



HAL
open science

Landau damping for the linearized Vlasov Poisson equation in a weakly collisional regime

Isabelle Tristani

► **To cite this version:**

Isabelle Tristani. Landau damping for the linearized Vlasov Poisson equation in a weakly collisional regime . 2016. hal-01292023v1

HAL Id: hal-01292023

<https://hal.science/hal-01292023v1>

Preprint submitted on 22 Mar 2016 (v1), last revised 31 Mar 2017 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LANDAU DAMPING FOR THE LINEARIZED VLASOV POISSON EQUATION IN A WEAKLY COLLISIONAL REGIME

ISABELLE TRISTANI

ABSTRACT. In this paper, we consider the linearized Vlasov-Poisson equation around an homogeneous Maxwellian equilibrium in a weakly collisional regime: there is a parameter ε in front of the collision operator which will tend to 0. Moreover, we study two cases of collision operators, linear Boltzmann and Fokker-Planck. We prove a result of Landau damping for those equations in Sobolev spaces uniformly with respect to the collision parameter ε as it goes to 0.

CONTENTS

1. Introduction	1
1.1. The model	2
1.2. Main result and known results	3
1.3. Notations and tools	5
1.4. Outline of the paper	6
2. Linear Boltzmann type collisions	6
2.1. The Penrose criterion	7
2.2. Proof of Theorem 2.1	9
3. Fokker-Planck type collisions	11
3.1. The Penrose criterion	12
3.2. The homogeneous Fokker-Planck equation	14
3.3. Proof of Theorem 3.1	14
References	17

1. INTRODUCTION

In this paper, we are interested in the phenomenon of Landau damping for a linear Vlasov Poisson equation in a weakly collisional regime. The physical motivation for this problem comes from plasma theory. Indeed, in kinetic theory, the Landau equation, which can be derived from the Boltzmann one, allows to describe collisions in plasmas. However, this description is only valid for large times. The Vlasov model, on the contrary, completely neglects collisions in plasmas. Between these two models, is an intermediate one which is a more realistic physical model and which can be written as follows:

$$(1.1) \quad \partial_t f + v \cdot \nabla_x f + F(t, x) \cdot \nabla_v f = \varepsilon Q_L(f, f)$$

2010 *Mathematics Subject Classification.* 35B35, 35Q83, 35Q20, 35Q84.

Key words and phrases. Landau damping, Vlasov-Poisson equation, Fokker-Planck operator, linear Boltzmann operator.

where $f = f(t, x, v)$ is the distribution function in phase space, the position x lies in the torus of size 2π denoted $\mathbb{T}^d := \mathbb{R}^d / (2\pi\mathbb{Z})^d$, the velocity v in \mathbb{R}^d . Note that one can reduce to the case where the size of the box is 2π by a rescaling argument. The operator Q_L is the Landau collision operator (that we do not describe precisely because we won't deal with this general model). The mean-field electrostatic force $F(t, x)$ is given by

$$F(t, x) = -\nabla W *_x \left(\rho_f(t, x) - \int_{\mathbb{T}^d} \rho_f(t, y) dy \right)$$

with $\rho_f(t, x) := \int_{\mathbb{R}^d} f(t, x, v) dv$. The potential $W(x)$ describes the mean-field interaction between particles, we consider here the case of Coulomb repulsive interactions (which occur between electrons in plasmas) for which

$$(1.2) \quad F = e_0 \nabla_x \Delta_x^{-1} \rho_f \quad \text{and} \quad \widehat{W}(k) = e_0 |k|^{-2}$$

where $e_0 > 0$ is the electron charge-to-mass ratio.

In classical plasma physics, the coefficient ε equals $(\log \Lambda)/(2\pi\Lambda)$ where Λ is the plasma parameter which is very large. It is thus coherent to study this type of model in the limit $\varepsilon \rightarrow 0$. We expect the behaviour of solutions to (1.1) to be complex and depend on the considered scales of time. When $\varepsilon > 0$ is fixed, one can expect the hypocoercive effect of collisions to be dominant over large time $O(1/\varepsilon)$, which should induce an exponential convergence towards the Maxwellian equilibrium. When $\varepsilon = 0$, Landau damping occurs in time of order $O(1)$. In this paper, we focus on Landau damping in the limit $\varepsilon \rightarrow 0$.

As pointed out by Villani in [27], the study of this problem is of great interest for several reasons: as we mentioned above, from a physical viewpoint, the model is more realistic than the pure Landau or Vlasov ones; it also permits quantification with respect to ε of the Landau damping; and finally, one could hope to bypass the obstacle of Gevrey regularity that faces the non collision model thanks to regularizing effects induced by collisions. Let us underline the fact that the last aspect is only significant in the nonlinear case that we won't explore in this paper. Our study, which is restricted to the linearized problem with some linear collision operators, can thus be seen as the starting point of a long-range program.

1.1. The model. As mentioned above, we only deal with collision operators denoted $\mathcal{C}(f)$ which are linear. To simplify the problem, we also suppose that

$$(1.3) \quad \text{Ker}(\mathcal{C}) = \text{Span}(M)$$

where M is the Maxwellian distribution

$$(1.4) \quad M(v) := (2\pi)^{-d/2} e^{-|v|^2/2}.$$

We consider the model

$$(1.5) \quad \partial_t f + v \cdot \nabla_x f + F(t, x) \cdot \nabla_v f = \varepsilon \mathcal{C}(f).$$

We write the solutions of (1.5) as perturbations of the Maxwellian distribution M :

$$f(t, x, v) = M(v) + h(t, x, v)$$

where h is a mean-zero perturbation if $\int_{\mathbb{T}^d \times \mathbb{R}^d} f_{in}(x, v) dx dv = \int_{\mathbb{T}^d \times \mathbb{R}^d} M(v) dx dv$, an assumption that we will suppose to hold throughout the paper. Let us emphasize here that we consider a linearization around the Maxwellian distribution M and not around some homogeneous distribution or other as it is usually done for the Vlasov-Poisson equation. It is explained by the fact that, under the assumption (1.3), M is also the unique equilibrium

of the collision operator \mathcal{C} satisfying $\int_{\mathbb{R}^d} M(v) dv = 1$. Then h satisfies the following equation (where we denote $\rho(t, x) := \int_{\mathbb{R}^d} h(t, x, v) dv$ the perturbation density):

$$(1.6) \quad \begin{cases} \partial_t h + v \cdot \nabla_x h + F(t, x) \cdot \nabla_v h + F(t, x) \cdot \nabla_v M = \varepsilon \mathcal{C}(h) \\ F(t, x) = -\nabla W *_x \rho(t, x) \\ h(0, x, v) = h_{in}(x, v). \end{cases}$$

where we have used that $\mathcal{C}(M) = 0$.

In this paper, we only work on the linearized problem associated to (1.6) which reads:

$$(1.7) \quad \begin{cases} \partial_t h + v \cdot \nabla_x h + F(t, x) \cdot \nabla_v M = \varepsilon \mathcal{C}(h) \\ F(t, x) = -\nabla W *_x \rho(t, x) \\ h(0, x, v) = h_{in}(x, v). \end{cases}$$

We end the description of the model we are concerned with by the definition of the collision operators \mathcal{C} that we shall consider. It can be a linear Boltzmann operator of type:

$$(1.8) \quad \mathcal{C}(h) := \rho M - h$$

or a Fokker-Planck operator defined through:

$$(1.9) \quad \mathcal{C}(h) := \Delta_v h + \operatorname{div}_v(vh).$$

These are classical operators of statistical physics that allow to describe the interaction between particles and a fixed background. Note that in the case of the collisional Vlasov equation (without the term coming from the mean-field interaction between particles):

$$\partial_t h + v \cdot \nabla_x h = \varepsilon \mathcal{C}(h)$$

for fixed $\varepsilon > 0$, it is clear that the hypocoercive effect is dominant for large times. We can for example apply the results of Dolbeault, Mouhot and Schmeiser [9] to obtain a result of decay at infinity with a rate of type $e^{-\varepsilon\lambda t}$ for some $\lambda > 0$ in some Hilbert space.

1.2. Main result and known results.

Main result and comments. In this paper, we focus on the limit $\varepsilon \rightarrow 0$ and we intend to obtain a result of Landau damping in the limit $\varepsilon \rightarrow 0$ uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$ for some $\varepsilon_0 > 0$ for the equation (1.7). Before stating a theorem which provides a summary of the main results that we obtain, we introduce the Sobolev spaces in which we are going to develop our analysis.

We fix for the rest of the paper $\ell > d/2$ and introduce the space $L_x^2 \mathcal{H}_v^n$, $n \in \mathbb{N}$ associated to the following norm:

$$\|h\|_{L_x^2 \mathcal{H}_v^n}^2 := \sum_{|\alpha| \leq n} \int_{\mathbb{T}^d \times \mathbb{R}^d} \langle v \rangle^{2\ell} |\partial_v^\alpha f|^2 dx dv$$

where we have used the notation $\langle v \rangle := (1 + |v|^2)^{1/2}$. The dependency of $L_x^2 \mathcal{H}_v^n$ with respect to ℓ is not included in the notation since ℓ is definitely fixed.

Here is our main result:

Theorem 1.1. *Consider a mean-zero distribution $h_{in} \in L_x^2 \mathcal{H}_v^n$ for $n \in \mathbb{N}$, $n \geq 3$. There exist $\varepsilon_0 > 0$, $\lambda_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$, the density $\rho = \rho(t, x)$ of the solution*

$h = h(t, x, v)$ of (1.7) satisfies the following estimate (the range of times for which the following estimates hold is given below):

$$\|\rho(t, \cdot)\|_{L_x^2} \leq \frac{C}{\langle t \rangle^n},$$

the solution $h = h(t, x, v)$ satisfies itself

$$\forall \xi \in \mathbb{R}^d, \quad |\widehat{h}(t, 0, \xi)| \leq C e^{-\varepsilon \lambda_0 t}$$

and

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \forall \xi \in \mathbb{R}^d, \quad |\widehat{h}(t, k, \xi)| \leq \frac{C(\xi)}{\langle t \rangle^{n-2}}$$

where $C(\xi) := C \langle \xi \rangle^{n-2}$ and C is a positive constant depending on $\|h_{in}\|_{L_x^2 \mathcal{H}_v^n}$. These estimates are valid for all times $t \geq 0$ in the case of the model associated to the linear Boltzmann collision operator (1.8) and for $t \in [0, 1/\varepsilon]$ for the model with Fokker-Planck collisions described by (1.9).

Remark 1.2. For $\varepsilon = 0$, it is well-known that

$$\forall \xi \in \mathbb{R}^d, \forall t \geq 0, \quad \widehat{h}(t, 0, \xi) = \widehat{h_{in}}(0, \xi),$$

it is not anymore the case for $\varepsilon > 0$ due to the presence of the collision operator.

Let us underline that our study fits into a Sobolev framework. We do not work with analytic or Gevrey regularity as in [22] or [1] in which more regularity is needed to handle the nonlinear problem. However, the study of the linearized problem could have been done in Sobolev regularity, this is what we are doing since we only deal with the linearized problem. To do that, we take advantage of the paper by Faou and Rousset [14] which consists in the analysis of Landau damping for the (nonlinear) Vlasov-HMF equation in a Sobolev setting. For this equation, the nonlinear study do not require analytic (or Gevrey regularity), indeed, due to the regularity of the interaction kernel, the control of the nonlinear effect ‘‘plasma-echo’’ is made easier. Finally, let us mention that our result can easily be adapted to an analytic framework. Also, for the nonlinear equation (1.6), it would be particularly interesting to investigate if the gain of regularity induced by collisions between particles could allow us to relax the hypothesis on the initial datum.

Another aspect that we point out is that, to the best of our knowledge, it is the first paper in which such a problematic is investigated for the Vlasov-Poisson equation in a weakly collisional regime (in the limit $\varepsilon \rightarrow 0$). It should be relevant to compare this kind of question with the one studied by Bedrossian, Masmoudi and Vicol [2] about the two-dimensional Euler equation where the equivalent of ε should be the inverse of the Reynolds number. Concerning this kind of problematic of uniform analysis of large time behaviour with respect to a small parameter, we can also mention the work of the author with Mischler [21] in which an analysis of large-time behaviour is developed for several homogeneous Fokker-Planck equations uniformly with respect to a small parameter which allows to go from a model to another.

Strategy of the proof. To deal with this problem, we use the classical strategy to analyse Landau damping. We use method of characteristics and Fourier transform to obtain a closed equation on $\widehat{\rho}(t, k)$ for each $k \in \mathbb{Z}^d$. The presence of the collisional operator raises additional difficulties. Indeed, in this equation, there is a term of the form

$$\int_0^t K_\varepsilon(t-s) \widehat{\rho}(s, k) ds$$

which is typical. However, K_ε splits into two parts, one coming from the Vlasov-Poisson term and the other coming from the collision term. In order to obtain our final result uniformly with respect to ε , we have to sharply estimate the Fourier transform in time of K_ε and this is the main technical issue of this paper. We are then able to check that a Penrose type criterion is satisfied uniformly with respect to ε for ε small enough. Taking advantage of the analysis developed in [14] in a Sobolev framework, we end up with an estimate on $\|\rho(t, \cdot)\|_{L_x^2}$. We then go back to the kinetic distribution h and estimate each mode of it separately thanks to the result obtained previously on ρ .

State of the art. The behaviour of solutions to the equation (1.7) has been largely studied for some fixed ε .

If $\varepsilon = 0$, we are in the framework of Mouhot and Villani [22], they prove Landau damping (linear and nonlinear) in analytic or Gevrey regularity. We also mention the work of Bedrossian, Masmoudi and Mouhot [1] in which a new simplified proof is given in Gevrey regularity. Let us also specify that some earlier results were obtained in [5] by Caglioti and Maffei and in [19] by Hwang and Velázquez. We refer to the works of Ryutov [25], Mouhot and Villani [22], Villani [28] for a detailed historical background on Landau damping and references therein.

If $\varepsilon > 0$ is fixed, one can expect that the collisional term induces a hypocoercive effect. Some results have been obtained on linear and/or nonlinear frameworks for the Vlasov-Poisson-Fokker-Planck equation. About existence results, we mention the papers of Victory and O'Dwyer in two dimensions [26], Rein and Weckler [24] in three dimensions for small data and Bouchut [3] in three dimensions. Concerning the long-time behaviour, we point out the papers of Bouchut and Dolbeault [4], Carrillo, Soler and Vásquez [6], Dolbeault [8] and Hérau and Thomman [18], in the latter, both Cauchy problem and long-time behaviour have been studied.

As far as the case of the linear Boltzmann is concerned, the literature is more scarce and we were not able to find any reference on it. However, let us emphasize the fact that a large literature is devoted to the study of the Vlasov-Poisson-Boltzmann equation with a general Boltzmann collision operator. On this topic, we call attention on the paper of Dolbeault and Desvillettes [7] which deals with the large time behaviour of solutions. On this matter, we also have to refer to a series of papers of Guo in which a robust energy method is developed to treat the global stability of global Maxwellians in a perturbative framework. There are many variants based on Guo's works and its energy method which allow to analyse the Cauchy problem and the large time behaviour of solutions. Let us mention two papers of Guo on the Vlasov-Poisson-Boltzmann equation [16, 17] the first one falls in a near-vacuum regime, the second one in a near Maxwellian setting. Let us give other references which are concerned with the Cauchy problem in a perturbative setting and/or the large time behaviour of solutions and the rates of convergence, among others, we mention the papers of Duan and Strain [12], Duan, Yang and Zhao [13], Duan and Liu [10], Xia, Xiong and Zhao [29]. From another viewpoint, an analysis of the spectrum has been carried out in [20] by Li, Yang and Zhong. There is another phenomenon which could occur when the initial datum of the equation approaches distinct global Maxwellians at far fields, the solution to the Cauchy problem does not converge anymore to a constant equilibrium state in large time but we do not enter into details in this regard and refer to the recent paper of Duan and Liu [11] and the references given therein.

1.3. Notations and tools. We here define some notations and state an elementary lemma on Fourier transform that we will make good use in what follows.

Fourier transform. For a function $f = f(x)$, $x \in \mathbb{T}^d$, we define its Fourier transform as follows:

$$\widehat{f}(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(x) e^{-ix \cdot k} dx, \quad k \in \mathbb{Z}^d.$$

Similarly, for a function $f = f(v)$, $v \in \mathbb{R}^d$, we define its Fourier transform by:

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(v) e^{-iv \cdot \xi} dv, \quad \xi \in \mathbb{R}^d.$$

Let us notice that the Maxwellian distribution M defined in (1.4) satisfies $\widehat{M} = M$, an equality that we will use all along the paper.

Finally, if $f = f(x, v)$, $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, we define its Fourier transform through the following formula:

$$\widehat{f}(k, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x, v) e^{-ik \cdot x - iv \cdot \xi} dx dv, \quad (k, \xi) \in \mathbb{Z}^d \times \mathbb{R}^d.$$

We shall also use the Fourier transform in time, if $f = f(t)$, $t \in \mathbb{R}$, we denote

$$(1.10) \quad \widetilde{f}(\tau) = \int_{\mathbb{R}} f(t) e^{-it\tau} dt, \quad \tau \in \mathbb{C}.$$

Elementary lemma. We here state a simple result which links Fourier transform and regularity of a function (one can check the proof in [14, Lemma 2.1] in dimension 1).

Lemma 1.3. *For any $n \in \mathbb{N}$, we have:*

$$\forall k \in \mathbb{Z}^d, \quad \forall \xi \in \mathbb{R}^d, \quad |\widehat{f}(k, \xi)| \leq C(n, \ell) \langle \xi \rangle^{-n} \|f\|_{L_x^2 \mathcal{H}_v^n}$$

where $C(n, \ell)$ is a positive constant depending only on ℓ and n .

Notation. Finally, let us specify that we shall use the same notation C for positive constants that may change from line to line, when C is a positive constant depending only on fixed number.

1.4. Outline of the paper. In Section 2, we deal with the behaviour of solutions of (1.7) with the linear Boltzmann operator (1.8) as a collision operator. In Section 3, we handle the problem with collisions given by a Fokker-Planck operator (1.9).

Acknowledgments. The author would like to thank Daniel Han-Kwan for enlightened discussions, his help and his suggestions. The author is supported by the *Fondation Mathématique Jacques Hadamard*.

2. LINEAR BOLTZMANN TYPE COLLISIONS

In this section, we deal with the following model:

$$(2.1) \quad \begin{cases} \partial_t h + v \cdot \nabla_x h + F(t, x) \cdot \nabla_v M = \varepsilon (\rho M - h) \\ F(t, x) = -\nabla W *_x \rho(t, x) \\ h(0, x, v) = h_{in}(x, v). \end{cases}$$

where we recall that $\rho(t, x) = \int_{\mathbb{R}^d} h(t, x, v) dv$. The main theorem reads:

Theorem 2.1. *Consider a mean-zero distribution $h_{in} \in L_x^2 \mathcal{H}_v^n$ for $n \in \mathbb{N}$, $n \geq 3$. There exists $\varepsilon_0 > 0$ such that the solution $h = h(t, x, v)$ of (2.1) and its associated density $\rho = \rho(t, x)$ satisfy the following estimates:*

$$(2.2) \quad \forall \varepsilon \in [0, \varepsilon_0], \forall t \geq 0, \quad \|\rho(t, \cdot)\|_{L_x^2} \leq \frac{C}{\langle t \rangle^n},$$

and

$$(2.3) \quad \begin{cases} \forall \varepsilon \geq 0, \forall \xi \in \mathbb{R}^d, \forall t \geq 0, & |\widehat{h}(t, 0, \xi)| \leq C e^{-\varepsilon t}, \\ \forall \varepsilon \in [0, \varepsilon_0], \forall k \in \mathbb{Z}^d \setminus \{0\}, \forall \xi \in \mathbb{R}^d, \forall t \geq 0, & |\widehat{h}(t, k, \xi)| \leq \frac{C(\xi)}{\langle t \rangle^{n-2}}, \end{cases}$$

where $C(\xi) := C \langle \xi \rangle^{n-2}$ and C is a positive constant depending on $\|h_{in}\|_{L_x^2 \mathcal{H}_v^n}$.

2.1. The Penrose criterion. Before going into the proof of Theorem 2.1, we start by giving a lemma which allows us to apply a result from [14]. To do that, we introduce the following notations: for $(t, k) \in \mathbb{R} \times \mathbb{Z}^d$, we define for $\varepsilon \geq 0$,

$$(2.4) \quad \begin{cases} K_\varepsilon^0(t, k) := \varepsilon e^{-\varepsilon t} \widehat{M}(kt) \mathbf{1}_{t \geq 0} \\ K_\varepsilon^1(t, k) := -\widehat{W}(k) e^{-\varepsilon t} \widehat{M}(kt) |k|^2 t \mathbf{1}_{t \geq 0} \\ K_\varepsilon(t, k) := K_\varepsilon^0(t, k) + K_\varepsilon^1(t, k). \end{cases}$$

Lemma 2.2. *There exists $\varepsilon_0 > 0$ such that the kernel K_ε defined in (2.4) satisfies the following condition:*

$$(H) \quad \exists \kappa > 0, \quad \forall \varepsilon \in [0, \varepsilon_0], \quad \inf_{k \in \mathbb{Z}^d \setminus \{0\}} \inf_{\text{Im } \tau \leq 0} |1 - \widetilde{K}_\varepsilon(\tau, k)| \geq \kappa$$

where we recall that $\widetilde{K}_\varepsilon(\tau, k)$ stands for the Fourier transform in time of K_ε defined in (1.10).

Proof of Lemma 2.2. We start by giving an estimate on $|1 - \widetilde{K}_0^1(\tau, k)|$. We recall that since all marginals of M are increasing/decreasing, one can check (see [28]) that the gaussian distribution M satisfies the following form of Penrose stability criterion [23] (given in [22] in this form):

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \forall w \in \mathbb{R} \text{ s.t. } (M_k)'(w) = 0, \quad \widehat{W}(k) \left(\text{p.v.} \int_{\mathbb{R}} \frac{(M_k)'(r)}{r - w} dr \right) < 1$$

where M_k denotes the marginals of M along $k \in \mathbb{Z}^d \setminus \{0\}$:

$$M_k(r) := \int_{kr/|k| + k^\perp} M(w) dw.$$

Consequently, we know (see [22] and [1] for a complete proof) that there exists $\lambda_\dagger > 0$ and $\kappa_0 > 0$ such that

$$(2.5) \quad \forall \xi \in \mathbb{C}, \quad \Re \xi < \lambda_\dagger, \quad \inf_{k \in \mathbb{Z}^d} |\mathcal{L}(k, \xi) - 1| \geq \kappa_0$$

where \mathcal{L} is defined through ($\bar{\xi}$ is the complex conjugate of ξ)

$$\mathcal{L}(k, \xi) = - \int_0^\infty e^{\bar{\xi}|k|t} \widehat{M}(kt) \widehat{W}(k) |k|^2 t dt, \quad (k, \xi) \in \mathbb{Z}^d \times \mathbb{R}^d,$$

note that this integral is well defined because it is absolutely convergent due to the decay property of $\widetilde{M} = M$. Then, one can check that if we denote $\tau = \lambda + i\zeta$ with $(\lambda, \zeta) \in \mathbb{R}^2$,

$$\widetilde{K}_0^1(\tau, k) = \mathcal{L} \left(k, \frac{\zeta + i\lambda}{|k|} \right)$$

and thus using (2.5), we deduce that for any $k \in \mathbb{Z}^d \setminus \{0\}$,

$$(2.6) \quad \forall \tau \in \mathbb{C}, \operatorname{Im} \tau \leq 0, \quad |\widetilde{K}_0^1(\tau, k) - 1| \geq \kappa_0.$$

Still using the notation $\tau = \lambda + i\zeta$, we also have

$$|\widetilde{K}_\varepsilon^0(\tau, k)| = \varepsilon \left| \int_0^\infty e^{-it\tau} e^{-\varepsilon t} \widehat{M}(kt) dt \right| \leq \varepsilon \int_0^\infty e^{\zeta t} e^{-\varepsilon t} M(kt) dt.$$

We deduce for any $k \in \mathbb{Z}^d \setminus \{0\}$,

$$(2.7) \quad \forall \tau \in \mathbb{C}, \operatorname{Im} \tau \leq 0, \quad |\widetilde{K}_\varepsilon^0(\tau, k)| \leq \varepsilon \int_0^\infty M(kt) dt \leq \frac{\varepsilon}{|k|} \int_0^\infty M(t) dt \leq C\varepsilon.$$

Then, it remains to estimate the difference coming from $K_\varepsilon^1 - K_0^1$. We have:

$$\begin{aligned} |\widetilde{K}_\varepsilon^1(\tau, k) - \widetilde{K}_0^1(\tau, k)| &= \left| \int_0^\infty e^{-it\tau} (K_\varepsilon^1(t, k) - K_0^1(t, k)) dt \right| \\ &= e_0 \left| \int_0^\infty e^{-it\tau} (1 - e^{-\varepsilon t}) \widehat{M}(kt) t dt \right| \\ &\leq C\varepsilon \int_0^\infty e^{\zeta t} t^2 e^{-\frac{|k|^2 t^2}{2}} dt \end{aligned}$$

where we used that $1 - e^{-\varepsilon t} \leq \varepsilon t$ for any $t \geq 0$. Performing a change of variable in the last integral, we get for any $k \in \mathbb{Z}^d \setminus \{0\}$,

$$(2.8) \quad \forall \tau \in \mathbb{C}, \operatorname{Im} \tau \leq 0, \quad |\widetilde{K}_\varepsilon^1(\tau, k) - \widetilde{K}_0^1(k, \tau)| \leq \frac{C}{|k|^3} \varepsilon \leq C\varepsilon.$$

We now go back to our full problem which consists in estimating the difference

$$|1 - \widetilde{K}_\varepsilon(\tau, k)| \quad \text{with} \quad K_\varepsilon = K_\varepsilon^0 + K_0^1 + K_\varepsilon^1 - K_0^1.$$

Gathering (2.6), (2.7) and (2.8), we conclude that for any $k \in \mathbb{Z}^d \setminus \{0\}$, for any $\tau \in \mathbb{C}$, $\operatorname{Im} \tau \leq 0$,

$$\begin{aligned} |\widetilde{K}_\varepsilon(\tau, k) - 1| &\geq |\widetilde{K}_0^1(\tau, k) - 1| - |\widetilde{K}_\varepsilon^1(\tau, k) - \widetilde{K}_0^1(k, \tau)| - |\widetilde{K}_\varepsilon^0(\tau, k)| \\ &\geq \kappa_0 - c_0\varepsilon, \end{aligned}$$

for some positive constant $c_0 > 0$ which only depends on a fixed number. We then choose $\varepsilon_0 > 0$ such that $\kappa_0 - c_0\varepsilon_0 > 0$. Denoting $\kappa := \kappa_0 - c_0\varepsilon_0$, we obtain

$$\forall \varepsilon \in [0, \varepsilon_0], \quad \inf_{k \in \mathbb{Z}^d \setminus \{0\}} \inf_{\operatorname{Im} \tau \leq 0} |\widetilde{K}_\varepsilon(\tau, k) - 1| \geq \kappa,$$

from which we conclude that **(H)** is fulfilled. \square

2.2. Proof of Theorem 2.1. We can now handle the proof of Theorem 2.1, we divide it into two parts, the first one being devoted to the proof of estimate (2.2) and the second one to (2.3).

Proof of the estimate (2.2). The first thing that we have to notice is that, due to the conservation properties of the operator $-v \cdot \nabla_x h - F(t, x) \cdot \nabla_v M + \varepsilon(\rho M - h)$, we know that for all times, $\widehat{\rho}(t, 0) = (2\pi)^{d/2} \widehat{h_{in}}(0, 0) = 0$ since h_{in} is a mean-zero distribution.

We then focus on the study of $\widehat{\rho}(t, k)$ for $k \in \mathbb{Z}^d \setminus \{0\}$. In the rest of the proof, we consider $\varepsilon \in [0, \varepsilon_0]$ where ε_0 is given in Lemma 2.2. Using the method of characteristics, we have

$$\begin{aligned} h(t, x, v) &= h_{in}(x - vt, v) e^{-\varepsilon t} + \varepsilon \int_0^t e^{-\varepsilon(t-s)} \rho(s, x - v(t-s)) M(v) ds \\ &\quad - \int_0^t e^{-\varepsilon(t-s)} (\nabla_v M \cdot F)(s, x - v(t-s), v) ds \end{aligned}$$

We then take the Fourier transform in both variables (x, v) of the previous equality:

$$\begin{aligned} \widehat{h}(t, k, \xi) &= e^{-\varepsilon t} \int_{\mathbb{T}^d \times \mathbb{R}^d} h_{in}(x - vt, v) e^{-ik \cdot x - iv \cdot \xi} dx dv \\ &+ \frac{\varepsilon}{(2\pi)^d} \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} e^{-\varepsilon(t-s)} \rho(s, x - v(t-s)) M(v) e^{-ik \cdot x - iv \cdot \xi} dx dv ds \\ &- \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} e^{-\varepsilon(t-s)} (\nabla_v M \cdot F)(s, x - v(t-s), v) e^{-ik \cdot x - iv \cdot \xi} dx dv ds. \end{aligned}$$

Performing change of variables, we obtain

$$\begin{aligned} \widehat{h}(t, k, \xi) &= e^{-\varepsilon t} \widehat{h_{in}}(k, \xi + kt) \\ &+ \frac{\varepsilon}{(2\pi)^d} \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} e^{-\varepsilon(t-s)} \rho(s, x) M(v) e^{-ik \cdot x - iv \cdot (\xi + k(t-s))} dx dv ds \\ &- \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} e^{-\varepsilon(t-s)} (\nabla_v M \cdot F)(s, x, v) e^{-ik \cdot x - iv \cdot (\xi + k(t-s))} dx dv ds, \end{aligned}$$

from which we deduce

$$\begin{aligned} \widehat{h}(t, k, \xi) &= e^{-\varepsilon t} \widehat{h_{in}}(k, \xi + kt) + \varepsilon \int_0^t e^{-\varepsilon(t-s)} \widehat{\rho}(s, k) \widehat{M}(\xi + k(t-s)) ds \\ (2.9) \quad &- \int_0^t e^{-\varepsilon(t-s)} k \cdot (\xi + k(t-s)) \widehat{W}(k) \widehat{\rho}(s, k) \widehat{M}(\xi + k(t-s)) ds. \end{aligned}$$

Taking $\xi = 0$ in (2.9), we obtain the closed equation on $\widehat{\rho}(t, k)$:

$$\begin{aligned} \widehat{\rho}(t, k) &= e^{-\varepsilon t} \widehat{h_{in}}(k, kt) + \varepsilon \int_0^t e^{-\varepsilon(t-s)} \widehat{\rho}(s, k) \widehat{M}(k(t-s)) ds \\ &- \int_0^t e^{-\varepsilon(t-s)} k \cdot (k(t-s)) \widehat{W}(k) \widehat{\rho}(s, k) \widehat{M}(k(t-s)) ds. \end{aligned}$$

that we can sum up in

$$\widehat{\rho}(t, k) = e^{-\varepsilon t} \widehat{h_{in}}(k, kt) + \int_0^t K_\varepsilon(t-s) \widehat{\rho}(s, k) ds$$

where we recall that K_ε is defined in (2.4). We can notice here that we are almost in the same framework as the one of [14, Lemma 2.3] for $\varepsilon \in [0, \varepsilon_0]$. Indeed, we have

$\langle \cdot \rangle^2 M \in L_x^2 \mathcal{H}_v^n$ for any $n \in \mathbb{N}$ and from Lemma 2.2, we know that K_ε satisfies the following condition:

$$(H) \quad \exists \kappa > 0, \quad \forall \varepsilon \in [0, \varepsilon_0], \quad \inf_{k \in \mathbb{Z}^d \setminus \{0\}} \inf_{\text{Im } \tau \leq 0} |1 - \widetilde{K}_\varepsilon(\tau, k)| \geq \kappa.$$

The only difference is that the proof in [14] is done only for the case $k = \pm 1$. But it is uncomplicated to extend the result for $|k| > 1$. We then obtain that for any $\gamma \geq 0$, for any $T \geq 0$:

$$(2.10) \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \sup_{t \in [0, T]} \langle kt \rangle^\gamma |\widehat{\rho}(t, k)| \leq \sup_{t \in [0, T]} \langle kt \rangle^\gamma e^{-\varepsilon t} |\widehat{h}_{in}(k, kt)| \leq \sup_{t \in [0, T]} \langle kt \rangle^\gamma |\widehat{h}_{in}(k, kt)|.$$

Using Lemma 1.3, there holds,

$$(2.11) \quad |\widehat{h}_{in}(k, kt)| \leq C(n) \langle kt \rangle^{-n} \|h_{in}\|_{L_x^2 \mathcal{H}_v^n}.$$

Combining (2.10) and (2.11) with $\gamma = n$, we obtain for any $T \geq 0$,

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \sup_{t \in [0, T]} \langle kt \rangle^n |\widehat{\rho}(t, k)| \leq C(n) \|h_{in}\|_{L_x^2 \mathcal{H}_v^n}.$$

from which we deduce

$$(2.12) \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \forall t \geq 0, \quad |\widehat{\rho}(t, k)| \leq \frac{C(n)}{\langle kt \rangle^n} \|h_{in}\|_{L_x^2 \mathcal{H}_v^n},$$

this yields the final result (2.2). \square

We now concentrate on the proof of estimate on the solution $h = h(t, x, v)$.

Proof of the estimate (2.3). The first part concerns the moment of order 0 in x . Contrary to the case of the linear Vlasov-Poisson equation, the latter is not preserved due to the presence of the collision operator. Indeed, noticing that since h is mean-zero, we have:

$$\partial_t \widehat{h}(t, 0, \xi) = -\varepsilon \widehat{h}(t, 0, \xi)$$

we then deduce that for any fixed $\xi \in \mathbb{R}^d$,

$$\forall \varepsilon \geq 0, \quad |\widehat{h}(t, 0, \xi)| = e^{-\varepsilon t} |\widehat{h}_{in}(0, \xi)| \leq C e^{-\varepsilon t} \|h_{in}\|_{L_x^2 \mathcal{H}_v^n}$$

where the last estimate comes from Lemma 1.3.

We now deal with the case $k \in \mathbb{Z}^d \setminus \{0\}$ and consider $\varepsilon \in [0, \varepsilon_0]$. From (2.9), we have

$$\begin{aligned} \widehat{h}(t, k, \xi - kt) &= e^{-\varepsilon t} \widehat{h}_{in}(k, \xi) + \varepsilon \int_0^t e^{-\varepsilon(t-s)} \widehat{\rho}(s, k) \widehat{M}(\xi - ks) ds \\ &\quad - \int_0^t e^{-\varepsilon(t-s)} k \cdot (\xi - ks) \widehat{W}(k) \widehat{\rho}(s, k) \widehat{M}(\xi - ks) ds \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

Since $h_{in} \in L_x^2 \mathcal{H}_v^n$, we have from Lemma 1.3 that

$$(2.13) \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \forall \xi \in \mathbb{R}^d, \quad |T_1| \leq C \langle \xi \rangle^{-n}.$$

To estimate T_2 and T_3 , we use the estimate (2.12) on ρ that we obtained previously combined with the decay property of $\widehat{M} = M$ which implies for example

$$\widehat{M}(\xi - ks) \leq C \langle \xi - ks \rangle^{2-n}$$

Concerning T_2 , we thus have

$$|T_2| \leq C \int_0^t \langle ks \rangle^{-n} \langle \xi - ks \rangle^{2-n} ds \leq C \int_0^t \langle ks \rangle^{-n} \langle \xi \rangle^{2-n} \langle ks \rangle^{n-2} ds$$

where we have used the estimate $\langle \xi - ks \rangle^{2-n} \leq C \langle \xi \rangle^{2-n} \langle ks \rangle^{n-2}$, we deduce

$$|T_2| \leq C \langle \xi \rangle^{2-n} \int_0^t \frac{ds}{\langle |k|s \rangle^2}.$$

Performing a change of variable, we obtain

$$(2.14) \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \forall \xi \in \mathbb{R}^d, \quad |T_2| \leq C \frac{1}{|k|} \langle \xi \rangle^{2-n} \leq C \langle \xi \rangle^{2-n}.$$

We treat T_3 in a similar way, noticing that $|k| \widehat{W}(k) = O(1)$, and that we have

$$|\xi - ks| \widehat{M}(\xi - ks) \leq C \langle \xi - ks \rangle^{2-n}.$$

We conclude this proof as for T_2 and we obtain

$$(2.15) \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \forall \xi \in \mathbb{R}^d, \quad \forall t \geq 0, \quad |T_3| \leq C \langle \xi \rangle^{2-n}.$$

Combining (2.13), (2.14) and (2.15), we obtain

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \forall \xi \in \mathbb{R}^d, \quad \forall t \geq 0, \quad |\widehat{h}(t, k, \xi - kt)| \leq \frac{C}{\langle \xi \rangle^{n-2}}.$$

In particular, we have for any fixed $\xi \in \mathbb{R}^d$,

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \forall t \geq 0, \quad |\widehat{h}(t, k, \xi)| \leq \frac{C}{\langle \xi + kt \rangle^{n-2}} \leq \frac{C(\xi)}{\langle t \rangle^{n-2}}$$

where we have used that $\langle \xi + kt \rangle^{2-n} \leq C \langle \xi \rangle^{n-2} \langle kt \rangle^{2-n} \leq C \langle \xi \rangle^{n-2} \langle t \rangle^{2-n}$ and this concludes the proof of (2.3). \square

3. FOKKER-PLANCK TYPE COLLISIONS

In this section, we deal with the following model:

$$(3.1) \quad \begin{cases} \partial_t h + v \cdot \nabla_x h + F(t, x) \cdot \nabla_v M = \varepsilon (\Delta_v h + \operatorname{div}_v(vh)) \\ F(t, x) = -\nabla W *_x \rho(t, x) \\ h(0, x, v) = h_{in}(x, v) \end{cases}$$

where we recall that $\rho(t, x) = \int_{\mathbb{R}^d} h(t, x, v) dv$. The main theorem reads:

Theorem 3.1. *Consider a mean-zero distribution $h_{in} \in L_x^2 \mathcal{H}_v^n$ for $n \in \mathbb{N}$, $n \geq 3$. There exists $\varepsilon_0 > 0$ such that the solution $h = h(t, x, v)$ of (3.1) and its associated density $\rho = \rho(t, x)$ satisfy the following estimate:*

$$(3.2) \quad \forall \varepsilon \in [0, \varepsilon_0], \quad \forall t \in [0, 1/\varepsilon], \quad \|\rho(t, \cdot)\|_{L_x^2} \leq \frac{C}{\langle t \rangle^n}.$$

and

$$(3.3) \quad \begin{cases} \forall \varepsilon \geq 0, \quad \forall \xi \in \mathbb{R}^d, \quad \forall t \geq 0, \quad |\widehat{h}(t, 0, \xi)| \leq C e^{-\varepsilon \lambda_* t}, \\ \forall \varepsilon \in [0, \varepsilon_0], \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \forall \xi \in \mathbb{R}^d, \quad \forall t \in [0, 1/\varepsilon], \quad |\widehat{h}(t, k, \xi)| \leq \frac{C(\xi)}{\langle t \rangle^{n-2}}, \end{cases}$$

where $\lambda_* > 0$ is the rate of decay for the homogeneous Fokker-Planck equation in $L^2(\langle v \rangle^\ell)$ (see Lemma 3.3) and $C(\xi) := C \langle \xi \rangle^{n-2}$ and C is a positive constant depending on $\|h_{in}\|_{L_x^2 \mathcal{H}_v^n}$.

3.1. The Penrose criterion. As in Section 2, we introduce a kernel K_ε which is going to be useful in the proof of Theorem 3.1. For $(t, k) \in \mathbb{R} \times \mathbb{Z}^d$ and $\varepsilon > 0$, we define

$$(3.4) \quad K_\varepsilon(t, k) := -\exp\left(-\varepsilon \int_0^t |\chi_\varepsilon(\sigma) k|^2 d\sigma\right) k \cdot (\chi_\varepsilon(t) k) \widehat{W}(k) \widehat{M}(\chi_\varepsilon(t) k) \mathbb{1}_{0 \leq t \leq 1/\varepsilon}$$

where we denoted for $\varepsilon > 0$,

$$\chi_\varepsilon(t) := \frac{1 - e^{-\varepsilon t}}{\varepsilon}, \quad t \in \mathbb{R}.$$

We also introduce a kernel K_0 to which we shall compare K_ε :

$$(3.5) \quad K_0(t, k) := -\widehat{W}(k) |k|^2 M(kt) t \mathbb{1}_{t \geq 0}.$$

The aim of the next part is to prove that K_ε satisfies a Penrose type criterion such as we did in Section 2, Lemma 2.2.

Lemma 3.2. *There exists $\varepsilon_0 > 0$ such that the kernel K_ε defined in (3.4)-(3.5) satisfies the following condition:*

$$(H) \quad \exists \kappa > 0, \quad \forall \varepsilon \in [0, \varepsilon_0], \quad \inf_{k \in \mathbb{Z}^d \setminus \{0\}} \inf_{\text{Im } \tau \leq 0} |1 - \widetilde{K}_\varepsilon(\tau, k)| \geq \kappa$$

where we recall that $\widetilde{K}_\varepsilon(\tau, k)$ stands for the Fourier transform in time of K_ε defined in (1.10).

Proof of Lemma 3.2. First, we recall that we have proved in the proof of Lemma 2.2 (see estimate (2.6)) that for any $k \in \mathbb{Z}^d \setminus \{0\}$,

$$(3.6) \quad \forall \tau \in \mathbb{C}, \quad \text{Im } \tau \leq 0, \quad |\widetilde{K}_0(\tau, k) - 1| \geq \kappa_0$$

for some $\kappa_0 > 0$.

We then estimate the difference $\widetilde{K}_\varepsilon - \widetilde{K}_0$ splitting it into several parts. For any $\tau = \lambda + i\zeta$, $\text{Im } \tau = \zeta \leq 0$, we have:

$$\begin{aligned} & |\widetilde{K}_\varepsilon(\tau, k) - \widetilde{K}_0(\tau, k)| \\ & \leq e_0 \int_0^\infty |\chi_\varepsilon(t) M(k\chi_\varepsilon(t)) - t M(kt)| \mathbb{1}_{0 \leq t \leq 1/\varepsilon} \exp\left(-\varepsilon \int_0^t \chi_\varepsilon^2(\sigma) |k|^2 d\sigma\right) |e^{-it\tau}| dt \\ & \quad + e_0 \int_0^\infty \left| \exp\left(-\varepsilon \int_0^t \chi_\varepsilon^2(\sigma) |k|^2 d\sigma\right) - 1 \right| t M(kt) \mathbb{1}_{0 \leq t \leq 1/\varepsilon} |e^{-it\tau}| dt \\ & \quad + e_0 \int_0^\infty t M(kt) \mathbb{1}_{t \geq 1/\varepsilon} |e^{-it\tau}| dt =: I_1 + I_2 + I_3. \end{aligned}$$

Concerning the term I_1 , we have since $\zeta = \text{Im } \tau \leq 0$,

$$\begin{aligned} I_1 & \leq C \int_0^{1/\varepsilon} |\chi_\varepsilon(t) M(k\chi_\varepsilon(t)) - t M(kt)| e^{\zeta t} dt \\ & \leq C \int_0^{1/\varepsilon} |\chi_\varepsilon(t) - t| M(kt) dt + C \int_0^{1/\varepsilon} |M(k\chi_\varepsilon(t)) - M(kt)| \chi_\varepsilon(t) dt \\ & =: I_{11} + I_{12}. \end{aligned}$$

First, we perform a straightforward analysis of the function $t \mapsto t - \chi_\varepsilon(t) - \varepsilon t^2/2$ to prove that

$$(3.7) \quad \forall t \in \mathbb{R}^+, \quad |\chi_\varepsilon(t) - t| = t - \chi_\varepsilon(t) \leq \varepsilon \frac{t^2}{2}.$$

We deduce that

$$\begin{aligned}
 (3.8) \quad I_{11} &\leq C \varepsilon \int_0^{1/\varepsilon} t^2 M(kt) dt \leq \frac{C \varepsilon}{|k|^2} \int_0^{1/\varepsilon} |k|^2 t^2 M(kt) dt \\
 &\leq \frac{C \varepsilon}{|k|^3} \int_0^{|k|/\varepsilon} t^2 e^{-t^2/2} dt \leq C \varepsilon \int_0^\infty t^2 e^{-t^2/2} dt \leq C \varepsilon
 \end{aligned}$$

where we performed a change of variable. Concerning I_{12} , we notice that

$$|M(k\chi_\varepsilon(t)) - M(kt)| = \left| M^{1/2}(k\chi_\varepsilon(t)) - M^{1/2}(kt) \right| \left(M^{1/2}(k\chi_\varepsilon(t)) + M^{1/2}(kt) \right).$$

But, one can easily prove that $\chi_\varepsilon(t) \geq e^{-1}t$ for any $t \in [0, 1/\varepsilon]$. We thus have, denoting $\mu := e^{-\frac{|k|^2}{4\varepsilon^2}}$,

$$M^{1/2}(k\chi_\varepsilon(t)) + M^{1/2}(kt) \leq C \mu(kt).$$

Also, the gradient of $M^{1/2}$ is bounded in \mathbb{R}^d , we deduce that for any $t \in [0, 1/\varepsilon]$,

$$\begin{aligned}
 |M(k\chi_\varepsilon(t)) - M(kt)| &\leq C \left| M^{1/2}(k\chi_\varepsilon(t)) - M^{1/2}(kt) \right| \mu(kt) \\
 &\leq C |k|(t - \chi_\varepsilon(t)) \mu(kt) \leq C \varepsilon |k| t^2 \mu(kt)
 \end{aligned}$$

where the last inequality comes from (3.7). This implies, since $\chi_\varepsilon(t) \leq t$ for any $t \geq 0$:

$$\begin{aligned}
 (3.9) \quad I_{12} &\leq C \varepsilon \int_0^{1/\varepsilon} |k| t^3 \mu(kt) dt \leq \frac{C \varepsilon}{|k|^3} \int_0^{|k|/\varepsilon} t^3 e^{-(\frac{t}{2\varepsilon})^2} dt \\
 &\leq \frac{C \varepsilon}{|k|^3} \int_0^\infty t^3 e^{-(\frac{t}{2\varepsilon})^2} dt \leq C \varepsilon.
 \end{aligned}$$

Concerning I_2 , we study the function $t \mapsto 1 - \exp\left(-\varepsilon \int_0^t \chi_\varepsilon^2(\sigma) |k|^2 d\sigma\right) - \varepsilon |k|^2 t^3/3$ to obtain

$$\begin{aligned}
 \forall t \in [0, 1/\varepsilon], \quad &\left| \exp\left(-\varepsilon \int_0^t \chi_\varepsilon^2(\sigma) |k|^2 d\sigma\right) - 1 \right| \\
 &= 1 - \exp\left(-\varepsilon \int_0^t \chi_\varepsilon^2(\sigma) |k|^2 d\sigma\right) \leq \varepsilon |k|^2 \frac{t^3}{3}.
 \end{aligned}$$

We deduce that for any $k \in \mathbb{Z}^d \setminus \{0\}$,

$$(3.10) \quad I_2 \leq C \varepsilon |k|^2 \int_0^\infty t^4 M(kt) e^{\zeta t} dt \leq \frac{C \varepsilon}{|k|^3} \int_0^\infty t^4 e^{-t^2/2} dt \leq C \varepsilon,$$

where we have performed a change of variable to get the last inequality but one. Finally, we control I_3 using that $\mathbf{1}_{t \geq 1/\varepsilon} \leq \varepsilon t$. We obtain by performing a change of variable,

$$(3.11) \quad I_3 \leq C \varepsilon \int_0^\infty t^2 M(kt) dt \leq \frac{C \varepsilon}{|k|^3} \int_0^\infty t^2 e^{-t^2/2} dt \leq C \varepsilon.$$

Gathering (3.8), (3.9), (3.10), (3.11) yields

$$(3.12) \quad |\tilde{K}_\varepsilon(\tau, k) - \tilde{K}_0(\tau, k)| \leq c_0 \varepsilon,$$

for some constant $c_0 > 0$. We then combine (3.6) with (3.12) to conclude that

$$\forall \tau \in \mathbb{C}, \quad \text{Im} \tau \leq 0, \quad |\tilde{K}_\varepsilon(\tau, k) - 1| \geq \kappa_0 - c_0 \varepsilon.$$

We choose $\varepsilon_0 > 0$ such that $\kappa_0 - c_0 \varepsilon_0 > 0$, set $\kappa := \kappa_0 - c_0 \varepsilon_0$ and it follows that **(H)** is satisfied. \square

3.2. The homogeneous Fokker-Planck equation. In this part, we recall some features of solutions to the homogeneous Fokker-Planck equation

$$(3.13) \quad \begin{cases} \partial_t f = \varepsilon (\Delta_v f + \operatorname{div}_v(vf)) \\ f(0, \cdot) = f_0 \end{cases}$$

that will be useful for the study of the evolution of $\widehat{h}(t, 0, \xi)$, $t \geq 0$ and $\xi \in \mathbb{R}^d$ in the proof of estimate (3.3).

We introduce $L_v^2(m)$ the weighted Lebesgue space associated to the norm

$$(3.14) \quad \|f\|_{L_v^2(m)}^2 := \int_{\mathbb{R}^d} f^2(v) m^2(v) dv, \quad m(v) = \langle v \rangle^\ell$$

where we recall that ℓ has been fixed in Subsection 1.3 and satisfies $\ell > d/2$.

The behaviour of solutions to the homogeneous Fokker-Planck equation has been largely studied in several types of functional spaces thanks to Poincaré inequality or Log-Sobolev inequality for example. In our case, we will make use of a result from Gualdani, Mischler and Mouhot [15, Theorem 3.1] which is stated in the following lemma, the main idea is that they have enlarged the space in which the solutions to the equation decay as time goes to infinity to various kind of weighted Lebesgue spaces and in particular to the space $L_v^2(m)$ defined above.

Lemma 3.3. *Consider $f_0 = f_0(v) \in L_v^2(m)$ such that $\int_{\mathbb{R}^d} f_0(v) dv = 0$. Then, there exist $C \geq 1$ and $\lambda_* > 0$ such that the solution $f = f(t, v)$ to the homogeneous Fokker-Planck equation (3.13) satisfies*

$$\|f(t, \cdot)\|_{L_v^2(m)} \leq C e^{-\varepsilon \lambda_* t} \|f_0\|_{L_v^2(m)}.$$

3.3. Proof of Theorem 3.1. We now handle the proof of Theorem 3.1 and we split it into two parts, we first look at the behaviour of the density $\rho = \rho(t, x)$ before going back to the solution $h = h(t, x, v)$ itself. Let us underline the fact that in the proof of Theorem 2.1, we are able to deal with the cases $\varepsilon = 0$ and $\varepsilon > 0$ in the same time, which is not the case here. However, we do not write the proof for $\varepsilon = 0$ since it is just a (simpler) adaptation of the case $\varepsilon > 0$.

Proof of the estimate (3.2). As in the proof of the estimate (2.2), we start by noticing that, due to the conservation properties of the operator $-v \cdot \nabla_x h - F(t, x) \cdot \nabla_v M + \varepsilon (\Delta_v h + \operatorname{div}_v(vh))$, we know that for all times, $\widehat{\rho}(t, 0) = (2\pi)^{d/2} \widehat{h}_{in}(0, 0) = 0$ since h is mean-zero.

We then concentrate on the study of $\widehat{\rho}(t, k)$ for $k \in \mathbb{Z}^d \setminus \{0\}$. In this part, we only consider $\varepsilon \in (0, \varepsilon_0]$ where ε_0 is given in Lemma 3.2 and times t satisfying $0 \leq t \leq 1/\varepsilon$. We first take the Fourier transform of the equation (3.1). It provides us the following equality:

$$\partial_t \widehat{h} - k \cdot \nabla_\xi \widehat{h} + k \cdot \xi \widehat{W}(k) \widehat{\rho}(t, k) \widehat{M}(\xi) + \varepsilon \xi \cdot \nabla_\xi \widehat{h} = -\varepsilon |\xi|^2 \widehat{h}$$

where we have denoted $\widehat{h} = \widehat{h}(t, k, \xi)$. We rewrite the latter equality as:

$$\partial_t \widehat{h} + (\varepsilon \xi - k) \cdot \nabla_\xi \widehat{h} = -\varepsilon |\xi|^2 \widehat{h} - k \cdot \xi \widehat{W}(k) \widehat{\rho}(t, k) \widehat{M}(\xi)$$

in order to consider $(\varepsilon \xi - k) \cdot \nabla_\xi$ as our “new” transport operator, the term $-\varepsilon |\xi|^2 \widehat{h}$ is seen as an absorption term and $-k \cdot \xi \widehat{W}(k) \widehat{\rho}(t, k) \widehat{M}(\xi)$ as a source one. Then using the

method of characteristics, we end up with the following formula:

$$(3.15) \quad \begin{aligned} \widehat{h}(t, k, \xi) &= \exp\left(-\varepsilon \int_0^t \left| e^{-\varepsilon(t-s)} \xi + \chi_\varepsilon(t-s) k \right|^2 ds\right) \widehat{h}_{in}(k, e^{-\varepsilon t} \xi + \chi_\varepsilon(t) k) \\ &- \int_0^t \exp\left(-\varepsilon \int_s^t \left| e^{-\varepsilon(t-\sigma)} \xi + \chi_\varepsilon(t-\sigma) k \right|^2 d\sigma\right) k \cdot \left(e^{-\varepsilon(t-s)} \xi + \chi_\varepsilon(t-s) k \right) \\ &\quad \widehat{W}(k) \widehat{\rho}(s, k) \widehat{M}\left(e^{-\varepsilon(t-s)} \xi + \chi_\varepsilon(t-s) k \right) ds. \end{aligned}$$

Taking $\xi = 0$ in the previous formula gives

$$\begin{aligned} \widehat{\rho}(t, k) &= \exp\left(-\varepsilon \int_0^t |\chi_\varepsilon(t-s) k|^2 ds\right) \widehat{h}_{in}(k, \chi_\varepsilon(t) k) \\ &- \int_0^t \exp\left(-\varepsilon \int_s^t |\chi_\varepsilon(t-\sigma) k|^2 d\sigma\right) k \cdot (\chi_\varepsilon(t-s) k) \widehat{W}(k) \widehat{\rho}(s, k) \widehat{M}(\chi_\varepsilon(t-s) k) ds. \end{aligned}$$

that we can sum up in: for any $t \in [0, 1/\varepsilon]$,

$$\widehat{\rho}(t, k) = \exp\left(-\varepsilon \int_0^t |\chi_\varepsilon(t-s) k|^2 ds\right) \widehat{h}_{in}(k, \chi_\varepsilon(t) k) + \int_0^t K_\varepsilon(t-s) \widehat{\rho}(s, k) ds$$

where we recall that K_ε is defined in (3.4) and we have used that

$$\mathbb{1}_{0 \leq t \leq 1/\varepsilon} \mathbb{1}_{0 \leq s \leq t} = \mathbb{1}_{0 \leq t \leq 1/\varepsilon} \mathbb{1}_{0 \leq t-s \leq 1/\varepsilon} \mathbb{1}_{s \geq 0}.$$

Then, using Lemma 3.2 and proceeding as in the proof of estimate (2.2), we obtain:

$$\begin{aligned} \forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \sup_{t \in [0, 1/\varepsilon]} \langle kt \rangle^n |\widehat{\rho}(t, k)| &\leq \sup_{t \in [0, 1/\varepsilon]} \langle kt \rangle^n |\widehat{h}_{in}(k, \chi_\varepsilon(t) k)| \\ &\leq \sup_{t \in [0, 1/\varepsilon]} \frac{\langle kt \rangle^n}{\langle \chi_\varepsilon(t) k \rangle^n} \|h_{in}\|_{L_x^2 \mathcal{H}_v^n}. \end{aligned}$$

But, we recall that $\chi_\varepsilon(t) \geq e^{-1}t$ for any $t \in [0, 1/\varepsilon]$. Consequently

$$(3.16) \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \sup_{t \in [0, 1/\varepsilon]} \langle kt \rangle^n |\widehat{\rho}(t, k)| \leq C(n) \|h_{in}\|_{L_x^2 \mathcal{H}_v^n},$$

this provides us the result (3.2). \square

We are now able to go back to the analysis of the behaviour of the solution $h = h(t, x, v)$ thanks to the previous study on the density $\rho = \rho(t, x)$.

Proof of the estimate (3.3). We first deal with the case $k = 0$ and consider $\varepsilon \geq 0$. One can notice that the moment of order 0 in x is not preserved due to the presence of the collision operator as in the Proof of estimate (2.3). Indeed, $\langle h \rangle := \int_{\mathbb{T}^d} h(t, x, \cdot) dx$ satisfies the following homogeneous Fokker-Planck equation:

$$\partial_t \langle h \rangle = \varepsilon (\Delta_v \langle h \rangle + \operatorname{div}_v(v(h))).$$

Note that since $h_{in} \in L_x^2 \mathcal{H}_v^n$, we have $\langle h_{in} \rangle \in L_v^2(m)$ where $L_v^2(m)$ is defined through its norm (3.14). Indeed, using Jensen inequality,

$$\begin{aligned} \int_{\mathbb{R}^d} \langle h_{in} \rangle^2 m^2(v) dv &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{T}^d} h_{in}(x, v) dx \right)^2 m^2(v) dv \\ &\leq C \int_{\mathbb{R}^d \times \mathbb{T}^d} h_{in}^2(x, v) \langle v \rangle^{2\ell} dx dv \leq C \|h_{in}\|_{L_x^2 \mathcal{H}_v^n}^2. \end{aligned}$$

Using Lemma 3.3, we have

$$\forall t \geq 0, \quad \|\langle h \rangle(t, \cdot)\|_{L_v^2(m)} \leq C e^{-\varepsilon \lambda_* t} \|\langle h_{in} \rangle\|_{L_v^2(m)}$$

where we used that h_{in} is mean-zero. Consequently, using Cauchy-Schwarz inequality combined with the fact that $m^{-2} = \langle \cdot \rangle^{-2\ell} \in L_v^1(\mathbb{R}^d)$, we have

$$\forall \xi \in \mathbb{R}^d, \forall t \geq 0,$$

$$\begin{aligned} |\widehat{h}(t, 0, \xi)| &= |\widehat{\langle h \rangle}(t, \xi)| = \left| \int_{\mathbb{R}^d} \langle h \rangle(t, v) m(v) m^{-1}(v) e^{-iv \cdot \xi} dv \right| \\ &\leq \|\langle h \rangle(t, \cdot)\|_{L_v^2(m)} \int_{\mathbb{R}^d} m^{-2}(v) dv \\ &\leq C e^{-\varepsilon \lambda_* t} \|\langle h_{in} \rangle\|_{L_v^2(m)} \leq C e^{-\varepsilon \lambda_* t} \|h_{in}\|_{L_x^2 \mathcal{H}_v^n} \end{aligned}$$

from which we readily conclude.

We now focus on the case $k \in \mathbb{Z}^d \setminus \{0\}$, consider $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, 1/\varepsilon]$. For sake of simplicity, we introduce the notation

$$\psi_\varepsilon(t) := \frac{e^{\varepsilon t} - 1}{\varepsilon} = e^{\varepsilon t} \chi_\varepsilon(t), \quad t \in \mathbb{R}.$$

Going back to the equality (3.15), we have:

$$\begin{aligned} \widehat{h}(t, k, e^{\varepsilon t} \xi - e^{\varepsilon t} \chi_\varepsilon(t) k) &= \exp\left(-\varepsilon \int_0^t |e^{\varepsilon s} \xi - \psi_\varepsilon(s) k|^2 ds\right) \widehat{h_{in}}(k, \xi) \\ &\quad - \int_0^t \exp\left(-\varepsilon \int_s^t |e^{\varepsilon \sigma} \xi - \psi_\varepsilon(\sigma) k|^2 d\sigma\right) k \cdot (e^{\varepsilon s} \xi - \psi_\varepsilon(s) k) \\ &\quad \widehat{W}(k) \widehat{\rho}(s, k) \widehat{M}(e^{\varepsilon s} \xi - \psi_\varepsilon(s) k) ds =: J_1 + J_2. \end{aligned}$$

The first term is easily treated using Lemma 1.3:

$$(3.17) \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \forall \xi \in \mathbb{R}^d, \quad |J_1| \leq |\widehat{h_{in}}(k, \xi)| \leq C \langle \xi \rangle^{-n}.$$

As far as J_2 is concerned, we use that $|k| \widehat{W}(k) = O(1)$ combined with inequality (3.16) and the decay property of $\widehat{M} = M$ which in particular implies:

$$|e^{\varepsilon s} \xi - \psi_\varepsilon(s) k| \widehat{M}(e^{\varepsilon s} \xi - \psi_\varepsilon(s) k) \leq C \langle e^{\varepsilon s} \xi - \psi_\varepsilon(s) k \rangle^{2-n}.$$

We then estimate for any $t \in [0, 1/\varepsilon]$:

$$|J_2| \leq C \int_0^t \langle sk \rangle^{-n} \langle e^{\varepsilon s} \xi - \psi_\varepsilon(s) k \rangle^{2-n} ds.$$

Then, we use the following series of simple (in)equalities:

$$\begin{aligned} \langle e^{\varepsilon s} \xi - \psi_\varepsilon(s) k \rangle^{2-n} &= (1 + |e^{\varepsilon s} \xi - \psi_\varepsilon(s) k|^2)^{(2-n)/2} \\ &= (1 + e^{\varepsilon s} |\xi - \chi_\varepsilon(s) k|^2)^{(2-n)/2} \leq (1 + |\xi - \chi_\varepsilon(s) k|^2)^{(2-n)/2} \\ &= \langle \xi - \chi_\varepsilon(s) k \rangle^{2-n} \leq C \langle \xi \rangle^{2-n} \langle \chi_\varepsilon(s) k \rangle^{n-2} \leq C \langle \xi \rangle^{2-n} \langle sk \rangle^{n-2}, \end{aligned}$$

where we used that $\chi_\varepsilon(s) \leq s$ for any $s \geq 0$. Consequently,

$$|J_2| \leq C \langle \xi \rangle^{2-n} \int_0^t \langle sk \rangle^{-2} ds$$

and performing a change of variable, we obtain

$$(3.18) \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \forall \xi \in \mathbb{R}^d, \quad |J_2| \leq C \frac{1}{|k|} \langle \xi \rangle^{2-n} \leq C \langle \xi \rangle^{2-n}.$$

Combining (3.17) and (3.18), we reach the following conclusion

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \forall \xi \in \mathbb{R}^d, \forall t \in [0, 1/\varepsilon], \quad |\widehat{h}(t, k, e^{\varepsilon t} \xi - e^{\varepsilon t} \chi_\varepsilon(t)k)| \leq \frac{C}{\langle \xi \rangle^{n-2}}.$$

In particular, we have

$$\begin{aligned} \forall k \in \mathbb{Z}^d \setminus \{0\}, \forall \xi \in \mathbb{R}^d, \forall t \in [0, 1/\varepsilon], \\ |\widehat{h}(t, k, \xi)| \leq \frac{C}{\langle e^{-\varepsilon t} \xi + \chi_\varepsilon(t)k \rangle^{n-2}} \leq C \frac{\langle e^{-\varepsilon t} \xi \rangle^{n-2}}{\langle \chi_\varepsilon(t)k \rangle^{n-2}} \leq C \frac{\langle \xi \rangle^{n-2}}{\langle \chi_\varepsilon(t)k \rangle^{n-2}}, \end{aligned}$$

where we have used that

$$\langle e^{-\varepsilon t} \xi + \chi_\varepsilon(t)k \rangle^{2-n} \leq C \langle e^{-\varepsilon t} \xi \rangle^{n-2} \langle \chi_\varepsilon(t)k \rangle^{2-n}.$$

We now take advantage of the fact that $\chi_\varepsilon(t) \geq e^{-1}t$ for $t \in [0, 1/\varepsilon]$ to get the inequality $\langle \chi_\varepsilon(t)k \rangle^{2-n} \leq C \langle kt \rangle^{2-n} \leq C \langle t \rangle^{2-n}$ and consequently,

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \forall \xi \in \mathbb{R}^d, \forall t \in [0, 1/\varepsilon], \quad |\widehat{h}(t, k, \xi)| \leq \frac{C(\xi)}{\langle t \rangle^{n-2}}$$

with $C(\xi) := C \langle \xi \rangle^{n-2}$ and this concludes the proof of (3.3). \square

REFERENCES

- [1] BEDROSSIAN, J., MASMOUDI, N., AND MOUHOT, C. Landau damping: paraproducts and Gevrey regularity. To appear in *Annals of PDE*.
- [2] BEDROSSIAN, J., MASMOUDI, N., AND VICOL, V. Enhanced Dissipation and Inviscid Damping in the Inviscid Limit of the Navier–Stokes Equations Near the Two Dimensional Couette Flow. *Arch. Ration. Mech. Anal.* 219, 3 (2016), 1087–1159.
- [3] BOUCHUT, F. Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions. *J. Funct. Anal.* 111, 1 (1993), 239–258.
- [4] BOUCHUT, F., AND DOLBEAULT, J. On long time asymptotics of the Vlasov-Fokker-Planck equation and of the Vlasov-Poisson-Fokker-Planck system with Coulombic and Newtonian potentials. *Differ. Integral Equ.* 8, 3 (1995), 487–514.
- [5] CAGLIOTI, E., AND MAFFEI, C. Time asymptotics for solutions of Vlasov-Poisson equation in a circle. *J. Stat. Phys.* 92, 1-2 (1998), 301–323.
- [6] CARRILLO, J. A., SOLER, J., AND VÁZQUEZ, J. L. Asymptotic behaviour and self-similarity for the three-dimensional Vlasov-Poisson-Fokker-Planck system. *J. Funct. Anal.* 141, 1 (1996), 99–132.
- [7] DESVILLETES, L., AND DOLBEAULT, J. On long time asymptotics of the Vlasov-Poisson-Boltzmann equation. *Comm. Partial Differential Equations* 16, 2-3 (1991), 451–489.
- [8] DOLBEAULT, J. Free energy and solutions of the Vlasov-Poisson-Fokker-Planck system: external potential and confinement (large time behavior and steady states). *J. Math. Pures Appl. (9)* 78, 2 (1999), 121–157.
- [9] DOLBEAULT, J., MOUHOT, C., AND SCHMEISER, C. Hypocoercivity for linear kinetic equations conserving mass. *Trans. Amer. Math. Soc.* 367, 6 (2015), 3807–3828.
- [10] DUAN, R., AND LIU, S. The Vlasov-Poisson-Boltzmann system without angular cutoff. *Comm. Math. Phys.* 324, 1 (2013), 1–45.
- [11] DUAN, R., AND LIU, S. Stability of the rarefaction wave of the Vlasov-Poisson-Boltzmann system. *SIAM J. Math. Anal.* 47, 5 (2015), 3585–3647.
- [12] DUAN, R., AND STRAIN, R. M. Optimal time decay of the Vlasov-Poisson-Boltzmann system in \mathbb{R}^3 . *Arch. Ration. Mech. Anal.* 199, 1 (2011), 291–328.

- [13] DUAN, R., YANG, T., AND ZHAO, H. The Vlasov-Poisson-Boltzmann system in the whole space: the hard potential case. *J. Differential Equations* 252, 12 (2012), 6356–6386.
- [14] FAOU, E., AND ROUSSET, F. Landau Damping in Sobolev Spaces for the Vlasov-HMF Model. *Arch. Ration. Mech. Anal.* 219, 2 (2016), 887–902.
- [15] GUALDANI, M., MISCHLER, S., AND MOUHOT, C. Factorization for non-symmetric operators and exponential H-Theorem. *Preprint arXiv:1006.5523*.
- [16] GUO, Y. The Vlasov-Poisson-Boltzmann system near vacuum. *Comm. Math. Phys.* 218, 2 (2001), 293–313.
- [17] GUO, Y. The Vlasov-Poisson-Boltzmann system near Maxwellians. *Comm. Pure Appl. Math.* 55, 9 (2002), 1104–1135.
- [18] HÉRAU, F., AND THOMANN, L. On global existence and trend to the equilibrium for the vlasov-poisson-fokker-planck system with exterior confining potential. *Preprint arXiv:1505.01698*.
- [19] HWANG, H. J., AND VELÁZQUEZ, J. J. L. On the existence of exponentially decreasing solutions of the nonlinear Landau damping problem. *Indiana Univ. Math. J.* 58, 6 (2009), 2623–2660.
- [20] LI, H.-L., YANG, T., AND ZHONG, M. Spectrum analysis for the Vlasov-Poisson-Boltzmann system. *Preprint arXiv:1402.3633*.
- [21] MISCHLER, S., AND TRISTANI, I. Uniform semigroup spectral analysis of the discrete, fractional & classical Fokker-Planck equations. *Preprint arXiv:1507.04861*.
- [22] MOUHOT, C., AND VILLANI, C. On Landau damping. *Acta Math.* 207, 1 (2011), 29–201.
- [23] PENROSE, O. Electrostatic instability of a uniform non-Maxwellian plasma. *Phys. Fluids* 3 (1960), 258–265.
- [24] REIN, G., AND WECKLER, J. Generic global classical solutions of the Vlasov-Fokker-Planck-Poisson system in three dimensions. *J. Differential Equations* 99, 1 (1992), 59–77.
- [25] RYUTOV, D. D. Landau damping: half a century with the great discovery. *Plasma Physics and Controlled Fusion* 41, 3A (1999), A1.
- [26] VICTORY, JR., H. D., AND O'DWYER, B. P. On classical solutions of Vlasov-Poisson Fokker-Planck systems. *Indiana Univ. Math. J.* 39, 1 (1990), 105–156.
- [27] VILLANI, C. (Ir)reversibility and entropy. *Séminaire Poincaré* (2010).
- [28] VILLANI, C. Landau damping. *Notes de cours, CEMRACS* (2010).
- [29] XIAO, Q., XIONG, L., AND ZHAO, H. The Vlasov-Poisson-Boltzmann system for non-cutoff hard potentials. *Sci. China Math.* 57, 3 (2014), 515–540.

CENTRE DE MATHÉMATIQUES LAURENT SCHWARTZ, ÉCOLE POLYTECHNIQUE, CNRS, UNIVERSITÉ PARIS-SACLAY, 91128 PALAISEAU CEDEX, FRANCE. E-MAIL: isabelle.tristani@polytechnique.edu