THE RATE OF CONVERGENCE FOR THE RENEWAL THEOREM IN R^d.

Jean-Baptiste Boyer

▶ **To cite this version:**

Jean-Baptiste Boyer. THE RATE OF CONVERGENCE FOR THE RENEWAL THEOREM IN R^d.. 2016. hal-01292084v1

HAL Id: hal-01292084

https://hal.science/hal-01292084v1

Preprint submitted on 22 Mar 2016 (v1), last revised 9 Jul 2016 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution - NonCommercial - ShareAlike 4.0 International License

THE RATE OF CONVERGENCE FOR THE RENEWAL THEOREM IN $\mathbb{R}^d$.

JEAN-BAPTISTE BOYER

IMB, Université de Bordeaux / MODAL'X, Université Paris-Ouest Nanterre

ABSTRACT. Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$. Consider the random walk (X_n) on $\mathbb{R}^d \setminus \{0\}$ defined by ρ : for any $x \in \mathbb{R}^d \setminus \{0\}$, we set $X_0 = x$ and $X_{n+1} = g_{n+1}X_n$ where (g_n) is an iid sequence of $\mathrm{SL}_d(\mathbb{R})-$valued random variables of law ρ. Guivarc'h and Raugi proved that under an assumption on the subgroup generated by the support of ρ (strong irreducibility and proximality), this walk is transient.

In particular, this proves that if f is a compactly supported continuous function on $\mathbb{R}^d$, then the function $Gf(x) := \mathbb{E}_x \sum_{k=0}^{+\infty} f(X_n)$ is well defined for any $x \in \mathbb{R}^d \setminus \{0\}$.

Guivarc'h and Le Page proved the renewal theorem in this situation : they study the possible limits of Gf at 0 and in this article, we study the rate of convergence in their renewal theorem.

To do so, we consider the family of operators $(P(it))_{t\in\mathbb{R}}$ defined for any continuous function f on the sphere $\mathbb{S}^{d-1}$ and any $x \in \mathbb{S}^{d-1}$ by

$$P(it)f(x) = \int_{\mathrm{SL}_d(\mathbb{R})} e^{-it\ln\frac{\|gx\|}{\|x\|}} f\left(\frac{gx}{\|gx\|}\right) \mathrm{d}\rho(g)$$

We prove that, adding an exponential moment condition to the strong irreducibility and proximality condition, we have that for some $L \in \mathbb{R}$ and any $t_0 \in \mathbb{R}_+^*$,

$$\sup_{\substack{t\in\mathbb{R}\\ |t|\geqslant t_0}} \frac{1}{|t|^L} \left\|(I_d - P(it))^{-1}\right\| \text{ is finite}$$

where the norm is taken in some space of hölder-continuous functions on the sphere.

CONTENTS

E-mail address: jeaboyer@math.cnrs.fr.
Date: March 22, 2016.

1. Introduction and main results

1.1. **Preliminaries.** Let ρ be a borelian probability measure on a second countable locally compact group $\mathbf{G}$ acting continuously on a topological space $\mathbf{X}$. We define a random walk on $\mathbf{X}$ starting at $x \in \mathbf{X}$ by

$$\left\{ \begin{array}{ccc} X_0 & = & x \\ X_{n+1} & = & g_{n+1} X_n \end{array} \right.$$

where $(g_n) \in \mathbf{G}^{\mathbb{N}}$ is a sequence of independent and identically distributed random variables of law ρ.

Moreover, we note P (and sometimes P_ρ to insist on the measure ρ) the Markov operator associated to ρ. This is the operator defined for any continuous function f on $\mathbf{X}$ and any $x \in \mathbf{X}$ by

$$Pf(x) = \int_{\mathbf{G}} f(gx) \mathrm{d}\rho(g)$$

In the case of $\mathbf{G} = \mathrm{SL}_d(\mathbb{R})$ acting on $\mathbb{R}^d \setminus \{0\}$, this defines a random walk that we intend to study in this article when the starting point goes to 0.

We say that a subgroup Γ of $\mathrm{SL}_d(\mathbb{R})$ acts strongly irreducibly and proximally on $\mathbb{R}^d$ (or that Γ is strongly irreducible and proximal) if it doesn't fix any finite union of proper subspaces of $\mathbb{R}^d$ and if there is some $\gamma \in \Gamma$ having an eigenvalue λ whose eigenspace V_1 is a line and such that the spectral radius of γ in the $\gamma-$invariant supplementary of V_1 in $\mathbb{R}^d$ is strictly smaller than $|\lambda|$.

If ρ has a finite first moment 1[1] and if it's support generates a strongly irreducible and proximal subgroup of $\mathrm{SL}_d(\mathbb{R})$, then, by a result of Furstenberg (see [Fur63] and also [GR85]) we have that, if $\|\,.\,\|$ is a norm on $\mathbb{R}^d$, then, for any $x \in \mathbb{R}^d \setminus \{0\}$,

$$\frac{1}{n}\ln\|g_n \dots g_1 x\| \to \lambda_\rho := \int_{\mathbf{G}}\int_{\mathbb{P}(\mathbb{R}^d)} \ln\|gx\| \mathrm{d}\rho(g)\mathrm{d}\nu(x) > 0 \quad \rho^{\otimes\mathbb{N}} - \text{a.e.} \tag{1.1}$$

where ν is a stationnary measure on $\mathbb{P}(\mathbb{R}^d)$ (that is actually unique in this case as shown in [GR85]).

In particular, this shows that the random walk on $\mathbb{R}^d \setminus \{0\}$ is transient. Thus, we can define an operator G (the Green kernel) on compactly supported continuous functions on $\mathbb{R}^d$ setting, for any $f \in \mathcal{C}_c^0(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d \setminus \{0\}$,

$$Gf(x) := \sum_{n=0}^{+\infty} P^n f(x) = \sum_{n=0}^{+\infty}\int_{\mathbf{G}} f(gx)\mathrm{d}\rho^{*n}(g) \tag{1.2}$$

The walk being transient and f having a compact support, it is clear that this function is well defined and even continuous on $\mathbb{R}^d \setminus \{0\}$ and we would like to study it's behaviour at 0. This is what we call the renewal theorem in $\mathbb{R}^d$ as an analogy to the situation on $\mathbb{R}$ (see [Bla48]).

Guivarc'h and Le Page proved in [GL12] that if T_ρ, the sub-semi-group generated by the support of ρ, fixes a proper convex cone in $\mathbb{R}^d$ then there are two stationary probability measures ν_1 and ν_2 on the sphere $\mathbb{S}^{d-1}$, the space of continuous $P-$invariant functions on the sphere has dimension two and we can choose a basis p_1, p_2 such that $p_1 + p_2 = 1$ et $p_i|_{\mathrm{supp}\,\nu_j} = \delta_{i,j}$ where we noted δ Kronecker's symbol ; on the other hand, if T_ρ doesn't fix any proper convex cone in $\mathbb{R}^d$, then there is a unique stationary measure ν_1 on $\mathbb{S}^{d-1}$ and we note p_1 the constant function that takes the value 1 on the sphere.

In both cases, we define an operator on the set of continuous functions on $\mathbb{R}^d$ vanishing at polynomial speed at infinity[2] noting, for such a function f and $x \in \mathbb{R}^d \setminus \{0\}$,

$$\Pi_0 f(x) = \sum_{i=1}^{r} p_i\left(\frac{x}{\|x\|}\right)\int_{\mathbb{S}^{d-1}}\int_{\|x\|}^{+\infty} f(uy)\frac{\mathrm{d}u}{u}\mathrm{d}\nu_i(y) \tag{1.3}$$

where, $r \in \{1, 2\}$ is the number of minimal $T_\rho-$invariant closed subsets of $\mathbb{S}^{d-1}$.

The renewal theorem becomes now the

Theorem 1.1 (Guivarc'h - Le Page in [GL12])**.** *Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ whose support generates a strongly irreducible and proximal subgroup.*

Then, for any $\gamma \in \mathbb{R}_+^*$ and any continuous function f on $\mathbb{R}^d$ such that

$$\sup_{x\in\mathbb{R}^d\setminus\{0\}} \frac{|f(x)|}{\|x\|^\gamma} \text{ and } \sup_{x\in\mathbb{R}^d}\|x\|^\gamma|f(x)| \text{ are finite}$$

we have that

$$\lim_{x\to 0}\left(G - \frac{1}{\lambda_\rho}\Pi_0\right)f(x) = 0$$

[1] i.e. $\int_{\mathrm{SL}_d(\mathbb{R})} \ln\|g\|\mathrm{d}\rho(g)$ is finite.

[2] There is $\alpha \in \mathbb{R}_+^*$ such that $\sup_{x\in\mathbb{R}^d}\|x\|^\alpha|f(x)|$ is finite.

Where λ_ρ, G and Π_0 are defined in equations 1.1, 1.2 and 1.3.

This theorem proves in particular that if f is a compactly sypported hölder continuous function on $\mathbb{R}^d$ such that $f(0) = 0$, then the function $(G - \frac{1}{\lambda_\rho}\Pi_0)f$ can be extended at 0 to a continuous function. So, the continuity of Gf at 0 is equivalent to the one of $\Pi_0 f$.

In particular, if there is a unique minimal $T_\rho-$invariant closed subset on the sphere, then

$$\lim_{x\to 0} \sum_{n=0}^{+\infty} P^n f(x) = \frac{1}{\lambda_\rho} \int_0^{+\infty} \int_{\mathbb{S}^{d-1}} f(uy) \mathrm{d}\nu(y) \frac{\mathrm{d}u}{u}$$

and in the second case, we just have a "directional" limit : for any $x \in \mathbb{R}^d \setminus \{0\}$,

$$\lim_{t\to 0^+} \sum_{n=0}^{+\infty} P^n f(tx) = \frac{1}{\lambda_\rho} \sum_{i=1}^{2} p_i \left(\frac{x}{\|x\|}\right) \int_0^{+\infty} \int_{\mathbb{S}^{d-1}} f(uy) \mathrm{d}\nu_i(y) \frac{\mathrm{d}u}{u}$$

In particular, in this case, the function Gf can't be extended to a continuous function at 0 in general.

Example 1.2. If T_ρ only contains matrices having non negative entries, then it preserves the cone $\mathcal{C}$ of vectors having non negative coefficients and also $-\mathcal{C}$. Therefore, considering an odd function f that is regular and strictly non negative on $\mathcal{C}$, we see that Gf can't be extended to a continuous function at 0.

We would like to study the modulus of continuity of Gf at 0 and to do so, we wan't to study the rate of convergence in Guivarc'h and Le Page's theorel. To simplify the study, we will consider $(G - \frac{1}{\lambda_\rho}\Pi_0)f$ and this will allow us not to care about the number of minimal $T_\rho-$invariant closed subsets of the sphere (and we will see in proposition 4.11 that this is more than a computational trick). Then, we will only have to study the modulus of continuity of $\Pi_0 f$ to get the one of Gf and, as we have a simple formula for $\Pi_0 f$, it will be very easy to get necessary and sufficient conditions for Gf to be extendable by continuity at 0.

A similar study was made in [BDP15] by Buraczewski, Damek and Przebinda ; however, their result assumes that T_ρ is actually (conjugated to) a subgroup of $\mathbb{R}_+^\star \times \mathcal{O}(d)$ and that some diophantine condition is satisfied by the projection on $\mathbb{R}_+^*$ of the measure ρ. They prove their result by going back to the situation in $\mathbb{R}$ (this is why they need this diophantine condition that is necessary in this case (see [Car83]) ; the equivalent of this condition will always hold for us as we will see in section 3).

Our study (and the one of Guivarc'h and Le Page) is in an opposit case where the subgroup generated by the support of ρ contains an element having a strictly dominant eigenvalue (this is what we called a proximal element).

More specifically, we will prove the

Theorem 1.3. *Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment[3] and whose support generates a strongly irreducible and proximal group.*

[3]There is $\varepsilon \in \mathbb{R}_+^\star$ such that $\int_{\mathrm{SL}_d(\mathbb{R})} \|g\|^\varepsilon \mathrm{d}\rho(g)$ is finite.

Then, for any $\gamma > 0$ small enough and any $M \in \mathbb{R}_+^\star$, there are $C, \alpha \in \mathbb{R}$ such that for any function $f \in \mathcal{C}^{0,\gamma}(\mathbb{R}^d)$ that vanishes on $\{x \in \mathbb{R}^d | \|x\| \geqslant M\}$ and such that $f(0) = 0$ we have that for any $x \in \mathbb{R}^d$,

$$\left| \left(G - \frac{1}{\lambda_\rho} \Pi_0 \right) f(x) \right| \leqslant \frac{C}{1 + |\ln \|x\||^\alpha} \|f\|_\gamma$$

Where λ_ρ, G and Π_0 are defined in equations 1.1, 1.2 and 1.3 and

$$\|f\|_\gamma := \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{\substack{x,y \in \mathbb{R}^d \\ x \neq y}} \frac{|f(x) - f(y)|}{\|x - y\|^\gamma}$$

If one studies the linear random walk on the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ defined by a probability measure on $\mathrm{SL}_d(\mathbb{Z})$, it appears that there are finite invariant subsets (e.g. the set 0). If $\mathbf{A}$ is one of them that is also minimal, then one can identify a neighbourhood of $\mathbf{A}$ in the torus to a neighbourhood of $\{0\} \times \mathbf{A}$ in $\mathbb{R}^d \times \mathbf{A}$.

This is why, from now on,, noting Γ_ρ the subgroup of $\mathrm{SL}_d(\mathbb{R})$ generated by the support of ρ, we study the renewal theorem on the product of $\mathbb{R}^d$ and a finite $\Gamma_\rho-$set $\mathbf{A}$ on which the walk defined by ρ is irreducible and aperiodic and we consider hölder continuous functions f on $\mathbb{R}^d \times \mathbf{A}$.

Remark that if $Gf(x,a) = \sum_{n=0}^{+\infty} P^n f(x,a)$ has a limit $g(a)$ as x converges to 0 then $(I_d - P)g(a) = f(0,a)$ and so g is a solution to the so called "Poisson's equation" for f restricted to $\mathbf{A}$ (in particular, we get that if $Gf(x,a)$ has a finite limit at $(0,a)$ then $\sum_{a \in \mathbf{A}} f(0,a) = 0$).

Remark also that for any $f \in \mathcal{C}^0(\mathbb{R}^d \times \mathbf{A})$ such that $\sum_{a \in \mathbf{A}} f(0,a) = 0$ and for any $a \in \mathbf{A}$, the function $\sum_{n=0}^{+\infty} P^n f(0,a)$ is well defined since the random walk on $\mathbf{A}$ is irreducible and aperiodic.

We are going to prove the

Theorem 1.4. *Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal subgroup Γ_ρ of $\mathrm{SL}_d(\mathbb{R})$.*

Let $\mathbf{A}$ be a non empty finite $\Gamma_\rho-$set on which the random walk defined by ρ is irreducible and aperiodic.

Then, for any $\gamma > 0$ small enough, there are $C \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+^*$ such that for any function f on $\mathbb{R}^d \times \mathbf{A}$ with

$$\|f\|_\gamma := \sup_{\substack{x,y \in \mathbb{R}^d \setminus \{0\} \\ a \in \mathbf{A}}} (1 + \|x\|)^\gamma (1 + \|y\|)^\gamma \frac{|f(x,a) - f(y,a)|}{\|x - y\|^\gamma} < +\infty,$$

and for any $a \in \mathbf{A}$,

$$\lim_{x \to +\infty} f(x,a) = 0 \text{ et } \sum_{a \in \mathbf{A}} f(0,a) = 0$$

We have that for any $a \in \mathbf{A}$,

$$\lim_{x\to 0}\left(G-\frac{1}{\lambda_\rho}\Pi_0\right)f(x,a)=\sum_{n=0}^{+\infty}P^n f(0,a)$$

Moreover, for any $x,y\in\mathbb{R}^d\setminus\{0\}$ *and any* $a\in\mathbf{A}$,

$$\left|\left(G-\frac{1}{\lambda_\rho}\Pi_0\right)f(x,a)-\left(G-\frac{1}{\lambda_\rho}\Pi_0\right)f(y,a)\right|\leqslant C\omega_0(x,y)^\alpha\|f\|_\gamma$$

where we noted, for $x,y\in\mathbb{R}^d\setminus\{0\}$,

$$\omega_0(x,y)=\frac{\sqrt{|\ln\|x\|-\ln\|y\||^2+\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2}}{(1+|\ln\|x\||)(1+|\ln\|y\||)}$$

Remark 1.5. The definition of ω_0 may seem complicate but we will see that it is a kind of conical distance on $\mathbb{R}^d$: we contract a neighbourhood of 0 and of infinity. The reader may go to section 4.4 to get more details.

Remark 1.6. The hypothesis on f is that there is a constant C such that for any $x,y\in\mathbb{R}^d$ and any $a\in\mathbf{A}$,

$$|f(x,a)-f(y,a)|\leqslant C\left(\frac{\|x-y\|}{(1+\|x\|)(1+\|y\|)}\right)^\gamma$$

In particular, compactly supported hölder continuous functions on $\mathbb{R}^d\times\mathbf{A}$ verify this condition. We do not study only these functions since the condition will become very natural after we identify $\mathbb{R}^d\setminus\{0\}$ with $\mathbb{R}\times\mathbb{S}^{d-1}$ in section 4.

Remark 1.7. As we already said, it is the continuity of Gf that interests us but it is easy to get the one of $\Pi_0 f$ and evaluate a modulus of continuity (if it is continuous).

For any metric space $(\mathbf{X},d)$ and any $\gamma\in]0,1]$, we note $\mathcal{C}^{0,\gamma}(\mathbf{X})$ the space of hölder-continuous functions on $\mathbf{X}$. These are the complex valued functions f on $\mathbf{X}$ such that

$$\|f\|_\gamma:=\|f\|_\infty+m_\gamma(f)$$

is finite where we set

$$\|f\|_\infty:=\sup_{x\in\mathbf{X}}|f(x)|\text{ and }m_\gamma(f):=\sup_{\substack{x,y\in\mathbf{X}\\x\neq y}}\frac{|f(x)-f(y)|}{d(x,y)^\gamma}$$

To prove this theorem, we will study the analytic family of operators (see section 4) on $\mathcal{C}^{0,\gamma}(\mathbb{S}^{d-1}\times\mathbf{A})$ defined for $z\in\mathbb{C}$ with $|\Re(z)|$ small enough, for any $f\in\mathcal{C}^{0,\gamma}(\mathbb{S}^{d-1}\times\mathbf{A})$ and any (x,a) in $\mathbb{S}^{d-1}\times\mathbf{A}$ by

$$P(z)f(x,a)=\int_{\mathbf{G}}e^{-z\ln\frac{\|gx\|}{\|x\|}}f(gx,ga)\mathrm{d}\rho(g)$$

Indeed, we will see in section 4 that the rate of convergence in the renewal theorem is closely related to the growth of $\|(I_d-P(z))^{-1}\|$ along the imaginary axis.

To get a control of $\|(I_d - P(it))^{-1}\|_{\mathcal{C}^{0,\gamma}(\mathbb{S}^{d-1}\times\mathbf{A})}$ for large values of t we use in section 2 the method employed by Dolgopyat in [Dol98] for Ruelle operators and we show proposition 2.23 that links $\|(I_d - P(it))^{-1}\|$ to the diophantine properties of the logarithm of the spectral radius of elements of T_ρ.

Then, we will prove that in a strongly irreducible and proximal subgroup of $\mathrm{SL}_d(\mathbb{R})$, we can construct elements whose spectral radius have good diophantine properties. This will be the aim of section 3 and more specifically of 3.16.

These two sections will allow us to prove the

Theorem (3.1)**.** *Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal subgroup Γ of $\mathrm{SL}_d(\mathbb{R})$.*

Let $(\mathbf{A}, \nu_{\mathbf{A}})$ be a non empty finite $\Gamma-$set endowed with the uniform probability measure and on which the random walk defined by ρ is irreducible and aperiodic.

Let, for any $t \in \mathbb{R}$, $P(it)$ be the operator defined on $\mathcal{C}^0(\mathbb{S}^{d-1} \times \mathbf{A})$ by

$$P(it)f(x,a) = \int_{\mathbf{G}} e^{-it \ln \frac{\|gx\|}{\|x\|}} f(gx, ga) \mathrm{d}\rho(g)$$

Then, for any $\gamma > 0$ small enough and any $t_0 \in \mathbb{R}_+^*$, there are $C, L \in \mathbb{R}_+$ such that for any $t \in \mathbb{R}$ with $|t| \geqslant t_0$,

$$\|(I_d - P(it))^{-1}\|_{\mathcal{C}^{0,\gamma}(\mathbb{S}^{d-1}\times\mathbf{A})} \leqslant C|t|^L$$

Moreover, the constant L only depend on $\mathbf{A}$ through the spectral gap[4] of P in $\mathrm{L}^2(\mathbf{A}, \nu_{\mathbf{A}})$.

Finally, in section 4, we will use this theorem to get the rate of convergence in the renewal theorem.

1.2. **Proofs of the main results using the results of the other sections.**
In this sub-section, we prove the results we stated in the introduction of this paper with the results we will prove in the following sections.

Proof of theorem 1.3 from theorem 1.4.

Let f be a compactly supported $\gamma-$hölder continuous function on $\mathbb{R}^d$.

Then,

$$\sup_{x,y\in\mathbb{R}^d\setminus\{0\}} (1+\|x\|)^\gamma (1+\|y\|)^\gamma \frac{|f(x)-f(y)|}{\|x-y\|^\gamma} \text{ is finite}$$

Therefore, we may apply theorem 1.4 to find constants $C, \alpha \in \mathbb{R}_+^\star$ (with C depending on M) such that for any compactly supported $\gamma-$hölder continuous function f on $\mathbb{R}^d$ such that $f(0) = 0$ and $f(x) = 0$ on $B(0,M)^c$ and any $x, y \in \mathbb{R}^d \setminus \{0\}$,

$$\left| \left(G - \frac{1}{\lambda_\rho}\Pi_0\right) f(x) - \left(G - \frac{1}{\lambda_\rho}\Pi_0\right) f(y) \right| \leqslant C\|f\|_\gamma \omega_0(x,y)^\alpha$$

and

$$\lim_{y\to 0} \left(G - \frac{1}{\lambda_\rho}\Pi_0\right) f(y) = 0$$

[4]The spectral gap of P is the spectral radius of P in the orthogonal of the constant functions in $\mathrm{L}^2(\mathbf{A}, \nu_{\mathbf{A}})$.

But, as we also have that

$$\lim_{y\to 0}\omega_0(x,y)=\frac{1}{1+|\ln\|x\||}$$

we get theorem 1.3. □

Proof of theorem 1.4.

This theorem is a straightforward application of theorem 4.1.

Indeed, noting $\mathbf{X}=\mathbb{S}^{d-1}\times\mathbf{A}$ and $\mathbf{H}=\{I_d,\vartheta\}$ where ϑ is the antipodal map on the sphere and identity on $\mathbf{A}$, we get that $\mathbf{H}$ acts isometrically on $\mathbf{X}\times\mathbf{A}$ and $(\mathbf{X}\times\mathbf{A})/\mathbf{H}$, that we identify to the product of the projective space and of $\mathbf{A}$ is $(\rho,\gamma,M,N)-$contracted over $\mathbf{A}$ (see example 2.4). Moreover, in section 3, we saw that the cocycle σ defined for $(g,\mathbb{R}x)\in \mathrm{SL}_d(\mathbb{R})\times\mathbb{P}(\mathbb{R}^d)$ by $\sigma(g,\mathbb{R}x)=\ln\frac{\|gx\|}{\|x\|}$ also belongs to $\mathcal{Z}^M(\mathbb{P}(\mathbb{R}^d))$ and the result of Furstenberg that we already saw implies that $\sigma_\rho>0$.

Moreover, we saw in theorem 3.1 that for any $b_0\in\mathbb{R}_+^*$ there are constants C,L such that for any $b\in\mathbb{R}$ with $|b|\geqslant b_0$,

$$\|(I_d-P(ib))^{-1}\|\leqslant C|b|^L$$

This proves that we really can apply theorem 4.1 to any function f that satisfies the condition of the theorem since any such function can be identified with a function on $\mathcal{C}^\gamma_\omega(\mathbf{X}\times\mathbb{R})$ such that $\sum_{a\in\mathcal{A}}\lim_{x\to-\infty}f(x,a)=0$ and $\lim_{x\to+\infty}f(x,a)=0$ through the application $(x,t)\mapsto e^tx$ from $\mathbb{S}^{d-1}\times\mathbb{R}$ to $\mathbb{R}^d\setminus\{0\}$. □

2. Unitary perturbations of Markov operators

Let ρ be a borelian probability measure on $\mathbb{R}$ having an exponential moment and a drift $\lambda=\int_{\mathbb{R}}y\mathrm{d}\rho(y)>0$.

In [Car83], Carlsson showed that the problem of the rate of convergence in the renewal theorem was linked to the problem of finding $l\in\mathbb{R}$ such that

$$\limsup_{t\to\pm\infty}\frac{1}{|t|^l}\left|1-\int_{\mathbb{R}}e^{ity}\mathrm{d}\rho(y)\right|^{-1}<+\infty$$

And this is closely related to to the diophantine properties of the support of ρ (see e.g. [Bre05] where a stronger but of the same kind of hypothesis is studied).

In particular, if such a parameter exists, then the speed in the renewal theorem is polynomial. If we can even take $l=0$ (which is the case if one of the powers of convolution of ρ is not singular with respect to Lebesgue's measure on $\mathbb{R}$), then we can have an exponential rate (cf. [BG07]).

In this section, we study a group $\mathbf{G}$ acting continuously on a compact metric space $(\mathbf{X},d)$, a function $\sigma:\mathbf{G}\times\mathbf{X}\to\mathbb{R}$ and we study the family of operators $(P(it))_{t\in\mathbb{R}}$ defined for any $t\in\mathbb{R}$, any continuous function f on $\mathbf{X}$ and any $x\in\mathbf{X}$ by

$$P(it)f(x)=\int_{\mathbf{G}}e^{it\sigma(g,x)}f(gx)\mathrm{d}\rho(g)$$

We simply note P or sometimes P_ρ the operator $P(0)$.

The existence of some $l \in \mathbb{R}$ such that

$$\limsup_{t\to\pm\infty} \frac{1}{|t|^l}\|(I_d - P(it))^{-1}\| \text{ is finite.}$$

will be the equivalent of Carlsson's assumption in our context. The norm will be taken for us in a space of hölder continuous functions on $\mathbf{X}$;

To do so, we adapt a result stated in [Dol98] for Ruelle's operators. This will be our proposition 2.23 which is the aim of this section.

2.1. **Preliminaries.**

To be able to state proposition 2.23, we introduce in this section many technical definitions.

2.1.1. *Contracting actions.* We say that a $\mathbf{G}$-space $\mathbf{X}$ is fibered over some other $\mathbf{G}$-space $\mathbf{A}$ is there is a continuous function $\pi_{\mathbf{A}} : \mathbf{X} \to \mathbf{A}$ that is $\mathbf{G}$-equivariant : for any x of $\mathbf{X}$ and any g of $\mathbf{G}$,

$$\pi_{\mathbf{A}}(gx) = g\pi_{\mathbf{A}}(x)$$

Definition 2.1 (Contracting actions)**.** Let $\mathbf{G}$ be a second countable locally compact group, $N : \mathbf{G} \to [1, +\infty[$ a sub-multiplicative function on $\mathbf{G}$ and $(\mathbf{X}, d)$ a compact metric space endowed with a continuous action of $\mathbf{G}$.

We assume that $\mathbf{X}$ is fibered over the finite $\mathbf{G}$-set $\mathbf{A}$.

Let ρ be a borelian probability measure on $\mathbf{G}$ and $\gamma, M \in \mathbb{R}_+^*$.

We say that $\mathbf{X}$ is $(\rho, \gamma, M, N)-$contracted over $\mathbf{A}$ if

(1) For any $g \in \mathbf{G}$ and any $x, y \in \mathbf{X}$,

$$d(gx, gy) \leqslant MN(g)^M d(x, y) \tag{2.1}$$

(2)

$$\int_G N(g)^{M\gamma} \mathrm{d}\rho(g) \text{ est finie} \tag{2.2}$$

(3) For some $n_0 \in \mathbb{N}^*$ we have that

$$\sup_{\substack{x,y\in\mathbf{X}\\ x\neq y\\ \pi_{\mathbf{A}}(x)=\pi_{\mathbf{A}}(y)}} \int_{\mathbf{G}} \frac{d(gx, gy)^\gamma}{d(x, y)^\gamma} \mathrm{d}\rho^{*n_0}(g) < 1$$

where $\pi_{\mathbf{A}} : \mathbf{X} \to \mathbf{A}$ is the $\mathbf{G}$-equivariant projection.

Remark 2.2. If $\mathbf{X}$ is $(\rho, \gamma, M, N)-$contracted over $\mathbf{A}$, then the operator P is continuous on the space $\mathcal{C}^{0,\gamma}(\mathbf{X})$ of $\gamma-$hölder-continuous functions on $\mathbf{X}$.

Remark 2.3. This notion is now classical in the study of random walk on reductive spaces and the reader will find more details in [BQ15].

We could have defined $N(g)$ as being the maximum $d(gx, gy)/d(x, y)$ (assuming that it is finite) since this defines a submultiplicative function on $\mathbf{G}$; however in our application, there will be natural function N associated to $\mathbf{G}$ (see lemma 3.2).

Example 2.4. Our main example will be the case where $\mathbf{G}$ is a strongly irreducible and proximal subgroup of $\mathrm{SL}_d(\mathbb{R})$, ρ is a borelian probability measure on $\mathbf{G}$ having an exponential moment and whose support generates $\mathbf{G}$ and $\mathbf{X}$ is the product of the projective space $\mathbb{P}(\mathbb{R}^d)$ (which is contracted according to the theorem V in [BL85]) and a finite $\mathbf{G}-$set $\mathbf{A}$ endowed with the discrete distance (for any $a, a' \in \mathbf{A}$, $d(a,a') = 0$ if $a = a'$ and 1 otherwise).

Remark that the sequence (u_n) defined for any $n \in \mathbb{N}$ by

$$u_n = \sup_{\substack{x,y\in\mathbf{X}\\ x\neq y\\ \pi_{\mathbf{A}}(x)=\pi_{\mathbf{A}}(y)}} \int_{\mathbf{G}} \frac{d(gx,gy)^\gamma}{d(x,y)^\gamma} \mathrm{d}\rho^{*n}(g)$$

being submultiplicative, if $\mathbf{X}$ is $(\rho,\gamma,M,N)-$contracted over $\mathbf{A}$, then there are constants $C_1, \delta \in \mathbb{R}_+^*$ such that for any $n \in \mathbb{N}$ and any $x, y \in \mathbf{X}$ with $\pi_{\mathbf{A}}(x) = \pi_{\mathbf{A}}(y)$,

$$\int_{\mathbf{G}} d(gx,gy)^\gamma \mathrm{d}\rho^{*n}(g) \leqslant C_1 e^{-\delta n} d(x,y)^\gamma \tag{2.3}$$

Remark also that if $\gamma' \in]0,\gamma]$ then the function $t \mapsto t^{\gamma'/\gamma}$ is concave on $[0, \mathrm{Diam}(\mathbf{X})]$ and so, if the space $\mathbf{X}$ is $(\rho,\gamma,M,N)-$contracted, it is also $(\rho,\gamma',M,N)-$contracted.

Let $\mathbf{X}$ be a compact metric space and P a continuous operator on $\mathcal{C}^0(\mathbf{X})$ such that for any non negative continuous function f on $\mathbf{X}$, Pf is non negative. We say that the operator P is *equicontinuous* if it is power bounded[5] and if for any $f \in \mathcal{C}^0(\mathbf{X})$, the sequence $(P^n f)_{n\in\mathbb{N}}$ is equicontinuous. We refer to [Rau92] (see also [BQ14]) for a study of these operators.

Proposition 2.5. *Let $\mathbf{G}$ be a second countable locally compact group, $N : \mathbf{G} \to [1, +\infty[$ a submultiplicative function on $\mathbf{G}$ and ρ a borelian probability measure on $\mathbf{G}$.*

Let $(\mathbf{X}, d)$ be a compact metric space endowed with a continuous action of $\mathbf{G}$ and which is $(\rho,\gamma,M,N)-$contracted over a finite $\mathbf{G}-$set $\mathbf{A}$.

Then, the Markov operator P associated to ρ is equicontinuous on $\mathcal{C}^0(\mathbf{X})$.

Moreover, if the random walk defined by ρ on $\mathbf{A}$ is irreducible and aperiodic then there is a unique $P_\rho-$stationary probability measure on $\mathbf{X}$ and 1 is the unique eigenvalue of P of modulus 1, and it's eigenspace is a line.

Before we prove the proposition, we recall a result on Markov chains defined by a group action on a finite state space.

Lemma 2.6. *Let $\mathbf{G}$ be a second countable, locally compact group acting on a finite set $\mathbf{A}$ and let ρ be a borelian probability measure on $\mathbf{G}$ such that then random walk on $\mathbf{A}$ defined by ρ is irreducible and aperiodic.*

Then, $\nu_{\mathbf{A}}$, the uniform probability measure on $\mathbf{A}$, is the unique $P_\rho-$stationary probability measure on $\mathbf{A}$ and the operator P has a spectral radius smaller than 1 in the orthogonal of constant functions in $\mathrm{L}^2(\mathbf{A}, \nu_{\mathbf{A}})$.

[5] i.e. $\sup_n \|P^n\|_\infty$ is finite.

Proof of proposition 2.5. The equicontinuity of P in $\mathcal{C}^0(\mathbf{X})$ can be proved as in the case of $\mathrm{SL}_d(\mathbb{R})$ acting on $\mathbb{P}(\mathbb{R}^d)$ studied in [BQ14]. We will prove it with more details in 2.12 where the space is only locally contracted.

Let f be a continuous function such that $Pf = \lambda f$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.
For any $x, y \in \mathbf{X}$ such that $\pi_{\mathbf{A}}(x) = \pi_{\mathbf{A}}(y)$, we have

$$\lambda^n(f(x) - f(y)) = P^n f(x) - P^n f(y) = \int_{\mathbf{G}} f(gx) - f(gy) \mathrm{d}\rho^{*n}(g)$$

But, $\mathbf{X}$ is contracted over $\mathbf{A}$ and $|\lambda| = 1$, so we get that for any $x, y \in \mathbf{X}$ with $\pi_{\mathbf{A}}(x) = \pi_{\mathbf{A}}(y)$, $f(x) = f(y)$.

Therefore, eigenvectors of P in $\mathcal{C}^0(\mathbf{X})$ associated to eigenvalues of modulus 1 can be identified to functions on $\mathbf{A}$. But, as we assumed that the random walk on $\mathbf{A}$ defined by ρ is irreducible and aperiodic, we get that the eigenvectors of P associated to eigenvalues of modulus 1 are constants on $\mathbf{X}$ (see lemma 2.6).

Finally, using propositions 3.2 and 3.3 of [Rau92], we get that the stationary measure ν is unique, that 1 is a simple eigenvalue and that there are no other eigenvalue of modulus 1. □

The previous lemma motivates the following

Definition 2.7 (Spectral gap)**.** Let $\mathbf{G}$ be a second countable locally compact group acting continuously on the probability space $(\mathbf{X}, \nu)$ preserving the measure ν and let ρ be a borelian probability measure on $\mathbf{G}$. Note P the Markov operator on $\mathbf{X}$ defined by the measure ρ.

Note $\mathrm{L}^2_0(\mathbf{X}, \nu) := \{f \in \mathrm{L}^2(\mathbf{X}, \nu) | \int f \mathrm{d}\nu = 0\}$ be the orthogonal of constant functions in $\mathrm{L}^2(\mathbf{X}, \nu)$.

Note r the spectral radius of P in $\mathrm{L}^2_0(\mathbf{X}, \nu)$.

We say that P has a spectral gap in $\mathrm{L}^2(\mathbf{X}, \nu)$ if $r < 1$ and in this case we call $1 - r$ the spectral gap of P.

We shall now extend to our context the theorem 2.5 in chapter V in [BL85] that shows that when the space is contracted, the operator has a spectral gap in the space of hölder continuous functions. This is the aim of the next

Proposition 2.8. *Let $\mathbf{G}$ be a second countable locally compact group, $N : \mathbf{G} \to [1, +\infty[$ a submultiplicative function on $\mathbf{G}$ and let ρ be a borelian probability measure on $\mathbf{G}$.*

Let $(\mathbf{X}, d)$ be a compact metric space endowed with a continuous action of $\mathbf{G}$ and which is $(\rho, \gamma, M, N)-$contracted over a finite $\mathbf{G}-$set $\mathbf{A}$ on which the random walk defined by ρ is irreducible and aperiodic.

Let ν be the unique $P_\rho-$stationary probability measure on $\mathbf{X}$ (given by proposition 2.5).

Then, there are constants $\kappa, C_0 \in \mathbb{R}^*_+$ such that for any $n \in \mathbb{N}$,

$$\left\| P^n_\rho - \Pi_\nu \right\|_{\mathcal{C}^{0,\gamma}(\mathbf{X})} \leqslant C_0 e^{-\kappa n}$$

where Π_ν is the operator defined on $\mathcal{C}^0(\mathbf{X})$ by

$$\Pi_\nu(f) = \int_{\mathbf{X}} f \mathrm{d}\nu$$

Moreover, κ depends on $\mathbf{A}$ only through the spectral radius of P in the orthogonal of the constants function in $\mathrm{L}^2(\mathbf{A})$.

Remark 2.9. This proposition could be seen as a corollary of the quasi-compactness of P in $\mathcal{C}^{0,\gamma}(\mathbf{X})$ that we will prove in proposition 2.12 and of the fact that in $\mathcal{C}^0(\mathbf{X})$, 1 is the only eigenvalue of modulus 1 and it's eigenspace is the set of constant functions on $\mathbf{X}$. However, we state it this way to get the link between the spectral radius of P in $\mathrm{L}^2(\mathbf{A})$ and the one of P in $\mathcal{C}^{0,\gamma}(\mathbf{X})$.

Proof. First, we note $C_{\mathbf{A}}, \kappa_{\mathbf{A}} \in \mathbb{R}_+^\star$ such that for any $f \in \mathrm{L}^\infty(\mathbf{A})$ and any $n \in \mathbb{N}$,

$$\left\| P^n f - \int f \mathrm{d}\nu_{\mathbf{A}} \right\|_\infty \leqslant C_{\mathbf{A}} e^{-\kappa_{\mathbf{A}} n} \|f\|_\infty$$

where $\nu_{\mathbf{A}}$ is the uniform probability measure on $\mathbf{A}$ (the existence of $C_{\mathbf{A}}$ and $\kappa_{\mathbf{A}}$ are given by lemma 2.6).

Let $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$, $x, y \in \mathbf{X}$ such that $\pi_{\mathbf{A}}(x) = \pi_{\mathbf{A}}(y)$ and $n \in \mathbb{N}$. We shall assume without any loss of generality that f is real-valued. Then, for any $n \in \mathbb{N}$, we compute

$$|P^n f(x) - P^n f(y)| \leqslant m_\gamma(f) \int_{\mathbf{G}} d(gx, gy)^\gamma \mathrm{d}\rho^{*n}(g) \leqslant m_\gamma(f) C_1 e^{-\delta n} d(x, y)^\gamma$$

where we noted C_1, δ the constants given in equation 2.3.

This proves that for any $n \in \mathbb{N}$,

$$m_\gamma(P^n f) \leqslant C_1 e^{-\delta n} m_\gamma(f)$$

Where we noted ν the unique $P-$stationary probability measure on $\mathbf{X}$ (given by proposition 2.5).

Moreover, for any $x \in \mathbf{X}$ and any non zero integer n we note ν_x the measure defined by

$$\int \varphi(y) \mathrm{d}\nu_x(y) = |\mathbf{A}| \int_{\mathbf{X}} \mathbf{1}_{\pi_{\mathbf{A}}(x) = \pi_{\mathbf{A}}(y)} \varphi(y) \mathrm{d}\nu(y)$$

and also for any function $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$, we note

$$f_1^n(x) = \int_{\mathbf{X}} P^n f(y) \mathrm{d}\nu_x(y) \text{ and } f_2^n(x) = P^n f(x) - f_1^n(x)$$

Then, as f_2^n is continuous, real-valued and $\int_{\mathbf{X}} f_2^n(y) \mathrm{d}\nu_x(y) = 0$, we have that for any $x \in \mathbf{X}$, there is $y \in \mathbf{X}$ such that $\pi_{\mathbf{A}}(x) = \pi_{\mathbf{A}}(y)$ and $f_2^n(y) = 0$. But,

$$m_\gamma(f_2^n) = m_\gamma(P^n f)$$

and so, noting $\mathrm{Diam}(\mathbf{X})$ the diameter of $\mathbf{X}$, we get that

$$\|f_2^n\|_\infty \leqslant \mathrm{Diam}(\mathbf{X})^\gamma m_\gamma(f_2^n) = \mathrm{Diam}(\mathbf{X})^\gamma m_\gamma(P^n f)$$

Moreover, as,

$$P^{2n} f(x) = P^n f_2^n(x) + P^n f_1^n(x)$$

We have the inequalities

$$\begin{aligned}\left|P^{2n}f(x)-\int_{\mathbf{A}}f_1^n(a)\mathrm{d}\nu_{\mathbf{A}}(a)\right| &\leqslant \|f_2^n\|_\infty+\left|P^nf_1^n(x)-\int_{\mathbf{A}}f_1^n(a)\mathrm{d}\nu_{\mathbf{A}}(a)\right|\\ &\leqslant \mathrm{Diam}(\mathbf{X})^\gamma C_1e^{-\delta n}m_\gamma(f)+C_{\mathbf{A}}e^{-\kappa_{\mathbf{A}}n}\|P^nf_1^n\|_\infty\\ &\leqslant \left(\mathrm{Diam}(\mathbf{X})^\gamma C_1e^{-\delta n}+C_{\mathbf{A}}e^{-\kappa_{\mathbf{A}}n}\right)\|f\|_\gamma\end{aligned}$$

And finally, using Fubini's theorem, we have

$$\int_{\mathbf{A}}f_1^n(a)\mathrm{d}\nu_{\mathbf{A}}(a)=\int_{\mathbf{X}}f(y)\mathrm{d}\nu(y)$$

This inequality ends the proof of the lemma since we also have that

$$m_\gamma(P^nf)\leqslant C_1e^{-\delta n}m_\gamma(f)$$

And so,

$$\left\|P^{2n}f-\int f\mathrm{d}\nu\right\|_\gamma\leqslant\left(CC_1e^{-\delta n}+C_1e^{-2\delta n}+C_{\mathbf{A}}e^{-\kappa_{\mathbf{A}}n}\right)\|f\|_\gamma$$

So we note $\kappa=\frac{1}{2}\min(\delta,\kappa_{\mathbf{A}})$ et $C_0=(1+C)C_1+1$. □

2.1.2. *Fibered actions over a contracting action.*

We now study the case where the space is only locally contracted and we try to recover the results of the previous section.

To study the action of $\mathrm{SL}_d(\mathbb{R})$ on the sphere and not only on the projective space, the notion of contracted actions is not enough anymore (since the sphere is not contracted). However, it is the only obstruction and if we note θ the antipodal map on the sphere (defined by $\vartheta(x)=-x$) we have that ϑ commutes to the action of $\mathbf{G}$ and so, noting $\mathbf{H}=\{I_d,\theta\}$, we have the identification $\mathbb{S}^{d-1}/\mathbf{H}\sim\mathbb{P}^d$ and the projective space is $(\rho,\gamma,M,N)-$contracted (if ρ has an exponential moment and supp ρ generated a strongly irreducible and proximal subgroup of $\mathrm{SL}_d(\mathbb{R})$) as we already saw in example 2.4.

This is why, from now on, we will consider a compact metric $\mathbf{G}-$space $\mathbf{X}$ endowed with an action of a finite group $\mathbf{H}$ that commutes to the action of $\mathbf{G}$ and such that the quotient space $\mathbf{X}/\mathbf{H}$ (endowed with the quotient metric) is contracted. At first glance, the reader can always assume that $\mathbf{G}=\mathrm{SL}_d(\mathbb{R})$, $\mathbf{X}=\mathbb{S}^{d-1}$, $\mathbf{H}=\{I_d,\theta\}$ and $\mathbf{X}/\mathbf{H}=\mathbb{P}(\mathbb{R}^d)$.

The first step is to recover proposition 2.5 and proposition 2.8.
To do so, we will use the following

Theorem 2.10 (Ionescu-Tulcea et Marinescu [ITM50])**.** *Let $(\mathcal{B},\|\,.\,\|_{\mathcal{B}})$ be a Banach space and assume that there is a norm $\|\,.\,\|$ on $\mathcal{B}$ such that the identity map from $(\mathcal{B},\|\,.\,\|_{\mathcal{B}})$ to $(\mathcal{B},\|\,.\,\|)$ is compact.*

Let P be a continuous operator on $\mathcal{B}$ such that for some $r,R\in\mathbb{R}_+$ we have that for any $f\in\mathcal{B}$,

$$\|Pf\|_{\mathcal{B}}\leqslant r\|f\|_{\mathcal{B}}+R\|f\|$$

Then, the essential spectral radius of P in $(\mathcal{B},\|\,.\,\|_{\mathcal{B}})$ is smaller than r.

Example 2.11. In our examples, $(\mathcal{B}, \|\,.\,\|_{\mathcal{B}})$ will be a space of hölder continuous functions endowed with it's Banach-space norm and $\|\,.\,\|$ will be the uniform norm.

Proposition 2.12. *Let $\mathbf{G}$ be a secong countable locally compact group, $N : \mathbf{G} \to [1, +\infty[$ a submultiplicative function on $\mathbf{G}$ and ρ a borelian probability measure on $\mathbf{G}$.*

Let $(\mathbf{X}, d)$ be a compact metric space endowed with a continuous action of $\mathbf{G}$ and of an action of finite group $\mathbf{H}$ that commutes to the $\mathbf{G}$-action and such that $\mathbf{X}/\mathbf{H}$ is $(\rho, \gamma, M, N)-$contracted over a finite $\mathbf{G}-$set $\mathbf{A}$.

Then, there are $C', \delta' \in \mathbb{R}_+^*$ such that for any function $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$ and any $n \in \mathbb{N}$,

$$m_\gamma(P^n f) \leqslant C' \left(e^{-\delta' n} m_\gamma(f) + \|f\|_\infty \right)$$

In particular, P is equicontinuous on $\mathcal{C}^0(\mathbf{X})$ and it's essential spectral radius in $\mathcal{C}^{0,\gamma}(\mathbf{X})$ is strictly smaller than 1.

Proof. We do not prove this here since we will do it in a more general setting in proposition 2.18 (the operator P being perturbed by a cocycle). □

Finally, we study the eigenvalues of P of modulus 1 in $\mathcal{C}^0(\mathbf{X})$. To do so, we begin by studying the $P-$stationary probability measures on $\mathbf{X}$ then, we will see that, contrary to the case where the space is contracted, there can be eigenvalue of modulus 1 that are not 1 and even invariant functions that are not constant.

This study will tell us why, in the renewal theorem, the cone hypothesis is necessary and where it comes from.

Lemma 2.13. *Let $\mathbf{G}$ be a second countable locally compact group, $N : \mathbf{G} \to [1, +\infty[$ a submultiplicative function on $\mathbf{G}$ and ρ a borelian probability measure on $\mathbf{G}$.*

Let $(\mathbf{X}, d)$ be a compact metric space endowed with a continuous action of $\mathbf{G}$ and of an action of a finite group $\mathbf{H}$ that commutes to the $\mathbf{G}$-action and such that $\mathbf{X}/\mathbf{H}$ is $(\rho, \gamma, M, N)-$contracted over a finite $\mathbf{G}-$set $\mathbf{A}$ on which the random walk defined by ρ is irreducible and aperiodic.

Then, there are at most $|\mathbf{H}|$ minimal closed subsets of $\mathbf{X}$ (for the action of T_ρ, the semi-group generated by the support of ρ) that we note $\Lambda_1, \dots, \Lambda_r$. Each is associated to a $P-$stationary probability measure ν_i with $\operatorname{supp} \nu_i = \Lambda_i$.

Moreover, for any $x \in \mathbf{X}$ and $\rho^{\otimes\mathbb{N}}-$a.e. $(g_n) \in \mathbf{G}^{\mathbb{N}}$, the sequence

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{g_k \dots g_1 x}$$

converges to one of the measures ν_i and if we note, for $i \in [1, r]$,

$$p_i(x) = \rho^{\otimes\mathbb{N}} \left(\left\{ (g_n) \middle| \frac{1}{n} \sum_{k=0}^{n-1} \delta_{g_k \dots g_1 x} \overset{*}{\rightharpoonup} \nu_i \right\} \right)$$

We have that the function p_i is continuous, $P-$invariant, $\sum_i p_i = 1$ and $p_i = \delta_{i,j}$ on Λ_j (where $\delta_{i,j}$ is Kronecker's symbol).

Finally, for any continuous function f on $\mathbf{X}$ and any $x \in \mathbf{X}$,

$$\frac{1}{n} \sum_{k=0}^{n-1} P^k f(x) \xrightarrow[n \to +\infty]{} \sum_{i=1}^{r} p_i(x) \int_{\mathbf{X}} f \mathrm{d}\nu_i$$

Proof. Let Λ be a minimal closed subset of $\mathbf{X}$ (there exists at least one since $\mathbf{X}$ is compact) and let $h \in \mathbf{H}$. Then,

$$P(\mathbf{1}_{h\Lambda})(x) = \int_{\mathbf{G}} \mathbf{1}_{h\Lambda}(gx)\mathrm{d}\rho(g) = \int_{\mathbf{G}} \mathbf{1}_{\Lambda}(gh^{-1}x) = P(\mathbf{1}_{\Lambda})(h^{-1}x)$$

So, $h\Lambda$ also is a minimal closed subset. This proves that $\mathbf{H}\Lambda$ is also $T_\rho-$invariant. But, this time it is $\mathbf{H}-$invariant and so $\pi_{\mathbf{H}}(\mathbf{H}\Lambda)$ is a minimal closed subset of P seen as an operator on $\mathcal{C}^{0,\gamma}(\mathbf{X}/\mathbf{H})$. But, this minimal closed subset is nique since P is contracting on $\mathbf{X}/\mathbf{H}$ avec $\mathbf{A}$ and the random walk defined by ρ on $\mathbf{A}$ is irreducible and aperiodic (see proposition 2.5). This proves that the set $\mathbf{H}\Lambda$ is unique and so, there are at most $|\mathbf{H}|$ minimal closed invariant subsets and $\mathbf{H}$ acts transitively on them.

We note $\Lambda_1, \dots, \Lambda_r$ these minimal closed subsets and Λ their reunion.

But, we saw in proposition 2.12 that P is equicontinuous and using propositions 3.2 and 3.3 of [Rau92], we get that there are at most r linearly independent continuous $P-$invariant functions $p_1, \dots p_r$ with $p_j = \delta_{i,j}$ on Λ_i. Moreover, if we note ν_i the $P-$stationary probability measure on Λ_i, we have that for any $f \in \mathcal{C}^0(\mathbf{X})$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} P^k f(x) = \sum_{i=1}^{r} p_i(x) \int f \mathrm{d}\nu_i$$

To conclude, we just need to prove that the functions p_i are indeed the ones we defined.

First, the fact that $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_k \dots g_1 x}$ converges a.e. for any $x \in \mathbf{X}$ to one of the ν_i is a consequence of the equicontinuity of P and of the propositions of Raugi that we already used.

And the fact that the function p_i that we defined is $P-$invariant comes from inequality 2.11 in [BQ14]. So, we shall conclude by unicity of the p_i. □

2.1.3. *Lazy random walks.* Let $\mathbf{G}$ be a second countable locally compact group. For a borelian probability measure ρ on $\mathbf{G}$, it will be useful to introduce the lazy random walk. This is the one associated to the measure :

$$\rho_{\mathbf{e}} = \frac{1}{2}\delta_{\mathbf{e}} + \frac{1}{2}\rho \tag{2.4}$$

The main interest of this measure is that $(\operatorname{supp} \rho_{\mathbf{e}}^{*n})_{n\in\mathbb{N}}$ is a non decreasing sequence. Moreover, for any $\lambda \in \mathbb{C}$,

$$\lambda I_d - P_{\rho_{\mathbf{e}}} = \frac{1}{2}((2\lambda - 1)I_d - P_\rho)$$

Thus, the spectral values of $P_{\rho_{\mathbf{e}}}$ and the ones of P_ρ are linked (in particular, for $\lambda = 1$, we get that $I_d - P_{\rho_{\mathbf{e}}} = \frac{1}{2}(I_d - P_\rho)$ therefore, $I_d - P_{\rho_{\mathbf{e}}}$ is invertible if and only if $I_d - P_\rho$ is).

The following lemma proves that the measure $\rho_{\mathbf{e}}$ keeps some other properties of ρ.

Lemma 2.14. *Let $\mathbf{G}$ be a second countable locally compact group and ρ a borelian probability measure on $\mathbf{G}$.*

Let $(\mathbf{X}, d)$ be a compact metric space endowed with a continuous action of $\mathbf{G}$ and which is contacted over a finite $\mathbf{G}$-set $\mathbf{A}$.

Then, $\mathbf{X}$ is also $(\rho_{\mathbf{e}}, \gamma, M, N)-$contracted over $\mathbf{A}$.

Proof. It is clear that the first two properties are satisfied by $\rho_{\mathbf{e}}$.

Moreover, for any $n \in \mathbb{N}$, we have that

$$\rho_{\mathbf{e}}^{*n} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \rho^{*k}$$

And so, for any $x, y \in \mathbf{X}$ such that $x \neq y$ and $\pi_{\mathbf{A}}(x) = \pi_{\mathbf{A}}(y)$ and any $n \in \mathbb{N}$,

$$\begin{aligned} \int_{\mathbf{G}} \frac{d(gx, gy)^\gamma}{d(x,y)^\gamma} \mathrm{d}\rho_{\mathbf{e}}^{*n}(g) &= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \int_{\mathbf{G}} \frac{d(gx, gy)^\gamma}{d(x,y)^\gamma} \mathrm{d}\rho^{*k}(g) \\ &\leqslant \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} C_1 e^{-\delta k} \leqslant C_1 \left(\frac{1 + e^{-\delta}}{2} \right)^n \end{aligned}$$

□

In the same way, one can prove the following

Lemma 2.15. *Let $\mathbf{G}$ be a second countable locally compact group and ρ a borelian probability measure on $\mathbf{G}$.*

Let $(\mathcal{B}, \|\,.\,\|_{\mathcal{B}})$ be a Banach space and $r : \mathbf{G} \to \mathrm{GL}(\mathcal{B})$ a representation such that $\begin{cases} \mathbf{G} \times \mathcal{B} & \to & \mathcal{B} \\ (g, b) & \mapsto & r(g)b \end{cases}$ *is continuous and $\int_{\mathbf{G}} \|r(g)\| \mathrm{d}\rho(g)$ is finite.*

We note P_ρ the operator $b \mapsto \int_{\mathbf{G}} r(g)(b) \mathrm{d}\rho(g)$.

We assume that there is a continuous operator N_0 on $\mathcal{B}$ and $C, \kappa \in \mathbb{R}$ such that for any integer n, $\|P_\rho^n - N_0\|_{\mathcal{B}} \leqslant Ce^{-\kappa n}$, then, for any $n \in \mathbb{N}$,

$$\|P_{\rho_{\mathbf{e}}}^n - N_0\|_{\mathcal{B}} \leqslant C \left(\frac{1 + e^{-\kappa}}{2} \right)^n$$

where $P_{\rho_{\mathbf{e}}}$ is the operator associated to $\rho_{\mathbf{e}} = \frac{1}{2}\delta_{\mathbf{e}} + \frac{1}{2}\rho$.

2.1.4. *Perturbation of Markov operators by cocycles.* In this section, $\mathbf{G}$ still is a second countable locally compact group acting continuously on a compact metric space $(\mathbf{X}, d)$ on which acts a finite group $\mathbf{H}$ whose action commutes to the $\mathbf{G}$-action and such that $\mathbf{X}/\mathbf{H}$ is contracted over a finite $\mathbf{G}-$set.

We are going to study perturbations of the Markov operator associated to a borelian probability measure ρ by a kernel of modulus 1. To simplify the study, we will only consider a kernel having a *cocycle* property :

Definition 2.16 (Cocycle)**.** Let $\mathbf{G}$ be a second countable locally compact group and $\mathbf{X}$ a topological space endowed with a continuous $\mathbf{G}$-action.

We say that a continuous function $\sigma : \mathbf{G} \times \mathbf{X} \to \mathbb{R}$ is a (continuous additive) *cocycle* if for any $g_1, g_2 \in \mathbf{G}$ and any $x \in \mathbf{X}$,

$$\sigma(g_2 g_1, x) = \sigma(g_2, g_1 x) + \sigma(g_1, x)$$

Let σ be a cocycle. We say that σ is a *coboundary* if there is a continuous function $\varphi : \mathbf{X} \to \mathbb{R}$ such that for any $(g, x) \in \mathbf{G} \times \mathbf{X}$, $\sigma(g, x) = \varphi(gx) - \varphi(x)$.

Remark 2.17. Let σ be a cocycle. The operator defined for any $f \in \mathcal{C}^0(\mathbf{X})$ and any $x \in \mathbf{X}$ by

$$P_{i\sigma} f(x) = \int_{\mathbf{G}} e^{i\sigma(g,x)} f(gx) \mathrm{d}\rho(g)$$

is continuous on $\mathcal{C}^0(\mathbf{X})$ and for any function $f \in \mathcal{C}^0(\mathbf{X})$, any $x \in \mathbf{X}$ and any $n \in \mathbb{N}$, we have

$$P^n_{i\sigma} f(x) = \int_{\mathbf{G}} e^{i\sigma(g,x)} f(gx) \mathrm{d}\rho^{*n}(g) \text{ and } \|P^n_{i\sigma} f\|_\infty \leqslant \|f\|_\infty$$

It is to get this equality that we only consider cocycles and not arbitrary functions on $\mathbf{G} \times \mathbf{X}$.

As we are going to study Markov chains on contracted spaces (and therefore hölder-continuous functions), we are looking for conditions that guarantee that $P_{i\sigma}$ preserves hölder continuous functions on $\mathbf{X}$.

For a cocycle σ and $g \in \mathbf{G}$, we note

$$\sigma_{\sup}(g) = \sup_{x \in \mathbf{X}} |\sigma(g,x)| \text{ and } \sigma_{\mathrm{Lip}}(g) = \sup_{\substack{x,y \in \mathbf{X} \\ \pi_{\mathbf{A}}(x) = \pi_{\mathbf{A}}(y) \\ x \neq y}} \frac{|\sigma(g,x) - \sigma(g,y)|}{d(x,y)}$$

Then, for any $x, y \in \mathbf{X}$ with $x \neq y$ and $\pi_{\mathbf{A}}(x) = \pi_{\mathbf{A}}(y)$,

$$\begin{aligned} 2^{\gamma-1} \left| e^{i\sigma(g,x)} - e^{i\sigma(g,y)} \right| \leqslant \left| e^{i\sigma(g,x)} - e^{i\sigma(g,y)} \right|^\gamma &\leqslant |\sigma(g,x) - \sigma(g,y)|^\gamma \\ &\leqslant \sigma^\gamma_{\mathrm{Lip}}(g) d(x,y)^\gamma \end{aligned}$$

So, if $\sigma_{\mathrm{Lip}}(g)$ is finite for any $g \in \mathbf{G}$, we have that the application $(x \mapsto e^{i\sigma(g,x)})$ is hölder-continuous.

We note, for any $M \in \mathbb{R}_+$,

$$\mathcal{Z}^M(\mathbf{X}) = \left\{ \sigma \text{ is a continuous additive cocycle} \middle| \sup_{g \in \mathbf{G}} \frac{\sigma_{\mathrm{Lip}}(g)}{N(g)^M} \text{ and } \sup_{g \in \mathbf{G}} \frac{e^{\sigma_{\sup}(g)}}{N(g)^M} \text{ are finite} \right\}$$

And, for $\sigma \in \mathcal{Z}^M(\mathbf{X})$, we note

$$[\sigma]_M = \sup_{g \in \mathbf{G}} \frac{\sigma_{\mathrm{Lip}}(g)}{N(g)^M} \text{ and } [\sigma]_\infty = \sup_{g \in \mathbf{G}} \frac{e^{\sigma_{\sup}(g)}}{N(g)^M} \tag{2.5}$$

The next proposition is an extension to our context of the corollary 3.21 in [GL12].

Proposition 2.18. *Let $\mathbf{G}$ be a second countable locally compact group, $N : \mathbf{G} \to [1, +\infty[$ a submultiplicative function and ρ a borelian probability measure on $\mathbf{G}$.*

Let $(\mathbf{X}, d)$ be a compact metric $\mathbf{G}$-space endowed with an action of a finite group $\mathbf{H}$ that commutes to the $\mathbf{G}-$action and such that $\mathbf{X}/\mathbf{H}$ is $(\rho, \gamma_0, M, N)-$contracted over a finite $\mathbf{G}-$set $\mathbf{A}$ on which the random walk defined by ρ is irreducible and aperiodic.

Then, there are $C_2, \delta_2 \in \mathbb{R}^*_+$ such that for any $\sigma \in \mathcal{Z}^M(\mathbf{X}/\mathbf{H})$, any $n \in \mathbb{N}$ and any function $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$, we have

$$m_\gamma(P^n_{i\sigma} f) \leqslant C_2 \left(\|f\|_\infty (1 + [\sigma]_M) + e^{-\delta_2 n} m_\gamma(f) \right)$$

In particular, the operator $P_{i\sigma}$ has an essential spectral radius smaller than $e^{-\delta_2}$ in $\mathcal{C}^{0,\gamma}(\mathbf{X})$.

Proof. Let $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$ and $x, y \in \mathbf{X}$ with $x \neq y$ and $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(y)$.

For any $n \in \mathbb{N}^*$, we have

$$\begin{aligned}
|P^n_{i\sigma} f(x) \ - P^n_{i\sigma} f(y)| &= \left| \int_{\mathbf{G}} e^{i\sigma(g,x)} f(gx) - e^{i\sigma(g,y)} f(gy) \mathrm{d}\rho^{*n}(g) \right| \\
&\leqslant \int_{\mathbf{G}} |f(gx) - f(gy)| \, \mathrm{d}\rho^{*n}(g) + \|f\|_\infty \int_{\mathbf{G}} \left| e^{i\sigma(g,x)} - e^{i\sigma(g,y)} \right| \mathrm{d}\rho^{*n}(g) \\
&\leqslant d(x,y)^\gamma m_\gamma(f) \int_{\mathbf{G}} \frac{d(gx,gy)^\gamma}{d(x,y)^\gamma} \mathrm{d}\rho^{*n}(g) \\
&\qquad + \|f\|_\infty 2^{1-\gamma} [\sigma]_M d(x,y)^\gamma \int_{\mathbf{G}} N^{\gamma M}(g) \mathrm{d}\rho^{*n}(g)
\end{aligned}$$

First, as N is submultiplicative, we have that

$$\int_{\mathbf{G}} N(g)^{\gamma M} \mathrm{d}\rho^{*n}(g) \leqslant \left(\int_{\mathbf{G}} N(g)^{\gamma M} \mathrm{d}\rho(g) \right)^n$$

Moreover, as the group $\mathbf{H}$ is finite, there is $d_0 \in \mathbb{R}_+^*$ such that for any $x, y \in \mathbf{X}$, if $d(x,y) \leqslant d_0$, then $d(x,y) = d(\pi_{\mathbf{H}}(x), \pi_{\mathbf{H}}(y))$.

So, for any $\varepsilon \in]0,1]$ and any $x, y \in \mathbf{X}$ such that $0 < d(x,y) \leqslant \varepsilon d_0$ and $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(y)$, we have

$$\begin{aligned}
I_n(x,y) :&= \int_{\mathbf{G}} d(gx,gy)^\gamma \mathrm{d}\rho^{*n}(g) \\
&= \int_{\mathbf{G}} \mathbf{1}_{d(gx,gy) \leqslant d_0} d(gx,gy)^\gamma + \mathbf{1}_{d(gx,gy) > d_0} d(gx,gy)^\gamma \mathrm{d}\rho^{*n}(g) \\
&= \int_{\mathbf{G}} \mathbf{1}_{d(gx,gy) \leqslant d_0} d(g\pi_{\mathbf{H}}x, g\pi_{\mathbf{H}}y)^\gamma + \mathbf{1}_{d(gx,gy) > d_0} d(gx,gy)^\gamma \mathrm{d}\rho^{*n}(g) \\
&\leqslant C_1 e^{-\delta n} d(x,y)^\gamma + d(x,y)^\gamma \int_{\mathbf{G}} \mathbf{1}_{d(gx,gy) > d_0} M^\gamma N(g)^{M\gamma} \mathrm{d}\rho^{*n}(g) \\
&\leqslant \left(C_1 e^{-\delta n} + M^\gamma \int_{\mathbf{G}} \mathbf{1}_{MN(g)^M \geqslant 1/\varepsilon} N(g)^{\gamma M} \mathrm{d}\rho^{*n}(g) \right) d(x,y)^\gamma
\end{aligned}$$

Thus, if we take n_0 with $C_1 e^{-\delta n_0} \leqslant 1/4$, as $\int_{\mathbf{G}} N(g)^{\gamma M} \mathrm{d}\rho^{*n_0}(g)$ is finite, we can also choose ε such that

$$\int_{\mathbf{G}} \mathbf{1}_{MN(g)^M \geqslant 1/\varepsilon} M N(g)^{\gamma M} \mathrm{d}\rho^{*n_0}(g) \leqslant 1/4$$

And so, for this choice of ε and n_0, we have that for any $x, y \in \mathbf{X}$ with $0 < d(x,y) \leqslant \varepsilon d_0$ and $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(y)$,

$$\int_{\mathbf{G}} d(gx,gy)^\gamma \mathrm{d}\rho^{*n_0}(g) \leqslant \frac{1}{2} d(x,y)^\gamma$$

This proves that for any $x, y \in \mathbf{X}$ with $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(y)$ and $d(x,y) \leqslant \varepsilon d_0$ and any function $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$,

$$\frac{|P_{i\sigma}^{n_0} f(x) - P_{i\sigma}^{n_0} f(y)|}{d(x,y)^\gamma} \leqslant \frac{1}{2} m_\gamma(f) + \|f\|_\infty 2^{1-\gamma} [\sigma]_M \left(\int_{\mathbf{G}} N^{\gamma_0 M}(g) \mathrm{d}\rho(g) \right)^n$$

But, since we also have, for x, y such that $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(y)$ and $d(x,y) \geqslant \varepsilon d_0$,

$$\frac{|P_{i\sigma}^{n_0} f(x) - P_{i\sigma}^{n_0} f(y)|}{d(x,y)^\gamma} \leqslant \frac{2\|f\|_\infty}{(\varepsilon d_0)^\gamma}$$

we finally obtain that for any $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$,

$$m_\gamma(P^{n_0} f) \leqslant \frac{1}{2} m_\gamma(f) + \left(\frac{2}{(\varepsilon d_0)^\gamma} + 2^{1-\gamma} [\sigma]_M \left(\int_{\mathbf{G}} N^{\gamma_0 M}(g) \mathrm{d}\rho(g) \right)^{n_0} \right) \|f\|_\infty$$

If we simplify notations, what we found is that there are $n_0 \in \mathbb{N}^*$ and a constant $C \in \mathbb{R}_+$ (depending on n_0) such that for any function $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$,

$$m_\gamma(P_{i\sigma}^{n_0} f) \leqslant \frac{1}{2} m_\gamma(f) + C(1 + [\sigma]_M) \|f\|_\infty$$

Iterating this inequality, we get that there are $C_2, \delta_2 \in \mathbb{R}_+^*$ such that for any $n \in \mathbb{N}$ and any function $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$,

$$m_\gamma(P_{i\sigma}^n f) \leqslant C_2 \left(e^{-\delta_2 n} m_\gamma(f) + (1 + [\sigma]_M) \|f\|_\infty \right)$$

This proves, using Ionescu-Tulcea and Marinescu's result (that we recalled in theorem 2.10), that the operator $P_{i\sigma}$ has an essential spectral radius smaller than $e^{-\delta_2}$ which is what we intended to prove. □

2.1.5. *Lower regularity of borelian measures on compact metric spaces.* Guivarc'h proved (cf. the 12th chapter of [BQ15]) that if ρ is a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal subgroup of $\mathrm{SL}_d(\mathbb{R})$, then there is a unique $P-$stationary probability measure ν on $\mathbb{P}(\mathbb{R}^d)$. Moreover, it exists $\Delta^+, C \in \mathbb{R}_+^*$ such that for any $x \in \mathbb{P}(\mathbb{R}^d)$ and any $r \in \mathbb{R}_+$,

$$\nu(B(x,r)) \leqslant C r^{\Delta^+}$$

This property of upper regularity of the measure ν means that ν is not too concentrated around points of it's support. Indeed, if ν had an atom x_0, we would have that for any $\Delta^+ \in \mathbb{R}_+^*$,

$$\lim_{r \to 0^+} \frac{\nu(B(x_0,r))}{r^{\Delta_+}} = +\infty.$$

Here, we will have to use the lower regularity of the measure ν : many times we will have to use the fact that a ball of radius r has a $\nu-$measure larger than some power of r. This leads us to make the following

Definition 2.19. Let $(\mathbf{X}, d)$ be a compact metric space and ν a borelian probability measure on $\mathbf{X}$.

Let $\Delta \in \mathbb{R}_+$ and $t, r \in \mathbb{R}_+^*$.

We say that a point $x \in \mathbf{X}$ is $(\Delta, t) - \nu$-*regular at scale* r if

$$\nu(B(x,r)) \geqslant t r^{\Delta}$$

In the same way, we say that a point x is $(\Delta, t) - \nu$-regular at every scales if

$$\inf_{r \in]0,1]} \frac{\nu(B(x,r))}{r^{\Delta}} \geqslant t$$

Finally, we say that it is $\Delta - \nu$-regular at scale r if it is $(\Delta, 1) - \nu$-regular at scale r.

Remark 2.20. Generic points of the measure ν are regular at every scales : indeed, if $\mathbf{X}$ has an Hausdorff dimension smaller than Δ then we have (cf. [Rud87]) that

$$\nu\left(\bigcup_{t \in \mathbb{R}_+^*} \{x \in \mathbf{X} | x \text{ is } (\Delta, t) - \nu - \text{regular at every scales}\}\right) = 1$$

We will sometimes make the following abuse of notations : if $\pi_0 : \mathbf{X} \to \mathbf{X}_0$ is a finite covering and ν is a borelian probability measure on $\mathbf{X}_0$, we will say that $x \in \mathbf{X}$ is $\Delta - \nu$-regular at scale r if so is $\pi_0(x)$.

2.1.6. *Isotypic decomposition.*

In this paragraph, we recall how to generalize the decomposition of a function between even and odd parts.

Let $\mathbf{H}$ be a finite group. For any irreducible unitary representation $\xi = (\rho, \mathbb{V})$ of $\mathbf{H}$, we endow $\mathrm{End}(\mathbb{V})$ of Hilbert-Schmidt's inner product defined for $A, B \in \mathrm{End}(\mathbb{V})$ by

$$\langle A, B \rangle_{HS} := \mathrm{tr} A^* B$$

We also endow it of the associated norm $|\,.\,|_{HS}$.

Let $(\mathbf{X}, d)$ be a compact metric space on which $\mathbf{H}$ acts by isometries (in particular, $\mathbf{H}$ preserve the space of hölder continuous functions on $\mathbf{X}$).

The action of $\mathbf{H}$ on $\mathbf{X}$ gives a representation of $\mathbf{H}$ in $\mathcal{C}^0(\mathbf{X})$ defined for any $h \in \mathbf{H}$, $f \in \mathcal{C}^0(\mathbf{X})$ and $x \in \mathbf{X}$ by

$$\rho_0(h) f(x) = f(h^{-1} x)$$

We note $\widehat{\mathbf{H}}$ a set of representatives of irreducible unitary representations of $\mathbf{H}$ up to isomorphism.

For $\xi = (\rho, \mathbb{V}) \in \widehat{\mathbf{H}}$, $f \in \mathcal{C}^0(\mathbf{X})$ and $x \in \mathbf{X}$, we note

$$\widehat{f}(x, \xi) = \frac{\dim \mathbb{V}}{|\mathbf{H}|} \sum_{h \in \mathbf{H}} f(h^{-1} x) \rho(h)^* \tag{2.6}$$

It is known (see theorem 8 in [Ser78]) that $\mathrm{tr} f(., \xi)$ is the projection of f on the isotypic component of ξ and that for any $x \in \mathbf{X}$,

$$f(x) = \sum_{\xi \in \widehat{\mathbf{H}}} \mathrm{tr} \widehat{f}(x, \xi) \tag{2.7}$$

However, we will need the following equivariant relation : for any $x \in \mathbf{X}$ and any $h \in \mathbf{H}$,

$$\widehat{f}(hx, \xi) = \frac{\dim \mathbb{V}}{|\mathbf{H}|} \sum_{h' \in \mathbf{H}} f((h^{-1} h')^{-1} x) \rho(h')^* = \widehat{f}(x, \xi) \rho(h)^* \tag{2.8}$$

Thus, for any function $f \in \mathcal{C}^0(\mathbf{X})$, any $x \in \mathbf{X}$, any $\xi \in \widehat{\mathbf{H}}$ and any $h \in \mathbf{H}$, we have that,

$$|\widehat{f}(hx,\xi)|_{HS} = |\widehat{f}(x,\xi)|_{HS}$$

Moreover, the function $(x \mapsto |\widehat{f}(x,\xi)|_{HS})$ can be identified to a continuous function on $\mathbf{X}/\mathbf{H}$.

The norm $|\,.\,|_{HS}$ allows one to define hölder continuous functions from $\mathbf{X}$ to $\mathrm{End}(\mathbb{V})$ so we note

$$\mathcal{C}^{0,\gamma}_{\xi}(\mathbf{X}) = \left\{ f \in \mathcal{C}^{0,\gamma}(\mathbf{X}, \mathrm{End}(\mathbb{V}_\xi)) \middle| \forall x \in \mathbf{X}\, \forall h \in \mathbf{H}\ f(hx) = f(x)\rho(h)^* \right\}$$

And we have the following

Lemma 2.21. *Let $\mathbf{H}$ be a group acting by isometries on a compact metric space $(\mathbf{X}, d)$.*

Then, the space $\mathcal{C}^{0,\gamma}(\mathbf{X})$ injects in $\prod_{\xi\in\widehat{\mathbf{H}}} \mathcal{C}^{0,\gamma}_{\xi}(\mathbf{X}, \mathrm{End}(\mathbb{V}))$. Moreover, for any $\xi \in \widehat{\mathbf{H}}$, the projection to $\mathcal{C}^{0,\gamma}_{\xi}(\mathbf{X})$ is given by equation 2.6 and the inverse map is given by equation 2.7.

We will have to quantify the irreducibility of representations of $\mathbf{H}$. To do so, we will use the following

Lemma 2.22. *Let $\mathbf{H}$ be a finite group and $\widehat{\mathbf{H}}$ a set of choice of representatives of representations of $\mathbf{H}$ up to isomorphism.*

Then, there is a constant $C_{\mathbf{H}}$ such that for any $\xi = (\rho, \mathbb{V}) \in \widehat{\mathbf{H}}$ and any $A, B \in \mathrm{End}(\mathbb{V})$,

$$\max_{h\in\mathbf{H}} |A\rho(h)B|_{HS} \geqslant C_{\mathbf{H}} |A|_{HS} |B|_{HS}$$

Proof. Let $\xi = (\rho, \mathbb{V}) \in \widehat{\mathbf{H}}$. We note $\mathcal{B} = \{A \in \mathrm{End}(\mathbb{V}) || A|_{HS} = 1\}$ and for any $A, B \in \mathcal{B}$, we define

$$\varphi(A,B) = \max_{h\in\mathbf{H}} |A\rho(h)B|_{HS}$$

Then, the function φ is continuous on the compact set $\mathcal{B}^2$ and moreover, for any $A, B \in \mathcal{B}$, we have that $\ker(A) \neq \mathbb{V}$ and $\mathrm{Im}(B) \neq \{0\}$ and so, as ξ is irreducible, there is $h \in \mathbf{H}$ such that $\rho(h)\mathrm{Im}(B) \not\subset \ker(A)$. This implies that $A\rho(h)B \neq 0$ and so $\varphi(A,B) > 0$. The function φ reaches a minimum on $\mathcal{B}^2$. This minimum depends on ξ but, as $\mathbf{H}$ has only finitely many irreducible unitary representations up to isomorphism (since it is finite), this finishes the proof of the lemma since the inequality also holds when $A = 0$ or $B = 0$ and the function φ satisfies that for any $A, B \in \mathrm{End}(\mathbb{V})$ and any $t \in \mathbb{R}_+$,

$$\varphi(tA, B) = \varphi(A, tB) = t\varphi(A, B)$$

□

2.2. **Control of the resolvent of the perturbed operator.**

2.2.1. *Statement of the proposition.*

We are now ready to state proposition 2.23, aim of this section.

We keep the notations we used earlier.

Let $\sigma : \mathbf{G} \times \mathbf{X} \to \mathbb{R}$ be an $\mathbf{H}-$invariant cocyle. We note $P_{i\sigma}$ (and sometimes $P_{\rho,i\sigma}$ to insist on the measure ρ) the operator defined for any continuous function f on $\mathbf{X}$ and any x of $\mathbf{X}$, by

$$P_{i\sigma} f(x) = \int_{\mathbf{G}} e^{i\sigma(g,x)} f(gx) \mathrm{d}\rho(g)$$

If $\rho_{\mathbf{e}} = \frac{1}{2}\delta_{\mathbf{e}} + \frac{1}{2}\rho$ is the probability measure associated to the lazy random walk, we will simply note $P_{\mathbf{e},i\sigma} = P_{\rho_{\mathbf{e}},i\sigma}$.

Finally, we set

$$[\sigma]_M = \sup_{g\in\mathbf{G}} \sup_{\substack{x,y\in\mathbf{X} \\ x\neq y \\ \pi_{\mathbf{A}}(x)=\pi_{\mathbf{A}}(y)}} \frac{|\sigma(g,x) - \sigma(g,y)|}{N(g)^M d(x,y)}$$

We saw in paragraph 2.1.4 that if $[\sigma]_M$ is finite and if $(g \mapsto N(g)^{M\gamma})$ is $\rho-$integrable then $P_{i\sigma}$ preserves the space of $\gamma-$hölder continuous functions on $\mathbf{X}$.

The aim of this section is to study the spectral properties of $P_{i\sigma}$ and to link them to the ones of P. A first case is when $e^{i\sigma}$ is a coboundary : $e^{i\sigma(g,x)} = \varphi(gx)\varphi(x)^{-1}$ where φ is a $\gamma-$holder continuous function taking it's values in $\{z \in \mathbb{C} | |z| = 1\}$. Indeed, in this case, the operator $P_{i\sigma}$ is conjugated to P by the multiplication by φ and so these two operators have the same spectral properties. In particular, $P_{i\sigma}$ has 1 as an eigenvalue (an associated eigenvector being φ^{-1}).

We will see that we can get some kind of reciprocal for this result. The next proposition proves that, under reasonable assumptions, if $I_d - P_{i\sigma}$ is not well invertible (the norm of the inverse operator is large) then $e^{i\sigma}$ is close to a coboundary.

Proposition 2.23. *Let $\mathbf{G}$ be a second countable locally compact group, $N : \mathbf{G} \to [1, +\infty[$ a submultiplicative function and ρ a borelian probability measure on $\mathbf{G}$.*

Let $(\mathbf{X}, d)$ be a compact metric $\mathbf{G}-$space endowed with an action by isometries of a finite group $\mathbf{H}$ that commutes to the $\mathbf{G}$-action and such that the quotient space $\mathbf{X}/\mathbf{H}$ is $(\rho, \gamma, M, N)-$contracted over a finite $\mathbf{G}-$set $\mathbf{A}$ on which the random walk defined by ρ is irreducible and aperiodic.

Fix a set $\widehat{\mathbf{H}}$ of representatives of irreducible unitary representations of $\mathbf{H}$ up to isomorphism.

Note $\kappa_{\mathbf{A}} \in \mathbb{R}_+^*$ and $C_{\mathbf{A}} \in \mathbb{R}$, $C_{\mathbf{A}} \geqslant |\mathbf{A}|$ such that for any function f on $\mathbf{A}$ and any $n \in \mathbb{N}$,

$$\left\| P_{\mathbf{e}}^n f - \int f \mathrm{d}\nu_{\mathbf{A}} \right\|_\infty \leqslant C_{\mathbf{A}} e^{-\kappa_{\mathbf{A}} n} \|f\|_\infty$$

where $P_{\mathbf{e}}$ is the operator associated to the lazy random walk and $\nu_{\mathbf{A}}$ is the uniform probability measure on $\mathbf{A}$ (see lemma 2.6 to get the existence of $\kappa_{\mathbf{A}}, C_{\mathbf{A}}$).

Let ν be the unique $P-$stationary probability measure on $\mathbf{X}/\mathbf{H}$ (given by proposition 2.5).

Then, for any $\gamma > 0$ small enough, any $\alpha_1, \beta \in \mathbb{R}_+^*$ there is $\alpha_2 \in \mathbb{R}_+$, such that for any $\Delta \in \mathbb{R}_+$ such that there is some $\Delta - \nu$-regular point at scale $2^{-\alpha_2}$ in $\in \mathbf{X}/\mathbf{H}$ (see

definition 2.19) we have that there is $\alpha_0, C \in \mathbb{R}_+$ *such that for any* $\sigma \in \mathcal{Z}^M(\mathbf{X}/\mathbf{H})$ *we have that if*

$$\|(I_d - P_{i\sigma})^{-1}\|_{\mathcal{C}^{0,\gamma}(\mathbf{X})} \geqslant C\left(C_{\mathbf{A}}(1+[\sigma]_M)\right)^{\alpha_0}$$

Then, there is a $\gamma-$*hölder continuous function* $\varphi : \mathbf{X} \to \mathbb{R}$ *with* $\|\varphi\|_\gamma \leqslant Cb$ *such that for any point* x *in* $\mathbf{X}$ *whose projection on* $\mathbf{X}/\mathbf{H}$ *is* $\Delta - \nu$*-regular at scale* $b^{-\alpha_2}$*, we have*

$$\max_{h\in\mathbf{H}} |\varphi(hx)| \geqslant \frac{1}{C}$$

and, for any $h \in \mathbf{H}$*,*

$$\int_{\mathbf{G}} \left| e^{i\sigma(g,x)}\varphi(ghx) - \varphi(hx)\right|^2 \mathrm{d}\rho_{\mathbf{e}}^{*n(\beta,b)}(g) \leqslant \frac{1}{b^{\alpha_1}}$$

where be noted

$$b = 2C_{\mathbf{A}}(1+[\sigma]_M) \text{ and } n(\beta,b) = \lfloor \beta \ln b \rfloor$$

and $\rho_{\mathbf{e}}$ *is the measure associated to the lazy random walk (see paragraph 2.1.3).*

Finally, the constants α_0, α_2 *and* C *only depend on* $\mathbf{A}$ *through* $\kappa_{\mathbf{A}}$*.*

Remark 2.24. Remark that in the conclusion of the proposition it is the measure $\rho_{\mathbf{e}}$ that appears and not ρ itself. This actually will be useful since the sequence $(\operatorname{supp}\rho_{\mathbf{e}}^{*n})$ is non decreasing.

Remark 2.25. If φ were invertible on $\mathbf{X}$, the proposition would imply that $e^{i\sigma}$ is close to the coboundary $\varphi(x)\varphi(gx)^{-1}$.

2.2.2. *Proof of proposition 2.23.* The proof of the proposition relies on a few lemma that we adapt from the ones of Dolgopyat for Ruelle operators used in [Dol98]

The first point is to remove the technical difficulty that $\mathbf{X}$ is only locally contracted. To do so, we will use the isotypic decomposition seen in lemma 2.21 and use the assumption that the actions of $\mathbf{H}$ and $\mathbf{G}$ on $\mathbf{X}$ commutes which implies that $\mathbf{G}$ preserves the decomposition of the lemma. Thus, as we also assumed that σ is $\mathbf{H}-$invarient, we are able to study $P_{i\sigma}$ on each $\mathcal{C}^{0,\gamma}_\xi(\mathbf{X})$ and on these spaces, we have the equivariant relation of equation 2.8.

Proposition 2.18 suggests that we shall normalize the norm of $\mathcal{C}^{0,\gamma}(\mathbf{X})$ to study $P_{i\sigma}$. This is why we make the following

Notation. Under the assumptions of proposition 2.23, we note C_2 the constant given by proposition 2.18 (we shall assume without any loss of generality that $1 \leqslant C_2$).

Let $\sigma \in \mathcal{Z}^M$. Then, for any function $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$, we note

$$\|f\|_\sigma = \max\left(\|f\|_\infty, \frac{m_\gamma(f)}{C_2(1+[\sigma]_M)}\right)$$

Remark that for any $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$,

$$\|f\|_\sigma \leqslant \|f\|_\gamma \leqslant (1 + C_2(1+[\sigma]_M))\,\|f\|_\sigma$$

So $(\mathcal{C}^{0,\gamma}(\mathbf{X}), \|\,.\,\|_\gamma)$ and $(\mathcal{C}^{0,\gamma}(\mathbf{X}), \|\,.\,\|_\sigma)$ are isomorphic. Moreover, the operator $P_{i\sigma}$ is better controlled with $\|\,.\,\|_\sigma$ as shown by next

Lemma 2.26. *Under the assumptions of proposition 2.23, for any $\sigma \in \mathcal{Z}^M$ and any $n \in \mathbb{N}$,*

$$\|P_{i\sigma}^n\|_\sigma \leqslant 2C_2$$

Proof. Let $f \in \mathcal{C}^{0,\gamma}$ be such that $\|f\|_\sigma \leqslant 1$. According to proposition 2.18, for any $n \in \mathbb{N}$, we have

$$m_\gamma(P_{i\sigma}^n f) \leqslant C_2 \left(\|f\|_\infty (1 + [\sigma]_M) + e^{-\delta n} m_\gamma(f) \right) \leqslant C_2 \left(1 + C_2 e^{-\delta n} \right) (1 + [\sigma]_M)$$

Moreover, we also have that

$$\|P_{i\sigma}^n f\|_\infty \leqslant \|f\|_\infty$$

And so,

$$\|P_{i\sigma}^n\|_\sigma \leqslant \max(1, 1 + C_2) \leqslant 2C_2$$

And this concludes the proof of the lemma. □

In the next lemmas, to try to simplify a little the notations, we will always note

$$b = 2C_{\mathbf{A}}(1 + [\sigma]_M) \tag{2.9}$$

The reader has to see that the "$1 + [\sigma]_M$" will allow us to study cocycles with $[\sigma]_M \leqslant 1$. The "$C_{\mathbf{A}}$" appearing here is useful to keep track of the dependence on $\mathbf{A}$. Finally, the "2" is here to have $b \geqslant 2$ and this will allow us to dominate any constant by an appropriate power of b.

For α_2, Δ, σ and ξ fixed, we are going to study the hypothesis

$\mathbf{H}(\alpha_1, \beta, \xi)$: For any function $f \in \mathcal{C}_\xi^{0,\gamma}(\mathbf{X})$ with $\|f\|_\sigma \leqslant 1$, there is $x_0 \in \mathbf{X}$ that is $\Delta - \nu$-regular at scale $b^{-\alpha_2}$ and $n \in [0, \lfloor \beta \ln(b) \rfloor]$ such that

$$|P_{\mathbf{e},i\sigma}^n f(x_0)|_{HS} \leqslant 1 - \frac{1}{b^{\alpha_1}}$$

Lemma 2.27. *Under the assumptions of proposition 2.23, for any $\alpha_1, \beta \in \mathbb{R}_+^*$ there are $\alpha_2 \in \mathbb{R}_+^*$ such that for any $\Delta \in \mathbb{R}_+^*$, there is α_0, C, depending on $\mathbf{A}$ only through $\kappa_{\mathbf{A}}$ such that for any $\sigma \in \mathcal{Z}^M(\mathbf{X}/\mathbf{H})$ and any $\xi \in \widehat{\mathbf{H}}$, we have that if the hypothesis $\mathbf{H}(\alpha_1, \beta, \xi)$ holds then*

$$\|(I_d - P_{i\sigma})^{-1}\|_{\mathcal{C}_\xi^{0,\gamma}(\mathbf{X})} \leqslant C \left(C_{\mathbf{A}}(1 + [\sigma]_M) \right)^{\alpha_0}$$

Proof. Let $f \in \mathcal{C}_\xi^{0,\gamma}(\mathbf{X})$ be a function such that $\|f\|_\sigma \leqslant 1$.

By assumption, there are $n \in [0, n(\beta, b)]$ and a point $x_0 \in \mathbf{X}$ whose projection on $\mathbf{X}/\mathbf{H}$ is $\Delta - \nu$-regular at scale $b^{-\alpha_2}$ such that

$$|P_{\mathbf{e},i\sigma}^n f(x_0)|_{HS} \leqslant 1 - \frac{1}{b^{\alpha_1}}$$

We are going to prove in a first time that we can extend this control at one point to a control of $P_{\mathbf{e},i\sigma}^m f$ on the whole space $\mathbf{X}$ for some $m \geqslant n$ and then, that this implies the expected result.

First, using the triangular inequality, we have, for any $m, n \in \mathbb{N}$ with $m \geqslant n$ and any $x \in \mathbf{X}$,

$$|P^m_{\mathbf{e},i\sigma} f(x)|_{HS} = \left| \int_{\mathbf{G}} e^{i\sigma(g,x)} P^n_{\mathbf{e},i\sigma} f(gx) \mathrm{d}\rho^{*m-n}_{\mathbf{e}}(g) \right|_{HS}$$
$$\leqslant \int_{\mathbf{G}} |P^n_{\mathbf{e},i\sigma} f(gx)|_{HS} \mathrm{d}\rho^{*m-n}_{\mathbf{e}}(g) = P^{m-n}_{\mathbf{e},0} |P^n_{\mathbf{e},i\sigma} f|_{HS}(x)$$

Moreover, as σ is $\mathbf{H}-$invariant and as the actions of $\mathbf{H}$ and $\mathbf{G}$ commute, we also have, by definition of $\mathcal{C}^{0,\gamma}_{\xi}(\mathbf{X})$, that for any $m \in \mathbb{N}$, the function $|P^m_{\mathbf{e},i\sigma} f|_{HS}$ is $\mathbf{H}-$invariant (see. lemma 2.21). This function can be identified to a function on $\mathbf{X}/\mathbf{H}$ and we have, according to proposition 2.8, that

$$P^{m-n}_{\mathbf{e},0} |P^n_{\mathbf{e},i\sigma} f|_{HS}(x) \leqslant \int_{\mathbf{X}/\mathbf{H}} |P^n_{\mathbf{e},i\sigma} f(y)|_{HS} \mathrm{d}\nu(y) + C_0 C_{\mathbf{A}} e^{-\kappa(m-n)} \|P^n_{\mathbf{e},i\sigma} f\|_{\gamma}$$

Moreover, using lemma 2.26 and the assumption that $\|f\|_{\sigma} \leqslant 1$ we compute

$$\|P^n_{\mathbf{e},i\sigma} f\|_{\gamma} = \|P^n_{\mathbf{e},i\sigma} f\|_{\infty} + m_{\gamma}(P^n_{\mathbf{e},i\sigma} f) \leqslant 1 + C_2 \left((1 + [\sigma]_M) \|f\|_{\infty} + e^{-\delta n} m_{\gamma}(f) \right)$$

But, $1 + [\sigma]_M = \frac{b}{2C_{\mathbf{A}}}$ and $m_{\gamma}(f) \leqslant C_2(1 + [\sigma]_M)$, so

$$\|P^n_{\mathbf{e},i\sigma} f\|_{\gamma} \leqslant 1 + C_2 \left(\frac{b}{2C_{\mathbf{A}}} + e^{-\delta n} C_2 \frac{b}{2C_{\mathbf{A}}} \right) \leqslant 1 + C_2^2 \frac{b}{C_{\mathbf{A}}} \leqslant \frac{2C_2^2 b}{C_{\mathbf{A}}}$$

since $b \geqslant 2C_{\mathbf{A}} \geqslant 2$ and $C_2 \geqslant 1$. Moreover, as $n \leqslant \beta \ln b$, we also have that

$$e^{\kappa n} b \leqslant e^{\kappa \beta \ln b} b = b^{1+\beta\kappa}$$

Finally, we get that for any $m \in \mathbb{N}$ larger than $\beta \ln b$ and any $x \in \mathbf{X}$,

$$|P^m_{\mathbf{e},i\sigma} f(x)|_{HS} \leqslant \int_{\mathbf{X}/\mathbf{H}} |P^n_{\mathbf{e},i\sigma} f(y)|_{HS} \mathrm{d}\nu(y) + 2C_0 C_2^2 e^{-\kappa m} b^{1+\beta\kappa}$$

Moreover, if $\mathbf{Z}$ is a borelian subset of $\mathbf{X}/\mathbf{H}$ and $M_{\mathbf{Z}} = \sup_{x \in \mathbf{Z}} |P^n_{\mathbf{e},i\sigma} f(x)|_{HS}$, then,

$$\int_{\mathbf{X}/\mathbf{H}} |P^n_{\mathbf{e},i\sigma} f(y)|_{HS} \mathrm{d}\nu(y) \leqslant M_{\mathbf{Z}} \nu(\mathbf{Z}) + \nu(\mathbf{Z}^c) \leqslant 1 + (M_{\mathbf{Z}} - 1)\nu(\mathbf{Z})$$

So, taking $\mathbf{Z} = B(\pi_{\mathbf{H}}(x_0), r)$ with $r = (C_{\mathbf{A}}/(4C_2^2 b^{\alpha_1+1}))^{1/\gamma}$, we get that

$$\sup_{x \in \mathbf{Z}} |P^n_{\mathbf{e},i\sigma} f(x)|_{HS} \leqslant |P^n_{\mathbf{e},i\sigma} f(\pi_{\mathbf{H}}(x_0))|_{HS} + \|P^n_{\mathbf{e},i\sigma} f\|_{\gamma} d(x, \pi_{\mathbf{H}}(x_0))^{\gamma}$$
$$\leqslant 1 - \frac{1}{b^{\alpha_1}} + \frac{2C_2^2 b}{C_{\mathbf{A}}} r^{\gamma} = 1 - \frac{1}{2b^{\alpha_1}}$$

And taking α_2 so large that $b^{-\alpha_2} \leqslant (1/(4C_2^2 b^{\alpha_1+1}))^{1/\gamma}$, we have that $b^{-\alpha_2} \leqslant r$ and so, as x is $\Delta - \nu-$regular at scale $b^{-\alpha_2}$, we have

$$\nu(\mathbf{Z}) \geqslant b^{-\alpha_2 \Delta}$$

Thus,

$$\int_{\mathbf{X}/\mathbf{H}} |P^n_{\mathbf{e},i\sigma} f(y)|_{HS} \mathrm{d}\nu(y) \leqslant 1 - \frac{1}{2b^{\alpha_1+\alpha_2\Delta}}$$

To sum-up, we proved that for any $m \in \mathbb{N}$ larger than $\beta \ln(b)$,

$$\|P^m_{\mathbf{e},i\sigma} f\|_\infty \leqslant 1 - \frac{1}{2b^{\alpha_1+\alpha_2\Delta}} + 2C_0C_2^2 e^{-\kappa m} b^{1+\beta\kappa} \tag{2.10}$$

To simplify the notations, we set $\alpha_3 = \alpha_1 + \alpha_2\Delta$.

Let $m = Kn(\beta, b)$ for some $K \in \mathbb{N}^\star$ that we will determine later. Then,

$$\|P^m_{\mathbf{e},i\sigma}\|_\infty \leqslant 1 - \frac{1}{2} b^{-\alpha_3} + 2C_0C_2^2 b^{-K\kappa\beta} b^{1+\beta\kappa}$$

And so, for K large enough (whose value doesn't depend on $C_{\mathbf{A}}$) we have that

$$\|P^m_{\mathbf{e},i\sigma} f\|_\infty \leqslant 1 - \frac{1}{4b^{\alpha_3}}$$

Moreover, for $l \in \mathbb{N}$ larger than m, using proposition 2.18, we get that,

$$\begin{aligned}
\frac{1}{C_2(1+[\sigma]_M)} m_\gamma(P^l_{\mathbf{e},i\sigma} f) &\leqslant \|P^m_{\mathbf{e},i\sigma} f\|_\infty + \frac{1}{1+[\sigma]_M} e^{-\delta(l-m)} m_\gamma(P^m_{\mathbf{e},i\sigma} f) \\
&\leqslant 1 - \frac{1}{4b^{\alpha_3}} + \frac{2C_{\mathbf{A}}}{b} e^{-\delta l} b^{K\beta\delta} \frac{2C_2^2 b}{C_{\mathbf{A}}} \\
&\leqslant 1 - \frac{1}{4b^{\alpha_3}} + 4C_2^2 e^{-\delta l} b^{K\beta\delta}
\end{aligned}$$

So taking $l = Lm = KLn(\beta, b)$, where $L \in \mathbb{N}$ is large enough, we get that

$$\frac{1}{C_2(1+[\sigma]_M)} m_\gamma(P^l_{\mathbf{e},i\sigma} f) \leqslant 1 - \frac{1}{8b^{\alpha_3}}$$

Remember that we also have that

$$\|P^l_{\mathbf{e},i\sigma}\|_\infty \leqslant \|P^m_{\mathbf{e},i\sigma}\|_\infty \leqslant 1 - \frac{1}{4b^{\alpha_3}}$$

So, what we proved is that, under the assumptions of the proposition, if $\mathbf{H}(\alpha_1, \beta, \xi)$ holds, then for any $f \in \mathcal{C}^{0,\gamma}_\xi(\mathbf{X})$ such that $\|f\|_\sigma \leqslant 1$,

$$\|P^l_{\mathbf{e},i\sigma} f\|_\sigma \leqslant 1 - \frac{1}{8b^{\alpha_3}}$$

And so, in $C^{0,\gamma}_\xi(\mathbf{X})$,

$$\|(I_d - P^l_{\mathbf{e},i\sigma})^{-1}\|_\sigma \leqslant 8b^{\alpha_3}$$

Moreover, as

$$(I_d - P_{\mathbf{e},i\sigma})^{-1} = \sum_{k=0}^{l-1} P^k_{\mathbf{e},i\sigma} (I_d - P^l_{\mathbf{e},i\sigma})^{-1} \text{ and } \frac{1}{2}(I_d - P_{i\sigma})^{-1} = 2(I_d - P_{\mathbf{e},i\sigma})^{-1},$$

we can compute

$$\|(I_d - P_{i\sigma})^{-1}\|_{(\mathcal{C}^{0,\gamma}_\xi(\mathbf{X}),\|.\|_\sigma)} \leqslant 2 \sum_{k=0}^{l-1} \|P^k_{\mathbf{e},i\sigma}\|_\sigma \|(I_d - P^l_{\mathbf{e},i\sigma})^{-1}\|_\sigma \leqslant 2C_2 l 8 b^{\alpha_3}$$

Finally, as for any function $f \in \mathcal{C}^{0,\gamma}_\xi(\mathbf{X})$,

$$\|f\|_\sigma \leqslant \|f\|_\gamma \leqslant (1 + C_2(1+[\sigma]_M))\|f\|_\sigma$$

we get that, in $\mathcal{C}^{0,\gamma}_{\xi}(\mathbf{X})$,

$$\|(I_d - P_{i\sigma})^{-1}\|_\gamma \leqslant (1 + C_2(1 + [\sigma]_M))16C_2 b^{1+\alpha_3} l$$

We conclude since l is bounded by the product of $\ln b$ and of some constant not depending on $C_{\mathbf{A}}$ and since $b = 2C_{\mathbf{A}}([\sigma]_M + 1)$ we get the expected result for $\alpha' > 1 + \alpha_3$. □

We saw in lemma 2.27 what happens if, under the assumptions of proposition 2.23, for any hölder continuous function f on $\mathbf{X}$ there is some point x_0 and some integer n such that $|P^n_{i\sigma} f(x_0)|_{HS}$ is far from 1. Now, we are going to study the other case : when there is a hölder-continuous function f on $\mathbf{X}$ such that for any x, $|P^n_{i\sigma} f(x)|_{HS}$ stays close to 1 for many values of n.

We will need the following lemma that controls the time of first return to a fixed ball in $\mathbf{X}/\mathbf{H}$.

Lemma 2.28. *Under the assumptions of proposition 2.23, there is a consant C depending only on γ such that for any $r \in]0,1]$, any $x, x_0 \in \mathbf{X}$ such that the projection of x_0 onto $\mathbf{X}/\mathbf{H}$ is $\Delta - \nu$-regular at scale r and any $n \in \mathbb{N}$,*

$$P^n(\mathbf{1}_{B(\mathbf{H}x_0, 2r)})(x) \geqslant r^\Delta - C_0 C \frac{e^{-\kappa n}}{r^\gamma}$$

Proof. For $x, x_0 \in \mathbf{X}$ and $r \in]0,1]$, we note

$$f(x) = 1 - \min\left(1, \frac{d(x, \mathbf{H}B(x_0, r))}{r}\right)$$

The function f is $1/r-$lipschitzian on $\mathbf{X}$ and for any $x \in \mathbf{H}B(x_0, r)$, $f(x) = 1$ and for $x \in \mathbf{H}B(x_0, 2r)^c$, $f(x) = 0$.

For $\gamma \in]0,1]$ and $x \in \mathbf{X}$, we note $f_\gamma(x) = f(x)^\gamma$. This function is $\gamma-$hölder continuous and $\|f_\gamma\|_\gamma \leqslant C_\gamma r^{-\gamma}$ for some constant C_γ depending only on γ. Moreover,

$$\mathbf{1}_{B(x_0,r)}(x) \leqslant f_\gamma(x) \leqslant \mathbf{1}_{B(x,2r)}(x)$$

So, using proposition 2.8, we have that for any $x \in \mathbf{X}$,

$$\begin{aligned} P^n\left(\mathbf{1}_{B(\mathbf{H}x_0,r)}\right)(x) &\geqslant P^n f_\gamma(x) \geqslant \int_{\mathbf{X}} f_\gamma \mathrm{d}\nu - C_0 e^{-\kappa n}\|f_\gamma\|_\gamma \\ &\geqslant \nu(B(\pi_{\mathbf{H}}(x_0), r)) - C_0 C_\gamma e^{-\kappa n} r^{-\gamma} \\ &\geqslant r^\Delta - C_0 C_\gamma e^{-\kappa n} r^{-\gamma} \end{aligned}$$

And this is what we intended to prove. □

Lemma 2.29. *Under the assumption of proposition 2.23.*

Let $\xi \in \widehat{\mathbf{H}}$, $f \in \mathcal{C}^{0,\gamma}_{\xi}(\mathbf{X})$ with $\|f\|_\infty \leqslant 1$ and let x in $\mathbf{X}$ and $L \in \mathbb{N}^*$.

Then, noting $t = 1 - \min(|f(x)|_{HS}, |P^L_{\mathbf{e},i\sigma} f(x)|_{HS}, |P^{2L}_{\mathbf{e},i\sigma} f(x)|_{HS})$, we have that for any $j \in \{0,1\}$,

$$\int_{\mathbf{G}} \left| e^{i\sigma(g,x)} P^{jL}_{\mathbf{e},i\sigma} f(gx) - P^{(j+1)L}_{\mathbf{e},i\sigma} f(x) \right|^2_{HS} \mathrm{d}\rho^{*L}_{\mathbf{e}}(g) \leqslant 2t$$

Moreover, for any $r \in]0,1]$ and any $x_0 \in \mathbf{X}$ whose projection on $\mathbf{X}/\mathbf{H}$ is $\Delta - \nu$-regular at scale r, if $r^\Delta - C_0 C e^{-\kappa L} r^{-\gamma} > 0$ then

$$|f(x_0)P^L_{\mathbf{e},i\sigma}f(x) - P^L_{i,\sigma}f(x_0)P^{2L}_{\mathbf{e},i\sigma}f(x)|^2_{HS} \leqslant 8\left(\frac{t}{r^\Delta - C_0 C e^{-\kappa L} r^{-\gamma}} + r^\gamma \|P^L_{\mathbf{e},i\sigma}f\|_\gamma\right)$$

Proof. Developing the following expression, we can compute, for $j \in \{0,1\}$,

$$\begin{aligned} I_j(x) :&= \int_{\mathbf{G}} \left|e^{i\sigma(g,x)}P^{jL}_{\mathbf{e},i\sigma}f(gx) - P^{(j+1)L}_{\mathbf{e},i\sigma}f(x)\right|^2_{HS} \mathrm{d}\rho^{*L}_{\mathbf{e}}(g) \\ &= P^L_{\mathbf{e},0}|P^{jL}_{\mathbf{e},i\sigma}f|^2_{HS}(x) + |P^{(j+1)L}_{\mathbf{e},i\sigma}f(x)|^2_{HS} - 2|P^{(j+1)L}_{\mathbf{e},i\sigma}f(x)|^2_{HS} \\ &\leqslant 1 - |P^{(j+1)L}_{\mathbf{e},i\sigma}f(x)|^2_{HS} \leqslant 2t \end{aligned}$$

This finishes the proof of the first part of the lemma.

To prove the last part of the lemma, we note, to simplify our notations, $Q = P^L_{\mathbf{e},i\sigma}$ and $Q_0 = P^L_{\mathbf{e},0}$. Then, we note

$$J(x) := \int_{\mathbf{G}} \left|Qf(gx)Qf(x) - f(gx)Q^2f(x)\right|^2_{HS} \mathrm{d}\rho^{*L}_{\mathbf{e}}(g)$$

Using the triangular inequality and the fact that Hilbert-Schmidt's norm is submultiplicative on the algebra $\mathrm{End}(\mathbb{V})$, we get that

$$\begin{aligned} \sqrt{J(x)} &\leqslant |Qf(x)|_{HS}\left(\int_{\mathbf{G}} \left|e^{i\sigma(g,x)}Qf(gx) - Q^2f(x)\right|^2_{HS} \mathrm{d}\rho^{*L}_{\mathbf{e}}(g)\right)^{1/2} \\ &\qquad + |Q^2f(x)|_{HS}\left(\int_{\mathbf{G}} \left|e^{i\sigma(g,x)}f(gx) - Qf(x)\right|^2_{HS} \mathrm{d}\rho^{*L}_{\mathbf{e}}(g)\right)^{1/2} \\ &\leqslant \sqrt{I_0(x)} + \sqrt{I_1(x)} \leqslant 2\sqrt{2t} \end{aligned}$$

This proves that

$$J(x) \leqslant 8t$$

Moreover, we can expand $J(x)$ to prove that for any $x_0 \in \mathbf{X}$,

$$\begin{aligned} J(x) &\geqslant \int_{\mathbf{G}} \mathbf{1}_{\mathbf{H}B(x_0,r)}(gx)\left|Qf(gx)Qf(x) - f(gx)Q^2f(x)\right|^2_{HS} \mathrm{d}\rho^{*L}_{\mathbf{e}}(g) \\ &\geqslant \int_{\mathbf{G}} \mathbf{1}_{\mathbf{H}B(x_0,r)}(gx)\mathrm{tr}\left(Qf(x)Qf(x)^*Qf(gx)^*Qf(gx)\right) \\ &\qquad + \int_{\mathbf{G}} \mathbf{1}_{\mathbf{H}B(x_0,r)}(gx)\mathrm{tr}\left(Q^2f(x)Q^2f(x)^*f(gx)^*f(gx)\right)\mathrm{d}\rho^{*L}_{\mathbf{e}}(g) \\ &\qquad - 2\int_{\mathbf{G}} \mathbf{1}_{\mathbf{H}B(x_0,r)}(gx)\mathrm{tr}\left(Q^2f(x)Qf(x)^*Qf(gx)^*f(gx)\right)\mathrm{d}\rho^{*L}_{\mathbf{e}}(g) \end{aligned}$$

Then, if the projection of x_0 on $\mathbf{X}/\mathbf{H}$ is $\Delta-\nu$-regular at scale r, then noting $A = f(x_0)$, $B = Qf(x_0)$, we get that for any $y \in B(x_0,r)$,

$$|Qf(x)Qf(x)^*\left(Qf(y)(Qf(y))^* - BB^*\right)|_{HS} \leqslant 2r^\gamma \|Qf\|_\gamma$$

As the function $Qf(Qf)^*$ is $\mathbf{H}-$invariant (since $f \in \mathcal{C}^{0,\gamma}_\xi(\mathbf{X})$ satisfies the equivariant relation 2.8), this inequality actually holds on $\mathbf{H}B(x_0,r)$.

Doing the same with $f(y)f(y)^*$ and $Qf(y)f(y)^*$, we get that for any $g \in \mathbf{G}$ such that $gx \in \mathbf{H}B(x_0, r)$,

$$|AQf(x) - BQ^2 f(x)|^2_{HS} P^L_{\mathbf{e}} \mathbf{1}_{\mathbf{H}B(x_0,r)}(x) \leqslant J(x) + 8r^\gamma \|Qf\|_\gamma P^L_{\mathbf{e}} \mathbf{1}_{B(\mathbf{H}x_0,r)}(x)$$

Where we noted $A = f(x_0)$ et $B = Qf(x_0)$.

To conclude we only need to control $P^L_{\mathbf{e}} \mathbf{1}_{\mathbf{H}B(x_0,r)}(x)$. To do so, we are going to use lemma 2.28. We assumed that the projection of x_0 on $\mathbf{X}/\mathbf{H}$ is $\Delta - \nu$-regular at scale r and so,

$$P^L_{\mathbf{e}} \mathbf{1}_{\mathbf{H}B(x,r_0)}(x) \geqslant r^\Delta - C_0 C e^{-\kappa L} r^{-\gamma}$$

And if $r^\Delta - C_0 C e^{-\kappa L} r^{-\gamma} > 0$, we get the expected result. □

Lemma 2.30. *Under the assumptions of proposition 2.23.*

Let $\xi \in \widehat{\mathbf{H}}$, $f \in \mathcal{C}^{0,\gamma}_\xi(\mathbf{X})$ such that $\|f\|_\infty \leqslant 1$ and let x in $\mathbf{X}$ and $K, L \in \mathbb{N}^*$.

Noting $t = 1 - \min(|f(x)|, |P^{KL}_{i\sigma} f(x)|, |P^{2KL}_{\mathbf{e},i\sigma} f(x)|)$ we assume that $t \neq 1$ and we have that for any x_0 that is $\Delta - \nu$-regular at scale r, if $r^\Delta - C_0 C e^{-\kappa KL} r^{-\gamma} > 0$ then

$$\left(\int_{\mathbf{G}} | \, e^{i\sigma(g,x)} f(x_0) P^{KL}_{\mathbf{e},i\sigma} f(gx) - f(x_0) P^{KL}_{\mathbf{e},i\sigma} f(x)|^2_{HS} \mathrm{d}\rho^{*L}_{\mathbf{e}}(g) \right)^{1/2}$$
$$\leqslant \frac{1}{C_{\mathbf{H}}(1-t)} \left(2^{KL/2+1} \sqrt{2t} + 2^{4+L/2} \left(\frac{t}{r^\Delta - C_0 C e^{-\kappa KL} r^{-\gamma}} + r^\gamma C_2 \|f\|_\gamma \right) \right)$$

Proof. Let's compute, using the triangular inequality,

$$\begin{aligned} I(x) :&= \left(\int_{\mathbf{G}} |e^{i\sigma(g,x)} P^{KL}_{\mathbf{e},i\sigma} f(x_0) P^{KL}_{\mathbf{e},i\sigma} f(gx) - f(x_0) P^{KL}_{\mathbf{e},i\sigma} f(x)|^2 \mathrm{d}\rho^{*L}_{\mathbf{e}}(g) \right)^{1/2} \\ &\leqslant \left(\int_{\mathbf{G}} |P^{KL}_{\mathbf{e},i\sigma} f(x_0) \left(e^{i\sigma(g,x)} P^{KL}_{\mathbf{e},i\sigma} f(gx) - P^{2KL}_{\mathbf{e},i\sigma} f(x) \right)|^2 \mathrm{d}\rho^{*L}_{\mathbf{e}}(g) \right)^{1/2} \\ &\qquad + |P^{KL}_{\mathbf{e},i\sigma} f(x_0) P^{2KL}_{\mathbf{e},i\sigma} f(x) - f(x_0) P^{KL}_{\mathbf{e},i\sigma} f(x)| \end{aligned}$$

But, according to lemma 2.29, we have

$$|P^{KL}_{\mathbf{e},i\sigma} f(x_0) P^{2KL}_{\mathbf{e},i\sigma} f(x) - f(x_0) P^{KL}_{\mathbf{e},i\sigma} f(x)| \leqslant 8 \left(\frac{t}{r^\Delta - C_0 C e^{-\kappa KL} r^{-\gamma}} + r^\gamma C_2 \|f\|_\gamma \right)$$

And according to the same lemma, we also have that

$$\rho_{\mathbf{e}}(\mathbf{e})^{(K-1)L} \int_{\mathbf{G}} |e^{i\sigma(g,x)} P^{KL}_{\mathbf{e},i\sigma} f(gx) - P^{2KL}_{\mathbf{e},i\sigma} f(x)|^2 \mathrm{d}\rho^{*L}_{\mathbf{e}}(g)$$
$$\leqslant \int_{\mathbf{G}} |e^{i\sigma(g,x)} P^{KL}_{\mathbf{e},i\sigma} f(gx) - P^{2KL}_{\mathbf{e},i\sigma} f(x)|^2 \mathrm{d}\rho^{*KL}_{\mathbf{e}}(g) \leqslant 2t$$

So, finally, we find, using that $\rho_{\mathbf{e}}(\mathbf{e}) \geqslant 1/2$, that

$$\begin{aligned} I(x) &\leqslant \sqrt{\frac{2t}{\rho_{\mathbf{e}}(\mathbf{e})^{(K-1)L}}} + 8 \left(\frac{t}{r^\Delta - C_0 C e^{-\kappa KL} r^{-\gamma}} + r^\gamma C_2 \|f\|_\gamma \right) \\ &\leqslant 2^{(K-1)L/2} \sqrt{2t} + 8 \left(\frac{t}{r^\Delta - C_0 C e^{-\kappa KL} r^{-\gamma}} + r^\gamma C_2 \|f\|_\gamma \right) \end{aligned}$$

To conclude, we have to see that $f(x_0)$ has to be close to $P^{KL}_{\mathbf{e},i\sigma}f(x_0)$. To do so, we are going to use the fact that, for the lazy random walk, at step L, the probability to have stayed at the same point (i.e. $\rho^{*L}_{\mathbf{e}}(\mathbf{e})$) is large. Indeed, we have that

$$\rho^{*L}_{\mathbf{e}}(\mathbf{e})\left|\left(P^{KL}_{\mathbf{e},i\sigma}f(x_0)-f(x_0)\right)P^{KL}_{\mathbf{e},i\sigma}f(x)\right|^2\leqslant I(x)^2$$

Doing the same computations with the hx_0 for $h\in\mathbf{H}$, we get that for any $h\in\mathbf{H}$,

$$\rho^{*L}_{\mathbf{e}}(\mathbf{e})\left|\left(P^{KL}_{\mathbf{e},i\sigma}f(x_0)-f(x_0)\right)\rho(h)^*P^{KL}_{\mathbf{e},i\sigma}f(x)\right|^2\leqslant I(x)^2$$

But, according to lemma 2.22

$$\max_{h\in\mathbf{H}}\left|\left(P^{KL}_{\mathbf{e},i\sigma}f(x_0)-f(x_0)\right)\rho(h)^*P^{KL}_{\mathbf{e},i\sigma}f(x)\right|\geqslant C_{\mathbf{H}}|P^{KL}_{\mathbf{e},i\sigma}f(x_0)-f(x_0)|_{HS}|P^{KL}_{\mathbf{e},i\sigma}f(x)|_{HS}$$

So,

$$|P^{KL}_{\mathbf{e},i\sigma}f(x_0)-f(x_0)|_{HS}\leqslant\frac{I(x)}{C_{\mathbf{H}}|P^{KL}_{\mathbf{e},i\sigma}f(x)|_{HS}\sqrt{\rho^{*L}_{\mathbf{e}}(\mathbf{e})}}$$

And, as

$$|P^{KL}_{\mathbf{e},i\sigma}f(x)|_{HS}\geqslant 1-t\text{ and }\rho^{*L}_{\mathbf{e}}(\mathbf{e})\geqslant 2^{-L},$$

we finally get that

$$|P^{KL}_{\mathbf{e},i\sigma}f(x_0)-f(x_0)|_{HS}\leqslant\frac{2^{L/2}I(x)}{C_{\mathbf{H}}(1-t)}$$

We just proved that

$$\begin{aligned}&\left(\int_{\mathbf{G}}|\ e^{i\sigma(g,x)}f(x_0)P^{KL}_{\mathbf{e},i\sigma}f(gx)-f(x_0)P^{KL}_{\mathbf{e},i\sigma}f(x)|^2\mathrm{d}\rho^{*L}_{\mathbf{e}}(g)\right)^{1/2}\\&\qquad\leqslant(1+\frac{2^{L/2}}{C_{\mathbf{H}}(1-t)})I(x)\\&\qquad\leqslant\frac{1}{C_{\mathbf{H}}(1-t)}\left(2^{KL/2+1}\sqrt{2t}+2^{4+L/2}\left(\frac{t}{r^{\Delta}-C_0Ce^{-\kappa KL}r^{-\gamma}}+r^{\gamma}C_2\|f\|_{\gamma}\right)\right)\end{aligned}$$

and this is what we intended to prove. □

Lemma 2.31. *Under the assumptions of proposition 2.23.*

For any α_1,β there is α_2 such that for any Δ there are α_1',β' such that for any $\sigma\in\mathcal{Z}^M$ and any $\xi\in\widehat{\mathbf{H}}$, if the hypothesis $\mathbf{H}(\alpha_1',\beta',\xi)$ is false, then there is an hölder continuous function $\varphi\in\mathcal{C}^{0,\gamma}(\mathbf{X})$ with $\|\varphi\|_{\sigma}\leqslant 2C_2$ such that for any point $x\in\mathbf{X}$ whose projection on $\mathbf{X}/\mathbf{H}$ is $\Delta-\nu$-regular at scale $b^{-\alpha_2}$, we have that

$$\max_{h\in\mathbf{H}}|\varphi(hx)|\geqslant 1-\frac{1}{b^{\alpha_1}}$$

and for any $h\in\mathbf{H}$,

$$\int_{\mathbf{G}}\left|e^{i\sigma(g,x)}\varphi(gx)-\varphi(x)\right|^2\mathrm{d}\rho^{n(\beta,b)}_{\mathbf{e}}(g)\leqslant\frac{1}{b^{\alpha_1}}$$

Proof. Fix $\alpha_1,\alpha_2,\beta,\Delta$ and take $\alpha_1',\beta'\in\mathbb{R}_+^*$ be constant that we will specify in the sequel.

If the assumption $\mathbf{H}(\alpha_1', \beta', \xi)$ is not satisfied, then there is $f \in \mathcal{C}_\chi^{0,\gamma}(\mathbf{X})$ with $\|f\|_\sigma \leqslant 1$ such that for any $n \in [0, n(\beta', b)]$ and any $x \in \mathbf{X}$ whose projection is $\Delta - \nu$-regular at scale $b^{-\alpha_2}$,

$$|P_{\mathbf{e},i\sigma}^n f(x)|_{HS} \geqslant 1 - \frac{1}{b^{\alpha_1'}}$$

As $b \geqslant 2$, we shall assume without any loss of generality that $\alpha_1' \geqslant \max(1, \alpha_1)$. So, we have that

$$1 - \frac{1}{b^{\alpha_1'}} \geqslant \max(\frac{1}{2}, 1 - \frac{1}{b^{\alpha_1}}).$$

We can now use lemma 2.30 to the function f with $L = n(\beta, b)$, $t = \frac{1}{b^{\alpha_1'}}$, $K \in \mathbb{N}^*$ and $\beta' \geqslant K\beta$ to find that for any point $\mathbf{X}$ whose projection is $\Delta - \nu$-regular at scale $b^{-\alpha_2}$,

$$\begin{aligned}\int_{\mathbf{G}} \left| e^{i\sigma(g,x)} f(x_0) P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(gx) - f(x_0) P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(x) \right|_{HS}^2 \mathrm{d}\rho_{\mathbf{e}}^{*n(\beta,b)}(g) \\ \leqslant \frac{2}{C_{\mathbf{H}}} \left(\frac{e^{Kn(\beta,b)\ln 2/2} + 2\ln 2}{b^{\alpha_1'/2}} + 16 b^{\beta \ln 2/2} \frac{b^{-\alpha_1'}}{b^{-\Delta\alpha_2} - C_0 C b^{-\kappa K\beta - \gamma\alpha_2}} + b^{-\gamma\alpha_2} \right)\end{aligned}$$

So, using the fact that $b \geqslant 2$ and taking α_2 large enough then K and finally α_1', we get that for any x whose projection is $\Delta - \nu$-regular at scale $b^{-\alpha_2}$,

$$\int_{\mathbf{G}} \left| e^{i\sigma(g,x)} f(x_0) P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(gx) - f(x_0) P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(x) \right|_{HS}^2 \mathrm{d}\rho_{\mathbf{e}}^{*n(\beta,b)}(g) \leqslant \frac{C_{\mathbf{H}}}{2|\mathbf{H}| b^{\alpha_1}}$$

Finally, we use the fact that we can do the same computations with all the hx_0 with $h \in \mathbf{H}$, to get, using that $f \in \mathcal{C}_\xi^{0,\gamma}(\mathbf{X})$, that

$$\int_{\mathbf{G}} \sum_{s \in \mathbf{H}} \left| f(x_0)\rho(h)^* \left(e^{i\sigma(g,x)} P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(gx) - P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(x) \right) \right|_{HS}^2 \mathrm{d}\rho_{\mathbf{e}}^{*n(\beta,b)}(g) \leqslant \frac{C_{\mathbf{H}}}{2b^{\alpha_1}}$$

We can now conclude using lemma 2.22 that allows us to make the following computation

$$\begin{aligned}\sum_{h \in \mathbf{H}} \left| f(x_0)\rho(s)^* \left(e^{i\sigma(g,x)} P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(gx) - P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(x) \right) \right|_{HS} \\ \geqslant C_{\mathbf{H}} |f(x_0)|_{HS} \left| e^{i\sigma(g,x)} P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(gx) - P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(x) \right|_{HS} \\ \geqslant \frac{C_{\mathbf{H}}}{2} \left| e^{i\sigma(g,x)} P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(gx) - P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(x) \right|_{HS}\end{aligned}$$

This proves that

$$\int_{\mathbf{G}} \left| e^{i\sigma(g,x)} \mathrm{tr} P_{e,i\sigma}^{Kn(\beta,b)} f(gx) - \mathrm{tr} P_{e,i\sigma}^{Kn(\beta,b)} \right|^2 \mathrm{d}\rho_{\mathbf{e}}^{*n(\beta,b)} \leqslant \frac{1}{b^{\alpha_1}}$$

Therefore, we get the expected result with $\varphi = \mathrm{tr} P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f$ and using that for any $x \in \mathbf{X}$,

$$P_{\mathbf{e},i\sigma}^{Kn(\beta,b)} f(x) = \frac{\dim \mathbb{V}}{\mathbf{H}} \sum_{h \in \mathbf{H}} \varphi(h^{-1}x)\rho(h)^*$$

So we really get that

$$\max_{h\in\mathbf{H}} |\varphi(hx)| \geqslant \left(1-\frac{1}{b^{\alpha_1}}\right)\frac{1}{\dim \mathbb{V}}$$

□

To prove proposition 2.23, we can now use the isomorphism given by lemma 2.21 and the lemma 2.27 and 2.31.

3. Diophantine properties in linear groups

In $\mathrm{SL}_d(\mathbb{R})$, the application mapping a matrix to it's spectral radius is not a morphism (for $d \geqslant 2$).

In a Zariski-dense subgroup Γ of $\mathrm{SL}_d(\mathbb{R})$, we can construct sequences of elements (g_n) and (h_n) for which we have a good control on the difference between the logarithm of the spectral radius of $g_n h_n$ and the sum of the ones of g_n and h_n (see [Qui05]).

In this section, we use this construction (that also works for strongly irreducible and proximal subgroups rather than Zariski-dense ones) to prove a technical result that will allow us to use proposition 2.23 and thus show theorem 3.1 which is needed in the study of the speed of convergence in Kesten's renewal theorem.

More specifically, studying the renewal theorem on $\mathbb{R}$ (see [Car83]), we get that some diophantine condition is needed and we are going to see that it's equivalent is always satisfied in $\mathrm{SL}_d(\mathbb{R})$ for measures whose support generate a strongly irreducible and proximal subgroup. This will be the

Proposition (3.16)**.** *Let ρ be a borelian probability measure on $\mathbf{G} = \mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal subgroup.*

Then, there are $\alpha, \beta \in \mathbb{R}_+^*$ such that

$$\liminf_{b\to\pm\infty} |b|^\alpha \int_{\mathbf{G}} \left|e^{ib\lambda_1(g)} - 1\right|^2 \mathrm{d}\rho^{*n(\beta,b)}(g) > 0$$

where we noted $\lambda_1(g)$ the logarithm of the spectral radius of g and

$$n(\beta, b) = \lfloor \beta \ln |b| \rfloor$$

We recall that an element g of $\mathrm{SL}_d(\mathbb{R})$ is called proximal it has a local attractive fixed point V_g^+ in the projective space $\mathbb{P}(\mathbb{R}^d)$ of $\mathbb{R}^d$.

We are also going to prove the generecity of lower-regular points which is an other condition we saw in the study of the perturbated operator in the

Proposition (3.18)**.** *Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal subgroup.*

Let ν be the unique $P-$stationary borelian probability measure on $\mathbb{P}(\mathbb{R}^d)$ (it's existence and uniqueness are proved in [GR85]).

Then, for any $M \in \mathbb{R}_+^*$, there is $n_0 \in \mathbb{N}$, $\Delta \in \mathbb{R}$ and $t \in \mathbb{R}_+^*$ such that for any $n \in \mathbb{N}$ with $n \geqslant n_0$,

$$\rho^{*n}\left(\left\{g \in \mathbf{G} \middle| g \text{ is proximal and } \nu\left(B\left(V_g^+, e^{-Mn}\right)\right) \geqslant e^{-\Delta Mn}\right\}\right) \geqslant 1 - e^{-tn}$$

where V_g^+ is the locally attractive fixed point of g in $\mathbb{P}(\mathbb{R}^d)$ and we endowed $\mathbb{P}(\mathbb{R}^d)$ of it's usual distance (see equation 3.1).

This two results and proposition 2.23 will allow us to prove the

Theorem 3.1. *Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and prowimal subgroup Γ.*

Let $(\mathbf{A}, \nu_{\mathbf{A}})$ be a non empty finite $\Gamma-$set endowed with the uniform probability measure and such that the random walk defined by ρ on $\mathbf{A}$ is irreducible and aperiodic.

Let $\sigma : \Gamma \times \mathbb{S}^{d-1} \to \mathbb{R}$ be the cocycle definded for $g \in \Gamma$ and $x \in \mathbb{S}^{d-1}$ by

$$\sigma(g, x) = \ln \frac{\|gx\|}{\|x\|}$$

and let, for any $t \in \mathbb{R}$, $P(it)$ be the operator on $\mathcal{C}^0(\mathbb{S}^{d-1} \times \mathbf{A})$ defined for $f \in \mathcal{C}^0(\mathbb{S}^{d-1} \times \mathbf{A})$ and $(x, a) \in \mathbb{S}^{d-1} \times \mathbf{A}$ by

$$P(it)f(x, a) = \int_{\mathbf{G}} e^{-it\sigma(g,x)} f(gx, ga) \mathrm{d}\rho(g)$$

Then, for any $\gamma > 0$ small enough and any $t_0 \in \mathbb{R}_+^*$, there are $C, L \in \mathbb{R}_+$ such that for any $t \in \mathbb{R}$ with $|t| \geqslant t_0$,

$$\|(I_d - P(it))^{-1}\|_{\mathcal{C}^{0,\gamma}(\mathbb{S}^{d-1} \times \mathbf{A})} \leqslant C|t|^L$$

Moreover, the constant L depends on $\mathbf{A}$ only through the spectral gap of P in $\mathrm{L}^2(\mathbf{A}, \nu_{\mathbf{A}})$ (see definition 2.7).

Proof. This is a straightforward application of proposition 2.23 (noting that in this case the group $\mathbf{H}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ so it's irreducible representations are of dimension 1), that we are allowed to apply thanks to lemma 3.2 and 3.8 whose assumptions hold as we saw in proposition 3.16 and in lemma 3.19. □

3.1. Notations and preliminaries.

We start by fixing the notations we are going to use in the whole section.

3.1.1. *Proximal elements of* $\mathrm{SL}_d(\mathbb{R})$. Let $(\mathbb{V}, \|\,.\,\|)$ be a finite dimensional real vector space endowed with an euclidean norm and an orthonormal basis $(e_1, \ldots, e_d)$.

Define a distance on $\mathbb{P}(\mathbb{V})$ noting, for $X = \mathbb{R}x, Y = \mathbb{R}y \in \mathbb{P}(\mathbb{V})$,

$$d(X, Y) = \frac{\|x \wedge y\|}{\|x\|\|y\|} \tag{3.1}$$

Where we extended the scalar product on $\mathbb{V}$ to $\wedge^2\mathbb{V}$ by asking the basis $(e_i \wedge e_j)_{1 \leqslant i < j \leqslant d}$ to be orthonormal.

We also define a pairing between $\mathbb{P}(\mathbb{V})$ and $\mathbb{P}(\mathbb{V}^*)$ noting, for $X = \mathbb{R}x \in \mathbb{V}$ and $Y = \mathbb{R}\varphi \in \mathbb{V}^*$,

$$\delta(X, Y) := \frac{|\varphi(x)|}{\|\varphi\|\|x\|} = \inf_{Y' \in Y^\perp} d(X, Y')$$

where $Y^\perp = \{Y' = \mathbb{R}y' \in Y | \varphi(y') = 0\}$.

We refer to the chapter 9 of [BQ15] for a proof the following lemma that proves that we are in the case studied in section 2.

Lemma 3.2. *For any $g \in \mathbf{G}$ and any $X, Y \in \mathbb{P}(\mathbb{R}^d)$,*

$$d(gX, gY) \leqslant \|g\|^{2d} d(X, Y)$$

Moreover, there is $C \in \mathbb{R}$ such that for any $X, Y \in \mathbb{P}(\mathbb{R}^d)$ and any $g \in \mathbf{G}$,

$$|\sigma(g, X) - \sigma(g, Y)| \leqslant C\|g\|^C d(X, Y)$$

where we noted, for $X = \mathbb{R}x \in \mathbb{P}(\mathbb{R}^d)$,

$$\sigma(g, X) = \ln \frac{\|gx\|}{\|x\|}$$

So, with the notations of section 2 we have that $\sigma \in \mathcal{Z}^C(\mathbb{P}(\mathbb{R}^d))$.

An element g of $\mathrm{SL}(\mathbb{V})$ is said to be *proximal* if it has a locally attractive fixed point in $\mathbb{P}(\mathbb{V})$. Equivalently, a proximal element is one such that there is a decomposition $\mathbb{V} = V_g^+ \oplus V_g^<$ of $\mathbb{V}$ into $g-$invariant subspaces such that V_g^+ is a line and the spectral radius of g in $V_g^<$ is strictly smaller than the one in V_g^+. We also remark that g is proximal if and only if ${}^t g$ is proximal when seen as acting on $\mathbb{V}^*$.

For a proximal element g of $\mathrm{SL}_d(\mathbb{R})$, we note $V_g^+ \in \mathbb{P}(\mathbb{V})$ the locally attractive fixed point in $\mathbb{P}(\mathbb{V})$ and $V_g^< \in \mathbb{P}(\mathbb{V}^*)$ the class of the $g-$invariant supplementary space of V_g^+ on $\mathbb{V}$ (it is a projective hyperplane that we can identify with the locally attractive fixed point of ${}^t g$ in $\mathbb{P}(\mathbb{V}^*)$. In the sequel, we will always note v_g^+ a representative of V_g^+ in $\mathbb{V}$ and $\varphi_g^<$ a representative of $V_g^<$ in $\mathbb{V}^*$ and we will always write our formulas in a way such that they do not depend on the choice of the representatives.

For an element g of $\mathrm{SL}_d(\mathbb{R})$, we note $\lambda_1(g), \dots, \lambda_d(g)$ the logarithms of the modulus of the eigenvalues of g in decreasing order and counted with multiplicities. If g is proximal, as V_g^+ is a line generated by v_g^+ and is $g-$invariant, we have by definition that $gv_g^+ = \varepsilon_1(g) e^{\lambda_1(g)} v_g^+$ with $\varepsilon_1(g) \in \{\pm 1\}$.

Finally, for an element g of $\mathrm{SL}_d(\mathbb{V})$, we choose a Cartan decomposition $g = k_g a_g l_g$. More specifically, we have that $k_g, l_g \in \mathcal{O}(\mathbb{V})$ and a_g is the diagonal matrix

$$a_g = \begin{pmatrix} \kappa_1(g) & & 0 \\ & \ddots & \\ 0 & & \kappa_d(g) \end{pmatrix}$$

where the $\kappa_i(g)$ are the *singular values* of g, and satisfy $\kappa_1(g) \geqslant \cdots \geqslant \kappa_d(g)$ and

$$\kappa_i(g) = \frac{\| \wedge^i g\|}{\| \wedge^{i-1} g\|} \tag{3.2}$$

where we noted $\wedge^i g$ the endomorphism defined by g in $\bigwedge^i \mathbb{V}$ endowed with the inner product extending the one of $\mathbb{V}$:

$$\wedge^i g(v_1 \wedge \cdots \wedge v_i) = (gv_1) \wedge \cdots \wedge (gv_i)$$

Moreover, we note

$$\kappa_{1,2}(g) = \frac{\kappa_2(g)}{\kappa_1(g)} = \frac{\| \wedge^2 g\|}{\|g\|^2}$$

Finally, for an element $g \in \mathbf{G}$ and a chosen Cartan decomposition $g = k_g a_g l_g$, we note

$$x_g^M = k_g e_1, \quad X_g^M = \mathbb{R}x_g^m, \quad y_g^m = {}^t l_g e_1^* \text{ and } Y_g^m = \mathbb{R}y_g^m$$

Lemma 3.3. *Let $\mathbb{V} = \mathbb{R}^d$ endowed with an euclidean norm, g an element of $\mathrm{SL}(\mathbb{V})$, $X = \mathbb{R}x \in \mathbb{P}(\mathbb{V})$, $Y = \mathbb{R}\varphi \in \mathbb{P}(\mathbb{V}^*)$.*

Then,

(1)

$$\delta(X, Y_g^m) \leqslant \frac{\|gx\|}{\|g\|\|x\|} \leqslant \delta(X, Y_g^m) + \kappa_{1,2}(g)$$

(2)

$$\delta(X_g^M, Y) \leqslant \frac{\|{}^t g\varphi\|}{\|g\|\|\varphi\|} \leqslant \delta(X_g^M, Y) + \kappa_{1,2}(g)$$

(3)

$$d(gX, X_g^M)\delta(X, Y_g^m) \leqslant \kappa_{1,2}(g)$$

Proof. The norm being euclidean, we shall assume without any loss of generality that g is the diagonal matrix $(\kappa_1(g), \dots, \kappa_d(g))$. We write $x = x_1 + x_2$ with $x_1 \in \mathrm{Vect}(e_1)$ and $x_2 \in \mathrm{Vect}(e_2, \dots, e_d)$.

Then,

$$gx_1 = \kappa_1(g)x_1 \text{ et } \|gx_2\| \leqslant \kappa_2(g)\|x_2\|$$

so, using the fact that $\kappa_1(g) = \|g\|$, we have that

$$\frac{\|x_1\|}{\|x\|} \leqslant \frac{\|gx\|}{\|g\|\|x\|} \leqslant \frac{\|x_1\|}{\|x\|} + \kappa_{1,2}(g)\frac{\|x_2\|}{\|x\|}$$

Moreover,

$$\frac{\|x_1\|}{\|x\|} = d(X, \mathrm{Vect}(e_2, \dots, e_d)) = d(X, Y_g^m)$$

and this proves the first inequalities.

The second ones are proved in the same way working in the dual space.

Finally, the last inequalities are proved after remarking that

$$d(gX, X_g^M)\delta(X, Y_g^m) = \frac{\|gx_2\|}{\|gx\|}\frac{\|x_1\|}{\|x\|} \leqslant \frac{\kappa_2(g)}{\kappa_1(g)} = \kappa_{1,2}(g)$$

□

In the sequel, we will have to control the Cartan decompositon of products of elements of $\mathbf{G}$. To do so, we will use the

Lemma 3.4. *For any $p \in \mathbb{N}$, $p \geqslant 2$, any $\varepsilon \in]0, 1/4]$, and any $g_1, \dots, g_p \in \mathbf{G}$ with $\kappa_{1,2}(g_i) \leqslant \varepsilon^3$, $\delta(X_{g_{i+1}}^M, Y_{g_i}^m) \geqslant 2\varepsilon$ and $\delta(X_{g_i}^M, Y_{g_{i+1}}^m) \geqslant 2\varepsilon$, we have that*

$$\kappa_1(g_p \dots g_1) \geqslant \varepsilon^{p-1}\kappa_1(g_p) \dots \kappa_1(g_1), \quad \kappa_{1,2}(g_p \dots g_1) \leqslant \frac{\kappa_{1,2}(g_1) \dots \kappa_{1,2}(g_p)}{\varepsilon^{2(p-1)}}$$

and,

$$d(X_{g_p \dots g_1}^M, X_{g_p}^M) \leqslant \frac{\kappa_{1,2}(g_p)}{\varepsilon}, \quad d(Y_{g_p \dots g_1}^m, Y_{g_1}^m) \leqslant \frac{\kappa_{1,2}(g_1)}{\varepsilon}$$

Proof. According to lemma 3.3, we have that for any $(g_i) \in \mathbf{G}^p$ and any $x \in \mathbb{R}^d \setminus \{0\}$, noting $X = \mathbb{R}x$,

$$\begin{aligned}\|g_p \dots g_1 x\| &\geqslant \|g_p\| \|g_{p-1} \dots g_1 x\| \delta(g_{p-1} \dots g_1 X, Y^m_{g_p}) \\ &\geqslant \|g_p\| \dots \|g_1\| \|x\| \delta(g_{p-1} \dots g_1 X, Y^m_{g_p}) \dots \delta(X, Y^m_{g_1})\end{aligned}$$

Moreover, taking x in the orthogonal of $Y^m_{g_1}$, we have that for any $l \in [1, p]$,

$$d(g_{l-1} \dots g_1 X, Y^m_{g_l}) \geqslant \frac{l+1}{l}\varepsilon$$

Indeed, this is true for $l = 1$ by assumption and, by induction, we have that for any $l \in [1, p-1]$,

$$d(g_l \dots g_1 X, X^M_{g_l}) \leqslant \frac{\kappa_{1,2}(g_l)}{d(g_{l-1} \dots g_1 X, Y^m_{g_l})} \leqslant \frac{l}{l+1}\varepsilon^2 \leqslant \frac{l}{l+1}\varepsilon$$

So,

$$\delta(g_l \dots g_1 X, Y^m_{g_{l+1}}) \geqslant \varepsilon\left(2 - \frac{l}{l+1}\right) = \frac{l+2}{l+1}\varepsilon$$

This proves that

$$\frac{\|g_p \dots g_1 x\|}{\|x\|} \geqslant \frac{p}{2}\|g_p\| \dots \|g_1\|\varepsilon^{p-1}$$

Thus

$$\kappa_1(g_p \dots g_1) \geqslant \frac{p}{2}\varepsilon^{p-1}\kappa_1(g_p) \dots \kappa_1(g_1)$$

Moreover, using the submultiplicativity of the function $\kappa_1\kappa_2$ (see equation 3.2 that proves that $\kappa_1(g)\kappa_2(g) = \| \wedge^2 g\|$) and that $\kappa_{1,2}(g_i) \leqslant \varepsilon^3$, we get that

$$\begin{aligned}\kappa_{1,2}(g_p \dots g_1) &= \frac{\kappa_2(g_p \dots g_1)\kappa_1(g_p \dots g_1)}{\kappa_1(g_p \dots g_1)^2} \leqslant \frac{4}{p^2\varepsilon^{2(p-1)}}\kappa_{1,2}(g_1) \dots \kappa_{1,2}(g_p) \\ &\leqslant \frac{4}{p^2}\varepsilon^{p-1}\kappa_{1,2}(g_p)\end{aligned}$$

Finally, using once again lemma 3.3, we get that

$$\delta(X, Y^m_{g_p \dots g_1}) + \kappa_{1,2}(g_p \dots g_1) \geqslant \frac{p}{2}\varepsilon^{p-1}$$

And so,

$$\delta(X, Y^m_{g_p \dots g_1}) \geqslant \frac{p}{2}\varepsilon^{p-1} - \frac{4}{p^2}\varepsilon^{p+2} = \frac{p}{2}\varepsilon^{p-1}\left(1 - \frac{8}{p^3}\varepsilon^3\right)$$

Thus

$$d(g_p \dots g_1 X, X^M_{g_p \dots g_1}) \leqslant \frac{\kappa_{1,2}(g_p \dots g_1)}{\delta(X, Y^m_{g_p \dots g_1})} \leqslant \frac{2}{p}\frac{\kappa_{1,2}(g_p)}{1 - 8\varepsilon^3/p^3}$$

This proves that

$$\begin{aligned}d(X^M_{g_p \dots g_1}, X^M_{g_p}) &\leqslant \frac{p}{p+1}\frac{\kappa_{1,2}(g_p)}{\varepsilon} + \frac{2}{p}\frac{\kappa_{1,2}(g_p)}{(1 - 8\varepsilon^3/p^3)} \\ &\leqslant \frac{\kappa_{1,2}(g)}{\varepsilon}\left(\frac{p}{p+1} + \frac{2\varepsilon}{p(1 - 8\varepsilon^3/p^3)}\right)\end{aligned}$$

And that proves the last inequality since for $\varepsilon \in]0,1/4]$ and $p \geqslant 2$, we have that $\frac{2\varepsilon}{p(1-8\varepsilon^3/p^3)} \leqslant \frac{1}{p+1}$.

To prove the control of $d(Y^m_{g_p\dots g_1}, Y^m_{g_1})$, we do the same computations in the dual space. □

The following lemma will allow us to, knowing the Cartan decomposition of an element g of $\mathrm{SL}_d(\mathbb{R})$, prove that it is proximal and to have a control on the distance between V_g^+ and $V_g^<$.

Lemma 3.5. *For any $\varepsilon \in]0,1/4]$ and any element g of $\mathrm{SL}_d(\mathbb{R})$, if $\kappa_{1,2}(g) \leqslant \varepsilon^3$ and $\delta(X_g^M, Y_g^m) \geqslant 2\varepsilon$ then g is proximal and*

$$d(V_g^+, X_g^M) \leqslant \frac{\kappa_{1,2}(g)}{\varepsilon}, \quad d(V_g^<, Y_g^m) \leqslant \frac{\kappa_{1,2}(g)}{\varepsilon}$$

Moreover,

$$e^{\lambda_1(g)} \geqslant \kappa_1(g)\delta(X_g^M, Y_g^m) \text{ et } \|g_{|_{V_g^<}}\| \leqslant \frac{\kappa_2(g)}{\varepsilon}$$

Proof. The first three inequalities come from lemma 13.14 in [BQ15].

To prove that the norm of g restricted to $V_g^<$ is controled by $\kappa_2(g)$, we remark that according to lemma 3.3, for any $x \in \mathbb{R}^d \setminus \{0\}$, noting $X = \mathbb{R}x$, we have that

$$\frac{\|gx\|}{\|x\|} \leqslant \kappa_1(g)\delta(X, Y_g^m) + \kappa_2(g)$$

But, for $X \in V_g^<$, we have that

$$\delta(X, Y_g^m) \leqslant d(V_g^<, Y_g^m) \leqslant \frac{\kappa_{1,2}(g)}{\delta(X_g^M, Y_g^m)}$$

And so, for any $x \in V_g^< \setminus \{0\}$,

$$\frac{\|gx\|}{\|x\|} \leqslant \kappa_2(g)\left(1 + \frac{1}{\delta(X_g^M, Y_g^m)}\right) \leqslant \frac{\kappa_{1,2}(g)}{\varepsilon}$$

And this proves the last expected inequality of the lemma. □

From now on, we note, for $g \in \mathrm{SL}_d(\mathbb{R})$ and $X = \mathbb{R}x \in \mathbb{P}(\mathbb{R}^d)$,

$$\sigma(g, X) = \ln \frac{\|gx\|}{\|x\|} \tag{3.3}$$

Lemma 3.6. *For any $\varepsilon \in]0,1/4]$, and any g of $\mathbf{G}$, if $\kappa_{1,2}(g) \leqslant \varepsilon^3$ and $d(X_g^M, Y_g^m) \geqslant 2\varepsilon$, we have that for any $X \in \mathbb{P}(\mathbb{R}^d)$ with $\delta(X, V_g^<) \geqslant 2\varepsilon$,*

$$\left|\sigma(g, X) - \lambda_1(g) - \ln \frac{\delta(X, V_g^<)}{\delta(V_g^+, V_g^<)}\right| \leqslant 2\frac{\kappa_{1,2}(g)}{\varepsilon^3}$$

Moreover, for any $X, Y \in \mathbb{P}(\mathbb{R}^d)$ with $\delta(X, V_g^<), \delta(Y, V_g^<) \geqslant 2\varepsilon$, we have that

$$d(gX, gY) \leqslant \frac{\kappa_{1,2}(g)}{4\varepsilon^4}$$

Proof. We recall that we noted $v_g^+, \varphi_g^<$ such that $V_g^+ = \mathbb{R}v_g^+$ and $V_g^< = \mathbb{R}\varphi_g^<$.

Then, according to the previous lemma,

$$\delta(V_g^+, V_g^<) \geqslant 2\varepsilon - 2\frac{\kappa_{1,2}(g)}{\varepsilon} \geqslant 2\varepsilon(1-\varepsilon^2) \geqslant \frac{3}{2}\varepsilon$$

For $x \in \mathbb{R}^d$, we can write

$$x = \frac{\varphi_g^<(x)}{\varphi_g^<(v_g^+)}v_g^+ + x - \frac{\varphi_g^<(x)}{\varphi_g^<(v_g^+)}v_g^+$$

and so, as $gv_g^+ = \varepsilon_1(g)e^{\lambda_1(g)}v_g^+$,

$$gx = \varepsilon_1(g)e^{\lambda_1(g)}\frac{\varphi_g^<(x)}{\varphi_g^<(v_g^+)}v_g^+ + g\left(x - \frac{\varphi_g^<(x)}{\varphi_g^<(v_g^+)}v_g^+\right)$$

But, $x - \frac{\varphi_g^<(x)}{\varphi_g^<(v_g^+)}v_g^+ \in V_g^<$ and so, according to lemma 3.5,

$$\left\|g\left(x - \frac{\varphi_g^<(x)}{\varphi_g^<(v_g^+)}v_g^+\right)\right\| \leqslant \frac{\kappa_2(g)}{\varepsilon}\left\|x - \frac{\varphi_g^<(x)}{\varphi_g^<(v_g^+)}v_g^+\right\| \leqslant \frac{\kappa_2(g)}{\varepsilon}\frac{2\|x\|}{\delta(V_g^+, V_g^<)}$$

Thus, if $x \neq 0$,

$$e^{\lambda_1(g)}\frac{\delta(X, V_g^<)}{\delta(V_g^+, V_g^<)}(1 - \|u\|) \leqslant \frac{\|gx\|}{\|x\|} \leqslant e^{\lambda_1(g)}\frac{\delta(X, V_g^<)}{\delta(V_g^+, V_g^<)}(1 + \|u\|)$$

with

$$u = \frac{\delta(V_g^+, V_g^<)}{e^{\lambda_1(g)}\delta(X, V_g^<)}g\left(x - \frac{\varphi_g^<(x)}{\varphi_g^<(v_g^+)}v_g^+\right) \text{ and } \|u\| \leqslant \frac{1}{e^{\lambda_1(g)}\delta(X, V_g^<)}\frac{2\kappa_2(g)}{\varepsilon}$$

But, according to lemma3.5, $e^{\lambda_1(g)} \geqslant 2\|g\|\varepsilon$ and so,

$$\frac{2\kappa_2(g)}{e^{\lambda_1(g)}\delta(X, V_g^<)\varepsilon} \leqslant \frac{\kappa_{1,2}(g)}{\varepsilon^2\delta(X, V_g^<)}$$

Thus, for X with $\delta(X, V_g^<) \geqslant 2\varepsilon$, we have

$$\|u\| \leqslant \frac{\kappa_{1,2}(g)}{2\varepsilon^3} \leqslant \frac{1}{2}$$

and

$$\ln(1 - \|u\|) \leqslant \sigma(g, X) - \lambda_1(g) - \ln\frac{\delta(X, V_g^<)}{\delta(V_g^+, V_g^<)} \leqslant \ln(1 + \|u\|)$$

This proves the first inequality if we use that

$$\|u\| \leqslant \min\left(\frac{\kappa_{1,2}(g)}{\varepsilon^3}, \frac{1}{2}\right)$$

Finally, for any $X, Y \in \mathbb{P}(\mathbb{R}^d)$ with $\delta(X, V_g^<), \delta(Y, V_g^<) \geqslant 2\varepsilon$,

$$d(gX, gY) = \frac{\| \wedge^2 g(x \wedge y)\|}{\|gx\|\|gy\|} \leqslant \frac{\kappa_1(g)\kappa_2(g)\|x\|\|y\|}{\|gx\|\|gy\|} \leqslant \frac{\kappa_1(g)\kappa_2(g)}{e^{-2\lambda_1(g)}} \frac{4\delta(V_g^+, V_g^<)^2}{\delta(X, V_g^<)\delta(Y, V_g^<)}$$
$$\leqslant \frac{\kappa_{1,2}(g)}{4\varepsilon^4}$$

□

3.1.2. *Genericity of proximal elements.* First of all, we recall that if ρ is a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having a finite first moment[6], then, there are $\lambda_1, \dots, \lambda_d \in \mathbb{R}$, called Lyapunov exponent of ρ, such that $\lambda_1 + \cdots + \lambda_d = 0$ and for any $i \in [1, d]$,

$$\frac{1}{n} \ln \| \wedge^i g_n \dots g_1\| \to \lambda_1 + \cdots + \lambda_i \ \rho^{\otimes \mathbb{N}} - \text{a.e.}$$

Moreover, if the support of ρ generates a strongly irreducible and proximal group, then $\lambda_1 > \lambda_2$ (see [GR85]).

In the sequel, we will have to produce elements g of $\mathrm{SL}_d(\mathbb{R})$ in the support of ρ^{*n} and having nice properties. To do so, we will use the following lemma and we refer to the chapter 12 in [BQ15] for a proof of this result.

Lemma 3.7. *Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal subgroup. Then, for any $\varepsilon \in \mathbb{R}_+^*$, there is $t \in \mathbb{R}_+^*$ and $n_0 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geqslant n_0$, we have that for any $X, Y \in \mathbb{P}(\mathbb{R}^d)$,*

$$\rho^{*n}\left(\left\{g \in \mathbf{G} \middle| \forall i \in [1, d],\ \left|\frac{1}{n}\kappa_i(g) - \lambda_i\right| \leqslant \varepsilon\right\}\right) \geqslant 1 - e^{-tn}$$

$$\rho^{*n}\left(\left\{g \in \mathbf{G} \middle| \delta(X, Y_g^m) \geqslant 2e^{-\varepsilon n}\right\}\right) \geqslant 1 - e^{-tn}$$

$$\rho^{*n}\left(\left\{g \in \mathbf{G} \middle| d(gX, X_g^M) \leqslant e^{-(\lambda_1 - \lambda_2 - \varepsilon)n}\right\}\right) \geqslant 1 - e^{-tn}$$

$$\rho^{*n}\left(\left\{g \in \mathbf{G} \middle| \delta(X_g^M, Y) \geqslant 2e^{-\varepsilon n}\right\}\right) \geqslant 1 - e^{-tn}$$

$$\rho^{*n}\left(\left\{g \in \mathbf{G} \middle| \delta(gX, Y) \geqslant 2e^{-\varepsilon n}\right\}\right) \geqslant 1 - e^{-tn}$$

$$\rho^{*n}\left(\left\{g \in \mathbf{G} \middle| \delta(X_g^M, Y_g^m) \geqslant 2e^{-\varepsilon n}\right\}\right) \geqslant 1 - e^{-tn}$$

Moreover, adding lemma 3.2, we get the

Lemma 3.8. *Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal subgroup. For any $g \in \mathbf{G}$, we note $N(g) := \|g\|$.*

Then, there is $\gamma \in \mathbb{R}_+^*$ such that $\mathbb{P}(\mathbb{R}^d)$ is $(\rho, \gamma, 2d, N)-$contracted.

Proof. We refer to [BL85] for a proof of this result. □

[6] $\int_{\mathbf{G}} \ln \|g\| \mathrm{d}\rho(g)$ is finite.

3.1.3. *Regularity of powers of convolution of measures.*

We state here a technical result that shows that if ρ is a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment, then for any $n \in \mathbb{N}$, very few points of $\operatorname{supp} \rho^{*n}$ have a small $\rho^{*n}-$measure.

Lemma 3.9. *Let ρ be a borelian probability measure on $\mathbf{G} = \mathrm{SL}_d(\mathbb{R})$ having an exponential moment.*

Then, for any $t_1, t_2 \in \mathbb{R}_+^*$, there are $n_0 \in \mathbb{N}$ and $t_3 \in \mathbb{R}_+^*$ such that for any $n \in \mathbb{N}$ with $n \geqslant n_0$, we have

$$\rho^{*n}\left(\left\{g \in \mathbf{G} \middle| \rho^{*n}\left(B(g, e^{-t_2 n})\right) \geqslant e^{-t_3 n}\right\}\right) \geqslant 1 - e^{-t_1 n}$$

Proof. Let $\varepsilon \in \mathbb{R}_+^*$ such that $\int_{\mathbf{G}} \|g\|^{\varepsilon} \mathrm{d}\rho(g)$ is finite and fix $n \in \mathbb{N}$. Using Markov's inequality, we have that for any $M \in \mathbb{R}_+$,

$$\rho^{*n}\left(\left\{g \in \mathbf{G} \middle| \|g\| \geqslant e^{Mn}\right\}\right) \leqslant e^{-\varepsilon M n} \int_{\mathbf{G}} \|g\|^{\varepsilon} \mathrm{d}\rho^{*n}(g) \leqslant \left(e^{-\varepsilon M} \int_{\mathbf{G}} \|g\|^{\varepsilon} \mathrm{d}\rho(g)\right)^n$$

So, noting $\widetilde{\Omega}_n = \left\{g \in \mathbf{G} \middle| \|g\| \leqslant e^{Mn}\right\}$, we have that

$$\rho^{*n}\left(\widetilde{\Omega}_n^c\right) \leqslant \left(e^{-\varepsilon M} \int_{\mathbf{G}} \|g\|^{\varepsilon} \mathrm{d}\rho(g)\right)^n$$

Moreover, there is a constant $C(d)$ depending only on the dimension d and $g_1, \dots g_L \in \widetilde{\Omega}_n$ such that

$$\widetilde{\Omega}_n \subset \bigcup_{i=1}^{L} B(g_i, e^{-t_2 n}/2)$$

and $L \leqslant C(d) e^{(M+t_2)d^2 n}$. Moreover, for $K \in \mathbb{R}_+^*$, we have

$$\mathbf{G}_n = \left\{g \in \{g_1, \dots, g_L\} \middle| \rho^{*n}(B(g, e^{-t_2 n}/2)) \geqslant e^{-Kn}\right\} \text{ and } \Omega_n = \bigcup_{g \in G_n} B(g, e^{-t_2 n}/2)$$

But, for any $h \in \Omega_n$, there is $g \in \mathbf{G}_n$ such that $d(g, h) \leqslant e^{-t_2 n}/2$ and this proves that

$$B(h, e^{-t_2 n}) \supset B(g, e^{-t_2 n}/2)$$

Thus,

$$\rho^{*n}(B(h, e^{-t_2 n})) \geqslant \rho^{*n}(B(g, e^{-t_2 n}/2)) \geqslant e^{-t_3 n}$$

Finally, as ρ is a probability measure we have

$$1 = \rho^{*n}\left(\widetilde{\Omega}_n^c\right) + \rho^{*n}\left(\Omega_n\right) + \rho^{*n}\left(\widetilde{\Omega}_n \setminus \Omega_n\right)$$

But, by definition,

$$\rho^{*n}\left(\widetilde{\Omega}_n \setminus \Omega_n\right) \leqslant L e^{-Kn}$$

So, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\rho^{*n}(\Omega_n) &\geqslant 1 - \left(e^{-\varepsilon M} \int_{\mathbf{G}} \|g\|^{\varepsilon} \mathrm{d}\rho(g)\right)^n - L e^{-Kn} \\
&\geqslant 1 - \left(e^{-\varepsilon M} \int_{\mathbf{G}} \|g\|^{\varepsilon} \mathrm{d}\rho(g)\right)^n - C(d) e^{-Kn} e^{(M+t_2)d^2 n}
\end{aligned}$$

And this is what we intended to prove since we can choose K and M as large as we want. □

3.2. **Diophantine properties of the lengths of translations.**

The aim of this section is to prove proposition 3.16 that shows that the logarithms of the spectral radii satisfy a diophantine condition of the same kind of the one used by Carlsson in [Car83] in his study of the renewal theorem on $\mathbb{R}$.

Let ρ be a borelian probability measure on $\mathbf{G} = \mathrm{SL}_d(\mathbb{R})$. For $g \in \mathbf{G}$, we note $\lambda_1(g)$ the logarithm of the spectral radius of g. As in the one-dimensional case, we will need a diophantine assumption to get the speed of convergence in the renewal theorem. Yet, we are going to prove, that this assumption always holds for measures having an exponential moment and whose support generates a strongly irreducible and proximal group. To do so, we will prove that we can construct elements such that the difference between $\lambda_1(gh)$ and $\lambda_1(g) + \lambda_1(h)$ is well controlled.

First of all, we are going to compute the difference between $\lambda_1(gh)$ and $\lambda_1(g) + \lambda_1(h)$ when g and h are proximal elements of $\mathbf{G}$ being in generic position.

Lemma 3.10. *There are $c_1, c_2 \in \mathbb{R}_+^*$ such that for any $\varepsilon \in]0, c_1]$ and any $g, h \in \mathbf{G}$ with $\kappa_{1,2}(g), \kappa_{1,2}(h) \leqslant \varepsilon^3$, $\delta(X_g^M, Y_g^m) \geqslant 2\varepsilon$, $\delta(X_h^M, Y_h^m) \geqslant 2\varepsilon$, $\delta(X_g^M, Y_h^m) \geqslant 2\varepsilon$ and $d(X_h^M, Y_g^m) \geqslant 2\varepsilon$, we have that*

$$\left| \lambda_1(g) + \lambda_1(h) - \lambda_1(gh) - \ln \frac{\delta(V_h^+, V_h^<)\delta(V_g^+, V_g^<)}{\delta(V_g^+, V_h^<)\delta(V_h^+, V_g^<)} \right| \leqslant c_2 \left(\frac{\kappa_{1,2}(g)}{\varepsilon^2} + \frac{\kappa_{1,2}(h)}{\varepsilon^2} \right)$$

Remark 3.11. According to lemma 3.7, there are many elements g and h satisfying the assumptions of the lemma in the support of ρ^{*n}.

Proof. We take at first $c_1 = 1/4$ and $c_2 = 1$.

First, according to lemma 3.4, we have

$$\kappa_{1,2}(gh) \leqslant \frac{\kappa_{1,2}(g)\kappa_{1,2}(h)}{\varepsilon^2}, \quad d(X_{gh}^M, X_g^M) \leqslant \frac{\kappa_{1,2}(g)}{\varepsilon} \text{ and } d(Y_{gh}^m, Y_h^m) \leqslant \frac{\kappa_{1,2}(h)}{\varepsilon}$$

So,

$$\kappa_{1,2}(gh) \leqslant \varepsilon^4 \text{ and } \delta(X_{gh}^M, Y_{gh}^m) \geqslant 2\varepsilon(1-\varepsilon) \geqslant \frac{3}{2}\varepsilon$$

We note $\varepsilon' = \frac{3}{4}\varepsilon$ and so we have that $d(X_{gh}^M, Y_{gh}^m) \geqslant 2\varepsilon'$ and $\kappa_{1,2}(gh) \leqslant \frac{1}{4}\left(\frac{4}{3}\right)^3 \varepsilon'^3 \leqslant \varepsilon'^3$.

Thus, according to lemma 3.5, gh is proximal and

$$d(V_{gh}^+, V_g^+) \leqslant d(V_{gh}^+, X_{gh}^M) + d(X_{gh}^M, X_g^M) + d(X_g^M, V_g^+) \leqslant \frac{\kappa_{1,2}(gh)}{\varepsilon'} + \frac{2\kappa_{1,2}(g)}{\varepsilon}$$

In the same way, we have

$$\begin{aligned} d(V_{gh}^<, V_h^<) &\leqslant d(V_{gh}^<, Y_{gh}^m) + d(Y_{gh}^m, Y_h^m) + d(Y_h^m, V_h^<) \leqslant \frac{\kappa_{1,2}(gh)}{\varepsilon'} + 2\frac{\kappa_{1,2}(h)}{\varepsilon} \\ &\leqslant \frac{4}{3}\varepsilon^3 + 2\varepsilon^2 \leqslant 3\varepsilon^2 \end{aligned}$$

And so,

$$\delta(V_h^+, V_{gh}^<) \geqslant 2\varepsilon - 3\varepsilon^2 = 2\varepsilon(1-3\varepsilon) \geqslant \frac{\varepsilon}{2}$$

So we note $\varepsilon'' = \frac{\varepsilon}{4}$ and so we have that $\delta(V_h^+, V_{gh}^<) \geqslant 2\varepsilon''$ and $\kappa_{1,2}(gh) \leqslant 64 c_1 \varepsilon''^3$. Therefore, assuming that $c_1 \leqslant \frac{1}{64}$, according to lemma 3.6, we have that

$$\left| \sigma(gh, V_h^+) - \lambda_1(gh) - \ln \frac{\delta(V_h^+, V_{gh}^<)}{\delta(V_{gh}^+, V_{gh}^<)} \right| \leqslant 2\frac{\kappa_{1,2}(gh)}{\varepsilon''^3} \leqslant 2\frac{\kappa_{1,2}(h)\varepsilon^3}{\varepsilon''^3}$$

Moreover, using the cocycle property of σ and the fact that, by definition of V_h^+, $\sigma(h, V_h^+) = \lambda_1(h)$, we have that

$$\sigma(gh, V_h^+) = \sigma(g, hV_h^+) + \sigma(h, V_h^+) = \sigma(g, V_h^+) + \lambda_1(h)$$

And finally we have, using once again lemma 3.6, that

$$\left| \sigma(g, V_h^+) - \lambda_1(g) - \ln \frac{\delta(V_h^+, V_g^<)}{\delta(V_g^+, V_g^<)} \right| \leqslant 2\frac{\kappa_{1,2}(g)}{\varepsilon^3}$$

And this is what we intended to prove. □

In next lemma, we prove a continuity result of the Cartan decomposition.

Lemma 3.12. *Let $g \in \mathbf{G}$. Then for any $h \in \mathbf{G}$,*

$$\|x_h^M - x_g^M\| \leqslant (2\|g-h\| + \kappa_{1,2}(g)), \quad \|y_h^m - y_g^m\| \leqslant (2\|g-h\| + \kappa_{1,2}(g))$$

Moreover, there are constants $c_1, c_2 \in \mathbb{R}_+^*$ such that for any $p \in \mathbb{N}^*$, any $\varepsilon \in]0, c_1]$, any g in $\mathbf{G}$ with $\kappa_{1,2}(g) \leqslant \varepsilon^3$ and $\delta(X_g^M, Y_g^m) \geqslant 2\varepsilon$, any $r \in \mathbb{R}_+^*$ with $r \leqslant \varepsilon^2 \left(\frac{\kappa_{1,2}(g)}{\varepsilon^2}\right)^p$ and any $f \in B(g^p, r)$, we have that

$$\delta(X_f^M, Y_f^m) \geqslant \varepsilon, \quad d(X_f^M, X_g^M) \leqslant 2\frac{\kappa_{1,2}(g)}{\varepsilon}, \quad d(Y_f^m, Y_g^m) \leqslant 2\frac{\kappa_{1,2}(g)}{\varepsilon}$$

$$\kappa_1(f) \geqslant \frac{1}{2}\varepsilon^{p-1}\kappa_1(g)^p, \quad \kappa_{1,2}(f) \leqslant \frac{16}{\varepsilon^{2(p-1)}}\kappa_{1,2}(g)^p$$

$$d(V_f^+, V_g^+) \leqslant c_2\frac{\kappa_{1,2}(g)^p}{\varepsilon^{2p-1}} \text{ and } d(V_f^<, V_g^<) \leqslant c_2\frac{\kappa_{1,2}(g)^p}{\varepsilon^{2p-1}}$$

Proof. We are going to give too expressions of $g^t l_h e_1$.

First of all,

$$\begin{aligned}\kappa_1(h)x_h^M &= h^t l_h e_1 = g^t l_h e_1 + (h-g)^t l_h e_1 \\ &= \kappa_1(g) <^t l_h e_1, {}^t l_g e_1 > x_g^M + u + (h-g)^t l_h e_1\end{aligned}$$

with u such that $\|u\| \leqslant \kappa_2(g)$.

So,

$$\begin{aligned}\kappa_1(h)\|x_h^M - <^t l_h e_1, {}^t l_g e_1 > x_g^M\| &\leqslant |\kappa_1(g) - \kappa_1(h)| + \|u\| + \|g-h\| \\ &\leqslant \kappa_2(g) + 2\|g-h\|\end{aligned}$$

But, as $\|x_h^M\| = 1 = \|x_g^M\|$, we can deduce the first part of the lemma since for $g \in \mathbf{G}$, $\kappa_1(g) \geqslant 1$.

To get the control of $y_g^m - y_h^m$, we do the same kind of computations in the dual space.

To prove the end of the lemma, we note that, according to lemma 3.4, we have that

$$\kappa_1(g^p) \geqslant \varepsilon^{p-1}\kappa_1(g)^p, \quad \kappa_{1,2}(g^p) \leqslant \frac{1}{\varepsilon^{2(p-1)}}\kappa_{1,2}(g)^p$$

and

$$d(X_{g^p}^M, X_g^M) \leqslant \frac{\kappa_{1,2}(g)}{\varepsilon}, \quad d(Y_{g^p}^m, Y_g^m) \leqslant \frac{\kappa_{1,2}(g)}{\varepsilon}$$

So, for any $f \in B(g^p, r)$,

$$\begin{aligned}\delta(X_f^M, Y_f^m) &\geqslant 2\varepsilon - d(X_{g^p}^M, X_g^M) - d(Y_{g^p}^m, Y_g^m) - d(X_f^M, X_{g^p}^M) - d(Y_f^m, Y_{g^p}^m)\\ &\geqslant 2\varepsilon - 2\frac{\kappa_{1,2}(g)}{\varepsilon} - 2\left(2r + \kappa_{1,2}(g^p)\right)\\ &\geqslant \varepsilon(2 - 6\varepsilon)\end{aligned}$$

This proves that (maybe for a smaller value of c_1) we have that

$$\delta(X_f^M, Y_f^m) \geqslant \varepsilon$$

Moreover,

$$\kappa_1(f) \geqslant \kappa_1(g^p) - r \geqslant \varepsilon^{p-1}\kappa_1(g)^p - \varepsilon^{p+2} \geqslant \varepsilon^{p-1}\kappa_1(g)^p\left(1 - \frac{\varepsilon^3}{\kappa_1(g)^p}\right)$$

And using the inequality $\kappa_1(g) \geqslant 1$ and $\varepsilon \leqslant c_1$, we have that for c_1 small enough,

$$\kappa_1(f) \geqslant \varepsilon^{p-1}\kappa_1(g)^p$$

Moreover, be definition, $\kappa_1(g)\kappa_2(g) = \kappa_1(\wedge^2 g)$, and so

$$\kappa_1(f)\kappa_2(f) \leqslant \kappa_1(g^p)\kappa_2(g^p) + \| \wedge^2 g^p - \wedge^2 f\| \leqslant \kappa_1(g^p)\kappa_2(g^p) + (\kappa_1(g^p) + \kappa_1(g^p))\|f - g^p\|$$

This proves that

$$\kappa_{1,2}(f) \leqslant \frac{\kappa_1(g^p)\kappa_2(g^p)}{\kappa_1(f)^2} + \frac{\kappa_1(g^p) + \kappa_1(f)}{\kappa_1(f)^2}\|g^p - f\| \leqslant \frac{16}{\varepsilon^{2(p-1)}}\kappa_{1,2}(g)^p \leqslant 16\varepsilon^{p+2}$$

And is c_1 is small enough, we get that $\kappa_{1,2}(f) \leqslant (\varepsilon/2)^3$, and so, according to lemma 3.5 we find that for any $f \in B(g^p, r)$, f is proximal and

$$d(V_f^+, X_f^M) \leqslant 2\frac{\kappa_{1,2}(f)}{\varepsilon} \text{ and } d(V_f^<, Y_f^m) \leqslant 2\frac{\kappa_{1,2}(f)}{\varepsilon}$$

Finally, using that $V_{g^p}^+ = V_g^+$, we find that

$$d(V_f^+, V_g^+) \leqslant d(V_f^+, X_f^M) + d(X_f^M, X_{g^p}^M) + d(X_{g^p}^M, V_g^+) \leqslant c_2\frac{\kappa_{1,2}(g)^p}{\varepsilon^{2p-1}}$$

for some universal constant c_2.

Working in the same way in the dual space, we can get the control of $d(V_f^<, V_g^<)$ and finish the proof of the lemma. □

We are now ready to compute the difference between the logarithms of the spectral radii of well chosen proximal elements of $\mathbf{G}$. To do so, we would like to choose elements that writes g^p and gh for generic elements g and h (as in [Qui05]). We cannot do that since the measure ρ is not assumed to be purely atomic and this is why we choose elements f and gh with f in a small neighbourhood of g^p. This is what we do in next

Lemma 3.13. *There are constants c_1, c_2, c_3 such that for any $p \in \mathbb{N}^*$, any $\varepsilon \in]0, c_1]$, any $g \in \mathbf{G}$ with $\kappa_{1,2}(g) \leqslant \varepsilon^5$ and $\delta(X_g^M, Y_g^m) \geqslant 2\varepsilon$ we have that for any $h \in \mathbf{G}$ with $\kappa_{1,2}(h) \leqslant \varepsilon^3$, $\delta(X_h^M, Y_h^m) \geqslant 2\varepsilon$, $\delta(X_h^M, Y_g^m) \geqslant 2\varepsilon$, $\delta(X_g^M, Y_h^m) \geqslant 2\varepsilon$ and any $f \in \mathbf{G}$ with $\|g^p - f\| \leqslant \varepsilon^2 \left(\frac{\kappa_{1,2}(g)}{\varepsilon}\right)^p$, we have that*

$$\left|\lambda_1(fgh) - \lambda_1(f) - \lambda_1(gh) - \ln \frac{\delta(V_g^+, V_g^<)\delta(gV_h^+, V_h^<)}{\delta(V_g^+, V_h^<)\delta(gV_h^+, V_g^<)}\right| \leqslant c_2 \left(\frac{\kappa_{1,2}(g)^p}{\varepsilon^{2p}} + \frac{\kappa_{1,2}(h)}{\varepsilon}\right)$$

Moreover, we note π_g the projection on $V_g^<$ parallel to V_g^+. And, we have that if $d(g\pi_g X_h^M, Y_h^m) \geqslant 2\varepsilon$, $d(X_g^M, X_h^M) \geqslant 2\varepsilon$ and $\kappa_{1,2}(h)\kappa_1(g) \leqslant \frac{1}{2}\varepsilon^3$, then

$$\frac{\varepsilon^3}{c_3\kappa_1(g)^d} \leqslant \left|\ln \frac{\delta(V_g^+, V_g^<)\delta(gV_h^+, V_h^<)}{\delta(V_g^+, V_h^<)\delta(gV_h^+, V_g^<)}\right| \leqslant c_3 \frac{\kappa_{1,2}(g)}{\varepsilon^4}$$

Proof. We want to apply lemma 3.10 to the elements f and gh. To do so, we are first going to prove that gh is proximal.

According to lemma 3.4, we have that

$$\kappa_{1,2}(gh) \leqslant \frac{\kappa_{1,2}(g)\kappa_{1,2}(h)}{\varepsilon^2} \leqslant \varepsilon^4$$

and

$$d(X_{gh}^M, X_g^M) \leqslant \frac{\kappa_{1,2}(g)}{\varepsilon} \text{ et } d(Y_{gh}^m, Y_h^m) \leqslant \frac{\kappa_{1,2}(h)}{\varepsilon}$$

Thus,

$$\delta(X_{gh}^M, Y_{gh}^m) \geqslant \delta(X_g^M, Y_h^m) - d(X_{gh}^M, X_g^M) - d(Y_h^m, Y_{gh}^m) \geqslant 2\varepsilon(1-\varepsilon) \geqslant \frac{3}{2}\varepsilon$$

So, for c_1 small enough, g and h satisfy to the assumptions of lemma 3.6 with $\varepsilon' = \varepsilon/2$. So, gh is proximal and

$$d(V_{gh}^+, gV_h^+) = d(ghV_{gh}^+, ghV_h^+) \leqslant \frac{\kappa_{1,2}(gh)}{4\varepsilon'^4} \leqslant c_2 \frac{\kappa_{1,2}(h)}{\varepsilon}$$

and

$$d(V_{gh}^<, V_h^<) \leqslant d(V_{gh}^<, Y_{gh}^m) + d(Y_{gh}^m, Y_h^m) + d(V_{gh}^<, Y_h^m) \leqslant 2\frac{\kappa_{1,2}(h)}{\varepsilon} + \frac{\kappa_{1,2}(gh)}{\varepsilon}$$

Moreover, according to lemma 3.12, for $f \in \mathbf{G}$ with $\|g^p - f\| \leqslant \varepsilon^2 \left(\frac{\kappa_{1,2}(g)}{\varepsilon}\right)^p$, we have that

$$\delta(X_f^M, Y_f^m) \geqslant \varepsilon, \quad d(X_f^M, X_g^M) \leqslant 2\frac{\kappa_{1,2}(g)}{\varepsilon}, \text{ and } d(Y_f^m, Y_g^m) \leqslant 2\frac{\kappa_{1,2}(g)}{\varepsilon}$$

Moreover,

$$\kappa_1(f) \geqslant \frac{1}{2}\varepsilon^{p-1}\kappa_1(g)^p, \quad \kappa_{1,2}(f) \leqslant \frac{16}{\varepsilon^{2(p-1)}}\kappa_{1,2}(g)^p$$

and

$$d(V_f^+, V_g^+) \leqslant c_2 \frac{\kappa_{1,2}(g)^p}{\varepsilon^{2p-1}} \text{ and } d(V_f^<, V_g^<) \leqslant c_2 \frac{\kappa_{1,2}(g)^p}{\varepsilon^{2p-1}}$$

Finally,

$$\begin{aligned}
\delta(X_f^M, Y_{gh}^m) &\geqslant \delta(X_g^M, Y_h^m) - d(X_f^M, X_g^M) - d(Y_h^m, Y_{gh}^m) \\
&\geqslant \varepsilon \left(2 - 2\frac{\kappa_{1,2}(g)}{\varepsilon} - \frac{\kappa_{1,2}(h)}{\varepsilon}\right) \\
&\geqslant \varepsilon \left(2 - 3\varepsilon^2\right)
\end{aligned}$$

So, choosing c_1 small enough, we get that $\delta(X_f^M, Y_{gh}^m) \geqslant \varepsilon$ and, in the same way, $\delta(X_{gh}^m, Y_g^m) \geqslant \varepsilon$. Thus, according to lemma 3.10, we have that

$$\left|\lambda_1(f) + \lambda_1(gh) - \lambda_1(fgh) - \ln \frac{\delta(V_{gh}^+, V_{gh}^<)\delta(V_f^+, V_f^<)}{\delta(V_f^+, V_{gh}^<)\delta(V_{gh}^+, V_f^<)}\right| \leqslant c_2 \left(\frac{\kappa_{1,2}(f)}{\varepsilon^2} + \frac{\kappa_{1,2}(gh)}{\varepsilon^2}\right)$$

And this finishes the proof of the first part of the lemma since we saw how to control $d(V_{gh}^+, gV_h^+)$, $d(V_f^+, V_g^+)$, $d(V_f^<, V_g^<)$ and $d(V_{gh}^<, V_h^<)$.

To prove the second part, note that we have the inequality

$$\begin{aligned}
\frac{\delta(V_g^+, V_g^<)\delta(gV_h^+, V_h^<)}{\delta(V_g^+, V_h^<)\delta(gV_h^+, V_g^<)} &= \left|\frac{\varphi_g^<(v_g^+)\varphi_h^<(gv_h^+)}{\varphi_h^<(v_g^+)\varphi_g^<(gv_h^+)}\right| \\
&= \left|1 + \frac{\varphi_g^<(v_g^+)\varphi_h^<(gv_h^+) - \varphi_h^<(v_g^+)\varphi_g^<(gv_h^+)}{\varphi_h^<(v_g^+)\varphi_g^<(gv_h^+)}\right| \qquad (3.4)
\end{aligned}$$

And as

$$v_h^+ = \frac{\varphi_g^<(v_h^+)}{\varphi_g^<(v_g^+)} v_g^+ + v_h^+ - \frac{\varphi_g^<(v_h^+)}{\varphi_g^<(v_g^+)} v_g^+ = \frac{\varphi_g^<(v_h^+)}{\varphi_g^<(v_g^+)} v_g^+ + \pi_g(v_h^+)$$

where we noted π_g the projection to $V_g^<$ parallel to V_g^+, we have that

$$\varphi_g^<(gv_h^+) = \varepsilon_1(g) e^{\lambda_1(g)} \varphi_g^<(v_h^+)$$

and

$$\varphi_h^<(gv_h^+) = \varepsilon_1(g) e^{\lambda_1(g)} \frac{\varphi_g^<(v_h^+)}{\varphi_g^<(v_g^+)} \varphi_h^<(v_g^+) + \varphi_h^<\left(g\pi_g(v_h^+)\right)$$

So,

$$\varphi_g^<(v_g^+)\varphi_h^<(gv_h^+) - \varphi_h^<(v_g^+)\varphi_g^<(gv_h^+) = \varphi_g^<(v_g^+)\varphi_h^<(g\pi_g v_h^+)$$

Using equation 3.4, we find that

$$\frac{\delta(V_g^+, V_g^<)\delta(gV_h^+, V_h^<)}{\delta(V_g^+, V_h^<)\delta(gV_h^+, V_g^<)} = \left|1 + \varepsilon_1(g) e^{-\lambda_1(g)} \frac{\varphi_g^<(v_g^+)\varphi_h^<(g\pi_g v_h^+)}{\varphi_h^<(v_g^+)\varphi_g^<(v_h^+)}\right|$$

Moreover, using the fact that $\pi_g v_h^+ \in V_g^<$, we have

$$\|g\pi_g v_h^+\| \leqslant \frac{\kappa_2(g)}{\varepsilon} \|\pi_g\| \|v_h^+\| \leqslant \frac{\kappa_2(g)}{\varepsilon} \frac{2}{\delta(V_g^+, V_g^<)} \|v_h^+\| \leqslant \frac{2\kappa_2(g)}{\varepsilon^2} \|v_h^+\|$$

So, using the inequality $e^{\lambda_1(g)} \geqslant 2\kappa_1(g)\varepsilon$ (that comes from lemma 3.5), we get that

$$\left| e^{-\lambda_1(g)} \frac{\varphi_g^<(v_g^+)\varphi_h^<(g\pi_g v_h^+)}{\varphi_h^<(v_g^+)\varphi_g^<(v_h^+)} \right| \leqslant \frac{\kappa_{1,2}(g)}{\varepsilon^2} \frac{\|\varphi_g^<\|\|v_g^+\|\|\varphi_h^<\|\|v_h^+\|}{|\varphi_h^<(v_g^+)\varphi_g^<(v_h^+)|} \leqslant \frac{\kappa_{1,2}(g)}{4\varepsilon^4}$$

Moreover, we can also compute

$$\begin{aligned}
\left| \frac{\varphi_g^<(v_g^+)\varphi_h^<(g\pi_g v_h^+)}{\varphi_h^<(v_g^+)\varphi_g^<(v_h^+)} \right| &= \delta(V_g^+, V_g^<)\delta(g\pi_g V_h^+, V_h^<) \frac{\|\varphi_g^<\|\|v_g^+\|\|\varphi_h^<\|\|g\pi_g v_h^+\|}{|\varphi_h^<(v_g^+)\varphi_g^<(v_h^+)|} \\
&\geqslant \delta(V_g^+, V_g^<)\delta(g\pi_g V_h^+, V_h^<) \frac{\|g\pi_g v_h^+\|}{\|v_h^+\|} \\
&\geqslant \delta(V_g^+, V_g^<)\delta(g\pi_g V_h^+, V_h^<) \frac{1}{\|g^{-1}\|} \frac{\|\pi_g v_h^+\|}{\|v_h^+\|} \qquad (3.5)
\end{aligned}$$

Finally, since $v_g^+ \wedge v_h^+ = v_g^+ \wedge \pi_g v_h^+$, we have

$$d(V_g^+, V_h^+) = \frac{\|v_g^+ \wedge v_h^+\|}{\|v_g^+\|\|v_h^+\|} = \frac{\|v_g^+ \wedge \pi_g v_h^+\|}{\|v_g^+\|\|v_h^+\|} \leqslant \frac{\|\pi_g v_h^+\|}{\|v_h^+\|}$$

and so, using inequality 3.5, we find that

$$\left| \frac{\varphi_g^<(v_g^+)\varphi_h^<(g\pi_g v_h^+)}{\varphi_h^<(v_g^+)\varphi_g^<(v_h^+)} \right| \geqslant \frac{1}{\|g^{-1}\|} \delta(V_g^+, V_g^<)\delta(g\pi_g V_h^+, V_h^<) d(V_g^+, V_h^+)$$

To conclude, we note that $\kappa_d(g)\dots\kappa_1(g) = 1$ and $\kappa_d(g) = \|g^{-1}\|^{-1}$ so $\|g^{-1}\|^{-1} \geqslant \kappa_1(g)^{1-d}$.

Moreover,

$$\begin{aligned}
\delta(g\pi_g V_h^+, V_h^<) &\geqslant \delta(g\pi_g x_h^M, y_h^m) - d(g\pi_g V_h^+, g\pi_g x_h^M) - d(V_h^<, y_h^m) \\
&\geqslant 2\varepsilon - \|g\|\|\pi_g\| d(V_h^+, x_h^M) - d(V_h^<, y_h^m) \\
&\geqslant 2\varepsilon - 2\frac{\kappa_1(g)}{\varepsilon}\frac{\kappa_{1,2}(h)}{\varepsilon} \geqslant 2\varepsilon\left(1 - \frac{\kappa_1(g)\kappa_{1,2}(h)}{\varepsilon^3}\right) \geqslant \varepsilon
\end{aligned}$$

And,

$$d(V_g^+, V_h^+) \geqslant d(X_g^M, X_h^M) - d(V_g^+, X_g^M) - d(V_h^+, X_h^M) \geqslant 2\varepsilon - \frac{\kappa_{1,2}(g)}{\varepsilon} - \frac{\kappa_{1,2}(h)}{\varepsilon} \geqslant \varepsilon$$

So, using that $e^{-\lambda_1(g)} \geqslant \kappa_1(g)^{-1}$, we get that

$$\left| e^{-\lambda_1(g)} \frac{\varphi_g^<(v_g^+)\varphi_h^<(g\pi_g v_h^+)}{\varphi_h^<(v_g^+)\varphi_g^<(v_h^+)} \right| \geqslant \frac{\varepsilon^3}{\kappa_1(g)^d}$$

And this is finally what we intended to prove. □

Lemma 3.13 proved that under good transversal assumptions on elements g and h, we have a good control on the difference between $\lambda_1(fgh)$ and $\lambda_1(f) + \lambda_1(gh)$ for f in a small neighbourhood of g^p. We can finally use lemma 3.7 to get that the elements g and h that satisfy these assumptions are generic to get the

Lemma 3.14. *Let ρ be a borelian probability measure on $\mathbf{G} = \mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal subgroup.*

Then, there are $n_0, p \in \mathbb{N}$ and $c_1, c_2, c_3 \in \mathbb{R}_+^*$ such that for any $n \in \mathbb{N}$ with $n \geqslant n_0$,

$$\rho^{*pn} \otimes \rho^{*pn} \left(\left\{ (g,h) \in \mathbf{G} \middle| e^{-c_1 n} \leqslant |\lambda_1(gh) - \lambda_1(g) - \lambda_1(h)| \leqslant e^{-c_2 n} \right\} \right) \geqslant e^{-c_3 n}$$

Proof. Recall that, according to [GR85], we have that $\lambda_1 > \lambda_2$.

We fix $\eta \in]0, (\lambda_1 - \lambda_2)/6[$. For any $n \in \mathbb{N}$, we note

$$G_n = \left\{ g \in \mathbf{G} \;\middle|\; \begin{array}{l} \delta(X_g^M, Y_g^m) \geqslant 2e^{-\eta n} \\ \forall i \in \{1,2\} \left| \frac{1}{n}\kappa_i(g) - \lambda_i \right| \leqslant \eta \end{array} \right\}$$

Then, for any element $g \in G_n$, we have that

$$\kappa_{1,2}(g) = \frac{\kappa_2(g)}{\kappa_1(g)} \leqslant e^{-(\lambda_1 - \lambda_2 - 2\eta)n}$$

So, noting $\varepsilon = e^{-\eta n}$, we have that for n large enough, g satisfy to the assumptions of lemma 3.13. Moreover, for any $g \in G_n$ and $p \in \mathbb{N}^*$, we note

$$H_n^p(g) = \left\{ h \in G_{pn} \;\middle|\; \begin{array}{l} \delta(X_h^M, Y_g^m) \geqslant 2e^{-\eta n},\ \delta(X_g^M, Y_h^m) \geqslant 2e^{-\eta n} \\ d(g\pi_g X_h^M, Y_h^m) \geqslant 2e^{-\eta n},\ d(X_g^M, X_h^M) \geqslant 2e^{-\eta n} \end{array} \right\}$$

where π_g is the projection onto $V_g^<$ parallel to V_g^+.

So, if p is such that $(p-1)(\lambda_1 - \lambda_2 - \eta) > \lambda_1 + \eta$ then the (g,h) satisfy to the assumptions of lemma 3.13 and we have that for any $f \in B(g^p, e^{-p(\lambda_1 - \lambda_2 - \eta)n})$,

$$\left| \lambda_1(fgh) - \lambda_1(f) - \lambda_1(gh) - \frac{d(V_g^+, V_g^<) d(gV_h^+, V_h^<)}{d(V_g^+, V_h^<) d(gV_h^+, V_g^<)} \right| \leqslant 2c_2 e^{-p(\lambda_1 - \lambda_2 - 4\eta)n}$$

and

$$\frac{e^{-(d\lambda_1 + 5\eta)n}}{c_3} \leqslant \frac{d(V_g^+, V_g^<) d(gV_h^+, V_h^<)}{d(V_g^+, V_h^<) d(gV_h^+, V_g^<)} \leqslant c_3 e^{-(\lambda_1 - \lambda_2 - 6\eta)n}$$

So, to prove the lemma, we take p such that $p(\lambda_1 - \lambda_2 - 4\eta) > d\lambda_1 + 5\eta$, we set $c_1 > d\lambda_1 + 5\eta$ and $c_2 < \lambda_1 - \lambda_2 - 6\eta$ and we have to prove that there is a constant $t \in \mathbb{R}_+^*$ such that for n large enough,

$$\int_{\mathbf{G}} \mathbf{1}_{\mathbf{G}_n}(g) \rho^{*(p-1)n}(H_n^{p-1})(g) \rho^{*(p-1)n}(B(g^p, e^{-p(\lambda_1 - \lambda_2 - \varepsilon)n})) \mathrm{d}\rho^{*n}(g) \geqslant e^{-tn}$$

But, according to lemma 3.7, we have that $h \in H_n^{p-1}(g)$ except on an a set having an exponentially small measure and $g \in G_n$ except on a set having an exponentially small measure. Therefore, we only need to prove that for some $t \in \mathbb{R}_+^*$,

$$\int_{\mathbf{G}} \rho^{*pn}(B(g^p, e^{-p(\lambda_1 - \lambda_2 - \varepsilon)n})) \mathrm{d}\rho^{*n}(g) \geqslant e^{-tn}$$

But, if $g_1, \dots g_p \in B(g, r)$, we have that

$$\|g_1 \dots g_p - g^p\| \leqslant pr\|g\|^{p-1}$$

So, taking

$$r = \frac{e^{-p(\lambda_1-\lambda_2-\varepsilon)n}}{p\|g\|^p} \geqslant \frac{e^{-p(\lambda_1-\lambda_2-\varepsilon)n}}{p} e^{-p(\lambda_1+\varepsilon)n} \geqslant e^{-2p(\lambda_1+\varepsilon)n}$$

(where we used that for $g \in G_n$, $\|g\| \leqslant e^{(\lambda_1+\varepsilon)n}$) we have that

$$\rho^{*pn}(B(g^p, e^{-p(\lambda_1-\lambda_2-\varepsilon)n})) \geqslant \left(\rho^{*n}(B(g, e^{-2p(\lambda_1+\varepsilon)n}))\right)^p$$

And we can finally conclude with lemma 3.9. □

We can now prove an " integrated version " of lemma 3.14.

Lemma 3.15. *Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal subgroup.*

There are $\alpha, \beta \in \mathbb{R}_+^*$ such that

$$\liminf_{b\to\pm\infty} |b|^\alpha \int_{\mathbf{G}^2} \left|e^{ib(\lambda_1(gh)-\lambda_1(g)-\lambda_1(h))} - 1\right|^2 \mathrm{d}\rho^{*n(\beta,b)}(g)\mathrm{d}\rho^{*n(\beta,b)}(h) > 0$$

Proof. Note, for any $n \in \mathbb{N}$,

$$G_n^2 := \left\{(g,h) \in \mathbf{G} \middle| e^{-c_1 n} \leqslant |\lambda_1(gh) - \lambda_1(g) - \lambda_1(h)| \leqslant e^{-c_2 n}\right\}$$

We choose $\alpha, \beta \in \mathbb{R}_+^*$ and we will specify their values later. For $b \in \mathbb{R}$, we note $n = \lfloor \beta \ln |b| \rfloor$.

Then, for $|b|$ large enough and any $(g,h) \in G_n^2$, we have that

$$|b|^{1-c_1\beta} \leqslant |b||\lambda_1(gh) - \lambda_1(g) - \lambda_1(g)| \leqslant |b|^{1-c_2\beta}$$

So, if $\beta > 1/c_2$, we can use the fact that

$$0 < \inf_{x\in[-1,1]} \frac{\left|e^{ix}-1\right|}{|x|} \leqslant \sup_{x\in[-1,1]} \frac{\left|e^{ix}-1\right|}{|x|} < +\infty$$

to get that for $|b|$ large enough and any $(g,h) \in G_n$,

$$\left|e^{ib(\lambda_1(gh)-\lambda_1(g)-\lambda_1(g))} - 1\right| \asymp |b||\lambda_1(gh) - \lambda_1(g) - \lambda_1(g)| \geqslant |b|^{1-c_1\beta}$$

So, we have that for $|b|$ large enough and any $(g,h) \in G_n^2$,

$$\left|e^{ib(\lambda_1(gh)-\lambda_1(g)-\lambda_1(g))} - 1\right| \geqslant |b|^{1-c_1\beta}$$

Moreover, according to lemma 3.14,

$$\rho^{*n(\beta,b)} \otimes \rho^{*n(\beta,b)}(G_b^2) \geqslant e^{-c_3 n(\beta,b)} \geqslant |b|^{-c_3\beta}$$

So, if α is such that $\alpha - c_3\beta + 2(1-c_1\beta) > 0$, we have that

$$\liminf_{b\to\pm\infty} |b|^\alpha \int_{\mathbf{G}} \left|e^{ib(\lambda_1(gh)-\lambda_1(g)-\lambda_1(h))} - 1\right|^2 \mathrm{d}\rho^{*n(\beta,b)}(g)\mathrm{d}\rho^{*n(\beta,b)}(h) > 0$$

And this is what we intended to prove. □

Finally, we can prove the diophantine control of the logarithms of the spectral radii of elements of $\mathrm{SL}_d(\mathbb{R})$.

Proposition 3.16. *Let ρ be a borelian probability measure on $\mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal group.*

Then, there are $\alpha, \beta \in \mathbb{R}_+^*$ such that

$$\liminf_{b\to\pm\infty} |b|^\alpha \int_{\mathbf{G}} \left| e^{ib\lambda_1(g)} - 1 \right|^2 \mathrm{d}\rho^{*n(\beta,b)}(g) > 0$$

where we noted $\lambda_1(g)$ the spectral radius of g and $n(\beta, b) = \lfloor \beta \ln |b| \rfloor$.

Proof. For any $\alpha, \beta \in \mathbb{R}_+^\star$, using the triangular inequality, we have that

$$\begin{aligned} I_b(\beta) :&= \left(\int_{\mathbf{G}^2} \left| e^{ib(\lambda_1(gh)-\lambda_1(g)-\lambda_1(h))} - 1 \right|^2 \mathrm{d}\rho^{*n(\beta,b)}(g) \mathrm{d}\rho^{*n(\beta,b)}(h) \right)^{1,2} \\ &\leqslant \left(\int_{\mathbf{G}} \left| e^{ib\lambda_1(g)} - 1 \right|^2 \mathrm{d}\rho^{*2n(\beta,b)}(g) \right)^{1,2} + 2 \left(\int_{\mathbf{G}} \left| e^{ib\lambda_1(g)} - 1 \right|^2 \mathrm{d}\rho^{*n(\beta,b)}(g) \right)^{1,2} \end{aligned}$$

and, according to lemma 3.15, there are $\alpha, \beta \in \mathbb{R}_+^\star$ such that,

$$\liminf_{b\to\pm\infty} |b|^\alpha I_b(\beta) > 0$$

and this proves the lemma. □

3.3. **Regular points of the projective space.**

In this paragraph, we study the lower regularity of the stationary measure on the projective space at the fixed points of proximal elements.

Let us first recall the following

Lemma 3.17. *Let ρ be a borelian probability measure on $\mathbf{G}$ whose support generates a strongly irreducible and proximal subsemi-group T_ρ.*

Then, there is a unique $P_\rho-$stationary measure ν on $\mathbb{P}(\mathbb{R}^d)$.

Moreover, for any proximal element $g \in T_\rho$, we have that $V_g^+ \in \operatorname{supp} \nu$.

Proof. The existence and uniqueness of ν come from [GR85].

To prove the last part of the lemma, note that for any $g \in T_\rho$, there is $X \in \operatorname{supp} \nu$ such that $X \notin V_g^<$. Indeed, if not, we would have some $g \in T_\rho$ such that $\operatorname{supp} \nu \subset V_g^<$ but this is impossible since T_ρ is strongly irreducible.

Moreover, for any $g \in T_\rho$ and any $X \notin V_g^<$, we have that

$$g^n X \to V_g^+$$

And, as $\operatorname{supp} \nu$ is closed and $T_\rho-$invariant, this proves that $V_g^+ \in \operatorname{supp} \nu$. □

We recall that for a metric space $(\mathbf{X}, d)$ endowed with a borelian probability measure ν, we say that some point x of $\mathbf{X}$ is $\Delta - \nu$-regular at scale r where $r \in \mathbb{R}_+^*$ and $\Delta \in \mathbb{R}_+$ if

$$\nu(B(x, r)) \geqslant r^\Delta$$

In our study of the perturbation of Markov operators on compact metric spaces (see section 2), we used this lower regularity assumption and we are about to prove that for a borelian probability measure ρ on $\mathrm{SL}_d(\mathbb{R})$, the fixed points of proximal elements of the subsemigroup generated by the support of ρ are lower regular for the unique stationary measure on $\mathbb{P}(\mathbb{R}^d)$.

This is the aim of next

Proposition 3.18. *Let ρ be a borelian probability measure on $\mathbf{G} = \mathrm{SL}_d(\mathbb{R})$ having an exponential moment and whose support generates a strongly irreducible and proximal subgroup.*

Let ν be the unique $P_\rho-$stationary borelian probability measure on $\mathbb{P}(\mathbb{R}^d)$.

Then, for any $M \in \mathbb{R}_+^*$, there are $n_0 \in \mathbb{N}$, $\Delta \in \mathbb{R}$ and $t \in \mathbb{R}_+^*$ such that for any $n \in \mathbb{N}$ with $n \geqslant n_0$,

$$\rho^{*n}\left(\left\{g \in \mathbf{G} \middle| g \text{ is proximal and } \nu\left(B\left(V_g^+, e^{-Mn}\right)\right) \geqslant e^{-\Delta Mn}\right\}\right) \geqslant 1 - e^{-tn}$$

Proof. Let $0 < \varepsilon < (\lambda_1 - \lambda_2)/3$. Using the $\rho-$stationarity of the measure ν and noting $\lambda = \lambda_1 - \lambda_2 - 3\varepsilon$, we have that for any $g \in \mathbf{G}$ such that $\kappa_{1,2}(g) \leqslant e^{-\lambda n}$ and $\delta(X_g^M, Y_g^m) \geqslant 2e^{-\varepsilon n}$,

$$\begin{aligned}
\nu(B(V_g^+, e^{-mn\lambda})) &= \int_{\mathbf{X}} \mathbf{1}_{B(V_g^+, e^{-mn\lambda})}(x) \mathrm{d}\nu(x) \\
&= \int_{\mathbf{X}} \int_{\mathbf{G}} \mathbf{1}_{B(V_g^+, e^{-mn\lambda})}(hx) \mathrm{d}\rho^{*mn}(h) \mathrm{d}\nu(x) \\
&= \int_{\mathbf{X}} \rho^{*mn}\left(\left\{h \middle| hx \in B(V_g^+, e^{-mn\lambda})\right\}\right) \mathrm{d}\nu(x) \\
&\geqslant \int_{\mathbf{X}} \mathbf{1}_{\delta(x, y_g^m) \geqslant 2e^{-\varepsilon n}} \rho^{*mn}\left(\left\{h \middle| hx \in B(V_g^+, e^{-mn\lambda})\right\}\right) \mathrm{d}\nu(x)
\end{aligned}$$

But, we saw in lemma 3.6 that if $x \in \mathbb{P}(\mathbb{R}^d)$ is such that $\delta(X, Y_g^m) \geqslant 2e^{-\varepsilon n}$, then

$$d(g^m X, V_g^+) = d(g^m X, g^m V_g^+) \leqslant e^{-\lambda n} d(g^{m-1} X, V_g^+) \leqslant e^{-\lambda mn}$$

where we used the fact that $gV_g^+ = V_g^+$.

Moreover, if $r \in]0, 1]$ then for any $h_1, \ldots, h_m \in B(g, r)$ we have that

$$\|g^m - h_1 \ldots h_m\| \leqslant \|g - h_1\| \|g\|^{m-1} + \|h_1\| \|g^{m-1} - h_2 \ldots h_m\| \leqslant m(2\|g\|)^{m-1} r$$

And so,

$$d(h_1 \ldots h_m X, V_g^+) \leqslant \|g^m - h_1 \ldots h_m\| + d(X, V_g^+) \leqslant m(2\|g\|)^{m-1} r + e^{-\lambda mn}$$

Thus, for any $X \in \mathbb{P}(\mathbb{R}^d)$ such that $\delta(X, Y_g^m) \geqslant 2e^{-\varepsilon n}$,

$$\rho^{*mn}\left(\left\{h \middle| hx \in B(V_g^+, e^{-mn\lambda})\right\}\right) \geqslant \left(\rho^{*n}\left(B\left(g, \frac{e^{-\lambda mn}}{m(2\|g\|)^{m-1}}\right)\right)\right)^m$$

From now on, we note

$$r_{n,m} = \frac{e^{-\lambda mn}}{m(2\|g\|)^{m-1}}$$

And we proved that for any $g \in \mathbf{G}$ such that $\delta(X_g^M, Y_g^m) \geqslant 2e^{-\varepsilon n}$ and $\kappa_{1,2}(g) \leqslant e^{-\lambda n}$, we have that

$$\begin{aligned}
\nu(B(V_g^+, e^{-mn\lambda})) &\geqslant (\rho^{*n} B(g, r_{n,m}))^m \, \nu\left(\left\{X \in \mathbb{P}(\mathbb{R}^d) \middle| \delta(X, Y_g^m) \geqslant 2e^{-\varepsilon n}\right\}\right) \\
&\geqslant (\rho^{*n} B(g, r_{n,m}))^m \left(1 - e^{-\varepsilon cn}\right)
\end{aligned}$$

where we used the upper regularity of the measure ν (cf chapter 12 in [BQ15]) to have that for some constant c, if n is large enough

$$\nu(X|\delta(X,Y_g^m)\geqslant e^{-\varepsilon n})\geqslant 1-e^{-\varepsilon cn}.$$

To conclude, we use the fact that such elements g with $\delta(X_g^M,Y_g^m)\geqslant 2e^{-\varepsilon n}$ and $\kappa_{1,2}(g)\leqslant e^{-\lambda n}$ are generic according to lemma 3.7 and so we proved that for some $t\in\mathbb{R}_+^*$ we have that for any integer n large enough,

$$\rho^{*n}\left(\left\{g\in\mathbf{G}\middle| g \text{ is proximal and } \nu(B(V_g^+,e^{-mn\lambda})\geqslant\frac{1}{2}\left(\rho^{*n}B(g,r_{n,m})\right)^m\right\}\right)\geqslant 1-e^{-tn}$$

To finish, we only need to get a lower bound of $\rho^{*n}(B(g,r_{n,m}))$. And to do so, we use lemma 3.9. □

To finish this section, we use this property of regularity of the measure to pass from a condition in proposition 2.23 where the action of $\mathbf{G}$ on $\mathbf{X}$ is considered to a condition only on the group $\mathbf{G}$.

Lemma 3.19. *Let ρ be a borelian probability measure on $\mathbf{G}=\mathrm{SL}_d(\mathbb{R})$, having an exponential moment and whose support generates a strongly irreducible and proximal subgroup.*

Let $\mathbf{X}=\mathbb{P}(\mathbb{R}^d)$ and $\sigma:\mathbf{G}\times\mathbf{X}\to\mathbb{R}$ be the cocycle defined for $g\in\mathbf{G}$ and $X=\mathbb{R}x\in\mathbb{P}(\mathbb{R}^d)$ by

$$\sigma(g,X)=\ln\frac{\|gx\|}{\|x\|}$$

Let ν be the unique $P_\rho-$stationary probability measure on $\mathbf{X}$ (see proposition 2.5).

For any α_1,β,α_2, there are Δ, α_1' such that for any $b\in\mathbb{R}$ with $|b|\geqslant b_1$, is there is a function φ on $\mathbf{X}$ such that for any x in $\mathbf{X}$ that is $\Delta-\nu$-regular at scale $(2|b|/b_1)^{-\alpha_2}$ we have that

$$|\varphi(x)|\geqslant\frac{1}{2}$$

and

$$\int_{\mathbf{G}}\left|e^{ib\sigma(g,x)}\varphi(gx)-\varphi(x)\right|^2\mathrm{d}\rho^{*n(\beta,2|b|/b_1)}(g)\leqslant\left(\frac{b_1}{2|b|}\right)^{\alpha_1'}$$

then,

$$\int_{\mathbf{G}}\left|e^{ib\lambda_1(g)}-1\right|^2\mathrm{d}\rho^{*n(\beta,2|b|/b_1)}(g)\leqslant\left(\frac{b_1}{2|b|}\right)^{\alpha_1}$$

where we noted $n(\beta,b)=\lfloor\beta\ln|b|\rfloor$.

Proof. The idea of this proof is to make the x in the integral depend of the element g chosen at random with the measure $\rho^{*n(\beta,2|b|/b_1)}$ to take $x=V_g^+$.

To do so, we first are going to get a control holding for any regular point x and any element g except on a set having a small measure, then chose the x we want and finally integrate again to get the lemma. The price we have to pay is to pass from α_1' to α_1.

We note, to simplify our notations, $n=n(\beta,2|b|/b_1)$.

We note $G_n = \{g \in \mathbf{G} | \rho^{*n}(B(g, e^{-t_2 n})) \geqslant e^{-t_3 n}\}$. Then, for any $g \in G_n$ and any x that is $\Delta - \nu$-regular at scale $b^{-\alpha_2}$, we have that

$$e^{-t_3 n}|e^{i\sigma(g,x)}\varphi(gx) - \varphi(x)| \leqslant \int_{\mathbf{G}} \left|e^{ib\lambda_1(g)} - 1\right|^2 \mathrm{d}\rho^{*n}(g) \leqslant \left(\frac{b_1}{2|b|}\right)^{\alpha_1'}$$

Moreover, according to proposition 3.18, for any $g \in G_n$ except on a set having an exponentially small measure, the point V_g^+ is $\Delta - \nu$-regular at scale $b^{-\alpha_2}$ and so, for any $g \in G_n$ except on a set having exponentially small measure,

$$|\varphi(V_g^+)| \left|e^{ib\lambda_1(g)} - 1\right|^2 = \left|e^{ib\sigma(g,V_g^+)}\varphi(V_g^+) - \varphi(V_g^+)\right| \leqslant e^{t_3 n} \left(\frac{b_1}{2|b|}\right)^{\alpha_1'}$$

So, for any $g \in G_n$ except on a set having exponentially small measure,

$$|e^{ib\lambda_1(g)} - 1|^2 \leqslant 2e^{t_3 n} \left(\frac{b_1}{2|b|}\right)^{\alpha_1'}$$

But, according to lemma 3.9, any g belong to G_n except a set of exponentially small measure and so,

$$\int_{\mathbf{G}} \left|e^{ib\lambda_1(g)} - 1\right|^2 \mathrm{d}\rho^{*n}(g) \leqslant \frac{1}{2} \left(\frac{b_1}{2|b|}\right)^{\alpha_1' - \beta t_3}$$

And this is what we intended to prove if we take $\alpha_1' > \alpha_1 + \beta t_3$. □

4. The renewal theorem

4.1. **Preliminaries.** Let $\mathbf{G}$ be a second countable locally compact group acting continuously on a metric space $\mathbf{X}$ and let $\sigma : \mathbf{G} \times \mathbf{X} \to \mathbb{R}$ be a (continuous) cocycle (see. definition 2.16). We can define an action of $\mathbf{G}$ on $\mathbf{X} \times \mathbb{R}$ by noting

$$g.(x,t) = (gx, t + \sigma(g,x))$$

If ρ is a borelian probability measure on $\mathbf{G}$, we can define a random walk on $\mathbf{X} \times \mathbb{R}$ whose associated Markov operator is defined for any continuous function f on $\mathbf{X} \times \mathbb{R}$ and any $(x,t) \in \mathbf{X} \times \mathbb{R}$ by

$$Pf(x,t) = \int_{\mathbf{G}} f(gx, t + \sigma(g,x)) \mathrm{d}\rho(g)$$

This operator commutes to the translations on $\mathbb{R}$ and this imply that for any $f \in \mathrm{L}^\infty(\mathbb{R})$ and any $g \in \mathrm{L}^1(\mathbb{R})$,

$$(Pf) * g = P(f * g)$$

We call renewal kernel the operator $G = \sum_{n=0}^{+\infty} P^n$ when it is defined.

Kesten studied in [Kes74] the properties of the operator G in a case where $\mathbf{X}$ is very general and this allows him to prove a renewal theorem used by Guivarc'h and Le Page in [GL12] to get the renewal theorem in $\mathbb{R}^d$ that we already stated in the introduction of this article.

In this section, we study the speed in Kesten's renewal theorem but we will not do so in a very general setting : we will only consider the case of a group contracting a compact metric space.

So, from now on, we fix a second countable locally compact group $\mathbf{G}$ and a borelian probability measure ρ on $\mathbf{G}$.

Let $\mathbf{X}$ be compact metric $\mathbf{G}-$space endowed with an action of a finite group $\mathbf{H}$ that commutes to the $\mathbf{G}-$action and such that $\mathbf{X}/\mathbf{H}$ is $(\rho,\gamma,M,N)-$contracted over a finite $\mathbf{G}-$set $\mathbf{A}$ on which the walk defined by ρ is irreducible and aperiodic (we defined these notions in section 2).

For technical reasons that we will explain in paragraph 4.4, we introduce the function $\omega : (\mathbf{X}\times\mathbb{R})^2 \to \mathbb{R}_+$ defined by

$$\omega((x,t),(x',t')) = \begin{cases} e^{-\frac{|t|+|t'|}{2}}\sqrt{d(x,x')^2 + \left(e^{(t-t')/2} - e^{(t'-t)/2}\right)^2} & \text{if } \pi_{\mathbf{A}}(x) = \pi_{\mathbf{A}}(x') \\ 1 & \text{if not} \end{cases}$$

and we set, for any $\gamma \in]0,1]$,

$$\mathcal{C}^{\gamma}_{\omega}(\mathbf{X}\times\mathbb{R}) = \left\{ f \in \mathcal{C}^0(\mathbf{X}\times\mathbb{R}) \middle| \|f\|_{\gamma,\omega} := \sup_{\substack{(x,t),(x',t')\in\mathbf{X}\times\mathbb{R} \\ (x,t)\neq(x',t')}} \frac{|f(x,t)-f(x',t')|}{\omega((x,t),(x',t'))^{\gamma}} \text{ is finite} \right\}$$

In the same way, we note, for $(x,t),(x',t') \in \mathbf{X}\times\mathbb{R}$,

$$\omega_0((x,t),(x',t')) = \frac{\sqrt{|t-t'|^2 + d(x,x')^2}}{(1+|t'|)(1+|t|)}$$

And we define $\mathcal{C}^{0,\gamma}_{\omega_0}(\mathbf{X}\times\mathbb{R})$ as $\mathcal{C}^{0,\gamma}_{\omega}(\mathbf{X}\times\mathbb{R})$.

We will see in paragraph 4.4 that for any function f in $\mathcal{C}^{0,\gamma}_{\omega}$, there are function $p^+(f), p^-(f)$ on $\mathbf{A}$ such that for any $x \in \mathbf{X}$,

$$p^-(f)(\pi_{\mathbf{A}}(x)) = \lim_{t\to-\infty} f(x,t) \text{ and } p^+(f)(\pi_{\mathbf{A}}(x)) = \lim_{t\to+\infty} f(x,t)$$

We are going to prove the

Theorem 4.1. *Let $\mathbf{G}$ be a secong countable locally compact group, $N : \mathbf{G} \to [1,+\infty[$ a submultiplicative function on $\mathbf{G}$ and ρ a borelian probability measure on $\mathbf{G}$.*

Let $(\mathbf{X},d)$ be a compact metric space endowed with a continuous action of $\mathbf{G}$ and of an action of finite group $\mathbf{H}$ that commutes to the $\mathbf{G}$-action and such that $\mathbf{X}/\mathbf{H}$ is $(\rho,\gamma,M,N)-$contracted over a finite $\mathbf{G}-$set $\mathbf{A}$ on which the random walk defined by ρ is irreducible and aperiodic.

Let $\sigma \in \mathcal{Z}^M(\mathbf{X}/\mathbf{H})$ and $\sigma_\rho = \int_{\mathbf{G}}\int_{\mathbf{X}/\mathbf{H}} \sigma(g,x)\mathrm{d}\nu(x)\mathrm{d}\rho(g)$ where ν is the unique $P-$stationary probability measure on $\mathbf{X}/\mathbf{H}$ given by proposition 2.5. We assume that $\sigma_\rho > 0$.

We also assume that there is $\gamma_0 \in]0,1]$ such that for any $\gamma \in]0,\gamma_0]$ and any $t_0 \in \mathbb{R}_+^*$, there are C_0, L such that for any $t \in \mathbb{R}$ with $|t| \geqslant t_0$,

$$\|(I_d - P(it))^{-1}\| \leqslant C_0|t|^L$$

We note Π_0 the operator defined for any $f \in \mathcal{C}^{\gamma}_{\omega}(\mathbf{X}\times\mathbb{R})$ such that $p^+(f) = 0$ by

$$\Pi_0 f(x,t) = \int_t^{+\infty} N_0 f(x,u)\mathrm{d}u$$

where N_0 is the projector on the space of $P-$invariant function in $\mathcal{C}^0(\mathbf{X})$ and we made the abuse of notions $N_0 f(x,u) = N_0 f_u(x)$ with $f_u = f(.,u)$.

Then, for any $\gamma > 0$ small enough, there are $\alpha, C \in \mathbb{R}_+^*$ such that for any $f \in \mathcal{C}^\gamma_\omega(\mathbf{X}\times\mathbb{R})$ such that $p^+(f) = 0 = \sum_{a\in\mathbf{A}} p^-(f)(a)$ and for any $x \in \mathbf{X}$,

$$\lim_{t\to-\infty}\left(G - \frac{1}{\sigma_\rho}\Pi_0\right) f(x,t) = \sum_{n\in\mathbb{N}} P^n p^-(f)(\pi_{\mathbf{A}}(x))$$

Moreover, $(G - \frac{1}{\sigma_\rho}\Pi_0)f \in \mathcal{C}^\alpha_{\omega_0}(\mathbf{X}\times\mathbb{R})$ and

$$\left\|\left(G - \frac{1}{\sigma_\rho}\Pi_0\right) f\right\|_{\alpha,\omega_0} \leqslant C\|f\|_{\gamma,\omega}$$

Proof. To prove this theorem, we use the decomposition given by lemma 4.16, corollary 4.18 and lemma 4.19 □

4.2. **Non-unitary perturbations of Markov operators by cocycles.**

In this paragraph, we study the inverse of the operator $I_d - P(z)$ and we prove proposition 4.2 which proves that a control of it's norm along the imaginary axis gives a control on the norm of it's derivatives on a neighbourhood of the imaginary axis having a shape that we control well.

Let $\mathbf{G}$ be a second countable locally compact group and ρ a borelian probability measure on $\mathbf{G}$. Let $\mathbf{X}$ be a compact metric space endowed with a continuous action of $\mathbf{G}$ and of a finite group $\mathbf{H}$ such that $\mathbf{X}/\mathbf{H}$ is $(\rho,\gamma,M,N)-$contracted over a finite $\mathbf{G}-$set $\mathbf{A}$.

For any cocycle σ on $\mathbf{X}$ (see definition 2.16) and any $g \in \mathbf{G}$, we noted

$$\sigma_{\sup}(g) = \sup_{x\in\mathbf{X}} |\sigma(g,x)| \text{ and } \sigma_{\mathrm{Lip}}(g) = \sup_{\substack{x,y\in\mathbf{X}\\ \pi_{\mathbf{A}}(x)=\pi_{\mathbf{A}}(y)\\ x\neq y}} \frac{|\sigma(g,x)-\sigma(g,y)|}{d(x,y)}$$

And, for $M \in \mathbb{R}_+$, we set

$$\mathcal{Z}^M(\mathbf{X}) = \left\{\sigma \text{ is a continuous cocycle on } \mathbf{X} \middle| \sup_{g\in\mathbf{G}} \frac{\sigma_{\mathrm{Lip}}(g)}{N(g)^M} \text{ and } \sup_{g\in\mathbf{G}} \frac{e^{\sigma_{\sup}(g)}}{N(g)^M} \text{ are finites}\right\}$$

Finally, for any $\sigma \in \mathcal{Z}^M(\mathbf{X})$, we noted

$$[\sigma]_M = \sup_{g\in\mathbf{G}} \frac{\sigma_{\mathrm{Lip}}(g)}{N(g)^M} \text{ and } [\sigma]_\infty = \sup_{g\in\mathbf{G}} \frac{e^{\sigma_{\sup}(g)}}{N(g)^M}$$

We note $\mathbb{C}_\gamma := \{z \in \mathbb{C} | |\Re(z)| < \gamma\}$. For $z \in \mathbb{C}_\gamma$ and $\sigma \in \mathcal{Z}^M(\mathbf{X}/\mathbf{H})$, we define an operator $P(z)$ on $\mathcal{C}^0(\mathbf{X})$ by

$$P(z)f(x) = \int_{\mathbf{G}} e^{-z\sigma(g,x)} f(gx) \mathrm{d}\rho(g)$$

This is a continuous operator since for any continuous function f on $\mathbf{X}$, any $z \in \mathbb{C}_\gamma$ and any $x \in \mathbf{X}$,

$$|P(z)f(x)| \leqslant \|f\|_\infty \int_{\mathbf{G}} e^{-\Re(z)\sigma(g,x)} \mathrm{d}\rho(g) \leqslant \|f\|_\infty \int_{\mathbf{G}} [\sigma]_\infty^\gamma N(g)^{\gamma M} \mathrm{d}\rho(g)$$

And $\int_{\mathbf{G}} N(g)^{\gamma M} \mathrm{d}\rho(g)$ is finite according to the definition of the contraction of the action.

We regroup the main results of this paragraph in next

Proposition 4.2. *Under the assumption of theorem 4.1.*

For any $\gamma > 0$ small enough, there are $\eta, C, t \in \mathbb{R}_+^*$ such that $(z \mapsto P(z))$ defines an analytic function from $\mathbb{C}_\eta$ to the space of continuous operators on $\mathcal{C}^{0,\gamma}(\mathbf{X})$. Moreover, for any $z \in \mathbb{C}_\eta$ with $\Re(z) \geqslant 0$ and any $n \in \mathbb{N}$,

$$\|P(z)^n\|_\gamma \leqslant C(1+|z|)e^{-t\Re(z)n}$$

Finally, noting

$$U(z) = (I_d - P(z))^{-1} - \frac{1}{\sigma_\rho z} N_0,$$

We have that $(z \mapsto U(z))$ (which is defined a priori *on a neighbourhood of the imaginary axis except at 0) can be extended to an analytic function taking it's values in the space of continuous operators on $\mathcal{C}^{0,\gamma}(\mathbf{X})$ and defined on*

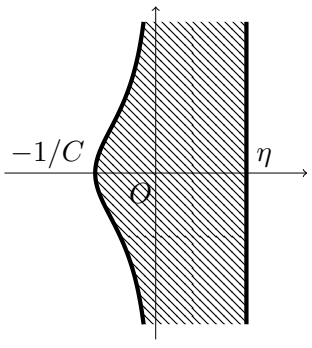
$$\mathcal{D}_{\eta,C,L} := \left\{ z \in \mathbb{C} \middle| \frac{-1}{C(1+|\Im z|)^{L+1}} < \Re(z) < \eta \right\}$$

Moreover, for any $n \in \mathbb{N}$ and any $z \in \mathcal{D}_{\eta,C,L}$

$$\|U^{(n)}(z)\|_\gamma \leqslant n! C^{n+1} (1+|\Im z|)^{(L+1)(n+1)}$$

Remark 4.3. This proposition only generalizes the situation in $\mathbb{R}$ when the operator $P(z)$ is the Fourier-Laplace transform $\widehat{\rho}(z)$ of the measure ρ. In this case, the same proposition can be obtained under the assumption "non-lattice of type p" used by Carlsson in [Car83].

Before we prove each point of the proposition, we draw the set $\mathcal{D}_{\eta,C,L}$.



$$\left\{ z \in \mathbb{C} \middle| \frac{-1}{C(1+|\Im z|)^{L+1}} < \Re(z) < \eta \right\}$$

FIGURE 1. Shape of $\mathcal{D}_{\eta,C,L}$

Lemma 4.4. *Under the assumptions of proposition 4.2, if $\sigma_\rho > 0$ then there are $\eta, t, C \in \mathbb{R}_+^*$ such that for any $s \in]0, \eta]$ and any $n \in \mathbb{N}$,*

$$\sup_{x \in \mathbf{X}} \int_{\mathbf{G}} e^{-s\sigma(g,x)} \mathrm{d}\rho^{*n}(g) \leqslant C e^{-tsn}$$

Proof. First, using Jensen's inequality, we have that for any $0 \leqslant s \leqslant \eta$,

$$\int_{\mathbf{G}} e^{-s\sigma(g,x)} \mathrm{d}\rho^{*n}(g) \leqslant \left(\int_{\mathbf{G}} e^{-\eta\sigma(g,x)} \mathrm{d}\rho^{*n}(g) \right)^{s/\eta} = (P(\eta)^n 1(x))^{s/\eta}$$

Moreover, as $\sigma_\rho > 0$, there are $\eta, t, C \in \mathbb{R}_+^*$ such that

$$\sup_x P(\eta)^n 1(x) \leqslant C e^{-tn}$$

And this is what we intended to prove. □

Lemma 4.5. *Under the assumptions of proposition 4.2, for any $\gamma > 0$ small enough, there is $\eta \in \mathbb{R}_+^*$ such that the function $(z \mapsto P(z))$ is analytic from $\mathcal{C}_\eta$ to the space of continuous operator on $\mathcal{C}^{0,\gamma}(\mathbf{X})$ and there are $t \in \mathbb{R}_+^*$, and $C \in \mathbb{R}$ such that for any $z \in \mathbb{C}_\eta$ with $\Re(z) \geqslant 0$, any function $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$ and any $n \in \mathbb{N}$,*

$$\|P(z)^n f\|_\gamma \leqslant C \left(e^{-tn} m_\gamma(f) + (1 + |z|) \|f\|_\infty \right)$$

Proof. To get that $(P(z))$ is an analytic family of operators, we refer to the lemma 10.16 of [BQ15].

Let $\gamma \in \mathbb{R}_+^*$ be small and $\eta \in \mathbb{R}_+^*$.

Let us compute, for $z \in \mathbb{C}_\eta$, $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$, $x, y \in \mathbf{X}$ with $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(y)$ and $n \in \mathbb{N}$,

$$\begin{aligned} &|P(z)^n f(x) - P(z)^n f(y)| \\ &\qquad \leqslant \int_{\mathbf{G}} \left| e^{-z\sigma(g,x)} f(gx) - e^{-z\sigma(g,y)} f(gy) \right| \mathrm{d}\rho^{*n}(g) \\ &\qquad \leqslant \int_{\mathbf{G}} e^{-\Re(z)\sigma(g,x)} |f(gx) - f(gy)| + \|f\|_\infty \left| e^{-z\sigma(g,x)} - e^{-z\sigma(g,y)} \right| \mathrm{d}\rho^{*n}(g) \end{aligned} \tag{4.1}$$

But, using that for any $a, b \in \mathbb{C}$ and any $\gamma \in]0, 1]$,

$$|e^a - e^b| \leqslant 2^{1-\gamma} (\max(|a|, |b|))^{1-\gamma} \max(e^{\Re(a)}, e^{\Re(b)}) |a - b|^\gamma$$

we have that

$$\left| e^{-z\sigma(g,x)} - e^{-z\sigma(g,y)} \right| \leqslant 2^{1-\gamma} |z| e^{\Re(z)\sigma_{\sup}(g)} (\sigma_{\sup}(g))^{1-\gamma} |\sigma(g,x) - \sigma(g,y)|^\gamma$$

Now, using the definition of $[\sigma]_M$ and $[\sigma]_\infty$ (see equation 2.5), we find that for any $\varepsilon \in \mathbb{R}_+^*$

$$\begin{aligned} \left| e^{-z\sigma(g,x)} - e^{-z\sigma(g,y)} \right| &\leqslant 2|z| [\sigma]_\infty^{|\Re(z)|} N(g)^{M(\gamma + |\Re(z)|)} \left(\ln([\sigma]_\infty N(g)^M) \right)^{1-\gamma} [\sigma]_M^\gamma d(x,y)^\gamma \\ &\leqslant 2C_\varepsilon |z| N(g)^{M(\gamma+\eta+\varepsilon)} [\sigma]_\infty [\sigma]_M d(x,y)^\gamma \end{aligned}$$

where we noted C_ε such that for any $x \in [1, +\infty[$, $x \leqslant C_\varepsilon e^{\varepsilon x}$. This proves that

$$\int_{\mathbf{G}} \left| e^{-z\sigma(g,x)} - e^{-z\sigma(g,y)} \right| \mathrm{d}\rho^{*n}(g) \leqslant C_\varepsilon 2^{1-\gamma} |z| d(x,y)^\gamma \int_{\mathbf{G}} N(g)^{M(\gamma+\eta+\varepsilon)} \mathrm{d}\rho^{*n}(g) \tag{4.2}$$

Moreover,

$$\int_{\mathbf{G}} e^{-\Re(z)\sigma(g,x)}|f(gx)-f(gy)|\mathrm{d}\rho^{*n}(g) \leqslant m_\gamma(f)[\sigma]_\infty^\eta \int_{\mathbf{G}} N(g)^{M\eta} d(gx,gy)^\gamma \mathrm{d}\rho^{*n}(g) \tag{4.3}$$

But, there is $d_0 \in \mathbb{R}_+^*$ such that if $d(x,y) \leqslant d_0$ then $d(x,y) = d(\pi_{\mathbf{H}}x, \pi_{\mathbf{H}}y)$. And so, for any $\varepsilon' \in \mathbb{R}_+^*$ and any x, y such that $0 < d(x,y) \leqslant \varepsilon' d_0$, we have that

$$\begin{aligned}
I_n(x,y) :&= \int_{\mathbf{G}} N(g)^{M\eta} d(gx,gy)^\gamma \mathrm{d}\rho^{*n}(g) \\
&\leqslant \int_{\mathbf{G}} N(g)^{M\eta} \mathbf{1}_{d(gx,gy)\leqslant d_0} d(gx,gy)^\gamma \mathrm{d}\rho^{*n}(g) \\
&\qquad + \int_{\mathbf{G}} N(g)^{M\eta} \mathbf{1}_{d(gx,gy)\geqslant d_0} d(gx,gy)^\gamma \mathrm{d}\rho^{*n}(g) \\
&\leqslant \int_{\mathbf{G}} N(g)^{M\eta} d(g\pi_{\mathbf{H}}x, g\pi_{\mathbf{H}}y)^\gamma \mathrm{d}\rho^{*n}(g) \\
&\qquad + \int_{\mathbf{G}} N(g)^{M\eta} \mathbf{1}_{MN(g)^M \geqslant 1/\varepsilon} M N(g)^{M\gamma} d(x,y)^\gamma \mathrm{d}\rho^{*n}(g) \\
&\leqslant d(x,y)^\gamma \left(\int_{\mathbf{G}} N(g)^{M\eta} \frac{d(g\pi_{\mathbf{H}}x, g\pi_{\mathbf{H}}y)^\gamma}{d(\pi_{\mathbf{H}}x, \pi_{\mathbf{H}}y)^\gamma} \mathrm{d}\rho^{*n}(g) \right. \\
&\qquad \left. + M \int_{\mathbf{G}} N(g)^{M(\eta+\gamma)} \mathbf{1}_{MN(g)^M \geqslant 1/\varepsilon} \mathrm{d}\rho^{*n}(g) \right)
\end{aligned} \tag{4.4}$$

But, according to Cauch-Schwarz's inequality and using the contraction of the action, we have that

$$\begin{aligned}
J_n(x,y) :&= \int_{\mathbf{G}} N(g)^{M\eta} \frac{d(g\pi_{\mathbf{H}}x, g\pi_{\mathbf{H}}y)^\gamma}{d(\pi_{\mathbf{H}}x, \pi_{\mathbf{H}}y)^\gamma} \mathrm{d}\rho^{*n}(g) \\
&\leqslant \left(\int_{\mathbf{G}} N(g)^{2M\eta} \mathrm{d}\rho^{*n}(g) \int_{\mathbf{G}} \frac{d(g\pi_{\mathbf{H}}x, g\pi_{\mathbf{H}}y)^{2\gamma}}{d(x,y)^{2\gamma}} \mathrm{d}\rho^{*n}(g) \right)^{1/2} \\
&\leqslant \sqrt{C_{2\gamma}} e^{-\delta_{2\gamma} n/2} \left(\int_{\mathbf{G}} N(g)^{2\eta M} \mathrm{d}\rho(g) \right)^{n/2}
\end{aligned} \tag{4.5}$$

We can now choose n such that $\sqrt{C_{2\gamma}} e^{-\delta_{2\gamma} n/4} \leqslant 1/4[\sigma]_\infty$ and then, for η such that

$$e^{-\delta_{2\gamma} n/4} \int_{\mathbf{G}} N(g)^{M\eta} \mathrm{d}\rho(g) \leqslant 1$$

we find, with equation 4.5, that

$$J_n(x,y) = \int_{\mathbf{G}} N(g)^{M\eta} \frac{d(g\pi_{\mathbf{H}}x, g\pi_{\mathbf{H}}y)^\gamma}{d(\pi_{\mathbf{H}}x, \pi_{\mathbf{H}}y)^\gamma} \mathrm{d}\rho^{*n}(g) \leqslant \frac{1}{4[\sigma]_\infty}$$

Moreover, this n being fixed, we can chose $\varepsilon' \in \mathbb{R}_+^*$ such that

$$M \int_{\mathbf{G}} N(g)^{M(\eta+\gamma)} \mathbf{1}_{MN(g)^M \geqslant 1/\varepsilon'} \mathrm{d}\rho^{*n}(g) \leqslant 1/4[\sigma]_\infty$$

and this proves, using equations 4.1, 4.2, 4.3 and 4.4, that for any $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$, any $x, y \in \mathbf{X}$ with $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(y)$ and $d(x,y) \leqslant \varepsilon' d_0$,

$$\frac{|P(z)^n f(x) - P(z)^n f(y)|}{d(x,y)^\gamma} \leqslant \frac{1}{2} m_\gamma(f) + 2\|f\|_\infty |z| [\sigma]_\infty [\sigma]_M \int_{\mathbf{G}} N(g)^{M(\gamma+\eta+\varepsilon)} \mathrm{d}\rho^{*n}(g)$$

Moreover, if $d(x,y) \geqslant \varepsilon' d_0$, then

$$|P^n(z)f(x) - P^n(z)f(y)| \leqslant 2\frac{d(x,y)^\gamma}{(\varepsilon' d_0)^\gamma} \|f\|_\infty [\sigma]_\infty \int_{\mathbf{G}} N(g)^{M\eta} \mathrm{d}\rho^{*n}(g)$$

What we proved is that for any $\gamma, \eta > 0$ small enough, there are $n \in \mathbb{N}^*$ and C (depending on n, σ and ρ) such that for any $z \in \mathbb{C}_\eta$ and any $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$,

$$m_\gamma(P(z)^n f) \leqslant \frac{1}{2} m_\gamma(f) + Ce^{Cn}(1 + |z|)\|f\|_\infty$$

Moreover, as we also have, according to lemma 4.4, that for $\Re(z) \geqslant 0$,

$$\|P(z)^n f\|_\infty \leqslant \|f\|_\infty \sup_{x \in \mathbf{X}} \int_{\mathbf{G}} e^{-\Re(z)\sigma(g,x)} \mathrm{d}\rho^{*n}(g) \leqslant C\|f\|_\infty e^{-\Re(z)tn}$$

we get the wanted inequalities by iterating this equations and we refer to [ITM50] for a proof that we can choose a constant C that doesn't depend on n nor on z. □

Let $\mathcal{U}$ be an open subset of $\mathbb{C}$ and $z_0 \in \mathcal{U}$. A family of operators $(P(z))_{z \in \mathcal{U} \setminus \{z_0\}}$ is said to be meromorphic at z_0 if there is $N \in \mathbb{N}$ such that the family $((z - z_0)^N P(z))$ can be extended to an analytic family of operators at z_0.

In the sequel, we will use an analytic Fredholm theorem for quasi-compact operators that we state in next

Theorem 4.6. *Let $(\mathcal{B}, \|\,.\,\|_{\mathcal{B}})$ be a Banach space.*

Let $\|\,.\,\|$ be a norm on $\mathcal{B}$ such that the unit ball of $\mathcal{B}$ for the norm $\|\,.\,\|_{\mathcal{B}}$ is relatively-compact for $\|\,.\,\|$. Let $\mathcal{U}$ be a connected open subset of $\mathbb{C}$.

Let $(P(z))_{z \in \mathcal{U}}$ be an analytic family of operators on $\mathcal{B}$ defined on $\mathcal{U}$ and such that for some $r \in [0,1[$ and some real-valued function $(z \mapsto R(z))$ we have that for any $f \in \mathcal{B}$ and any $z \in \mathcal{U}$,

$$\|P(z)f\|_{\mathcal{B}} \leqslant r\|f\|_{\mathcal{B}} + R(z)\|f\|$$

Then, the following alternative holds

- *The operator $I_d - P(z)$ is invertible for no $z \in \mathcal{U}$.*
- *The function $(z \mapsto (I_d - P(z))^{-1})$ is meromorphic on $\mathcal{U}$.*

Proof. The proof comes from the functional calculus in Banach algebras when we remark that according to Ionescu-Tulcea and Marinescu's theorem (that we recalled in theorem 2.10) we have a control on the essential spectral radius of the operator $P(z)$ that is uniform in z. □

Lemma 4.7. *Under the assumptions of proposition 4.2, the family $((I_d - P(z))^{-1})_{z \in \mathbb{C}_\eta}$ is meromorphic on an open neighbourhood of the imaginary axis and it's only singularity in $[0, \eta'[\oplus i\mathbb{R}$ is at 0.*

Moreover, we can choose η such that there are $C, t \in \mathbb{R}_+^\star$ such that for any $z \in]0, \eta[\oplus i\mathbb{R}$ and any $n \in \mathbb{N}$,

$$\|P(z)^n\|_\gamma \leqslant C(1 + |z|)e^{-t\Re(z)n}$$

Finally, if $\sigma_\rho > 0$ and if η' is small enough, we can write for $z \in B(0,\eta') \setminus \{0\}$,

$$(I_d - P(z))^{-1} = \frac{1}{\sigma_\rho z} N_0 + U(z)$$

where N_0 is the projection on the space of $P-$invariant continuous functions and $(U(z))$ is an analytic family of continuous operators on $\mathcal{C}^{0,\gamma}(\mathbf{X})$ defined on $B(0,\eta')$.

Proof. First, for any $z \in [0,\eta] \oplus i\mathbb{R}$, any $n \in \mathbb{N}$ and any function $f \in \mathcal{C}^0(\mathbf{X})$, we have, according to lemma 4.4, that

$$\|P(z)^n f\|_\infty \leqslant \|f\|_\infty \sup_{x\in\mathbf{X}} \int_{\mathbf{G}} e^{-\Re(z)\sigma(g,x)} \mathrm{d}\rho^{*n}(g) \leqslant C\|f\|_\infty e^{-t\Re(z)n}$$

And so, according to lemma 4.5, for any function $f \in \mathcal{C}^{0,\gamma}(\mathbf{X})$ and any $m,n \in \mathbb{N}$, (we can assume without any loss of generality that the constants t given by lemma 4.5 and 4.4 are the same and idem for C)

$$\begin{aligned}
\|P(z)^{2n} f\|_\gamma &\leqslant C\left(e^{-tn}\|P^n f\|_\gamma + (1+|z|)\|P^n f\|_\infty\right) \\
&\leqslant C(1+|z|)\left(e^{-tn} C(m_\gamma(f) + \|f\|_\infty) + Ce^{-t\Re(z)n}\|f\|_\infty\right) \\
&\leqslant C^2(1+|z|)\|f\|_\gamma \left(e^{-tn} + e^{-t\Re(z)n}\right)
\end{aligned}$$

Which is what we intended to prove.

Moreover, this proves that $(I_d - P(z))^{-1}$ has no pole in $]0,\eta[\oplus i\mathbb{R}$.

Then, lemma 4.5 proves that the essential spectral radius of $P(z)$ is uniformly bounded by e^{-t} on $\mathbb{C}_\eta$ and so, we can use theorem 4.6 to prove that the family of operators $(I_d - P(z))^{-1}$ is meromorphic on $\mathbb{C}_\eta$ since we just saw that $I_d - P(z)$ is invertible for $z \in \mathbb{C}_\eta$ with $\Re(z) > 0$.

Finally, we refer to lemma 3.2 in [BL85] or to lemma 10.17 in [BQ15] to get the development of $(I_d - P(z))^{-1}$ around 0. Indeed, adapting the demonstration, we find that there are analytic families of operators $(N(z)), (U_1(z))$ defined on a neighbourhood of 0 and an analytic function λ such that for any $z \neq 0$ in a neighbourhood of 0 in $\mathbb{C}$,

$$(I_d - P(z))^{-1} = \frac{1}{1-\lambda(z)} N(z) + U_1(z)$$

This finishes the proof of the lemma since λ and N sare analytic, $\lambda(0) = 1$, $\lambda'(0) = -\sigma_\rho \neq 0$ and $N(0)$ is the projector on $\ker(I_d - P)$. □

Lemma 4.8. *Under the assumptions of proposition 4.2*

We note

$$U(z) = (I_d - P(z))^{-1} - \frac{1}{z} N_0$$

Then, for any $\gamma, \eta > 0$ small enough, there are C, L such that for any $z \in \mathbb{C}$ with

$$\frac{-1}{C(1+|\Im z|)^L} < \Re(z) < \eta$$

and any $n \in \mathbb{N}$, we have that

$$\|U^{(n)}(z)\| \leqslant C^{n+1} n! (1+|z|)^{(L+1)(n+1)}$$

Proof. First of all, we note that, according to the previous lemma and to the growth assumption we made on $\|(I_d - P(it))^{-1}\|$, $(z \mapsto U(z))$ is an analytic family of continuous operators and that

$$\sup_{t\in\mathbb{R}} \frac{1}{1+|t|^L}\|U(it)\|_\gamma < +\infty$$

Moreover, using that

$$\sup_{z\in\mathbb{C}_\eta} \|P'(z)\|_\gamma < +\infty$$

we can prove that there is a constant C such that for $z \in \mathbb{C}$ with

$$|\Re(z)| \leqslant \frac{1}{C(1+|\Im(z)|)^L}$$

we have that U is analytic at z and

$$\|U(z)\| \leqslant C(1+|z|)^L$$

But, according to lemma 4.7, for any $z \in \mathbb{C}_\eta$ with $\Re(z) \geqslant 1/C(1+|\Im(z)|)^L$, we have that for any $n \in \mathbb{N}$,

$$\|P(z)^n\| \leqslant C(1+|z|)e^{-t\Re(z)n}$$

and so,

$$\|(I_d - P(z))^{-1}\| \leqslant \frac{C(1+|z|)}{1-e^{-t_0\Re(z)}} \leqslant \frac{C(1+|z|)}{t_0\Re(z)} \leqslant \frac{C^2(1+|z|)^{L+1}}{t_0}$$

This proves that the function U is analytic on $]0,\eta[\oplus i\mathbb{R}$.

We proved that for any $\eta > 0$ small enough and any $z \in \mathbb{C}$ with

$$\frac{-1}{C(1+|\Im(z)|)^L} < \Re(z) < \eta$$

we have, for some constant C',

$$\|U(z)\| \leqslant C'(1+|z|)^{L+1}$$

To conclude, we do the same computations than Gelfand and Shilov in the proof of the theorem 15 in [GC64] to get a control of the derivatives of U on some domain $D_{\eta,C'',L+1}$ for some constant C''. □

4.3. **The renewal theorem for regular functions.**

In this paragraph, we prove a result of representation of the renewal kernel applied to regular functions and use it to study the rate of convergence in the renewal theorem for these functions.

Let $\gamma \in \mathbb{R}_+^*$. For any $f \in \mathcal{C}^0(\mathbf{X}\times\mathbb{R})$, we note

$$m_{\gamma,\mathcal{E}}(f) = \sup_{t\in\mathbb{R}} \sup_{\substack{x,x'\in\mathbf{X}\\ x\neq x'\\ \pi_{\mathbf{A}}(x)=\pi_{\mathbf{A}}(x')}} e^{\gamma|t|}\frac{|f(x,t)-f(x',t)|}{d(x,x')^\gamma}$$

and

$$\|f\|_{\gamma,\infty} = \sup_{x\in\mathbf{X}}\sup_{t\in\mathbb{R}} e^{\gamma|t|}|f(x,t)|$$

Moreover, we note

$$\|f\|_{\gamma,\mathcal{E}} = \|f\|_{\gamma,\infty} + m_{\gamma,\mathcal{E}}(f)$$

Finally, we set

$$E_0^{\gamma,k}(\mathbf{X}\times\mathbb{R}) = \left\{ f \in \mathcal{C}^k(\mathbb{R}, \mathcal{C}^0(\mathbf{X})) \middle| \forall m \in [0,k] \|f^{(m)}\|_{\gamma,\mathcal{E}} \text{ is finite} \right\} \tag{4.6}$$

where we noted

$$f^{(k)}(x,t) = \frac{\partial^k f}{\partial t^k}(x,t)$$

And, for any $f \in \mathcal{E}_0^{\gamma,k}(\mathbf{X}\times\mathbb{R})$, we set

$$\|f\|_{\gamma,k} = \max_{m\in[0,k]} \|f^{(m)}\|_{\gamma,\mathcal{E}}$$

If $f \in \mathcal{E}_0^{\gamma,0}(\mathbf{X}\times\mathbb{R})$, then, we note, for any $x \in \mathbf{X}$ and $\xi \in \mathbb{R}$,

$$\widehat{f}(x,\xi) = \int_{\mathbb{R}} e^{-i\xi t} f(x,t) \mathrm{d}t$$

It is clear that, for fixed ξ, $\widehat{f}(x,\xi)$ is an hölder-continuous function on $\mathbf{X}$.

Moreover,

$$\frac{\partial^l \widehat{f}}{\partial \xi^l}(x,\xi) = \int_{\mathbb{R}} (-it)^l e^{-i\xi t} f(x,t) \mathrm{d}t$$

Integrating by parts, we find that for any $\xi \in \mathbb{R}$ and any $m \in \mathbb{N}$,

$$\xi^m \widehat{f}(x,\xi) = (i)^m \int_{\mathbb{R}} e^{-i\xi t} f^{(m)}(x,t) \mathrm{d}t$$

So, if $f \in \mathcal{E}_0^{\gamma,k}(\mathbf{X}\times\mathbb{R})$, then

$$(1+|\xi|^k)|\widehat{f}(x,\xi)| \leqslant \int_{\mathbb{R}} |f(x,t)| \mathrm{d}t + \int_{\mathbb{R}} |f^{(k)}(x,t)| \mathrm{d}t \leqslant 2\|f\|_{\gamma,k} \int_{\mathbb{R}} e^{-\gamma|t|} \mathrm{d}t$$

In the same way, if $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x')$, then

$$(1+|\xi|^k)|\widehat{f}(x,\xi) - \widehat{f}(x',x)| \leqslant 2\|f\|_{\gamma,k} d(x,x')^\gamma \int_{\mathbb{R}} e^{-\gamma|t|} \mathrm{d}t$$

As we can do the same kind of computations for the functions $\frac{\partial^l \widehat{f}}{\partial \xi^l}$, we just proved the

Lemma 4.9. *Let $k, l \in \mathbb{N}$. There is a constant C such that for any $f \in \mathcal{E}_0^{\gamma,k}(\mathbf{X}\times\mathbb{R})$ and any $\xi \in \mathbb{R}$, we have that*

$$\left\| \frac{\partial^l \widehat{f}}{\partial \xi^l}(\,.\,,\xi) \right\|_\gamma \leqslant C \frac{\|f\|_{\gamma,k}}{1+|\xi|^k}$$

We recover in this way the fact that the Fourier-transform exchanges regularity and decrease at infinity.

Before we continue, we are going to prove that convolution by functions of $\mathcal{E}_0^{\gamma,k}$ regularizes functions of $\mathcal{E}^{\gamma,0}$. As we will not use this in such generality, we only prove this lemma with a particular function in $\mathcal{E}^{\gamma,k}$.

Lemma 4.10. *Let, for $k \in \mathbb{N}$, φ_k be the function defined for any $t \in \mathbb{R}$ by*

$$\varphi_k(t) = t^k e^{-t} \mathbf{1}_{\mathbb{R}_+}(t)$$

Then, for any $\gamma \in]0,1[$ and any $k \in \mathbb{N}$, there is a constant C_k such that for any $f \in \mathcal{E}_0^{\gamma,0}(\mathbf{X} \times \mathbb{R})$,

$$\varphi_{k+1} * f \in \mathcal{E}_0^{\gamma,k}(\mathbf{X} \times \mathbb{R})$$

and

$$\|\varphi_{k+1} * f\|_{\gamma,k} \leqslant C_k \|f\|_{\gamma,0}$$

Proof. The usual properties of the convolution proves that, as $\varphi_{k+1} \in \mathcal{C}^k(\mathbb{R})$, so does the function $f * \varphi_{k+1}$ (since f is continuous) and for any $x \in \mathbf{X}$, any $t \in \mathbb{R}$ and any $m \in [0,k]$

$$(\varphi_{k+1} * f)^{(m)}(x,t) = \varphi_{k+1}^{(m)} * f(x,t)$$

But,

$$\varphi_{k+1}^{(m)}(t) = \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \frac{(k+1)!}{(k+1-l)!} t^{k+1-l} e^{-t} \mathbf{1}_{\mathbb{R}_+}(t)$$

Thus,

$$(\varphi_{k+1} * f)^{(m)}(x,t) = \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \frac{(k+1)!}{(k+1-l)!} \int_{\mathbb{R}_+} u^{k+1-l} f(x,t-u) e^{-u} \mathrm{d}u$$

And so, if $x, x' \in \mathbf{X}$ are such that $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x')$, we have that

$$\begin{aligned} I_{k,m,x,x',t} &:= \left| (\varphi_{k+1} * f)^{(m)}(x,t) - (\varphi_{k+1} * f)^{(m)}(x',t) \right| \\ &\leqslant \sum_{l=0}^{m} \binom{m}{l} \frac{(k+1)!}{(k+1-l)!} \int_{\mathbb{R}_+} u^{k+1-l} e^{-\gamma|t-u|} d(x,x')^{\gamma} \|f\|_{\gamma,0} e^{-u} \mathrm{d}u \\ &\leqslant e^{-\gamma|t|} d(x,x')^{\gamma} \|f\|_{\gamma,0} \sum_{l=0}^{m} \binom{m}{l} \frac{(k+1)!}{(k+1-l)!} \int_{\mathbb{R}_+} e^{\gamma|u|} u^{k+1-m} e^{-u} \mathrm{d}u \end{aligned}$$

where we used the fact that for any $v, w \in \mathbb{R}$,

$$e^{|v|-|v+w|} \leqslant e^{|w|}$$

Moreover,

$$J_{k,m,x,t} = \left| (\varphi_{k+1} * f)^{(m)}(x,t) \right| \leqslant C_{k,m} \|f\|_{\gamma,0} e^{-\gamma|t|}$$

for some constant $C_{k,m}$ and this finishes the proof of the lemma. □

We are now ready to prove the representation of the renewal kernel in next

Proposition 4.11. *Under the assumptions of proposition 4.1, for any $\gamma > 0$ small enough, there is a constant K such that for any $f \in \mathcal{E}_0^{\gamma,K}(\mathbf{X} \times \mathbb{R})$, any $x \in \mathbf{X}$ and any $t \in \mathbb{R}$,*

$$\sum_{n=0}^{+\infty} P^n f(x,t) = \frac{1}{\sigma_\rho} \Pi_0 f(x,t) + \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} U(-i\xi) \widehat{f}(x,\xi) \mathrm{d}\xi$$

Where $U(z)$ is the operator defined in proposition 4.2 and we made the abuse of notations $U(-i\xi)\widehat{f}(x,\xi) = U(-i\xi)\widehat{f}_\xi(x)$ with $\widehat{f}_\xi = \widehat{f}(.,\xi)$.

Proof. For $s \in \mathbb{R}_+^*$, we note P_s the operator defined by

$$P_s f(x,t) = \int_{\mathbf{G}} e^{-s\sigma(g,x)} f(gx, t+\sigma(g,x)) \mathrm{d}\rho(g)$$

We are first going to prove that for any non negative function f,

$$\sum_{n=0}^{+\infty} P^n f(x,t) = \lim_{s\to 0^+} \sum_{n=0}^{+\infty} P_s^n f(x,t)$$

and then that

$$\sum_{n=0}^{+\infty} P_s^n f(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} (I_d - P(s-i\xi))^{-1} \widehat{f}(x,\xi) \mathrm{d}\xi$$

To prove the first equality, note that

$$\int_{\mathbf{G}} e^{-s\sigma(g,x)} f(g.(x,t)) \mathrm{d}\rho^{*n}(g) = \int_{\mathbf{G}} e^{-s\sigma(g,x)} (\mathbf{1}_{\sigma(g,x)\leqslant 0} + \mathbf{1}_{\sigma(g,x)>0}) f(g.(x,t)) \mathrm{d}\rho^{*n}(g)$$

And the monotone convergence theorem proves that

$$\lim_{s\to 0^+} \sum_{n=0}^{+\infty} \int_{\mathbf{G}} e^{-s\sigma(g,x)} \mathbf{1}_{\sigma(g,x)>0} f(g.(x,t)) \mathrm{d}\rho^{*n}(g) = \sum_{n=0}^{+\infty} \int_{\mathbf{G}} \mathbf{1}_{\sigma(g,x)>0} f(g.(x,t)) \mathrm{d}\rho^{*n}(g)$$

Moreover, lemma 4.4 and Bienaymé-Tchebytchev's inequality prove that for any $x \in \mathbf{X}$,

$$\rho^{*n}(g \in \mathbf{G} | \sigma(g,x) \leqslant 0) \leqslant \int_{\mathbf{G}} e^{-s\sigma(g,x)} \mathrm{d}\rho^{*n}(g) \leqslant Ce^{-tn}$$

And so, the dominated convergence theorem proves that

$$\lim_{s\to 0^+} \sum_{n=0}^{+\infty} \int_{\mathbf{G}} e^{-s\sigma(g,x)} \mathbf{1}_{\sigma(g,x)\leqslant 0} f(g.(x,t)) \mathrm{d}\rho^{*n}(g) = \sum_{n=0}^{+\infty} \int_{\mathbf{G}} \mathbf{1}_{\sigma(g,x)\leqslant 0} f(g.(x,t)) \mathrm{d}\rho^{*n}(g)$$

and this finishes the proof of the first inequality.

Moreover, proposition 4.2 proves that we can use Fubini's theorem to make the following computation

$$\begin{aligned}
\sum_{n=0}^{+\infty} P_s^n f(t,x) &= \sum_{n=0}^{+\infty} \int_{\mathbf{G}} e^{-s\sigma(g,x)} f(gx, t+\sigma(g,x)) \mathrm{d}\rho^{*n}(g) \\
&= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \int_{\mathbf{G}} e^{-s\sigma(g,x)} \int_{\mathbb{R}} e^{i\xi(t+\sigma(g,x)} \widehat{f}(gx,\xi) \mathrm{d}\xi \mathrm{d}\rho^{*n}(g) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} \sum_{n=0}^{+\infty} \int_{\mathbf{G}} e^{-(s-i\xi)\sigma(g,x)} \widehat{f}(gx,\xi) \mathrm{d}\rho^{*n}(g) \mathrm{d}\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} \sum_{n=0}^{+\infty} P(s-i\xi)^n \widehat{f}(x,\xi) \mathrm{d}\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} (I_d - P(s-i\xi))^{-1} \widehat{f}(x,\xi) \mathrm{d}\xi
\end{aligned}$$

And this proves the second equality.

Finally, we noted $U(z)$ the family of operators defined by

$$(I_d - P(z))^{-1} = \frac{1}{\sigma_\rho z} N_0 + U(z)$$

and we saw in proposition 4.2 that $(U(z))$ is an analytic family of continuous operators on $\mathcal{C}^{0,\gamma}(\mathbf{X})$.

And so,

$$\int_{\mathbb{R}} e^{i\xi t} (I_d - P(s-i\xi))^{-1} \widehat{f}(x,\xi) \mathrm{d}\xi = \int_{\mathbb{R}} \frac{e^{i\xi t}}{s-i\xi} N_0 \widehat{f}(x,\xi) \frac{\mathrm{d}\xi}{\sigma_\rho} + \int_{\mathbb{R}} e^{i\xi t} U(s-i\xi) \widehat{f}(x,\xi) \mathrm{d}\xi$$

But, for any continuous function f on $\mathbf{X}$, according to lemma 2.13, we can write

$$N_0 f(x) = \sum_{i=1}^{r} p_i(x) \int f \mathrm{d}\nu_i$$

where the p_i are $P-$invariant functions on $\mathbf{X}$ and ν_i are $P-$stationary measures on $\mathbf{X}$. So,

$$\begin{aligned}
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi t}}{s-i\xi} N_0 \widehat{f}(x,\xi) \mathrm{d}\xi &= \frac{1}{2\pi} \sum_{i=1}^{r} p_i(x) \int_{\mathbf{X}} \int_{\mathbb{R}} \frac{e^{i\xi t}}{s-i\xi} \widehat{f}(y,\xi) \mathrm{d}\xi \mathrm{d}\nu_i(y) \\
&= \sum_{i=1}^{r} p_i(x) \int_{\mathbf{X}} \int_0^{+\infty} f(y, t+u) e^{-su} \mathrm{d}u \mathrm{d}\nu_i(y) \\
&= \int_0^{+\infty} N_0 f(x, t+u) e^{-su} \mathrm{d}u
\end{aligned}$$

So, using the definition of Π_0, we get that

$$\lim_{s \to 0^+} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi t}}{s-i\xi} N_0 \widehat{f}(x,\xi) \mathrm{d}\xi = \int_t^{+\infty} N_0 f(x,u) \mathrm{d}u = \Pi_0 f(x,t)$$

This proves that

$$Gf(x,t) = \sum_{n=0}^{+\infty} P^n f(x,t) = \frac{1}{\sigma_\rho}\Pi_0 f(x,t) + \lim_{s\to 0^+} \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t} U(s-i\xi)\widehat{f}(x,\xi)\mathrm{d}\xi$$

But, for fixed ξ, we have, according to proposition 4.2, that

$$\|U(s-i\xi)\widehat{f}(x,\xi)\|_\infty \leqslant \|U(s-i\xi)\|_\gamma \|\widehat{f}(x,\xi)\|_\gamma \leqslant C(1+|\xi|)^{L+1}\|\widehat{f}(x,\xi)\|_\gamma$$

and, as $f \in \mathcal{E}^{\gamma,K}(\mathbf{X}\times\mathbb{R})$, we can conclude with the dominated convergence theorem and lemma 4.9 if we take $K = L+3$. □

Corollary 4.12. *Under the assumptions of proposition 4.2, for any $\gamma > 0$ small enough, there are constants $C, K \in \mathbb{N}$ such that for any $f \in \mathcal{E}^{\gamma,K}$, any $x, x' \in \mathbf{X}$ and any $t, t' \in \mathbb{R}$ we have that*

$$\left|\left(G - \frac{1}{\sigma_\rho}\Pi_0\right) f(x,t) - \left(G - \frac{1}{\sigma_\rho}\Pi_0\right) f(x',t')\right| \leqslant C\|f\|_{\gamma,K}\omega_0((x,t),(x',t'))^\gamma$$

where,

$$\omega_0((x,t),(x',t')) = \frac{\sqrt{|t-t'|^2 + d(x,x')^2}}{(1+|t'|)(1+|t|)}$$

Remark 4.13. If we make t' go to $+\infty$ and if we use Riemann-Lebesgue's lemma and proposition 4.11 to see that under the assumptions of the corollary,

$$\lim_{t\to\pm\infty}\left(G - \frac{1}{\sigma_\rho}\Pi_0\right) f(x,t) = 0$$

then we get that for any $f \in \mathcal{E}_0^{\gamma,K}(\mathbf{X}\times\mathbb{R})$, any $x \in \mathbf{X}$ and any $t \in \mathbb{R}$,

$$\left|Gf(x,t) - \frac{1}{\sigma_\rho}\Pi_0 f(x,t)\right| \leqslant \frac{C}{(1+|t|)^\gamma}\|f\|_{\gamma,K}$$

Proof. According to proposition 4.11, we have that

$$\left(G - \frac{1}{\sigma_\rho}\Pi_0\right) f(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t} U(-i\xi)\widehat{f}(x,\xi)\mathrm{d}\xi$$

So, integrating by parts and noting $\psi(x,\xi) = U(-i\xi)\widehat{f}(x,\xi)$, we find that for any $t \in \mathbb{R}^*$,

$$(G - \frac{1}{\sigma_\rho}\Pi_0) f(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i\xi t}}{-t^2}\psi''(x,\xi)\mathrm{d}\xi$$

So, for any $x, x' \in \mathbf{X}$ such that $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x')$ and any $t, t' \in \mathbb{R}^*$,

$$\begin{aligned} I(x,t,x',t') &:= (G - \frac{1}{\sigma_\rho}\Pi_0) f(x,t) - (G - \frac{1}{\sigma_\rho}\Pi_0) f(x',t') \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i\xi t}}{-t^2}\psi''(x,\xi)\mathrm{d}\xi - \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i\xi t'}}{-t'^2}\psi''(x',\xi)\mathrm{d}\xi \\ &= \int_{\mathbb{R}} \left(\frac{e^{i\xi t}}{-t^2} - \frac{e^{i\xi t'}}{-t'^2}\right)\psi''(x,\xi) - \int_{\mathbb{R}} \frac{e^{i\xi t'}}{t'^2}\left(\psi''(x,\xi) - \psi''(x',\xi)\right)\frac{\mathrm{d}\xi}{2\pi} \end{aligned}$$

Thus,

$$\left|I(x,t,x',t')\right| \leqslant \int_{\mathbb{R}} \left|\frac{e^{i\xi t}}{t^2} - \frac{e^{i\xi t'}}{t'^2}\right| |\psi''(x,\xi)| \mathrm{d}\xi + \frac{1}{|t'|^2} \int_{\mathbb{R}} \left|\psi''(x,\xi) - \psi''(x',\xi)\right| \mathrm{d}\xi$$

But, assuming that $|t'| \geqslant |t| \geqslant 1$, we have that

$$\left|\frac{e^{i\xi t}}{t^2} - \frac{e^{i\xi t'}}{t'^2}\right| \leqslant \frac{|t^2 - t'^2|}{t^2 t'^2} + |\xi| \frac{|t - t'|}{t'^2} \leqslant \frac{|t - t'|}{|t||t'|} (2 + |\xi|)$$

And, as we also have, for any $t \in \mathbb{R}$ with $|t| \geqslant 1$ that

$$\frac{1}{|t|} \leqslant \frac{2}{1 + |t|},$$

what we get is that

$$\int_{\mathbb{R}} \left|\frac{e^{i\xi t}}{t^2} - \frac{e^{i\xi t'}}{t'^2}\right| |\psi''(x,\xi)| \mathrm{d}\xi \leqslant \frac{4|t - t'|}{(1 + |t|)(1 + |t'|)} \int_{\mathbb{R}} (2 + |\xi|) |\psi''(x,\xi)| \mathrm{d}\xi$$

Moreover,

$$\frac{1}{|t'|^2} \int_{\mathbb{R}} \left|\psi''(x,\xi) - \psi''(x',\xi)\right| \mathrm{d}\xi \leqslant \frac{4d(x,y)^\gamma}{(1 + |t|)(1 + |t'|)} \int_{\mathbb{R}} m_\gamma(\psi''(.,\xi)) \mathrm{d}\xi$$

So, we only need to study the integrability of $\|\psi''(\,.\,,\xi)\|_\gamma$. But

$$\psi''(x,\xi) = U''(-i\xi)\widehat{f}(x,\xi) - 2iU'(-i\xi)\widehat{f}'(x,\xi) + U(-i\xi)\widehat{f}''(x,\xi)$$

and, according to proposition 4.2 there is a constant C such that for any $m \in \{0,1,2\}$,

$$\|U^{(m)}(-i\xi)\|_\gamma \leqslant C^{m+1} m! (1 + |\xi|)^{(L+1)(m+1)}$$

Moreover, according to lemma 4.9, for any $k \in \mathbb{N}$, there is a constant C such that for any $l \in \{0,1,2\}$,

$$\left\| \frac{\partial^l \widehat{f}}{\partial \xi^l}(x,\xi) \right\|_\gamma \leqslant C \frac{\|f\|_{\gamma,k}}{1 + |\xi|^k}$$

This proves that there is a constant C such that for any function $f \in \mathcal{E}_0^{\gamma,K}(\mathbf{X} \times \mathbb{R})$, any $x, x' \in \mathbf{X}$ with $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x')$ and any $t, t' \in \mathbb{R}^*$ with $|t|, |t'| \geqslant 1$,

$$\left|I(x,t,x',t')\right| \leqslant C\|f\|_{\gamma,K} \frac{|t - t'| + d(x,x')^\gamma}{(1 + |t|)(1 + |t'|)} \tag{4.7}$$

We can now use the fact that for some constants C_γ, C, we have that for any $t, t' \in \mathbb{R}$ and any $x, x' \in \mathbf{X}$,

$$\frac{|t - t'| + d(x,x')^\gamma}{(1 + |t|)(1 + |t'|)} \leqslant C_\gamma \left(\frac{|t - t'| + d(x,x')}{(1 + |t|)(1 + |t'|)}\right)^\gamma \leqslant C C_\gamma \left(\frac{\sqrt{|t - t'|^2 + d(x,x')^2}}{(1 + |t|)(1 + |t'|)}\right)^\gamma$$

To prove equation 4.7 for $|t| \leqslant 1$, we are going to use that the objects we study behave well with the translations on $\mathbb{R}$.

Indeed, if $f \in \mathcal{E}^{\gamma,k}(\mathbf{X}\times\mathbb{R})$ and if we note, for $s\in\mathbb{R}$, $f_s(x,t)=f(x,t-s)$ then, for any $x\in\mathbf{X}$ and $t\in\mathbb{R}$,

$$\left(G-\frac{1}{\sigma_\rho}\Pi_0\right)f(x,t)=\left(G-\frac{1}{\sigma_\rho}\Pi_0\right)f_s(x,t+s)$$

and

$$\|f_s\|_{\gamma,k}\leqslant e^{\gamma|s|}\|f\|_{\gamma,k}$$

and so, if $|t|\leqslant 1$, we can take $s\in[-10,10]$ such that $1\leqslant|t+s|\leqslant|t'+s|$ and we get that for some new constant C,

$$|I(x,t,x',t')|\leqslant 4Ce^{10\gamma}\|f\|_{\gamma,k}\left(\frac{\sqrt{|t-t'|^2+d(x,x')^2}}{(1+|t|)(1+|t'|)}\right)^\gamma$$

And this is finally what we intended to prove. □

4.4. **Renewal theorem for hölder-continuous functions.** Until now, we proved the renewal theorem only for regular functions. Yet, we are interested in functions on $\mathbb{R}^d$ that will only be hölder-continuous.

This is why we are going to regularize by convolving them by regular ones and then, use tauberian theorems to get the expected result.

This method is already used in [BDP15] to study the renewal theorem in $\mathbb{R}^d$ for a borelian probability measure ρ on $\mathrm{SL}_d(\mathbb{R})$ whose support generates a subsemigroup that is (conjugated to) a subgroup of $\mathbb{R}_+^\star\times\mathcal{O}(d)$. Like them, we will use a result about remainder terms in the renewal theorem proved by Frennemo in [Fre65] (see also appendix A).

So, we are going to define a new class of continuous functions on $\mathbf{X}\times\mathbb{R}$ that we will use in the sequel.

Example 4.14. On $\mathbb{R}^d$, we are interested in functions having as modulus of continuity (a power of)

$$\omega(x,y)=\frac{\|x-y\|}{(1+\|x\|)(1+\|y\|)}$$

Using the application $\Phi:\mathbb{S}^d\times\mathbb{R}\to\mathbb{R}^d\setminus\{0\}$ that maps (x,t) onto e^tx and that identifies $\mathbb{R}^d\setminus\{0\}$ to $\mathbb{S}^d\times\mathbb{R}$, our function ω writes

$$\omega((x,t),(x',t'))=\frac{\|e^tx-e^{t'}x'\|}{(1+e^t)(1+e^{t'})}$$

But,

$$\begin{aligned}\|e^tx-e^{t'}x'\|^2&=e^{2t}+e^{2t'}-2e^{t+t'}<x,x'>=\left(e^t-e^{t'}\right)^2+e^{t+t'}\|x-x'\|^2\\&=e^{t+t'}\left(\left(e^{(t-t')/2}-e^{(t'-t)/2}\right)^2+\|x-x'\|^2\right)\end{aligned}$$

Therefore,

$$\omega((x,t),(x',t'))=\frac{e^{t/2}}{1+e^t}\frac{e^{t'/2}}{1+e^{t'}}\sqrt{d(x,x')^2+\left(e^{(t-t')/2}-e^{(t'-t)/2}\right)^2}$$

Moreover, as

$$\frac{e^{t/2}}{1+e^t} \asymp e^{-|t|/2},$$

This leads us to define, if $(\mathbf{X}, d)$ is a compact metric space and $(x,t),(x',t') \in \mathbf{X}\times\mathbb{R}$,

$$\omega((x,t),(x',t')) = e^{-|t|/2-|t'|/2}\sqrt{d(x,x')^2 + (e^{(t-t')/2} - e^{(t'-t)/2})^2}$$

Then, we note

$$\mathcal{C}^\gamma_\omega(\mathbf{X}\times\mathbb{R}) := \left\{ f \in \mathcal{C}^0(\mathbf{X}\times\mathbb{R}) \middle| \sup_{\substack{(x,t),(x',t')\in\mathbf{X}\times\mathbb{R} \\ (x,t)\neq(x',t')}} \frac{|f(x,t)-f(x',t')|}{\omega((x,t),(x',t'))^\gamma} \text{ is finite} \right\}$$

The following lemma proves that functions of $\mathcal{C}^\gamma_\omega$ can be extended to functions on $\mathbf{X}\times\overline{\mathbb{R}}$.

Lemma 4.15. *Let $f \in \mathcal{C}^\gamma_\omega(\mathbf{X}\times\mathbb{R})$. Then, there are $p^+, p^- \in \mathrm{L}^\infty(\mathbf{A})$ such that for any $x \in \mathbf{X}$ and any $t \in \mathbb{R}$,*

$$|f(x,t) - p^+(f)(\pi_{\mathbf{A}}\circ\pi_{\mathbf{H}}(x))| \leqslant e^{\gamma t} \text{ et } |f(x,t) - p^-(f)(\pi_{\mathbf{A}}\circ\pi_{\mathbf{H}}(x))| \leqslant e^{-\gamma t}$$

Proof. Note at first that for any $x \in \mathbf{X}$, $(t \mapsto f(x,t))$ has a limit at $-\infty$ since $f \in \mathcal{C}^\gamma_\omega(\mathbf{X}\times\mathbb{R})$.

So, we can set $p^-(f)(x) = \lim_{t\to-\infty} f(x,t)$ and then, by definition of $\mathcal{C}^\gamma_\omega$, we have that for any $(x,t),(x',t') \in \mathbf{X}\times\mathbb{R}$,

$$\begin{aligned} |f(x,t) - p^-(f)(x')| &\leqslant |f(x,t) - f(x',t')| + |f(x',t') - p^-(f)(x')| \\ &\leqslant \|f\|_{\gamma,\omega}\omega((x,t),(x',t'))^\gamma + |f(x',t') - p^-(f)(x')| \end{aligned}$$

Thus, letting t' go to $-\infty$, we find that for any $t \in \mathbb{R}$ and any $x, x' \in \mathbf{X}$ with $\pi_{\mathbf{A}}\circ\pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}}\circ\pi_{\mathbf{H}}(x')$,

$$|f(x,t) - p^-(f)(x')| \leqslant e^{\gamma t}$$

This proves that $p^-(f)(x) = p^-(f)(x')$ and that

$$|f(x) - p^-(f)(x)| \leqslant e^{\gamma t}$$

The proof of the existence of p^+ and of the rate of convergence of f to p^+ is done in the same way. □

In the sequel, we will need a function ψ on $\mathbb{R}$ that is regular and such that

$$\lim_{t\to-\infty} \psi(t) = 1 \text{ and } \lim_{t\to+\infty} \psi(t) = 0$$

The space of function in $\mathcal{C}^\gamma_\omega(\mathbf{X}\times\mathbb{R})$ such that $p^-(f) = 0 = p^+(f)$ being of finite codimension, the function ψ will allow us to control the projection on it.

From now on, we note, for any $t \in \mathbb{R}$,

$$\psi(t) = \frac{1}{\sqrt{2\pi}} \int_t^{+\infty} e^{-u^2/2} \mathrm{d}u \tag{4.8}$$

The choice of this particular function ψ is arbitrary but will simplify the computations to come.

In the following lemma, we prove that the projection of a function of $\mathcal{C}^{\gamma}_{\omega}(\mathbf{X}\times\mathbb{R})$ on functions such that $p^+(f)=0=p^-(f)$ belong to the space $\mathcal{E}^{\gamma,0}(\mathbf{X}\times\mathbb{R})$ that we defined in equation 4.6.

Lemma 4.16. *For any $\gamma\in]0,1]$ there is $C\in\mathbb{R}$ depending only on γ such that for any function $f\in\mathcal{C}^{0,\gamma}_{\omega}(\mathbf{X}\times\mathbb{R})$, $\varphi\in\mathcal{E}^{\gamma,0}(\mathbf{X}\times\mathbb{R})$ and*

$$\|\varphi\|_{\gamma,0}\leqslant C\|f\|_{\gamma,\omega}$$

where we noted, for any $(x,t)\in\mathbf{X}\times\mathbb{R}$,

$$\varphi(x,t)=f(x,t)-p^-(f)(x)\psi(t)-p^+(f)(x)(1-\psi(t))$$

and ψ is the function defined in equation 4.8.

Proof. It is clear that φ is continuous on $\mathbf{X}\times\mathbb{R}$.

Moreover, using that

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-u^2/2}\mathrm{d}u=1,$$

we find that for any $t\in\mathbb{R}$ and any $x\in\mathbf{X}$,

$$\begin{aligned}|\varphi(x,t)|&\leqslant|f(x,t)-p^-(f)(x)|\psi(t)+|f(x,t)-p^+(f)(x)|(1-\psi(t))\\&\leqslant\|f\|_{\gamma,\omega}\left(e^{\gamma t}\psi(t)+e^{-\gamma t}(1-\psi(t))\right)\end{aligned}$$

And so,

$$e^{\gamma|t|}|\varphi(x,t)|\leqslant\|f\|_{\gamma,\omega}\left(e^{\gamma(|t|+t)}\psi(t)+e^{\gamma(|t|-t)}(1-\psi(t))\right)$$

Thus, there is a constant $C\in\mathbb{R}$ depending only on γ such that

$$\sup_t e^{\gamma|t|}|\varphi(x,t)|\leqslant C\|f\|_{\gamma,\omega}$$

In the same way, for $x,x'\in\mathbf{X}$ such that $\pi_{\mathbf{A}}\circ\pi_{\mathbf{H}}(x)=\pi_{\mathbf{A}}\circ\pi_{\mathbf{H}}(x')$, we have that

$$|\varphi(x,t)-\varphi(x',t)|=|f(x,t)-f(x',t)|\leqslant e^{-\gamma|t|}d(x,x')^{\gamma}\|f\|_{\gamma,\omega}$$

And this finishes the proof of the lemma. □

To apply Frennemo's result about remainder terms in the renewal theorem, we will need to know the modulus of uniform continuity of the function $\sum_n P^n(x,.)$. We are going to study it in next lemma and this is where we do use that f belong to $\mathcal{C}^{0,\gamma}_{\omega}(\mathbf{X}\times\mathbb{R})$ and not only to $\mathcal{E}^{\gamma,0}(\mathbf{X}\times\mathbb{R})$. To get the renewal theorem without speed, we could have only consider functions of $\mathcal{E}^{\gamma,0}(\mathbf{X}\times\mathbb{R})$.

Lemma 4.17. *Under the assumptions of theorem 4.1, for any function $f\in\mathcal{C}^{0,\gamma}_{\omega}(\mathbf{X}\times\mathbb{R})$, any $x\in\mathbf{X}$ and any $t,t'\in\mathbb{R}$,*

$$\left|\sum_{n=0}^{+\infty}P^nf(x,t)-\sum_{n=0}^{+\infty}P^nf(x,t')\right|\leqslant C\|f\|_{\omega,\gamma}\left(e^{|t'-t|}-1\right)^{\gamma}$$

Proof. Let $f \in \mathcal{C}^{0,\gamma}_{\omega}(\mathbf{X}\times\mathbb{R})$, $x \in \mathbf{X}$ and $t, t' \in \mathbb{R}$.

Then,

$$\begin{aligned} I(t,t',x) :&= \left|\sum_{n=0}^{+\infty} P^n f(t,x) - \sum_{n=0}^{+\infty} P^n f(t',x)\right| \\ &\leqslant \sum_{n=0}^{+\infty} \int_{\mathbf{G}} |f(t+\sigma(g,x),gx) - f(t'+\sigma(g,x),gx)| \mathrm{d}\rho^{*n}(g) \\ &\leqslant \|f\|_{\omega,\gamma} \sum_{n=0}^{+\infty} \int_{\mathbf{G}} \omega(gx, t+\sigma(g,x), gx, t'+\sigma(g,x))^{\gamma} \mathrm{d}\rho^{*n}(g) \\ &\leqslant \|f\|_{\omega,\gamma} \left|e^{(t-t')/2} - e^{(t'-t)/2}\right|^{\gamma} \sum_{n=0}^{+\infty} \int_{\mathbf{G}} e^{-\gamma|t+\sigma(g,x)|/2 - \gamma|t'+\sigma(g,x)|/2} \mathrm{d}\rho^{*n}(g) \end{aligned}$$

Moreover,

$$|e^{(t-t')/2} - e^{(t'-t)/2}| = |e^{|t-t'|/2} - e^{-|t-t'|/2}| = e^{-|t-t'|/2}\left|e^{|t-t'|} - 1\right| \leqslant e^{|t-t'|} - 1$$

So,

$$\begin{aligned} I(t,t',x) &\leqslant \|f\|_{\omega,\gamma} \left|e^{|t-t'|} - 1\right|^{\gamma} \sum_{n=0}^{+\infty} \int_{\mathbf{G}} e^{-\gamma|t+\sigma(g,x)|/2} \mathrm{d}\rho^{*n}(g) \\ &\leqslant \|f\|_{\omega,\gamma} \left|e^{|t-t'|} - 1\right|^{\gamma} \sum_{n=0}^{+\infty} P^n \varphi(x,t) \end{aligned}$$

with $\varphi(t) = e^{-\gamma|t|/2}$.

To prove the lemma, we only need to prove that the series $\sum_{n=0}^{+\infty} P^n\varphi$ is bounded on $\mathbf{X}\times\mathbb{R}$. We would like to apply the renewal theorem (that proves that the considered sum has finite limits at $\pm\infty$ and so is bounded) to the function φ but we cannot do this yet since φ is not regular enough. Therefore, we set

$$\varphi_1(t) = \frac{2}{\sqrt{2\pi}} e^{\gamma t} \int_t^{+\infty} e^{-u^2/2} \mathrm{d}u \text{ et } \varphi_2(t) = \frac{2}{\sqrt{2\pi}} e^{-\gamma t} \int_{-\infty}^{t} e^{-u^2/2} \mathrm{d}u$$

to have that

$$\varphi_1(t) \geqslant \mathbf{1}_{\mathbb{R}_-} e^{\gamma t} \text{ et } \varphi_2(t) \geqslant e^{-\gamma t} \mathbf{1}_{\mathbb{R}_+}(t)$$

Then, we have that

$$\sum_{n=0}^{+\infty} \int_{\mathbf{G}} e^{-\gamma|t+\sigma(g,x)|} \mathrm{d}\rho^{*n}(g) \leqslant \sum_{n=0}^{+\infty} P^n\varphi_1(x,t) + \sum_{n=0}^{+\infty} P^n\varphi_2(x,t)$$

And this proves, using the renewal theorem for regular functions that we already proved (see corollary 4.12) and Riemann-Lebesgue's lemma that

$$\lim_{t\to+\infty} \sum_{n=0}^{+\infty} P^n\varphi_1(x,t) = 0 \text{ and } \lim_{t\to-\infty} \sum_{n=0}^{+\infty} P^n\varphi_1(x,t) = N_0 1 \int_{\mathbb{R}} \varphi_1(t)\mathrm{d}t$$

So, $\sum_{n=0}^{+\infty} P^n\varphi_1$ is bounded on $\mathbf{X}\times\mathbb{R}$. Doing the same for $\sum_{n=0}^{+\infty} P^n\varphi_2$ we finish the proof of the lemma. □

For $x, x' \in \mathbf{X}$ such that $\pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x) = \pi_{\mathbf{A}} \circ \pi_{\mathbf{H}}(x')$ and $t, t' \in \mathbb{R}$, we note

$$\omega_0((x,t),(x',t')) = \frac{\sqrt{|t-t'|^2 + d(x,x')^2}}{(1+|t|)(1+|t'|)}$$

We are finally able to prove the renewal theorem

Corollary 4.18. *Under the assumption of theorem 4.1, there are constants $C, K \in \mathbb{R}$ such that for any function $f \in \mathcal{C}^{0,\gamma}_{\omega}(\mathbf{X}\times\mathbb{R})$ such that $p^+(f) = 0 = p^-(f)$, for any $x, x' \in \mathbf{X}$ and any $t, t' \in \mathbb{R}$,*

$$\left|\left(G - \frac{1}{\sigma_\rho}\Pi_0\right) f(x,t) - \left(G - \frac{1}{\sigma_\rho}\Pi_0\right) f(x',t')\right| \leqslant C\|f\|_{\gamma,\omega}\omega_0((x,t),(x',t'))^{\gamma/(\gamma+K)}$$

Moreover, for any $x \in \mathbf{X}$,

$$\lim_{t\to\pm\infty}\left(G - \frac{1}{\sigma_\rho}\Pi_0\right) f(x,t) = 0$$

Proof. Let, for $k \in \mathbb{N}$, and $t \in \mathbb{R}$, $\varphi_k(t) = t^k e^{-t}\mathbf{1}_{\mathbb{R}_+}(t)$.

Then, we already saw in lemma 4.10 that there is a constant C_k such that for any function $f \in \mathcal{E}^{\gamma,0}(\mathbf{X}\times\mathbb{R})$, $f * \varphi_{k+1} \in \mathcal{E}^{\gamma,k}(\mathbf{X}\times\mathbb{R})$ and

$$\|f * \varphi_{k+1}\|_{\gamma,k} \leqslant C_k\|f\|_{\gamma,0}$$

In particular, for any $f \in \mathcal{C}^{\gamma}_{\omega}$ with $p^+(f) = p^-(f) = 0$, lemma 4.16 proves that

$$\|f * \varphi_{k+1}\|_{\gamma,k} \leqslant C_k\|f\|_{\gamma,\omega}$$

Moreover,

$$\varphi_{k+1} * (G - \frac{1}{\sigma_\rho}\Pi_0)f(x,t) = (G - \frac{1}{\sigma_\rho}\Pi_0)(f * \varphi_{k+1})(x,t)$$

So, for $k = K$, corollary 4.12 proves that

$$\begin{aligned} I(k,x,t,x',t',f) :&= \left|\varphi_{k+1} * (G - \frac{1}{\sigma_\rho}\Pi_0)f(x,t) - \varphi_{k+1} * (G - \frac{1}{\sigma_\rho}\Pi_0)f(x',t')\right| \\ &\leqslant CC_k\omega_0((x,t),(x',t'))^{\gamma}\|f\|_{\gamma,\omega} \end{aligned}$$

Moreover, as we also have that

$$|(G - \frac{1}{\sigma_\rho}\Pi_0)f(x,t) - (G - \frac{1}{\sigma_\rho}\Pi_0)f(x,t')| \leqslant C\|f\|_{\gamma,\omega}\left(e^{|t-t'|} - 1\right)^{\gamma},$$

we can conclude with the corollary of Frennemo's result stated in appendix A. □

We are now able to study functions of $\mathcal{C}^{\gamma}_{\omega}(\mathbf{X}\times\mathbb{R})$ that vanish at $\pm\infty$. This set will contain for instance functions on $\mathbf{A}\times\mathbb{R}^d$ that are compactly supported and that take the value 0 at $(a,0)$ for any $a \in \mathbf{A}$. Yet, we would like to study functions that doesn't vanish at $(a,0)$ but just satisfy that $\sum_{a\in\mathbf{A}} f(a,0) = 0$. This is why we prove the following

Lemma 4.19. *Under the assumptions of theorem 4.1, there is $C \in \mathbb{R}$ such that for any $p \in \mathrm{L}^\infty(\mathbf{A})$ with $\sum_{a\in\mathbf{A}} p(a) = 0$, any $x \in \mathbf{X}$ and any $t \in \mathbb{R}$,*

$$\left|(G - \frac{1}{\sigma_\rho}\Pi_0)f(x,t) - (G - \frac{1}{\sigma_\rho}\Pi_0)f(x',t')\right| \leqslant C\|p\|_\infty \omega_0((x,t),(x',t'))$$

where we noted

$$f(x,t) = p(\pi_{\mathbf{A}}(x))\psi(t)$$

and ψ the function defined in equation 4.8.

Moreover, for any $x \in \mathbf{X}$,

$$\lim_{t\to-\infty}\left(G - \frac{1}{\sigma_\rho}\Pi_0\right) f(x,t) = \sum_{n=0}^{+\infty} P^n p(\pi_{\mathbf{A}}(x))$$

Proof. Since P commutes to the derivation, we have that

$$\sum_{n=0}^{+\infty} P^n f(x,t) = -\int_t^{+\infty} \sum_{n=0}^{+\infty} P^n f'(x,u)\mathrm{d}u$$

But,

$$\sum_{n=0}^{+\infty} P^n f'(x,u) = \int_u^{+\infty} N_0 f'(x,s)\mathrm{d}s + \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi u}U(-i\xi)p(x)\widehat{f'}(\xi)\mathrm{d}\xi$$

And,

$$\widehat{f'}(\xi) = -e^{-\xi^2/2}, \text{ so } \int_u^{+\infty} N_0 f'(x,s)\mathrm{d}s = -N_0 f(x,u)$$

Therefore,

$$\sum_{n=0}^{+\infty} P^n f'(x,u) = -N_0 f(x,u) + \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi u}U(-i\xi)p(x)e^{-\xi^2/2}\mathrm{d}\xi$$

So,

$$\sum_{n=0}^{+\infty} P^n f(x,t) = \int_t^{+\infty} N_0 f(x,u)\mathrm{d}u - \frac{1}{2\pi}\int_t^{+\infty}\int_{\mathbb{R}} e^{i\xi u}U(-i\xi)p(x)e^{-\xi^2/2}\mathrm{d}\xi\mathrm{d}u$$

Moreover, noting $U(z) = N_1 + zV(z)$, we have that the second term of the previous equality is the sum of

$$\frac{1}{2\pi}N_1 p(x)\int_t^{+\infty}\int_{\mathbb{R}} e^{i\xi u}e^{-\xi^2/2}\mathrm{d}\xi\mathrm{d}u = -N_1 p(x)\varphi(t)$$

and of

$$\frac{1}{2\pi}\int_t^{+\infty}\int_{\mathbb{R}} e^{i\xi u}(-i\xi)V(-i\xi)p(x)e^{-\xi^2/2}\mathrm{d}\xi\mathrm{d}u = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}V(-i\xi)p(x)e^{-\xi^2/2}\mathrm{d}\xi$$

(to get this equality, we differentiate on both sides) So, finally, we get that

$$\sum_{n=0}^{+\infty} P^n f(x,t) = \int_t^{+\infty} N_0 f(x,u)\mathrm{d}u + N_1 f(x,t) - \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}V(-i\xi)p(x)e^{-\xi^2/2}\mathrm{d}\xi$$

And so,

$$\lim_{t\to-\infty}(G-\frac{1}{\sigma_\rho}\Pi_0)f(x,t)=\lim_{n\to-\infty}N_1f(x,t)=N_1f(x,-\infty)$$

To conclude, we only need to see that

$$N_1f(x,-\infty)=\sum_{n=0}^{+\infty}P^nf(\pi_{\mathbf{A}}\circ\pi_{\mathbf{H}}(x))$$

To prove this, we use that these to functions are solutions of $g-Pg=f$ and, as the random walk on $\mathbf{A}$ is irreducible and aperiodic, there is a constant $C\in\mathbb{R}$ such that for any $a\in\mathbf{A}$,

$$N_1p(a)=\sum_{n=0}^{+\infty}P^np(a)+C$$

But,

$$\sum_{a\in\mathbf{A}}N_1p(a)=\sum_{a\in\mathbf{A}}\sum_{n=0}^{+\infty}P^np(a)+|\mathbf{A}|C=|\mathbf{A}|C$$

And so, $C=0$ since $\sum_{a\in\mathbf{A}}N_1p(a)=0=\sum_{a\in\mathbf{A}}\sum_{n=0}^{+\infty}P^np(a)$.

Finally, using proposition 4.2, we have that

$$\left|\int_{\mathbb{R}}e^{i\xi t}V(-i\xi)p(x)e^{-\xi^2/2}\mathrm{d}\xi\right|\leqslant\|p\|_\infty\int_{\mathbb{R}}C(1+|\xi|)^Le^{-\xi^2/2}\mathrm{d}\xi$$

and that

$$|t|^2\left|\int_{\mathbb{R}}e^{i\xi t}V(-i\xi)p(x)e^{-\xi^2/2}\mathrm{d}\xi\right|\leqslant\|p\|_\infty C\int_{\mathbb{R}}(1+|\xi|)^{3L}e^{-\xi^2/2}\mathrm{d}\xi$$

This finishes the proof of the lemma (since $p(x)$ only depend of the projection of the point x on $\mathbf{A}$). □

Appendix A. Remainder terms in the tauberian theorem

In this section, we prove a theorem that controls the remainder term in the tauberian theorem.

The aim of this section is to study the following problem : given two functions f,φ on $\mathbb{R}$ for which we know the rate of convergence to 0 at infinity of $|f\star\varphi|$, can we get the rate of convergence of $|f|$ to 0 at infinity ?

The first related result is a corollary of a tauberion theorem of Wiener that states that if the Fourier transform of φ doesn't vanish on $i\mathbb{R}$ and if $f\star\varphi$ converges to 0 at $\pm\infty$, then so does f.

The interest for us of this kind of results is that in the study of the rate of convergence in the renewal theorem, we will always be able to regularise the functions we study (which is necessary for our method using the Fourier transform) but we will have a rate of convergence anyway.

Definition A.1. Let f be a uniformly continuous function on $\mathbb{R}$.

We say that a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, continuous at 0 and with $\omega(0) = 0$ is a modulus of uniform continuity for f if for any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leqslant \omega(|x - y|)$$

The following theorem is an adaptation of a result of Frennemo in [Fre65].

Theorem A.2. *Let $k \in \mathbb{N}^\star$.*

Let φ_k be the function on $\mathbb{R}$ defined by $\varphi_k(x) = x^k e^{-x} \mathbf{1}_{\mathbb{R}_+}(x)$

Then, there is a constant C depending only on k such that for any uniformly continuous bounded function f on $\mathbb{R}$ and any $x \in \mathbb{R}$,

$$|f(t)| \leqslant C \inf_{V \in \mathbb{R}_+^\star} \left(\omega_f\left(\frac{1}{V}\right) + \frac{\|f\|_\infty}{V} + (1+V)^k \sup_{t' \in \mathbb{R}} e^{-|t'|} |\varphi_k \star f(t - t')| \right)$$

where ω_f is a modulus of uniform continuity (that we shall assume non decreasing without any loss of generality) of f.

To prove this theorem, we will use the following

Lemma A.3. *There is a constant C such that for any integrable and uniformly continuous function f on $\mathbb{R}$,*

$$\sup_{x \in \mathbb{R}} |f(x)| \leqslant C \inf_{V \in \mathbb{R}_+^\star} \omega_f\left(\frac{1}{V}\right) + \sup_\tau \left| \int_{-V}^{V} e^{i\xi\tau} \left(1 - \frac{|\xi|}{V}\right) \widehat{f}(\xi) \mathrm{d}\xi \right|$$

where ω_f is a (non decreasing) modulus of uniform continuity for f.

Proof. The proof is a minor modification of the one of Frennemo's lemma using that if f is uniformly continuous and ω_f is a non decreasing modulus of uniform continuity for f, then

$$-\inf_{x \leqslant t \leqslant x + 1/V} f(t) - f(x) \leqslant \sup_{x \leqslant t \leqslant x + 1/V} \omega_f(|x - t|) \leqslant \omega_f(1/V)$$

□

Proof of theorem A.2. For $s \in \mathbb{R}$, let u_s be the function defined by $u_s(t) = f(t) e^{-\frac{1}{2}(t-s)^2}$.

Then, for any $t, t' \in \mathbb{R}$,

$$\begin{aligned} |u_s(t) - u_s(t')| &= \left| e^{-\frac{1}{2}(t-s)^2} (f(t) - f(t')) + f(t') (e^{-\frac{1}{2}(t-s)^2} - e^{-\frac{1}{2}(t'-s)^2}) \right| \\ &\leqslant \omega_f(|t - t'|) + \|f\|_\infty |t - t'| \sup_{u \in \mathbb{R}} |u| e^{-u^2/2} \end{aligned}$$

Thus, the function u_s is uniformly continuous on. As it is also integrable, we get, according to the lemma, that for any $V \in \mathbb{R}_+^\star$ and any $s \in \mathbb{R}$,

$$|f(s)| \leqslant C \left(\omega_f\left(\frac{1}{V}\right) + \frac{\|f\|_\infty}{V} \sup_{u \in \mathbb{R}} |u| e^{-u^2/2} + \sup_\tau \left| \int_{-V}^{V} e^{i\xi\tau} \left(1 - \frac{|\xi|}{V}\right) \widehat{u_s}(\xi) \mathrm{d}\xi \right| \right)$$

But Frennemo proves that

$$\sup_\tau \left| \int_{-V}^{V} e^{i\xi\tau} \left(1 - \frac{|\xi|}{V}\right) \widehat{u_s}(\xi) \mathrm{d}\xi \right| \leqslant C(1+V)^k \sup_{t' \in \mathbb{R}} e^{-|t'|} |\varphi_k \star f(t - t')|$$

and this finishes the proof of the theorem. □

Let $(\mathbf{X}, d)$ be a compact metric space and $\gamma \in]0,1]$. For $(x,t),(x',t') \in \mathbf{X} \times \mathbb{R}$, we note

$$\omega((x,t),(x',t')) = \frac{\sqrt{|t-t'|^2 + d(x,x')^2}}{(1+|t'|)(1+|t|)}$$

Corollary A.4. *Let $(\mathbf{X}, d)$ be a compact metric space and $\gamma \in]0,1]$.*

For any $k \in \mathbb{N}$, there is are constants $C_k \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}_+^\star$ such that for any bounded function f on $\mathbf{X} \times \mathbb{R}$ with

$$|f(x,t) - f(x,t')| \leqslant \left(e^{|t-t'|} - 1\right)^\gamma C(f),\ \|f\|_\infty \leqslant C(f)$$

and

$$|\varphi_k \star f(x,t) - \varphi_k \star f(x',t')| \leqslant C(f)\omega((x,t),(x',t'))^\gamma$$

for some constant $C(f)$ we have that

$$|f(x,t) - f(x',t')| \leqslant C_k C(f)\omega((x,t),(x',t'))^\alpha$$

Proof. For $x, x' \in \mathbf{X}$ and $s \in \mathbb{R}$, we note

$$f_{x,x',s}(t) = f(x,t) - f(x,t+s)$$

Then, for any $t, t' \in \mathbb{R}$,

$$\begin{aligned} |f_{x,x',s}(t) - f_{x,x',s}(t')| &= |f(x,t) - f(x',t+s) - f(x,t') + f(x',t'+s)| \\ &\leqslant 2C_0 \left(e^{|t-t'|} - 1\right)^\gamma \end{aligned}$$

and

$$|\varphi_k \star f_{x,x',s}(t)| = |\varphi_k \star f(x,t) - \varphi_k \star f(x',t+s)| \leqslant C(f)\omega((x,t),(x',t+s))$$

So, according to Frennemo's theorem, for any $x, x' \in \mathbf{X}$ and any $t, s \in \mathbb{R}$,

$$\begin{aligned} |f_{x,x',s}(t)| \leqslant CC(f) \Bigg(&\inf_{V \in \mathbb{R}_+^\star} 2(e^{1/V} - 1)^\gamma + \frac{2}{V} \\ &+ (1+V)^k \sup_{t' \in \mathbb{R}} e^{-|t'|}\omega((x,t-t'),(x',t-t'+s))^\gamma \Bigg) \end{aligned}$$

But, for any $t, t' \in \mathbb{R}$,

$$\frac{1}{1+|t-t'|} \leqslant \frac{1+|t'|}{1+|t|}$$

And so

$$\sup_{t' \in \mathbb{R}} e^{-|t'|}\omega((x,t-t'),(x',t-t'+s)) \leqslant \omega((x,t),(x',t+s)) \sup_{t' \in \mathbb{R}} e^{-|t'|}(1+|t'|)$$

Thus, for maybe another constant C not depending on f, we get that

$$|f_{x,x',s}(t)| \leqslant CC(f) \inf_{V \in [1,+\infty[} \frac{1}{V^\gamma} + (1+V)^k \omega((x,t),(x',t+s))^\gamma$$

Noting now, for $\delta \in \mathbb{R}_+^\star$ that we will specify later,

$$V = \omega((x,t),(x',t+s))^{-\delta}$$

we get that for maybe another constant C still non depending on f,

$$|f(x,x',s)(t)| \leqslant CC(f)\left(\omega^{\gamma\delta} + (1+\omega^{-\gamma\delta})^k\omega^{\gamma}\right) \leqslant CC(f)\omega^{\alpha}$$

for δ small enough and some choice of α where we used that ω is bounded $\mathbf{X}\times\mathbb{R}$.

And this finishes the proof of the corollary. □

References

[BDP15] Dariusz Buraczewski, Ewa Damek, and Tomasz Przebinda, *On the rate of convergence in the Kesten renewal theorem*, Electron. J. Probab. **20** (2015), no. 22, 35. MR 3325092 4, 67

[BG07] J. Blanchet and P. Glynn, *Uniform renewal theory with applications to expansions of random geometric sums*, Adv. in Appl. Probab. **39** (2007), no. 4, 1070–1097. MR 2381589 (2009e:60191) 8

[BL85] Philippe Bougerol and Jean Lacroix, *Products of random matrices with applications to Schrödinger operators*, Progress in Probability and Statistics, vol. 8, Birkhäuser Boston, Inc., Boston, MA, 1985. MR 886674 (88f:60013) 10, 11, 39, 59

[Bla48] David Blackwell, *A renewal theorem*, Duke Math. J. **15** (1948), 145–150. MR 0024093 (9,452a) 3

[BQ14] Yves Benoist and Jean-François Quint, *Random walks on projective spaces*, Compos. Math. **150** (2014), no. 9, 1579–1606. MR 3260142 10, 11, 15

[BQ15] ———, *Random walks on reductives groups*, Preprint (2015), . 9, 19, 34, 37, 39, 51, 56, 59

[Bre05] E. Breuillard, *Distributions diophantiennes et théorème limite local sur* $\mathbb{R}^d$, Probab. Theory Related Fields **132** (2005), no. 1, 39–73. MR 2136866 (2006b:60093) 8

[Car83] Hasse Carlsson, *Remainder term estimates of the renewal function*, Ann. Probab. **11** (1983), no. 1, 143–157. MR 682805 (84e:60127) 4, 8, 32, 41, 55

[Dol98] Dmitry Dolgopyat, *Prevalence of rapid mixing in hyperbolic flows*, Ergodic Theory and Dynamical Systems **18** (1998), 1097–1114. 7, 9, 23

[Fre65] Lennart Frennemo, *On general Tauberian remainder theorems*, Math. Scand. **17** (1965), 77–88. MR 0193439 (33 #1659) 67, 74

[Fur63] Harry Furstenberg, *Noncommuting random products*, Trans. Amer. Math. Soc. **108** (1963), 377–428. MR 0163345 (29 #648) 3

[GC64] I. M. Guelfand and G. E. Chilov, *Les distributions. Tome 2: Espaces fondamentaux*, Traduit par Serge Vasilach. Collection Universitaire de Mathématiques, XV, Dunod, Paris, 1964. MR 0161145 (28 #4354) 60

[GL12] Y. Guivarc'h and E. Le Page, *Spectral gap properties for linear random walks and Pareto's asymptotics for affine stochastic recursions*, ArXiv e-prints (2012), . 3, 17, 52

[GR85] Y. Guivarc'h and A. Raugi, *Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence*, Z. Wahrsch. Verw. Gebiete **69** (1985), no. 2, 187–242. MR 779457 3, 32, 39, 47, 49

[ITM50] C. T. Ionescu Tulcea and G. Marinescu, *Théorie ergodique pour des classes d'opérations non complètement continues*, Ann. of Math. (2) **52** (1950), 140–147. MR 0037469 (12,266g) 13, 58

[Kes74] Harry Kesten, *Renewal theory for functionals of a markov chain with general state space*, Ann. Probab. **2** (1974), no. 3, 355–386. 52

[Qui05] Jean-François Quint, *Groupes de Schottky et comptage*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 2, 373–429. MR 2147895 (2006g:37033) 32, 44

[Rau92] Albert Raugi, *Théorie spectrale d'un opérateur de transition sur un espace métrique compact*, Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques **28** (1992), no. 2, 281–309 (fre). 10, 11, 15

[Rud87] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR 924157 (88k:00002) 20

[Ser78] Jean-Pierre Serre, *Représentations linéaires des groupes finis*, revised ed., Hermann, Paris, 1978. MR 543841 20