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# APPLICATION OF STOCHASTIC FLOWS TO THE STICKY BROWNIAN MOTION EQUATION

HATEM HAJRI <sup>(1)</sup> AND MARC ARNAUDON <sup>(2)</sup>

ABSTRACT. We show how the theory of stochastic flows allows to recover in an elementary way a well known result of Warren on the sticky Brownian motion equation.

## 1. INTRODUCTION

A  $\theta$ -sticky Brownian on the half line  $[0, \infty[$  is a diffusion with generator

$$(\mathcal{A}f)(x) = \begin{cases} \frac{1}{2}f''(x) & \text{if } x > 0 \\ \theta f'(0+) & \text{if } x = 0 \end{cases}$$

and domain

$$\mathcal{D}(\mathcal{A}) = \left\{ f \in C^2(0, \infty) : f \in C_0([0, \infty)), f'(0^+), \right. \\ \left. f''(0^+) \text{ exist, } f''(0^+) = 2\theta f'(0^+), \lim_{x \rightarrow \infty} f''(x) = 0 \right\}$$

where  $\theta > 0$  is the stickiness parameter. This is a special case of Feller one dimensional diffusions introduced by Feller by means of their infinitesimal generators [4]. Sticky Brownian motion has an intermediate behavior, depending on  $\theta$ , between Brownian motion absorbed at 0 and reflected Brownian motion. One possible path construction of a  $\theta$ -sticky Brownian motion  $X$  started from 0 consists in slowing down a reflected Brownian motion  $R$  started from 0 whenever it is at 0 in the following way

$$X_t = R_{\inf\{u: u + \frac{1}{\theta}L_u > t\}}$$

where  $L_t = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{0 \leq R_s \leq \epsilon\}} ds$  is the local time of  $R$  [8, 7]. As a consequence of this construction, the amount of time spent at 0 by  $X$  up to  $t$ ,  $\int_0^t 1_{\{X_s=0\}} ds$ , has

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positive probability of being greater than 0. More precisely, the following equality holds in law

$$\int_0^t 1_{\{X_s=0\}} ds \stackrel{\text{law}}{=} \frac{|N|}{\theta} \sqrt{t + \frac{N^2}{4\theta^4}} - \frac{N^2}{2\theta^2}$$

where  $N \sim \mathcal{N}(0, 1)$  (see Proposition 5 in [7]).

In this paper, we are interested in sticky Brownian motion as solution of the following stochastic differential equation

$$(1) \quad X_t = x + \int_0^t 1_{\{X_s>0\}} dW_s + \theta \int_0^t 1_{\{X_s=0\}} ds$$

driven by a standard Brownian motion  $W$  and where  $\theta > 0$  is a given constant and  $x \in [0, \infty[$  is a given initial condition.

It has been proved by Chitashvili [2] that (1) has a weak solution  $X$  which is a  $\theta$ -sticky Brownian motion started from  $x$ , the law of  $(X, W)$  is unique but  $X$  is not a strong solution to (1). Later on, Warren [14] derived the following remarkable result describing the law of  $X_t$  given  $W$  (the form given here follows Theorem 2 [12]).

**Theorem 1.1.** *For all  $t \geq 0$  and all measurable bounded  $f$ ,*

$$\mathbb{E}[f(X_t) | \mathcal{F}^W] = G_f(W_t^+)$$

where  $W_t^+ = W_t - \min_{0 \leq u \leq t} W_u$  and  $G_f(y) = \mathbb{E}[f((y - T)^+)]$  with  $T$  an exponential variable with mean  $\frac{1}{2\theta}$ .

This theorem shows, in particular, that  $X$  can not be a strong solution to (1). Subsequently, Warren [12] described all couplings of solutions to (1) which leave the diagonal. Before going on, we mention the work of Engelbert and Peskir [3] where a third proof of the non strong solvability of (1) and a two sided version of it can be found (see also [1]).

A remarkable and attractive fact in Warren's conditional law identity is that it involves the well known and habitual process  $W^+$  strong solution to

$$(2) \quad Y_t = W_t + L_t(Y)$$

where  $L_t(Y) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{0 \leq Y_s \leq \epsilon\}} ds$ . This raises the question whether there is a link between (1) and (2) explaining Theorem 1.1.

In this paper, it is shown that stochastic flows of kernels [10] provide an answer to the previous question. More precisely, define

$$(3) \quad \varphi_{s,t}(x) = (x + W_t - W_s) 1_{\{t \leq \tau_s(x)\}} + W_{s,t}^+ 1_{\{t > \tau_s(x)\}}$$

where  $\tau_s(x) = \inf\{u \geq s : x + W_u - W_s = 0\}$ . Then  $\varphi$  is a stochastic flow of maps which solves the flow version of (2)

$$\varphi_{s,t}(x) = x + W_t - W_s + L_{s,t}(x)$$

where  $L_{s,t}(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_s^t 1_{\{0 \leq \varphi_{s,u}(x) \leq \epsilon\}} du$ . Now, let

$$(4) \quad K_{s,t}f(x) = \begin{cases} f(\varphi_{s,t}(x)) & \text{if } s \leq t \leq \tau_s(x) \\ G_f(\varphi_{s,t}(x)) & \text{if } t > \tau_s(x) \end{cases}$$

Then  $K$  is a stochastic flow of kernels which is a strong solution to the flow of kernels version of (1): for all  $t \geq s$ ,  $f \in \mathcal{D}(A)$  and  $x \geq 0$  a.s

$$(5) \quad K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(f'1_{(0,\infty)})(x)dW_u + \frac{1}{2} \int_s^t K_{s,u}f''(x)du$$

$K$ , called the Wiener solution of (5) in [10], is characterized by being the unique (up to modification), strong solution of (5). This leads to Theorem (1.1) as the conditional law of  $X_t$  given  $W$  should coincide with  $K_{0,t}(0, dy)$ .

The next section gives details and proofs of the previously claimed facts. It can be remarked that the proofs only rely on the definition of stochastic flows with no additional results of the theory and can be taught to various students. The present paper provides, in particular, a direct application of stochastic flows to the study of weak solutions (see [6] for another recent application by the same authors).

## 2. PROOF OF THEOREM 1.1

### 2.1. The generalized sticky Brownian motion equation.

Let us now recall the definition of stochastic flows from [10]. In this definition  $\mathcal{P}(\mathbb{R}_+)$  denotes the space of all probability measures on  $\mathbb{R}_+$  and  $\mathcal{B}(E)$  indicates the Borel  $\sigma$ -field of  $E$ .

**Definition 2.1.** *A stochastic flow of kernels  $K$  on  $\mathbb{R}_+$ , defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , is a family  $(K_{s,t})_{s \leq t}$  such that*

- (1)  $(s, t, x, \omega) \mapsto K_{s,t}(x, \omega)$  is measurable;
- (2) For all  $h \in \mathbb{R}$ ,  $s \leq t$ ,  $K_{s+h, t+h}$  is distributed like  $K_{s,t}$ ;
- (3) For all  $s_1 \leq t_1 \leq \dots \leq s_n \leq t_n$ , the family  $\{K_{s_i, t_i}, 1 \leq i \leq n\}$  is independent;
- (4) For all  $s \leq t \leq u$  and all  $x \in \mathbb{R}_+$ , a.s.  $K_{s,u}(x) = K_{s,t}K_{t,u}(x)$  and  $K_{s,s}(x) = \delta_x$ ;
- (5) For all  $f \in C_0(\mathbb{R}_+)$ ,  $t \geq 0$ ,

$$\lim_{t \rightarrow 0^+} \mathbb{E}[(K_{0,t}f(x) - f(x))^2] = 0;$$

- (6) For all  $f \in C_0(\mathbb{R}_+)$ ,  $x \in \mathbb{R}_+$ ,  $t \geq 0$ ,

$$\lim_{y \rightarrow x} \mathbb{E}[(K_{0,t}f(y) - K_{0,t}f(x))^2] = 0;$$

- (7) For all  $t \geq 0$ ,  $f \in C_0(\mathbb{R}_+)$ ,  $\lim_{x \rightarrow \infty} \mathbb{E}[(K_{0,t}f(x))^2] = 0$ .

We say that  $\varphi$  is a stochastic flow of mappings on  $\mathbb{R}_+$  if  $K_{s,t}(x) = \delta_{\varphi_{s,t}(x)}$  is a stochastic flow of kernels on  $\mathbb{R}_+$ .

For  $K$ , a stochastic flow of kernels on  $\mathbb{R}_+$ ,

$$(6) \quad P_t^n f(x_1, \dots, x_n) = \mathbb{E} \left[ \int_{\mathbb{R}_+^n} f(y_1, \dots, y_n) K_{0,t}(x_1, dy_1) \cdots K_{0,t}(x_n, dy_n) \right]$$

defines a Feller semigroup on  $\mathbb{R}_+^n$ . Moreover  $(P^n)_{n \geq 1}$  is a compatible family (in a sense explained in [10]) of Feller semigroups acting respectively on  $C_0(\mathbb{R}_+^n)$  that uniquely characterize the law of  $K$ . Conversely, it has been proved in [10] that to each family of compatible Feller semigroups  $(P^n)_{n \geq 1}$  is associated a (law unique) stochastic flow of kernels such that (6) holds for every  $n \geq 1$ .

**Definition 2.2.** (*Real white noise*) A family  $(W_{s,t})_{s \leq t}$  is called a real white noise if there exists a Brownian motion on the real line  $(W_t)_{t \in \mathbb{R}}$ , that is  $(W_t)_{t \geq 0}$  and  $(W_{-t})_{t \geq 0}$  are two independent standard Brownian motions such that for all  $s \leq t$ ,  $W_{s,t} = W_t - W_s$  (in particular, when  $t \geq 0$ ,  $W_t = W_{0,t}$  and  $W_{-t} = -W_{-t,0}$ ).

For a family of random variables  $Z = (Z_{s,t})_{s \leq t}$ , define  $\mathcal{F}_{s,t}^Z = \sigma(Z_{u,v}, s \leq u \leq v \leq t)$  for all  $s \leq t$ .

**Definition 2.3.** Let  $K$  be a stochastic flow of kernels and  $W$  be a real white noise defined on the same probability space. We say that  $(K, W)$  is a (generalized) solution of the sticky equation if for all  $s \leq t$  and for all  $f \in \mathcal{D}(\mathcal{A})$ ,  $t \geq 0$  and  $x \in \mathbb{R}$  a.s

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(f'1_{(0,\infty)})(x)dW_u + \frac{1}{2} \int_s^t K_{s,u}f''(x)du$$

If  $\mathcal{F}_{s,t}^K \subset \mathcal{F}_{s,t}^W$  for all  $s \leq t$ , then  $K$  is called Wiener solution of the sticky equation.

The Wiener solution was introduced in [9] (it is called statistical solution there) by means of its Wiener chaos expansion with respect to  $W$  only depending on the semigroup of the diffusion (sticky Brownian motion here). Interestingly, this solution exists and is unique under weak assumptions.

Let us explain the link between the generalized SDE and the original sticky equation (1). We start with the following

**Lemma 2.4.** If  $(K, W)$  is a solution of the generalized sticky equation, then  $\mathcal{F}_{s,t}^W \subset \mathcal{F}_{s,t}^K$  for all  $s \leq t$ .

In view of this lemma, we may sometimes say  $K$  is a solution of the generalized sticky equation without specifying the white noise since it is determined by  $K$ .

*Proof.* Define  $\tau_s(x) = \inf\{u \geq s : x + W_{s,u} = 0\}$ , then following Lemma 3.1 in [11], one proves that  $K_{s,t}(x) = \delta_{x+W_{s,t}}$  for all  $s \leq t \leq \tau_s(x)$ . Since this holds for  $x$  arbitrarily distant from 0, the lemma follows.

□

Assume  $(K, W)$  satisfies Definition (2.3) and set

$$Q_t(f \otimes g)(x, w) = \mathbb{E}[K_{0,t}f(x)g(w + W_t)]$$

Then  $(Q_t)_t$  defines a Feller semigroup. Denote by  $\mathcal{L}$  its generator and  $\mathcal{D}(\mathcal{L})$  its domain. A simple application of Itô's formula shows that  $\mathcal{D}_1 \otimes C_K^2(\mathbb{R}_+) \subset \mathcal{D}(\mathcal{L})$  where  $\mathcal{D}_1 = \{f \in \mathcal{D}(\mathcal{A}) : f'(0+) = 0\}$  and  $C_K^2(\mathbb{R}_+)$  denotes the space of  $C^2$  functions on  $\mathbb{R}_+$  with compact supports. Moreover for all  $f \in \mathcal{D}_1$  and  $g \in C_K^2(\mathbb{R}_+)$ ,

$$\mathcal{L}(f \otimes g)(x, w) = \frac{1}{2}f(x)g''(w) + \frac{1}{2}g(w)f''(x) + f'(x)g'(w).$$

Let  $(X, B)$  be the Markov process associated to  $(Q_t)_t$  and started from  $(x, 0)$ . Then  $X$  is a  $\theta$ -sticky Brownian motion started from  $x$  and  $B$  is a standard Brownian motion started from 0. The first claim holds because  $(Q_t)_t$  is the unique Feller semigroup satisfying

$$Q_t f = f + \int_0^t Q_u(\mathcal{A}f)du, \quad f \in \mathcal{D}(\mathcal{A})$$

Now for  $f \in \mathcal{D}_1$  and  $g \in C_K^2(\mathbb{R}_+)$ ,

$$(7) \quad f(X_t)g(B_t) - \int_0^t \mathcal{L}(f \otimes g)(X_s, B_s) ds \text{ is a martingale.}$$

As  $X$  is a  $\theta$ -sticky Brownian motion, it satisfies  $X_t = x + M_t + \theta \int_0^t 1_{\{X_s=0\}} ds$  with  $M$  a martingale with quadratic variation  $\langle M \rangle_t = \int_0^t 1_{\{X_s>0\}} ds$ . Writing Itô's formulas for  $f(X)g(B)$  and using (7) shows that

$$\int_0^t f'(X_s)g'(B_s)d\langle M, B \rangle_s = \int_0^t f'(X_s)g'(B_s)ds.$$

Now by an approximation argument  $1_{\{X_s>0\}}d\langle M, B \rangle_s = 1_{\{X_s>0\}}ds$  and consequently  $M_t = \int_0^t 1_{\{X_s>0\}}dB_s$  in  $L^2(\mathbb{P})$ . So finally,  $(X, B)$  is a weak solution to the sticky equation.

More generally, considering the semigroups  $Q_t^n(f \otimes g)(x, w) = \mathbb{E}[K_{0,t}^{\otimes n} f(x)g(w + W_t)]$  for  $n \geq 1$ , one can prove that there exists a one to one correspondance between the laws of stochastic flows of kernels satisfying Definition (2.3) and compatible weak solutions to the sticky equation (see Proposition 2.1 in [5] for more details in a similar context).

To close this paragraph, we mention that if  $\varphi$  is a flow of mappings such that  $K = \delta_\varphi$  satisfies Definition (2.3), then necessarily  $\varphi$  is a flow of mappings solution of

$$\varphi_{s,t}(x) = x + \int_s^t 1_{\{\varphi_{s,u}(x)>0\}}dW_u + \theta \int_s^t 1_{\{\varphi_{s,u}(x)=0\}}ds$$

and vice versa. Warren [13] proved that such a flow  $\varphi$  exists and its law is uniquely determined. This flow can also be constructed by applying the general results of [10].

**Proposition 2.5.** *Let  $(K^1, W)$  and  $(K^2, W)$  be two Wiener solutions of the sticky equation relatively to the same  $W$ . Then for all  $s \leq t$  and  $x \in \mathbb{R}$ , with probability one,  $K_{s,t}^1(x) = K_{s,t}^2(x)$ .*

*Proof.* We follow the proof of Proposition 4.2 [5]. Note that  $K_{s,t}(x, y) = K_{s,t}^1(x) \otimes K_{s,t}^2(y)$  is a stochastic flow of kernels on  $\mathbb{R}_+^2$  and

$$Q_t(f \otimes g \otimes h)(x, y, w) = \mathbb{E}[K_{0,t}^1 f(x) K_{0,t}^2 g(y) h(w + W_t)]$$

is a Feller semigroup on  $(\mathbb{R}_+)^2 \times \mathbb{R}$ . Fix  $x \in \mathbb{R}_+$  and let  $(X^1, X^2, B)$  be the Markov process associated to  $Q$  started from  $(x, x, 0)$ , then  $B$  is a standard Brownian motion and  $X^1, X^2$  are two  $\theta$ -sticky Brownian motions. Moreover  $X^1, X^2$  are solutions of the sticky equation driven by  $B$  and in particular  $(X^1, B)$  and  $(X^2, B)$  have the same law. Since  $K^1$  and  $K^2$  are two Wiener solutions, there exist two measurable functions  $F_{t,x}^1, F_{t,x}^2 : C([0, t], \mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$  such that  $K_{0,t}^1(x) = F_{t,x}^1(W_u, u \leq t)$ ,  $K_{0,t}^2(x) = F_{t,x}^2(W_u, u \leq t)$ . Let  $N_{0,t}^1(x) = F_{t,x}^1(B_u, u \leq t)$  and  $N_{0,t}^2(x) = F_{t,x}^2(B_u, u \leq t)$ . We will prove that for all measurable bounded  $f : \mathbb{R} \rightarrow \mathbb{R}$  a.s.

$$(8) \quad N_{0,t}^i f(x) = \mathbb{E}[f(X_t^i) | \sigma(B_u, u \leq t)], \quad i = 1, 2$$

To prove (8), we will check by induction on  $n$  that for all  $t_1 \leq \dots \leq t_{n-1} \leq t_n = t$  and all bounded functions  $f, g_1, \dots, g_n : \mathbb{R} \rightarrow \mathbb{R}$ , we have

$$(9) \quad \mathbb{E}[K_{0,t}^i f(x) \prod_{j=1}^n g_j(W_{t_j})] = \mathbb{E}[f(X_t^i) \prod_{j=1}^n g_j(B_{t_j})], \quad i = 1, 2.$$

Let us prove this for  $i = 1$  and set  $Q_t^1(f \otimes g) = Q_t(f \otimes \text{Id} \otimes g)$ . For  $n = 1$ , (9) is immediate from the definition of  $Q$ . Let us prove (9) for  $n = 2$ . We have

$$\mathbb{E}[K_{0,t}^1 f(x) g_1(W_{t_1}) g_2(W_t)] = \mathbb{E}[K_{0,t_1}^1 (Q_{t-t_1}^1(f \otimes g_2)(\cdot, W_{t_1}))(x) g_1(W_{t_1})].$$

On the other hand

$$\mathbb{E}[f(X_{t_1}^1) g_1(B_{t_1}) g_2(B_t)] = \mathbb{E}[Q_{t-t_1}^1(f \otimes g_2)(X_{t_1}^1, B_{t_1}) g_1(B_{t_1})].$$

Now (9) holds using a uniform approximation of  $Q_{t-t_1}^1(f \otimes g)$  by a linear combination of functions of the form  $h \otimes k$ ,  $h, k \in C_0(\mathbb{R}_+)$ . It is clear now from (8), that  $N_{0,t}^1(x) = N_{0,t}^2(x)$  since  $(X, B)$  and  $(Y, B)$  have the same law.  $\square$

Now in the rest of the paper, we take  $W$  a real white noise and will check that  $K$  defined in (4) is the Wiener solution of the generalized sticky equation. This gives Theorem (1.1) in vue of (8).

**Proposition 2.6.** *K is the, unique up to modification, Wiener stochastic flow of kernels solution of the generalized sticky equation driven by W.*

*Proof.* To check that  $K$  is a stochastic flow of kernels, we will only check the flow property for all  $f \in C_0(\mathbb{R}_+)$ ,  $s \leq t \leq u$  and  $x \in \mathbb{R}_+$ , with probability 1,

$$(10) \quad K_{s,u}f(x) = K_{s,t}K_{t,u}f(x)$$

The other claims in Definition 2.1 are easy to verify. Let us now check (10). For this, we will use the fact that  $\varphi$  defined in (3) is a stochastic flow of mappings (for non specialists of stochastic flows, this can be proved following the same steps (a), (b), (c) given below). Moreover this is a coalescing flow : for all  $x \geq 0$ , the paths  $\varphi_{s,\cdot}(x)$  and  $\varphi_{s,\cdot}(0)$  meet for the first time at  $\tau_s(x)$  and then finish together.

Note that  $G_f$  writes as (with  $\lambda = 2\theta$ )

$$(11) \quad G_f(y) = f(0)e^{-\lambda y} + \lambda e^{-\lambda y} \int_0^y f(u)e^{\lambda u} du$$

We first check (10) for  $x = 0$ . By the flow property of  $\varphi$ ,  $K_{s,u}f(0) = G_f(\varphi_{s,u}(0)) = G_f(\varphi_{t,u} \circ \varphi_{s,t}(0))$  and  $K_{s,t}K_{t,u}f(0) = G_{K_{t,u}f}(\varphi_{s,t}(0))$ . Using the independence of increments of  $\varphi$ , it suffices to prove that for all  $y \geq 0$ , a.s  $G_f(\varphi_{t,u}(y)) = G_{K_{t,u}f}(y)$  which is equivalent to

$$e^{\lambda y} G_f(\varphi_{t,u}(y)) = G_f(\varphi_{t,u}(0)) + \lambda \int_0^y K_{t,u}f(a)e^{\lambda a} da$$

To prove this identity, note that for all  $y > 0$ ,  $z \mapsto \varphi_{t,u}(z)$  is differentiable at  $y$  with derivative at  $y$  given by  $1_{\{u < \tau_t(y)\}}$ . Thus by a simple calculation, the derivative of  $z \mapsto e^{\lambda y} \varphi_{t,u}(z)$  at  $y$  coincides with  $\lambda e^{\lambda y} K_{t,u}f(y)$ . This proves (10) for  $x = 0$ .

Now take  $x > 0$  and let  $y = \varphi_{s,t}(x)$ .

- (a) On the event  $\{u \leq \tau_s(x)\}$ , we have  $\tau_s(x) = \tau_t(y)$ ,  $K_{s,u}f(x) = f(\varphi_{s,u}(x))$ ,  $K_{s,t}(K_{t,u}f)(x) = (K_{t,u}f)(y) = f(\varphi_{t,u}(y))$  since  $u \leq \tau_t(y)$  and so (10) holds by the flow property of  $\varphi$ .
- (b) On the event  $\{t \leq \tau_s(x) < u\}$ , we still have  $\tau_s(x) = \tau_t(y)$  and  $K_{s,u}f(x) = G_f(\varphi_{s,u}(x)) = G_f(\varphi_{t,u}(y))$ . Moreover  $K_{s,t}(K_{t,u}f)(x) = G_{K_{t,u}f}(y)$  and so the flow property holds by the calculations above.
- (c) On the event  $\{\tau_s(x) \leq t\}$ , we have  $K_{s,u}f(x) = G_f(\varphi_{s,u}(x)) = G_f(\varphi_{s,u}(0)) = K_{s,u}f(0)$ . Moreover  $K_{s,t}(K_{t,u}f)(x) = G_{K_{t,u}f}(y) = G_{K_{t,u}f}(\varphi_{s,t}(0)) = K_{s,t}(K_{t,u}f)(0)$  and the flow property holds again from the case  $x = 0$ .

Thus  $K$  is a stochastic flow of kernels. Note that  $\mathcal{F}_{s,t}^K \subset \mathcal{F}_{s,t}^W$  for all  $s \leq t$ . It remains now to check that  $K$  solves the generalized equation. We take  $s = 0$  and first  $x = 0$ . Denote  $W_{0,t}^+$  simply by  $W_t^+$ . Let

$$\mathcal{D} = \{g \in C^2(0, \infty) : g \in C_0[0, \infty), g'(0^+) = 0, g''(0^+) \text{ exists}\}$$



By Itô's formula, for all  $g \in \mathcal{D}$

$$g(W_t^+) = g(0) + \int_0^t g'(W_u^+) dW_u + \frac{1}{2} \int_0^t g''(W_u^+) du$$

Let  $f \in \mathcal{D}(\mathcal{A})$  and set  $g(y) = G_f(y)$ . Then  $g$  is continuous on  $\mathbb{R}_+$ ,  $C^2$  on  $\mathbb{R}_+^*$  and

$$(12) \quad g'(y) = -\lambda f(0)e^{-\lambda y} - \lambda^2 e^{-\lambda y} \int_0^y f(u)e^{\lambda u} du + \lambda f(y)$$

In particular  $g'(0+) = 0$ . Moreover

$$g''(y) = \lambda^2 f(0)e^{-\lambda y} + \lambda^3 e^{-\lambda y} \int_0^y f(u)e^{\lambda u} du - \lambda^2 f(y) + \lambda f'(y)$$

and so  $G_f \in \mathcal{D}$ . Consequently for all  $f \in \mathcal{D}(\mathcal{A})$ ,

$$\begin{aligned} G_f(W_t^+) &= G_f(0) + \int_0^t (G_f)'(W_u^+) dW_u + \frac{1}{2} \int_0^t (G_f)''(W_u^+) du \\ &= f(0) + \int_0^t (G_f)'(W_u^+) dW_u + \frac{1}{2} \int_0^t (G_f)''(W_u^+) du \end{aligned}$$

We now check that for all  $f \in \mathcal{D}(\mathcal{A})$  and  $y \geq 0$ ,

$$G_{f'1_{(0,\infty)}}(y) = (G_f)'(y) \text{ and } G_{f''}(y) = (G_f)''(y)$$

Using (11), we see that

$$G_{f'1_{(0,\infty)}}(y) = \lambda e^{-\lambda y} \int_0^y f'(u)e^{\lambda u} du$$

which is also equal to  $(G_f)'(y)$  given in (12) by a simple integration by parts. Again from (11), we have

$$G_{f''}(y) = f''(0)e^{-\lambda y} + \lambda e^{-\lambda y} \int_0^y f''(u)e^{\lambda u} du$$

Integrating twice by parts, we see that

$$G_{f''}(y) = f''(0)e^{-\lambda y} - \lambda f'(0)e^{-\lambda y} + \lambda f'(y) - \lambda^2 f(y) + \lambda^2 f(0)e^{-\lambda y} + \lambda^3 e^{-\lambda y} \int_0^y f(u)e^{\lambda u} du$$

which is the same as  $(G_f)''(y)$  using the hypothesis  $f''(0) = \lambda f'(0)$  as  $f \in \mathcal{D}(\mathcal{A})$ .

Finally for all  $f \in \mathcal{D}_A$ ,

$$G_f(W_t^+) = f(0) + \int_0^t G_{f'1_{(0,\infty)}}(W_u^+) dW_u + \frac{1}{2} \int_0^t G_{f''}(W_u^+) du$$

or equivalently

$$K_{0,t}f(0) = f(0) + \int_0^t K_{0,u}(f'1_{(0,\infty)})(0) dW_u + \frac{1}{2} \int_0^t K_{0,u}f''(0) du$$

Now the case  $x > 0$  holds by discussing  $t \leq \tau_0(x)$  and  $t > \tau_0(x)$  and using the fact that  $K_{0,t}(x) = K_{0,t}(0)$  for  $t \geq \tau_0(x)$ .  $\square$

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