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On the paper: Numerical radius preserving linear maps on Banach algebras

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Abstract. In this note, we give a counterexample disproving two results in the above paper.

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Let $(A, \|\cdot\|)$ be a complex normed algebra with an identity e such that $\|e\| = 1$. Let $D(A, e) = \{f \in A' : f(e) = \|f\| = 1\}$, where A' is the dual space of A . The elements of $D(A, e)$ are called normalized states on A . For $a \in A$, let $V(A, a) = \{f(a) : f \in D(A, e)\}$, $V(A, a)$ is called the numerical range of a . Let $sp(a)$ be the spectrum of $a \in A$, and let $co(sp(a))$ be the convex hull of $sp(a)$. We denote by $M(A)$ the set of all non-zero continuous multiplicative linear functionals on A .

In [4], F. Golfarshchi and A. A. Khalilzadeh proved the following results:

[4, Theorem 2]. Let A be a unital complex Banach algebra, and let f be a linear functional on A . Then f is a normalized state on A if and only if $f(a) \in co(sp(a))$ for all $a \in A$.

[4, Theorem 3]. Let A be a unital commutative complex Banach algebra. Then each extreme normalized state on A is multiplicative.

Counterexample. Let $(A, \|\cdot\|)$ be a non-zero commutative radical complex Banach algebra [6, p.316]. Let $A_e = \{a + \lambda e : a \in A, \lambda \in C\}$ be the unitization of A with the identity e , and the norm $\|a + \lambda e\|_1 = \|a\| + |\lambda|$ for all $a + \lambda e \in A_e$. $(A_e, \|\cdot\|_1)$ is a unital commutative complex Banach algebra, and $M(A_e) = \{\varphi_\infty\}$, where φ_∞ is the continuous multiplicative linear functional on A_e defined by $\varphi_\infty(a + \lambda e) = \lambda$ for all $a + \lambda e \in A_e$.

(1) Let a be a non-zero element of A , $V(A_e, a) = \{z \in C : |z| \leq \|a\|\}$ by [2, Remark 3.8], and $sp(a) = \{\varphi_\infty(a)\} = \{0\}$, hence $co(sp(a)) = \{0\} \subsetneq V(A_e, a)$ since $\|a\| \neq 0$. Therefore the direct implication of [4, Theorem 2] doesn't hold.

(2) By [1, lemma 1.10.3], $D(A_e, e)$ is a nonempty weak* compact convex subset of A'_e , then $ext(D(A_e, e)) \neq \emptyset$. Assume that each extreme normalized state on A_e is multiplicative, then $ext(D(A_e, e)) = \{\varphi_\infty\}$. Let a be a non-zero element of A , by [1, Corollary 1.10.15] there exists $f \in D(A_e, e)$ such that $f(a) \neq 0 = \varphi_\infty(a)$. Therefore $\overline{co}(ext(D(A_e, e))) = \{\varphi_\infty\} \subsetneq D(A_e, e)$, which contradicts the Krein-Milman Theorem. This shows that [4, theorem 3] is not valid.

Remark. Theorems 5 and 6 in [4] are called into question since the authors used [4, Theorem 3] to prove these results.

Let $(A, \|\cdot\|)$ be a non-unital complex Banach algebra, and let $A_e = \{a + \lambda e : a \in A, \lambda \in C\}$ be the unitization of A with the identity e . Let $\|a + \lambda e\|_{op} = \sup\{\|(a + \lambda e)x\|, \|x(a + \lambda e)\| : x \in A, \|x\| \leq 1\}$ for all $a + \lambda e \in A_e$, $\|\cdot\|_{op}$ is an algebra seminorm on A_e . We say that $\|\cdot\|$ is regular if $\|\cdot\|_{op} = \|\cdot\|$ on A . If $\|\cdot\|$ is regular, it is well known that $(A_e, \|\cdot\|_{op})$ is a complex Banach algebra. In [3], the following question was asked: If $(A_e, \|\cdot\|_{op})$ is a complex Banach algebra, is the norm $\|\cdot\|$ regular ?

In [5], A. Orenstein tried to give an answer to this question in the commutative case, but his proof is not correct since it is essentially based on the direct implication of [4, Theorem 2].

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