = C_{ijkl}(\mathbf{y}) \varepsilon_{k,l}(\mathbf{x}, \mathbf{y}),$$

where  $\sigma_{ij}$ ,  $\varepsilon_{ij}$  are the stress and strain tensors respectively, related to the small parameter  $\alpha$ . Assuming zero body forces the elastic equilibrium equation is

$$\sigma_{ij,j} = 0 \quad \text{in } \Omega,$$

where the subscript comma denotes partial differentiation. The gradient equations, which are the strain-displacement equations

$$\varepsilon_{kl} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right),$$

where  $u_i = u_i(\mathbf{x}, \mathbf{y})$  are the components of the mechanic displacement related to the small parameter  $\alpha$ . Replacing and into a system of partial differential equations with rapidly oscillating coefficients can be obtained

$$\left( C_{ijkl}(\mathbf{y}) u_{k,l}(\mathbf{x}, \mathbf{y}) \right)_{,j} = 0 \quad \text{in } \Omega.$$

Equation represents a system of equations for finding  $u_i^\alpha$ . For a complete solution, it is necessary to assign suitable boundary and interface conditions, for instance

$$u_i = u_i^0 \quad \text{on } \partial\Omega_u; \quad \sigma_{ij} n_j = S_i^0 \quad \text{on } \partial\Omega_T,$$

where  $\partial\Omega \setminus \partial\Omega_u = \partial\Omega_T$  and  $u_i^0$ ,  $S_i^0$  are the prescribed displacement, force on the boundary of the composite.

An adhesive with monoclinic symmetry and constant elastic moduli is considered in the homogenization analysis via a layer of mechanical springs of zero thickness. The spring constants  $\mathbf{K} \equiv \mathbf{K}^{33}$  are the measures for the magnitude of the associated continuities on  $\Gamma$ , where  $\tilde{K}_n$ ,  $\tilde{K}_t$  and  $\tilde{K}_s$  are the diagonal spring constant parameters, which have dimension of stress divided by length, as seen in previous sections. These constants are called the interface parameters and they are written with another notation for the monoclinic case in the diagonal matrix  $\mathbf{K}^{33}$  given in page. 7. It is seen that infinite values of the parameters imply vanishing of displacement jumps and therefore perfect interface conditions. At the other extremity, zero values of the parameters imply vanishing of interface tractions and therefore disbond. Any finite position values of the interface parameters define an imperfect interface. This may be due to the presence of an interphase but also due to interface bond deterioration caused for different reasons, such as fatigue damage or environmental and chemical effects. Within this approach, the composite is modeled as a two-phase material with imperfect interface conditions.

The vanishing value of  $\tilde{K}_n$ ,  $\tilde{K}_t$  and  $\tilde{K}_s$  corresponds to pure debonding (normal perfect debonding), in-plane pure sliding, and out-of-plane pure sliding, respectively. The status of the mechanical bonding is completely determined by appropriate values of these constants. For large enough values of the constants, the perfect bonding interface is achieved.

Using the vector notation and defining the mechanical displacement  $\mathbf{u}$ , the traction vector  $\mathbf{T}$  and the spring stiffness matrix  $\mathbf{K}$  in the following manner,

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_n \\ \mathbf{u}_t \\ \mathbf{u}_s \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \mathbf{T}_n \\ \mathbf{T}_t \\ \mathbf{T}_s \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} \tilde{K}_n & 0 & 0 \\ 0 & \tilde{K}_t & 0 \\ 0 & 0 & \tilde{K}_s \end{pmatrix},$$

the mechanical imperfect condition ([10], [11]), in general, may be expressed as:

$$\mathbf{T}^{(1)} + \mathbf{T}^{(2)} = 0, \quad \mathbf{T}^{(\gamma)} = (-1)^{\gamma+1} \mathbf{K} \llbracket \mathbf{u} \rrbracket \quad \text{on } \Gamma.$$

In the relation, the symbol  $\llbracket \mathbf{g} \rrbracket = \mathbf{g}^{(1)} - \mathbf{g}^{(2)}$  indicates the jump in the quantity  $\mathbf{g}$  at the common interface denoted by  $\Gamma$  between the fiber and the matrix. The superscripts in brackets  $(\gamma)$ ,  $\gamma = 1, 2$  denote the matrix (1) and fiber (2), respectively. The magnitudes  $u_n$ ,  $u_t$ ,  $u_s$  are the tangential and normal components of the mechanical displacement vector;

$T_n$ ,  $T_t$ ,  $T_s$  are the tangential and normal components of the traction vector  $\mathbf{T}$  ( $T_i = u_{ij}n_j$ ) and  $\mathbf{n}$  is the outward unit normal on  $\Gamma$ .

In order to study the imperfect contact conditions, the relations between the mechanic displacement and traction vectors are related to Cartesian mechanic displacement ( $u_i$ ) and the traction ( $T_i$ ) vectors respectively (Fig. 3) by the following expressions,

$$\begin{pmatrix} u_n \\ u_t \\ u_s \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \begin{pmatrix} T_n \\ T_t \\ T_s \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}.$$

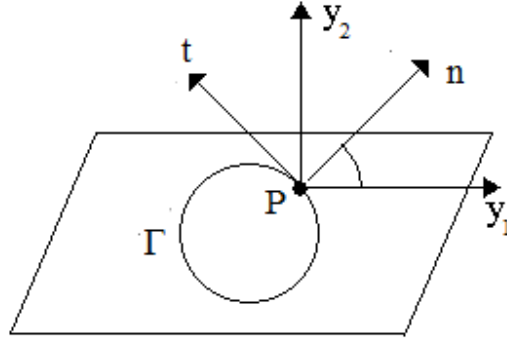


Fig. 3. The coordinate system in each point  $P \in \Gamma$  of the interface.

### Two scale asymptotic homogenization method

By means of the asymptotic homogenization method, the original constitutive relations with rapidly oscillating material coefficients - are transformed into equivalent system

$$C_{ijkl}^* u_{l,kj}^{(0)} = 0 \quad \text{on } \Omega,$$

with the boundary conditions

$$u_i^{(0)} = u_i^0 \quad \text{on } \partial\Omega_u; \quad \sigma_{ij}^{(0)} n_j = S_i^0 \quad \text{on } \partial\Omega_T,$$

with constant coefficients  $C^*$ , which represent the elastic properties of an equivalent homogeneous medium. They are called as effective coefficients of the composite  $\Omega$ .

In order to obtain average coefficients  $C^*$  the periodic solutions of six  $_{pq}L$  local problems on  $S$  in terms of the fast variable  $\mathbf{y}$ , where  $p, q = 1, 2, 3$  are found. Each local problem uncouples into sets of equations, i.e. plane-strain and antiplane-strain systems.

The  ${}_{pq}L$  problem consists to find the displacements  ${}_{pq}N^{(\gamma)}(\mathbf{y})$  in  $S_\gamma$ ,  $\gamma=1,2$  (double periodic functions with periods  $w_1, w_2$ ) as solution of the following system of partial differential equations,

$${}_{pq}\sigma_{i\delta,\delta}^{(\gamma)} = 0 \quad \text{in } S_\gamma, \quad (1)$$

where,

$${}_{pq}\sigma_{i\delta,\delta}^{(\gamma)} = C_{i\delta k\lambda}^{(\gamma)} {}_{pq}N_{k,\lambda}^{(\gamma)}, \quad (2)$$

the comma notation denotes a partial derivative relative to the  $y_\delta$  component, i.e.,  $U_{,\delta} \equiv \partial U / \partial y_\delta$  the summation convention is also understood for Greek indices, which run from 1 to 2; no summation is carried out over upper case indices, whether Latin or Greek.

Thus, the eq. on  $\Gamma$  for the  ${}_{pq}L$  problem can be expressed in the following indicial form,

$${}_{pq}T^{(1)} + {}_{pq}T^{(2)} = 0, \quad (3)$$

$${}_{pq}T_n^{(\gamma)} = (-1)^{\gamma+1} \tilde{K}_n \llbracket {}_{pq}N_n \rrbracket, \quad {}_{pq}T_t^{(\gamma)} = (-1)^{\gamma+1} \tilde{K}_t \llbracket {}_{pq}N_t \rrbracket, \quad {}_{pq}T_s^{(\gamma)} = (-1)^{\gamma+1} \tilde{K}_s \llbracket {}_{pq}N_s \rrbracket, \quad (4)$$

where  ${}_{pq}N_n$ ,  ${}_{pq}N_t$ ,  ${}_{pq}N_s$  and  ${}_{pq}T_n$ ,  ${}_{pq}T_t$ ,  ${}_{pq}T_s$  have the same meaning as , but adequate to  ${}_{pq}L$  problems. To assure the only one solution of the  ${}_{pq}L$  problems, the functions also

satisfy that  $\langle {}_{pq}N_k \rangle = 0$ . In the calculations the following relations are used:  $\tilde{K}_n = \frac{K_n C_{1212}^{(1)}}{R}$ ,

$\tilde{K}_t = \frac{K_t C_{1212}^{(1)}}{R}$  and  $\tilde{K}_s = \frac{K_s C_{1212}^{(1)}}{R}$  where  $\tilde{K}_n$ ,  $\tilde{K}_t$  and  $\tilde{K}_s$  are dimensionless imperfect

parameters. The symmetry between the indices  $p$  and  $q$  shows right away that at most six problems needs to be considered. Once the local problems are solved, the homogenized moduli  $C_{ijpq}^*$  may be determined by using the following formulae

$$C_{ijpq}^* = \langle C_{ijpq} + C_{ijkl} {}_{pq}N_{k,l} \rangle.$$

The potential method of complex variables  $z = y_1 + iy_2$ ,  $(y_1, y_2) \in S$  and the properties of

doubly periodic Weierstrass  $\wp(z) = \frac{1}{z^2} + \sum' \left\{ \frac{1}{(z - P_{st})^2} - \frac{1}{P_{st}^2} \right\}$ ,  $P_{st} = s + it$ ,

$s, t = 0, \pm 1, \pm 2, \dots$  and related functions (Z- function  $\zeta(z) = -\wp'(z)$  and Natanzon's function

$Q(z)$ ) are used for the solution of the local problems (1)-(4). Hence, the non-zero solution  ${}_{pq}N_k^{(\alpha)}$  in  $S_\gamma$  of the problem defined by equations (1)-(4) must be found among doubly periodic functions of half periods  $w_1, w_2$  (Fig. 2). Each local problem (1)-(4) uncouples into sets of equations. An in-plane strain system for  ${}_{pq}N_\lambda^{(\gamma)}$ ,  $\lambda = 1, 2$  and an out-of-plane strain Laplace's equation for  ${}_{pq}N_3^{(\gamma)}$  has to be solved. Then the solution of the in-plane (out-of-plane) strain problems involves the determination of the in-plane (out-of-plane) displacements, strains and stresses over each phase  $S_\gamma$  of the composite. Due to the non-vanishing components of the elastic tensors  $C_{ijpq}^{(\gamma)}$ , the only non-homogeneous problems, that have a non-zero solution, correspond to the four in-plane strain problems  ${}_{\beta\beta}L$ , ( $\beta = 1, 2, 3$ ) and  ${}_{12}L$ , and the two out-of-plane strain ones  ${}_{23}L$  and  ${}_{13}L$ . Therefore, the solutions of both (in-plane and out-of-plane) local problems ([11], [12]) (six  ${}_{pq}L$  problems need to be considered due to the symmetry between  $p$  and  $q$ ) lead to obtain the average coefficients of the composite given in Fig. 2. Imperfect contact for out-of-plane problem is well described in [3]-[4] where the imperfect parameter  $\tilde{K}_s = \frac{K_s C_{1212}^{(1)}}{R}$  is considered. Therefore, from now on, only the solution of the plane local problems  ${}_{\beta\beta}L$ , ( $\beta = 1, 2, 3$ ) and  ${}_{12}L$  are studied where the imperfect parameters  $\tilde{K}_n = \frac{K_n C_{1212}^{(1)}}{R}$ ,  $\tilde{K}_t = \frac{K_t C_{1212}^{(1)}}{R}$  are involved..

### **Solution of plane problems ${}_{\beta\beta}L$ and ${}_{12}L$ . Effective coefficients**

The aim of this work is to obtain the coefficients  $C_{1111}^*$ ,  $C_{1122}^*$ ,  $C_{1133}^*$ ,  $C_{2211}^*$ ,  $C_{2222}^*$ ,  $C_{2233}^*$ ,  $C_{3311}^*$ ,  $C_{3322}^*$ ,  $C_{3333}^*$ ,  $C_{1211}^*$ ,  $C_{1222}^*$  and  $C_{1233}^*$  derived of the  ${}_{\beta\beta}L$  local in-plane problems and  $C_{1211}^*$ ,  $C_{1222}^*$ ,  $C_{1233}^*$  and  $C_{1212}^*$  from the  ${}_{12}L$  for composites with parallelogram arrays.

Therefore, the methods of a complex variable  $z$  in terms of two harmonic functions  $\varphi_\gamma(z)$  and  $\psi_\gamma(z)$ , the Kolosov–Muskhelishvili complex potentials, are applicable. The potentials are related to the displacement and stress components by means of the formulae

$$\begin{cases} 2C_{1212}^{(\gamma)} [N_1^{(\gamma)}(z) + iN_2^{(\gamma)}(z)] = \chi^{(\gamma)}\varphi_\gamma(z) - z\bar{\varphi}'_\gamma(z) - \bar{\psi}_\gamma(z), \\ \sigma_{11}^{(\gamma)} + \sigma_{22}^{(\gamma)} = 2[\varphi'_\gamma(z) + \bar{\varphi}'_\gamma(z)], \quad \sigma_{22}^{(\gamma)} - \sigma_{11}^{(\gamma)} + 2i\sigma_{12}^{(\gamma)} = 2[\bar{z}\varphi''_\gamma(z) + \psi'_\gamma(z)], \end{cases}$$

where  $\chi^{(\gamma)} = 3 - 4\nu^{(\gamma)}$ ,  $\nu^{(\gamma)} = \frac{C_{1122}^{(\gamma)}}{C_{1111}^{(\gamma)} + C_{1122}^{(\gamma)}}$ .

A relevant representation of the complex potentials  $\varphi_\gamma(z)$  and  $\psi_\gamma(z)$  of periods  $w_1$  and  $w_2$  is

$$\begin{cases} \varphi_1(z) = \frac{z}{R}a_0 + \sum_{p=1}^{\infty} * \left(\frac{R}{z}\right)^p a_p + \sum_{k=1}^{\infty} * \sum_{p=1}^{\infty} * \left(\frac{z}{R}\right)^p \eta_{kp} a_k, & \varphi_2(z) = \sum_{k=1}^{\infty} * \left(\frac{z}{R}\right)^k c_k, \\ \psi_1(z) = \frac{z}{R}b_0 + \sum_{p=1}^{\infty} * \left(\frac{R}{z}\right)^p b_p + \sum_{k=1}^{\infty} * \sum_{p=1}^{\infty} * \left(\frac{z}{R}\right)^p \eta_{kp} b_k, & \psi_2(z) = \sum_{k=1}^{\infty} * \left(\frac{z}{R}\right)^k d_k. \end{cases}$$

After some algebraic manipulations, a system of infinite equations is obtained for  ${}_{\beta\beta} \mathbf{a}_1$  and  ${}_{12} \mathbf{a}_1$ , that is

$$\begin{cases} {}_{\beta\beta} \mathbf{a}_p + \hat{H}_{1p \beta\beta} \mathbf{a}_1 + \hat{H}_{2p \beta\beta} \bar{\mathbf{a}}_1 + \sum_{k=1}^{\infty} * \hat{M}_{kp \beta\beta} \bar{\mathbf{a}}_k + \sum_{k=1}^{\infty} * \hat{W}_{kp \beta\beta} \mathbf{a}_k = \hat{H}_{3p}(\beta), \quad \text{for } {}_{\beta\beta} L, \\ {}_{12} \mathbf{a}_p + \hat{H}_{1p 12} \mathbf{a}_1 + \hat{H}_{2p 12} \bar{\mathbf{a}}_1 + \sum_{k=1}^{\infty} * \hat{M}_{kp 12} \bar{\mathbf{a}}_k + \sum_{k=1}^{\infty} * \hat{W}_{kp 12} \mathbf{a}_k = \hat{H}_{4p}, \quad \text{for } {}_{12} L, \end{cases}$$

where

$$\frac{{}_{\beta\beta} \mathbf{a}}{\gamma_2(\beta) \mathbf{R}} = {}_{\beta\beta} \mathbf{a}, \quad \frac{{}_{12} \mathbf{a}}{\mathbf{R} \llbracket m \rrbracket} = {}_{12} \mathbf{a}, \quad \chi^* = \frac{C_{1212}^{(2)}}{C_{1212}^{(1)}}, \quad A_1 = \frac{\bar{w}_1 \delta_2 - \bar{w}_2 \delta_1}{\bar{w}_1 \omega_2 - \omega_1 \bar{w}_2}, \quad \bar{A}_2 = \frac{\bar{w}_1 \bar{\delta}_2 - \bar{w}_2 \bar{\delta}_1}{\bar{w}_1 w_2 - w_1 \bar{w}_2},$$

$$\bar{A}_3 = \frac{w_1 \gamma_2 - w_2 \gamma_1}{\bar{w}_1 w_2 - w_1 \bar{w}_2}, \quad \bar{A}_4 = \frac{w_1 \gamma_2 - w_2 \gamma_1}{\bar{w}_1 w_2 - w_1 \bar{w}_2}, \quad \hat{H}_{1p} = \hat{B} A_3 \chi^{(1)} \mathbf{R}^2 \delta_{1p} - A_1 (\bar{\eta}_{1p} - \bar{A}_1 \mathbf{R}^2 \delta_{1p}) \hat{C} \mathbf{R}^2,$$

$$\hat{H}_{2p} = \bar{A}_4 \hat{B} \mathbf{R}^2 \delta_{1p} - (\bar{\eta}_{1p} - \bar{A}_1 \mathbf{R}^2 \delta_{1p}) \bar{A}_1 \hat{C} \mathbf{R}^2, \quad \hat{W}_{kp} = (\bar{\eta}_{1p} - \bar{A}_1 \mathbf{R}^2 \delta_{1p}) \eta_{k1} \hat{C} - \hat{B} C_k^0 r_{kp},$$

$$\hat{M}_{kp} = (\bar{\eta}_{1p} - \bar{A}_1 \mathbf{R}^2 \delta_{1p}) \bar{\eta}_{k1} \hat{C} + \hat{B} (G_{kp} - D_p^0 \bar{\eta}_{k+2p}) + \frac{K_n B_p^0}{A_p^0} \bar{\eta}_{k(p+2)},$$

$$\hat{H}_{3p}(\beta) = \frac{2\chi^* \gamma_3(\beta) - (K_n - 2\chi^* p) \gamma_1(\beta)}{A_p^0 \gamma_2(\beta)} R \delta_{1p} + (\bar{\eta}_{1p} - \bar{A}_1 R^2 \delta_{1p}) \hat{P},$$

$$\hat{H}_{4p} = - \left( 1 - \frac{2\chi^*}{K_n} \right) (A_p^0)^{-1} \delta_{1p}, \quad \hat{B} = \left( 1 - \chi^* - \frac{2\chi^* p}{K_n} \right) (A_p^0)^{-1},$$

$$\hat{C} = \hat{B} \frac{1 + \chi^* \chi^{(1)} - \chi^* - \chi^{(2)} + 4\chi^*/K_n}{2\hat{\alpha}_0}, \quad \hat{P}(\beta) = \frac{\hat{B}(\chi^{(2)} - 1)}{2\hat{\alpha}_0} - \frac{2\hat{B}\chi^*}{K_n \hat{\alpha}_0} \left( 1 + \frac{\gamma_4(\beta)}{\gamma_2(\beta)} \right),$$

$$G_{kp} = (p+2) \bar{\eta}_{k(p+2)} + k \eta_{k+2p} + k R^{p+k} C_{p+k}^p \bar{T}_{p+k}, \quad r_{kp} = \sum_{i=1}^{\infty} \eta_{k(i+2)} \bar{\eta}_{i+2p},$$

$$D_k^0 = \frac{K_n K_t \chi^* (\chi^{(1)} + 1) D_k}{E_k},$$

$$A_p^0 = \frac{K_n K_t \chi^* (\chi^{(1)} + 1) [K_n (\chi^{(2)} + \chi^*) - 2\chi^* (p+2)] D_p}{E_p} + K_n (\chi^* \chi^{(1)} + 1) - 2\chi^* p,$$

$$B_p^0 = \chi^* (\chi^{(1)} + 1) \left[ \frac{K_n K_t [K_n (\chi^{(2)} + \chi^*) - 2\chi^* (p+2)] B_p}{E_p} + K_n \right],$$

$$C_k^0 = \frac{K_n K_t \chi^* (\chi^{(1)} + 1) B_k}{E_k} + 1,$$

$$\tilde{A}_p = K_n K_t [p+2 - \chi^* (p+1) - \chi^*] + [(K_n - K_t)p + (K_n + K_t)] \chi^* (p+2),$$

$$B_p = K_t K_n (1 - \chi^*) + (K_n - K_t) \chi^* p,$$

$$C_p = K_n K_t (\chi^{(2)} + \chi^*) + [(K_n + K_t)p + K_n - K_t] (p+2) \chi^*, \quad D_p = (K_n + K_t) \chi^* p,$$

$$E_p = \begin{vmatrix} \tilde{A}_p & B_p \\ C_p & D_p \end{vmatrix}, \quad \beta_0 = (\chi^{(2)} + 1) \left[ \frac{\text{Re}(A_3) R^2}{\chi^{(1)} - 1} + \frac{1}{2} \right] - i \chi^* \text{Im}(A_3) R^2,$$

$$\hat{\alpha}_0 = \chi^* [1 - \text{Re}(A_3) R^2] + \frac{1}{\hat{F}} \left[ \frac{\text{Re}(A_3) R^2}{\chi^{(1)} - 1} + \frac{1}{2} \right], \quad \hat{F} = \frac{1}{\chi^{(2)} - 1 - 4\chi^*/K_n},$$

$$2\gamma_1(\beta) = C_{22\beta\beta}^{(1)} - C_{22\beta\beta}^{(2)} + C_{11\beta\beta}^{(2)} - C_{11\beta\beta}^{(1)}, \quad 2\gamma_2(\beta) = C_{11\beta\beta}^{(1)} - C_{11\beta\beta}^{(2)} + C_{22\beta\beta}^{(1)} - C_{22\beta\beta}^{(2)},$$

$$2\gamma_3(\beta) = C_{22\beta\beta}^{(2)} - C_{11\beta\beta}^{(2)} \quad \text{y} \quad 2\gamma_4(\beta) = C_{22\beta\beta}^{(2)} + C_{11\beta\beta}^{(2)}.$$

We can write from the expression,



$$\begin{cases} C_{11\beta\beta}^* + C_{22\beta\beta}^* = \langle C_{11\beta\beta} + C_{22\beta\beta} \rangle + \langle (C_{1111} + C_{1122})(\beta\beta N_{1,1} + \beta\beta N_{2,2}) \rangle, \\ C_{22\beta\beta}^* - C_{11\beta\beta}^* + 2iC_{12\beta\beta}^* = \langle C_{22\beta\beta} - C_{11\beta\beta} \rangle + \langle (C_{1122} - C_{1111})(\beta\beta N_{1,1} - \beta\beta N_{2,2}) \rangle + \\ \quad + 2i \langle C_{1212}(\beta\beta N_{2,1} + \beta\beta N_{1,2}) \rangle, \\ C_{33\beta\beta}^* = \langle C_{33\beta\beta} \rangle + \langle C_{1133} \beta\beta N_{1,1} + C_{2233} \beta\beta N_{2,2} \rangle. \end{cases}$$

$$\begin{cases} C_{1112}^* = \langle C_{1111\ 12} N_{1,1} + C_{1122\ 12} N_{2,2} \rangle, & C_{2212}^* = \langle C_{1122\ 12} N_{1,1} + C_{2222\ 12} N_{2,2} \rangle, \\ C_{3312}^* = \langle C_{1133\ 12} N_{1,1} + C_{2233\ 12} N_{2,2} \rangle, & C_{1212}^* = \langle C_{1212} \rangle + \langle C_{1212\ 12} N_{1,2} + C_{1212\ 12} N_{2,1} \rangle, \end{cases}$$

and the following effective coefficients can be calculated using the average operator, applying the Green theorem and considering  $\Upsilon(\beta) = \chi^{(2)}\Delta_1(\beta) - \bar{\Delta}_1(\beta)$ ,  $\beta = 1, 2, 3$ ,

$2k_\gamma = C_{1111}^{(\gamma)} + C_{1122}^{(\gamma)}$  and  $m_\gamma = C_{1212}^{(\gamma)}$  we obtain

$$C_{1111}^* = \langle C_{1111} \rangle - V_2 \llbracket \mathbf{k} \rrbracket \operatorname{Re} \left\{ \frac{\llbracket \mathbf{k} \rrbracket \Upsilon(1)}{m_1} + (\chi^{(1)} + 1)_{11} \mathbf{a}_1 - \frac{\llbracket \mathbf{m} \rrbracket}{\llbracket \mathbf{k} \rrbracket} \right\} + \frac{2V_2 \llbracket \mathbf{k} \rrbracket k_1}{K_n m_1} \left\{ 2\chi^* \operatorname{Re}[\Delta_1(1)] + \frac{k_2}{\llbracket \mathbf{k} \rrbracket} \right\},$$

$$C_{2211}^* = \langle C_{2211} \rangle - V_2 \llbracket \mathbf{k} \rrbracket \operatorname{Re} \left\{ \frac{\llbracket \mathbf{k} \rrbracket \Upsilon(1)}{m_1} - (\chi^{(1)} + 1)_{11} \mathbf{a}_1 + \frac{\llbracket \mathbf{m} \rrbracket}{\llbracket \mathbf{k} \rrbracket} \right\} + \frac{2V_2 \llbracket \mathbf{k} \rrbracket k_1}{K_n m_1} \left\{ 2\chi^* \operatorname{Re}[\Delta_1(1)] + \frac{k_2}{\llbracket \mathbf{k} \rrbracket} \right\},$$

$$C_{3311}^* = \langle C_{1133} \rangle - V_2 \llbracket C_{1133} \rrbracket \llbracket \mathbf{k} \rrbracket \operatorname{Re} \left[ \frac{\Upsilon(1)}{m_1} \right] + \frac{2V_2 \llbracket \mathbf{k} \rrbracket C_{1133}^{(1)}}{K_n m_1} \left\{ 2\chi^* \operatorname{Re}[\bar{\Delta}_1(1)] + \frac{k_2}{\llbracket \mathbf{k} \rrbracket} \right\},$$

$$C_{1211}^* = V_2 \llbracket \mathbf{k} \rrbracket \operatorname{Im} \left[ (\chi^{(1)} + 1)_{11} \mathbf{a}_1 \right],$$

$$C_{1122}^* = \langle C_{1122} \rangle - V_2 \llbracket \mathbf{k} \rrbracket \operatorname{Re} \left\{ \frac{\llbracket \mathbf{k} \rrbracket \Upsilon(2)}{m_1} + (\chi^{(1)} + 1)_{22} \mathbf{a}_1 + \frac{\llbracket \mathbf{m} \rrbracket}{\llbracket \mathbf{k} \rrbracket} \right\} + \frac{2V_2 \llbracket \mathbf{k} \rrbracket k_1}{K_n m_1} \left\{ 2\chi^* \operatorname{Re}[\Delta_1(2)] + \frac{k_2}{\llbracket \mathbf{k} \rrbracket} \right\},$$

$$C_{2222}^* = \langle C_{2222} \rangle - V_2 \llbracket \mathbf{k} \rrbracket \operatorname{Re} \left\{ \frac{\llbracket \mathbf{k} \rrbracket \Upsilon(2)}{m_1} - (\chi^{(1)} + 1)_{22} \mathbf{a}_1 - \frac{\llbracket \mathbf{m} \rrbracket}{\llbracket \mathbf{k} \rrbracket} \right\} + \frac{2V_2 \llbracket \mathbf{k} \rrbracket k_1}{K_n m_1} \left\{ 2\chi^* \operatorname{Re}[\Delta_1(2)] + \frac{k_2}{\llbracket \mathbf{k} \rrbracket} \right\},$$

$$C_{3322}^* = \langle C_{3322} \rangle - V_2 \llbracket C_{2233} \rrbracket \llbracket \mathbf{k} \rrbracket \operatorname{Re} \left[ \frac{\Upsilon(2)}{m_1} \right] + \frac{2V_2 \llbracket \mathbf{k} \rrbracket C_{1133}^{(1)}}{K_n m_1} \left\{ 2\chi^* \operatorname{Re}[\bar{\Delta}_1(2)] + \frac{k_2}{\llbracket \mathbf{k} \rrbracket} \right\},$$

$$C_{1222}^* = V_2 \llbracket \mathbf{k} \rrbracket (\chi^{(1)} + 1) \text{Im}({}_{22} \mathbf{a}_1),$$

$$C_{1133}^* = \langle C_{1133} \rangle - V_2 \llbracket C_{1133} \rrbracket \text{Re} \left\{ \frac{\llbracket \mathbf{k} \rrbracket \Upsilon(3)}{m_1} + (\chi^{(1)} + 1) {}_{33} \mathbf{a}_1 \right\} + \frac{2V_2 \llbracket C_{1133} \rrbracket k_1}{K_n m_1} \left\{ 2\chi^* \text{Re}[\Delta_1(3)] + \frac{C_{1133}^{(2)}}{\llbracket C_{1133} \rrbracket} \right\},$$

$$C_{2233}^* = \langle C_{2233} \rangle - V_2 \llbracket C_{2233} \rrbracket \text{Re} \left[ \frac{\llbracket \mathbf{k} \rrbracket \Upsilon(3)}{m_1} - (\chi^{(1)} + 1) {}_{33} \mathbf{a}_1 \right] + \frac{2V_2 \llbracket C_{2233} \rrbracket k_1}{K_n m_1} \left\{ 2\chi^* \text{Re}[\Delta_1(3)] + \frac{C_{2233}^{(2)}}{\llbracket C_{2233} \rrbracket} \right\},$$

$$C_{3333}^* = \langle C_{3333} \rangle - V_2 \llbracket C_{1133} \rrbracket^2 \text{Re} \left[ \frac{\Upsilon(3)}{m_1} \right] + \frac{2V_2 C_{1133}^{(1)}}{K_n m_1} \text{Re} \left[ 2\chi^* \bar{\Delta}_1(3) + \frac{C_{1133}^{(2)}}{\llbracket C_{1133} \rrbracket} \right],$$

$$C_{1233}^* = V_2 \llbracket C_{1133} \rrbracket (\chi^{(1)} + 1) \text{Im}({}_{33} \mathbf{a}_1),$$

$$C_{1112}^* = \llbracket \mathbf{m} \rrbracket V_2 \text{Re} \left[ (\chi^{(1)} + 1) {}_{12} \mathbf{a}_1 + \frac{\llbracket \mathbf{k} \rrbracket (\chi^{(2)} \Delta_2 - \bar{\Delta}_2)}{m_1} \right] + \frac{4k_2 \llbracket \mathbf{m} \rrbracket \chi^* V_2 \text{Re}(\Delta_2)}{K_n m_1},$$

$$C_{2212}^* = -\llbracket \mathbf{m} \rrbracket V_2 \text{Re} \left[ (\chi^{(1)} + 1) {}_{12} \mathbf{a}_1 - \frac{\llbracket \mathbf{k} \rrbracket (\chi^{(2)} \Delta_2 - \bar{\Delta}_2)}{m_1} \right] + \frac{4k_2 \llbracket \mathbf{m} \rrbracket \chi^* V_2 \text{Re}(\Delta_2)}{K_n m_1},$$

$$C_{3312}^* = \llbracket \mathbf{m} \rrbracket \llbracket \mathbf{I} \rrbracket V_2 \text{Re} \left( \frac{\chi^{(2)} \Delta_2 - \bar{\Delta}_2}{m_1} \right) + \frac{4 \llbracket \mathbf{m} \rrbracket \chi^* I_2 V_2 \text{Re}(\Delta_2)}{K_n m_1},$$

$$C_{1212}^* = C_{1212}^{(1)} - \llbracket \mathbf{m} \rrbracket V_2 \text{Im} \left[ (\chi^{(1)} + 1) {}_{12} \mathbf{a}_1 \right],$$

where

$$\Delta_1(\beta) = \frac{1}{2\chi^* (\chi^{(2)} + 1)} \left[ \left( \chi^{(2)} + 1 + \chi^* \chi^{(1)} + \chi^* + \frac{2\beta_0 \hat{C}}{\hat{B}} \right) \left( \sum_{k=1}^{\infty} \eta_{k1\beta\beta}^* \mathbf{a}_k - \mathbf{A}_1 \mathbf{R}^2_{\beta\beta} \mathbf{a}_1 \right) + \left( \chi^{(2)} + 1 - \chi^* \chi^{(1)} - \chi^* + \frac{2\beta_0 \hat{C}}{\hat{B}} \right) \left( \sum_{k=1}^{\infty} \bar{\eta}_{k1\beta\beta}^* \bar{\mathbf{a}}_k - \bar{\mathbf{A}}_1 \mathbf{R}^2_{\beta\beta} \bar{\mathbf{a}}_1 \right) + \left( \chi^{(2)} + 1 - 2\beta_0 \frac{\hat{P}}{\hat{B}} \right) \right],$$

$$\Delta_2 = \frac{1}{2\chi^*(\chi^{(2)}+1)} \left\{ \begin{aligned} & \left( \chi^{(2)} + 1 + \chi^* \chi^{(1)} + \chi^* + \frac{2\beta_0 \hat{C}}{\hat{B}} \right) \left( -A_1 R^2_{12} \mathbf{a}_1 + \sum_{k=1}^{\infty} \eta_{k112}^* \mathbf{a}_k \right) + \\ & + \left( \chi^{(2)} + 1 - \chi^* \chi^{(1)} - \chi^* + \frac{2\beta_0 \hat{C}}{\hat{B}} \right) \left( -\bar{A}_1 R^2_{12} \bar{\mathbf{a}}_1 + \sum_{k=1}^{\infty} \bar{\eta}_{k112}^* \bar{\mathbf{a}}_k \right) \end{aligned} \right\}.$$

## Numerical results

As a validation of the present model in Fig. 4, a comparison between the present model (AHM) and the differential scheme (see details Guinovart-Diaz et al. 2013 [12]) is given. A particular two-phase composite with  $C_{66}^{(2)}/C_{66}^{(1)}=10$  under imperfect spring contact and parallelogram cell with  $75^\circ$  is analyzed. Both methods illustrate similar behavior for the normalized effective coefficients  $C_{66}^*$  and  $\kappa^*$  by the matrix property.

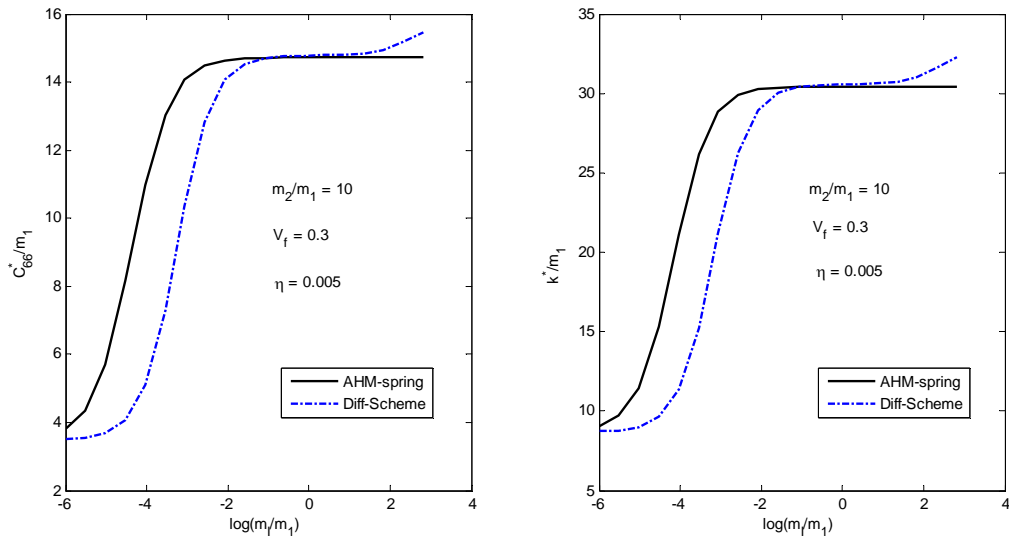


Fig. 4. Comparison between the AHM-spring model and differential scheme reported in [13].

The material parameters used in the calculations are given in Table 1.

	$C_{1111}$	$C_{1122}$	$C_{1133}$	$C_{1212}$
Matrix - Al	94.23	40.38	40.38	26.92
Fibra - SiC	483.68	99.07	99.07	192.31

Table 1. Material constants of the constituents used in the composite taken from Otero et al. 2010 [14].

In Tables 2-5 we consider the effective coefficients of the composite. Table 2 shows the behavior of the composite with perfect contact ( $K_n = 10^6$  and  $K_t = 10^6$ ), different volume fraction and different parallelogram cell. The effect of angles of the cell in the value of the effective coefficients are observed except for the effective coefficient  $C_{3333}^*$  because it is not affected for the in-plane imperfect parameter  $K_n$  and  $K_t$ .

Tables 3-5 show the performance of the composite for different values of in-plane imperfect parameters for diverse volume fractions and different angle of the cell. Similar effect of the angle of the cell in the value of the effective coefficients is observed.

$K_n = 10^6$ , $K_t = 10^6$						
Vf		0.05	0.2	0.35	0.55	0.75
$C_{1111}^*$	$90^0$	1.055613	1.260705	1.543585	2.114488	3.126401
	$75^0$	1.05538	1.256187	1.527928	2.070327	3.027767
$C_{1122}^*$	$90^0$	1.040484	1.156486	1.259936	1.395765	1.785151
	$75^0$	1.041084	1.167927	1.299503	1.497467	1.895254
$C_{1133}^*$	$90^0$	1.022294	1.100150	1.200130	1.392355	1.752533
	$75^0$	1.022300	1.100267	1.200527	1.392180	1.736814
$C_{3333}^*$	$90^0$	1.200995	1.805214	2.411915	3.227463	4.061863
	$75^0$	1.200995	1.805213	2.411901	3.227181	4.059612
$C_{1212}^*$	$90^0$	1.062476	1.261943	1.504107	2.005303	3.335493
	$75^0$	1.063301	1.274627	1.543077	2.097136	3.344825

Table 2. Behavior of the composite with perfect contact ( $K_n = 10^6$  and  $K_t = 10^6$ ), different volume fraction and different parallelogram cell.

$K_n = 10$ , $K_t = 10^6$						
Vf		0.05	0.2	0.35	0.55	0.75
$C_{1111}^*$	$90^\circ$	1.026958	1.123008	1.244121	1.439684	1.662482
	$75^\circ$	1.026775	1.119934	1.235330	1.425826	1.666418
$C_{1122}^*$	$90^\circ$	0.986175	0.915688	0.797154	0.578442	0.322619
	$75^\circ$	0.986613	0.923068	0.818406	0.612930	0.316467
$C_{1133}^*$	$90^\circ$	0.991006	0.962850	0.932782	0.889236	0.840845
	$75^\circ$	0.991004	0.962812	0.932646	0.888842	0.840290
$C_{3333}^*$	$90^\circ$	1.194071	1.776466	2.359163	3.136634	3.914866
	$75^\circ$	1.194071	1.776466	2.359162	3.136621	3.914810
$C_{1212}^*$	$90^\circ$	1.054247	1.227442	1.438317	1.875361	3.091240
	$75^\circ$	1.054857	1.236100	1.461128	1.904661	2.939453

Table 3. Compartment of composite for different values of in-plane imperfect parameters for diverse volume fractions and different angle of the cell.

$K_n = 10^6$ , $K_t = 50$						
Vf		0.05	0.2	0.35	0.55	0.75
$C_{1111}^*$	$90^\circ$	1.054961	1.256744	1.533481	2.090600	3.088266
	$75^\circ$	1.054739	1.252523	1.518770	2.048378	2.985805
$C_{1122}^*$	$90^\circ$	1.042007	1.165717	1.283256	1.446467	1.837369
	$75^\circ$	1.042570	1.176434	1.320523	1.544647	1.951860
$C_{1133}^*$	$90^\circ$	1.022294	1.100148	1.200096	1.391695	1.747719
	$75^\circ$	1.022300	1.100262	1.200482	1.391651	1.731405

$C_{3333}^*$	$90^\circ$	1.200995	1.805213	2.411911	3.227390	4.061323
	$75^\circ$	1.200995	1.805213	2.411900	3.227138	4.058847
$C_{1212}^*$	$90^\circ$	1.060471	1.252743	1.482812	1.941781	2.998318
	$75^\circ$	1.061245	1.264700	1.519784	2.033492	3.076328

Table 4. Comportment of composite for different values of in-plane imperfect parameters for diverse volume fractions and different angle of the cell.

$K_n = 10$ , $K_t = 50$						
Vf		0.05	0.2	0.35	0.55	0.75
$C_{1111}^*$	$90^\circ$	1.026128	1.118309	1.233205	1.418272	1.628926
	$75^\circ$	1.025962	1.115538	1.225215	1.404864	1.625590
$C_{1122}^*$	$90^\circ$	0.988111	0.926652	0.822606	0.628191	0.400022
	$75^\circ$	0.988509	0.933313	0.841944	0.661499	0.411230
$C_{1133}^*$	$90^\circ$	0.991006	0.962850	0.932785	0.889275	0.841009
	$75^\circ$	0.991004	0.962814	0.932658	0.888904	0.840381
$C_{3333}^*$	$90^\circ$	1.194071	1.776466	2.359162	3.136628	3.914840
	$75^\circ$	1.194071	1.776466	2.359161	3.136618	3.914793
$C_{1212}^*$	$90^\circ$	1.051650	1.215785	1.412449	1.803788	2.699235
	$75^\circ$	1.052204	1.223664	1.433339	1.833875	2.647238

Table 5. Comportment of composite for different values of in-plane imperfect parameters for diverse volume fractions and different angle of the cell.

## Conclusions

Two-scale asymptotic homogenization technique was used for effective coefficients calculation of an circular elastic fiber reinforced composite with imperfect bonding

between constituents and parallelogram periodic cell. Analytical expressions are obtained for all effective coefficients considering two different  $K_n$  and  $K_t$  (normal and tangential) values of imperfect mechanic parameters. The numerical computations for all effective coefficients are shown for different values of the aforementioned imperfect parameters and inclination angle of the cell. The influence in the effective coefficient behavior of the imperfect adhesion and the inclination angle of the cell is significant and this is reported. In order to generalize the present work to more general fibers shape, the study of fibers with elliptic shape could be the object of a future work.

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