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## On the paper: Numerical radius preserving linear maps on Banach algebras

M. El Azhari

**Abstract.** We give an example of a unital commutative complex Banach algebra having a normalized state which is not a spectral state and admitting an extreme normalized state which is not multiplicative. This disproves two results by Golfarshchi and Khalilzadeh.

*Keywords and phrases:* Banach algebra, regular norm, normalized state, spectral state.

*(2010) Mathematics Subject Classification:* 46H05.

### 1. Preliminaries

Let  $(A, \|\cdot\|)$  be a complex normed algebra with an identity  $e$  such that  $\|e\| = 1$ . Let  $D(A, e) = \{f \in A' : f(e) = \|f\| = 1\}$ , where  $A'$  is the dual space of  $A$ . The elements of  $D(A, e)$  are called normalized states on  $A$ . For  $a \in A$ , let  $V(A, a) = \{f(a) : f \in D(A, e)\}$ ,  $V(A, a)$  is called the numerical range of  $a$ . Let  $sp(a)$  be the spectrum of  $a \in A$ , and let  $co(sp(a))$  be the convex hull of  $sp(a)$ . We say that a linear functional  $f$  on  $A$  is a spectral state if  $f(a) \in co(sp(a))$  for all  $a \in A$ . We denote by  $M(A)$  the set of all non-zero continuous multiplicative linear functionals on  $A$ .

### 2. Result

#### 2.1. Counterexample

Golfarshchi and Khalilzadeh proved the following results [4]:

[4, **Theorem 2**]. Let  $A$  be a unital complex Banach algebra, and let  $f$  be a linear functional on  $A$ . Then  $f$  is a normalized state on  $A$  if and only if  $f(a) \in co(sp(a))$  for all  $a \in A$ .

[4, **Theorem 3**]. Let  $A$  be a unital commutative complex Banach algebra. Then each extreme normalized state on  $A$  is multiplicative.

Here we give a counterexample disproving the above results. We also remark that Theorems 5 and 6 [4] are called into question since the authors used Theorem 3 [4] to prove these results.

Let  $(A, \|\cdot\|)$  be a non-zero commutative radical complex Banach algebra [6, p.316]. Let  $A_e = \{a + \lambda e : a \in A, \lambda \in C\}$  be the unitization of  $A$  with the identity  $e$ , and the norm  $\|a + \lambda e\|_1 = \|a\| + |\lambda|$  for all  $a + \lambda e \in A_e$ .  $(A_e, \|\cdot\|_1)$  is a unital commutative complex Banach algebra, and  $M(A_e) = \{\varphi_\infty\}$ , where  $\varphi_\infty$  is the continuous multiplicative linear functional on  $A_e$  defined by  $\varphi_\infty(a + \lambda e) = \lambda$  for all  $a + \lambda e \in A_e$ .

(1) Let  $a$  be a non-zero element of  $A$ ,  $V(A_e, a) = \{z \in C : |z| \leq \|a\|\}$  by [2, Remark 3.8], and  $sp(a) = \{\varphi_\infty(a)\} = \{0\}$ , hence  $co(sp(a)) = \{0\}$  is strictly included in  $V(A_e, a)$  since  $\|a\| \neq 0$ . Therefore the direct implication of [4, Theorem 2] does not hold.

(2) By [1, lemma 1.10.3],  $D(A_e, e)$  is a non-empty weak\* compact convex subset of  $A_e$ , then  $ext(D(A_e, e))$  is a non-empty set. Assume that each extreme normalized state on  $A_e$  is multiplicative, then  $ext(D(A_e, e)) = \{\varphi_\infty\}$ . Let  $a$  be a non-zero element of  $A$ , by [1, Corollary 1.10.15] there exists  $f \in D(A_e, e)$  such that  $f(a) \neq 0 = \varphi_\infty(a)$ . Therefore  $\bar{co}(ext(D(A_e, e))) = \{\varphi_\infty\}$  is strictly included in  $D(A_e, e)$ , which contradicts the Krein-Milman Theorem. This shows that [4, theorem 3] is not valid.

## 2.2. Regular norm and the operator seminorm

Let  $(A, \|\cdot\|)$  be a non-unital complex Banach algebra, and let  $A_e = \{a + \lambda e : a \in A, \lambda \in C\}$  be the unitization of  $A$  with the identity  $e$ . Let  $\|a + \lambda e\|_{op} = \sup\{\|(a + \lambda e)x\|, \|x(a + \lambda e)\| : x \in A, \|x\| \leq 1\}$  for all  $a + \lambda e \in A_e$ ,  $\|\cdot\|_{op}$  is an algebra seminorm on  $A_e$ . We say that  $\|\cdot\|$  is regular if  $\|\cdot\|_{op} = \|\cdot\|$  on  $A$ . If  $\|\cdot\|$  is regular, it is well known that  $(A_e, \|\cdot\|_{op})$  is a complex Banach algebra. The following question was asked [3]: If  $(A_e, \|\cdot\|_{op})$  is a complex Banach algebra, is the norm  $\|\cdot\|$  regular ?

Orenstein tried to give an answer to this question in the commutative case [5], but his proof is not correct since it is essentially based on the direct implication of [4, Theorem 2].

## 3. Conclusion

In this note, we show that Theorems 2 and 3 [4] are false by giving a counterexample. We also remark that Theorems 5 and 6 [4] and Theorem 1.1 [5] are called into question since the authors used Theorems 2 or 3 [4] to prove these results.

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