THE FOURIER EXPANSION OF $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$

CHRISTIAN KASSEL AND CHRISTOPHE REUTENAUER

ABSTRACT. We compute the Fourier coefficients of the weight one modular form $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$ in terms of the number of representations of an integer as a sum of two squares. We deduce a relation between this modular form and translates of the modular form $\eta(z)^4/\eta(2z)^2$.

1. INTRODUCTION

In this note we consider the η-product

$$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)} = \prod_{n\geqslant 1} \frac{(1-q^n)^2}{1-q^n+q^{2n}}. \tag{1.1}$$

where $q = e^{2\pi iz}$. Recall that $\eta(z)$ is Dedekind's eta function

$$\eta(z) = e^{\pi iz/12} \prod_{n\geqslant 1} (1-q^n).$$

The η-product $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$ is a modular form of weight 1 and level 6. Since it is invariant under the transformation $z \mapsto z+1$, it has a Fourier expansion of the form

$$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)} = \sum_{n\geqslant 0} a_6(n)\, q^n, \tag{1.2}$$

where the Fourier coefficients $a_6(n)$ are integers. For general information on η-products, see [4, Sect. 2.1].

Our first result expresses $a_6(n)$ in terms of the number $r(n)$ of representations of n as the sum of two squares, i.e the number of elements $(x,y) \in \mathbb{Z}^2$ such that $x^2+y^2 = n$. Observe that $r(n)$ is divisible by 4 for all $n \geqslant 1$ (for $n=0$ we have $r(0)=1$). The sequence $r(n)$ appears as Sequence A004018 in [5].

Theorem 1.1. *For all non-negative integers m we have*

$$\begin{aligned} a_6(3m) &= (-1)^m\, r(3m), \\ a_6(3m+1) &= (-1)^{m+1}\, \frac{r(3m+1)}{4}, \\ a_6(3m+2) &= (-1)^{m+1}\, \frac{r(3m+2)}{2}. \end{aligned}$$

We next relate $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$ to the weight one modular form $\eta(z)^4/\eta(2z)^2$ and two of its translates.

2010 *Mathematics Subject Classification.* (Primary) 11F11, 11F20, 14C05, 14G15, 14N10.
Key words and phrases. Dedekind eta function, eta products, Fourier coefficient, punctual Hilbert scheme.

Theorem 1.2. *Set $j = e^{2\pi i/3}$. We have the following linear relation between weight one modular forms:*

$$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)} = \frac{1}{4}\frac{\eta(z)^4}{\eta(2z)^2} + \frac{1-j}{4}\frac{\eta(z+1/3)^4}{\eta(2z+2/3)^2} + \frac{1-j^2}{4}\frac{\eta(z+2/3)^4}{\eta(2z+1/3)^2}.$$

Both modular forms $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$ and $\eta(z)^4/\eta(2z)^2$ came up naturally in [3], where we computed the number $C_n(q)$ of ideals of codimension n of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of Laurent polynomials in two variables over a finite field $\mathbb{F}_q$ of cardinality q. Equivalently, $C_n(q)$ is the number of $\mathbb{F}_q$-points of the Hilbert scheme of n points on a two-dimensional torus. We proved that $C_n(q)$ is the value at q of a palindromic one-variable polynomial $C_n(x) \in \mathbb{Z}[x]$ with integer coefficients, which we computed completely (see [3, Th. 1.3]).

We also showed (see [3, Cor. 6.2]) that the generating function of the polynomials $C_n(x)$ can be expressed as the following infinite product:

$$1 + \sum_{n\geqslant 1} \frac{C_n(x)}{x^n} q^n = \prod_{n\geqslant 1} \frac{(1-q^n)^2}{1-(x+x^{-1})q^n + q^{2n}}. \tag{1.3}$$

It follows from the previous equality that $C_n(1) = 0$. Actually, we proved (see [3, Th. 1.3 and 1.4]) that there exists a polynomial $P_n(x) \in \mathbb{Z}[x]$ such that $C_n(x) = (x-1)^2 P_n(x)$. Moreover, $P_n(x)$ is palindromic, has non-negative coefficients and its value at $x = 1$ is equal to the sum of divisors of n: $P_n(1) = \sum_{d|n} d$.

When $x = e^{2i\pi/k}$ with $k = 2, 3, 4$, or 6, then $x + x^{-1} = 2\cos(2\pi/k)$ is an integer. For such an integer k, we define the sequence $a_k(n)$ by

$$\sum_{n\geqslant 0} a_k(n)\, q^n = \prod_{n\geqslant 1} \frac{(1-q^n)^2}{1-2\cos(2\pi/k)\, q^n + q^{2n}}. \tag{1.4}$$

Since $2\cos(2\pi/k)$ is an integer, so is each $a_k(n)$. It follows from (1.3) that these integers are related to the polynomials $C_n(x)$ by

$$C_n(e^{2i\pi/k}) = a_k(n)\, e^{2ni\pi/k}.$$

In [3] we computed $a_2(n)$, $a_3(n)$, and $a_4(n)$ explicitly in terms of well-known arithmetical functions. In particular, we established the equality

$$a_2(n) = (-1)^n\, r(n), \tag{1.5}$$

where $r(n)$ is the number of representations of n as the sum of two squares.

We also observed in [3, (1.8)] that

$$\sum_{n\geqslant 0} a_2(n)\, q^n = \frac{\eta(z)^4}{\eta(2z)^2} \quad \text{and} \quad \sum_{n\geqslant 0} a_6(n)\, q^n = \frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)}. \tag{1.6}$$

The question of finding an explicit expression for $a_6(n)$ had been left open in [3]. This is now solved with Theorem 1.1 of this note. In view of this theorem, of (1.5), and of (1.6), for all $m \geqslant 0$ we obtain

$$\begin{cases} a_6(3m) = & a_2(3m), \\ a_6(3m+1) = & \dfrac{a_2(3m+1)}{4}, \\ a_6(3m+2) = & -\dfrac{a_2(3m+2)}{2}. \end{cases} \tag{1.7}$$

We had experimentally observed (see [3, Note 7]) that $a_6(n) = 0$ whenever $a_2(n) = 0$. As a consequence of (1.7) we can now state that $a_6(n) = 0$ if and only if $a_2(n) = 0$, i.e. if and only n is not the sum of two squares.

Theorems 1.1 and 1.2 will be proved in the next two sections.

Remarks 1.3. (a) The sequence $a_6(n)$ is Sequence A258210 in [5]. The sequence $a_6(3n+1)$ is probably the opposite of Sequence A258277 in *loc. cit.*

(b) It can be seen from Table 1 that $a_6(n)$ is not a multiplicative function. Indeed, $a_6(10) \neq a_6(2)a_6(5)$ or $a_6(18) \neq a_6(2)a_6(9)$ or $a_6(20) \neq a_6(4)a_6(5)$.

TABLE 1. *First values of* $a_6(n)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$a_6(n)$	−1	−2	0	1	4	0	0	−2	−4	2	0	0	−2	0	0	1	4	4	0	−4

2. PROOF OF THEOREM 1.1

2.1. For any odd integer m we set $\xi(m) = -2\sin(m\pi/6)$. Because of the well-known properties of the sine function, $\xi(m)$ depends only on the class of m modulo 12 and we have the following equalities for all odd m:

$$\xi(-m) = -\xi(m) \quad \text{and} \quad \xi(m+6) = -\xi(m), \tag{2.1}$$

which is equivalent to $\xi(-m) = -\xi(m)$ and $\xi(6-m) = \xi(m)$.

We have

$$\xi(m) = \begin{cases} -1 & \text{if } m \equiv 1,5 \pmod{12}, \\ -2 & \text{if } m \equiv 3 \pmod{12}, \\ 1 & \text{if } m \equiv 7,11 \pmod{12}, \\ 2 & \text{if } m \equiv 9 \pmod{12}. \end{cases} \tag{2.2}$$

Next consider the excess function $E_1(n;4)$ defined by

$$E_1(n;4) = \sum_{d|n,\, d\equiv 1 \,(\mathrm{mod}\ 4)} 1 - \sum_{d|n,\, d\equiv -1 \,(\mathrm{mod}\ 4)} 1.$$

It is a multiplicative function, i.e., $E_1(mn;4) = E_1(m;4)\,E_1(n;4)$ whenever m and n are coprime. It is well known that the excess function can be computed in terms of the prime decomposition of n. Write $n = 2^c p_1^{a_1} p_2^{a_2} \cdots q_1^{b_1} q_2^{b_2} \cdots$, where p_i, q_i's are distinct prime numbers such that $p_i \equiv 1 \pmod 4$ et $q_i \equiv 3 \pmod 4$. Then $E_1(n;4) = 0$ if and only if one of the exponents b_i is odd. If all b_i's are even, then

$$E_1(n;4) = (1+a_1)(1+a_2)\cdots\,. \tag{2.3}$$

In the sequel we will need the following result.

Lemma 2.1. *Let n be a positive integer which is not divisible by* 3*. We have*

$$\sum_{d|n,\, d\ \mathrm{odd}} \xi(d) = -E_1(n;4) \quad \textit{and} \quad \sum_{d|n,\, d\ \mathrm{odd}} \xi(3d) = -2E_1(n;4).$$

Proof. Let d be an odd divisor of n; it is not divisible by 3 since n is not. Therefore, $d \equiv 1, 5, 7$ or $11 \pmod{12}$. Observe that $d \equiv 1, 5 \pmod{12}$ if and only if $d \equiv 1 \pmod 4$ since $d \equiv 3 \pmod{12}$ is excluded. Similarly, $d \equiv 7, 11 \pmod{12}$ if and only if $d \equiv 3 \pmod 4$. Now, $\xi(d) = -1$ if $d \equiv 1, 5$ and $\xi(d) = 1$ if $d \equiv 7, 11 \pmod{12}$. Consequently,

$$\sum_{d|n\,,\, d \text{ odd}} \xi(d) = \sum_{d|n\,,\, d\equiv 3 \pmod 4} 1 - \sum_{d|n\,,\, d\equiv 1 \pmod 4} 1 = -E_1(n;4).$$

Similarly, $\xi(3d) = \xi(3) = -2$ if $d \equiv 1, 5$ and $\xi(3d) = \xi(9) = 2$ if $d \equiv 7, 11 \pmod{12}$. Therefore,

$$\sum_{d|n\,,\, d \text{ odd}} \xi(3d) = \sum_{d|n\,,\, d\equiv 3 \pmod 4} 2 - \sum_{d|n\,,\, d\equiv 1 \pmod 4} -2 = -2E_1(n;4).$$

□

2.2. We now express $a_6(n)$ in terms of the function ξ introduced above.

Proposition 2.2. *We have*

$$a_6(n) = \sum_{d|n\,,\, d \text{ odd}} \xi\left(\frac{2n}{d} - d\right). \tag{2.4}$$

Note that $2n/d - d$ is an odd integer since d is an odd divisor of n.

Proof. Set $u = \pi/k$ and $\omega = d$ in Formula (9.3) of [2, p. 10]. It becomes

$$\sum_{n\geqslant 0} a_k(n)\, q^n = 1 - 4\sin(\pi/k) \sum_{n\geqslant 1} \left(\sum_{d|n\,,\, d \text{ odd}} \sin\left(\left(\frac{2n}{d} - d\right)\frac{\pi}{k}\right)\right) q^n. \tag{2.5}$$

Consider the special case $k = 6$ of (2.5). Since $\sin(\pi/6) = 1/2$, Equality (2.5) becomes

$$\begin{aligned}\sum_{n\geqslant 0} a_6(n)\, q^n &= 1 - 2\sum_{n\geqslant 1} \left(\sum_{d|n\,,\, d \text{ odd}} \sin\left(\left(\frac{2n}{d} - d\right)\frac{\pi}{6}\right)\right) q^n \\ &= 1 + \sum_{n\geqslant 1} \left(\sum_{d|n\,,\, d \text{ odd}} \xi\left(\frac{2n}{d} - d\right)\right) q^n.\end{aligned}$$

The formula for $a_6(n)$ follows. □

Proof of Theorem 1.1. Let us first mention the following well-known fact (see [1, § 51, Th. 65]): the number $r(n)$ of representations of n as a sum of two squares is related to the excess function $E_1(n;4)$ by

$$r(n) = 4\,E_1(n;4) \tag{2.6}$$

for all $n \geqslant 0$. It follows from this fact and from (1.5) that

$$a_2(n) = (-1)^n 4\,E_1(n;4). \tag{2.7}$$

We now distinguish three cases according to the residue of n modulo 3.

(a) We start with the case $n \equiv 1 \pmod 3$. We have $n = 3\ell + 1$ for some non-negative integer ℓ. Since the odd divisors d of n are not divisible by 3, they must satisfy $d \equiv 1, 5, 7$ or $11 \pmod{12}$. Such divisors are invertible $\pmod{12}$ et we have $d^2 \equiv 1 \pmod{12}$. Consequently,

$$\frac{2n}{d} - d = \frac{2nd^2}{d} - d = 2nd - d \pmod{12}.$$

Hence,

$$\xi\left(\frac{2n}{d} - d\right) = \xi(2nd - d) = \xi(6d\ell + d) = ((-1)^d)^\ell \xi(d) = (-1)^\ell \xi(d)$$

in view of (2.1). Therefore, by Proposition 2.2,

$$a_6(n) = (-1)^\ell \sum_{d|n\,,\, d \text{ odd}} \xi(d).$$

Together with Lemma 2.1 and (2.7), this implies

$$a_6(n) \quad = \quad (-1)^{\ell+1} E_1(n;4) = (-1)^{n+\ell+1} a_2(n)/4.$$

Finally observe that n is odd (resp. even) if ℓ is even (resp. odd). Therefore, $a_6(n) = a_2(n)/4$.

(b) Now consider the case $n \equiv 2 \pmod 3$. We have $n = 3\ell + 2$ for some non-negative integer ℓ. Again the odd divisors d of n must satisfy $d \equiv 1, 5, 7, 11 \pmod{12}$ since they are not divisible by 3. Consequently, as above,

$$\xi\left(\frac{2n}{d} - d\right) = \xi(2nd - d) = \xi(6d\ell + 3d) = (-1)^\ell \xi(3d).$$

By Lemma 2.1 and (2.7), we obtain

$$\begin{aligned} a_6(n) &= (-1)^\ell \sum_{d|n\,,\, d \text{ odd}} \xi(3d) \\ &= (-1)^{\ell+1}\, 2E_1(n;4) = (-1)^{n+\ell+1} a_2(n)/2. \end{aligned}$$

Since n and ℓ are of the same parity, we have $a_6(n) = -a_2(n)/2$.

(c) Finally we consider the case when n is divisible by 3. We write $n = 3^N m$, where m is coprime to 3 and $N \geqslant 1$. Any odd divisor d of n is of the form $d = 3^r s$ for some odd divisor s of m and $0 \leqslant r \leqslant N$. Since m and its divisors s are not divisible by 3 and since s is odd, we again have $s \equiv 1, 5, 7$ or $11 \pmod{12}$. Recall that for such s we have $s^2 \equiv 1 \pmod{12}$. Thus, for $d = 3^r s$, we obtain

$$\frac{2n}{d} - d \equiv \left(2 \cdot 3^{N-r} m - 3^r\right) s \pmod{12}.$$

If $r = 0$, then $2n/d - d \equiv (6 \cdot 3^{N-1} m - 1)s \pmod{12}$. Therefore,

$$\xi\left(\frac{2n}{d} - d\right) = \xi\left((6 \cdot 3^{N-1} m - 1)s\right) = (-1)^m \xi(-s) = (-1)^{m-1} \xi(s).$$

in view of (2.1).

If $0 < r < N$, then $2n/d - d \equiv (6 \cdot 3^{N-r-1} m - 3^r)s \pmod{12}$. Therefore,

$$\xi\left(\frac{2n}{d} - d\right) = \xi\left((6 \cdot 3^{N-r-1} m - 3^r)s\right) = (-1)^m\, \xi(-3^r s) = (-1)^{m-1}\, \xi(3^r s).$$

Now, $3^r \equiv 3 \pmod{12}$ if r is odd, and $3^r \equiv -3$ if $r > 0$ is even. Then by (2.1),

$$\xi\left(\frac{2n}{d} - d\right) = (-1)^{m-r}\, \xi(3s).$$

Now consider the case $r = N$. If N is odd, then $3^N \equiv 3 \pmod{12}$ and

$$\xi\left(\frac{2n}{d} - d\right) = \xi\left((2m - 3^N)s\right) = \xi((2m-3)s).$$

Now, if m is odd, then $m \equiv 1, 5, 7$ or $11 \pmod{12}$. We have $2m - 3 \equiv 7$ or $11 \pmod{12}$ and the multiplication by 7 or by 11 exchanges the sets $\{1, 5\}$ and $\{7, 11\}$. Since by (2.2) ξ takes opposite values on such sets, we have $\xi((2m-3)s) = -\xi(s)$. Consequently, $\xi(2n/d - d) = -\xi(s)$ when m is odd.

If m is even, then $m \equiv 2, 4, 8$ or $10 \pmod{12}$. Then $2m - 3 \equiv 1$ or $5 \pmod{12}$. The multiplication by 1 or by 5 preserves each set $\{1, 5\}$ and $\{7, 11\}$, so that by (2.2) we have $\xi((2m-3)s) = \xi(s)$. In conclusion,

$$\xi(2n/d - d) = (-1)^m\, \xi(s)$$

when $r = N$ is odd.

If $r = N$ is even, then $3^N \equiv -3 \pmod{12}$ and $\xi(2n/d - d) = \xi((2m - 3^N)s) = \xi((2m+3)s)$. A reasoning as in the odd N case shows that when N is even we have

$$\xi(2n/d - d) = (-1)^{m-1}\, \xi(s).$$

We can now compute $a_6(n)$. We start with the case of odd N. Putting the above information together, we obtain

$$\begin{aligned}
a_6(n) &= \sum_{d|n,\, d \text{ odd}} \xi\left(\frac{2n}{d} - d\right) = \sum_{s|m,\, s \text{ odd}} \sum_{r=0}^{N} \xi\left(\frac{2\cdot 3^N m}{3^r s} - d\right) \\
&= \sum_{s|m,\, s \text{ odd}} \left((-1)^{m-1}\xi(s) + \left(\sum_{r=1}^{N-1} (-1)^{m-r}\right) \xi(3s) + (-1)^m\, \xi(s) \right) \\
&= \left((-1)^{m-1} + (-1)^m\right) \sum_{s|m,\, s \text{ odd}} \xi(s) = 0.
\end{aligned}$$

On the other hand, since the power of 3 in n is odd, then by (2.3) we have $a_2(n) = (-1)^n 4\, E_1(n, 4) = 0$. Therefore, $a_6(n) = a_2(n)$ in this case.

If N is even, then

$$\begin{aligned}
a_6(n) &= \sum_{d|n,\, d \text{ odd}} \xi\left(\frac{2n}{d} - d\right) = \sum_{s|m,\, s \text{ odd}} \sum_{r=0}^{N} \xi\left(\frac{2n}{d} - d\right) \\
&= \sum_{s|m,\, s \text{ odd}} \left((-1)^{m-1}\xi(s) + \left(\sum_{r=1}^{N-1} (-1)^{m-r}\right) \xi(3s) + (-1)^{m-1}\xi(s) \right) \\
&= \sum_{s|m,\, s \text{ odd}} \left((-1)^{m-1} 2\,\xi(s) + (-1)^{m-1}\xi(3s)\right) \\
&= (-1)^{m-1} \left(2 \sum_{s|m,\, s \text{ odd}} \xi(s) + \sum_{s|m,\, s \text{ odd}} \xi(3s) \right) \\
&= (-1)^m 4\, E_1(m; 4)
\end{aligned}$$

by Lemma 2.1. Now, by multiplicativity of the excess fonction,

$$E_1(n; 4) = E_1(3^N; 4)\, E_1(m; 4) = E_1(m; 4)$$

since $E_1(3^N;4) = 1$ for even N. Finally, m and n being of the same parity, we have

$$a_6(n) = (-1)^m\, 4\, E_1(m;4)) = (-1)^n\, 4\, E_1(n;4)) = a_2(n).$$

Q.e.d. $\square$

3. Proof of Theorem 1.2

Set $f(q) = \eta(z)\eta(2z)\eta(3z)/\eta(6z) = \sum_{n\geqslant 0} a_6(n)\, q^n$ and $g(q) = \eta(z)^4/\eta(2z)^2 = \sum_{n\geqslant 0} a_2(n)\, q^n$; see (1.6). To prove Theorem 1.2 it suffices to check that

$$f(q) = ag(q) + bg(jq) + cg(j^2q),$$

where $a = 1/4$, $b = (1-j)/4$, and $c = (1-j^2)/4$. Now,

$$\begin{aligned} ag(q) + bg(jq) + cg(j^2q) &= a\sum_{n\geqslant 0} a_2(n)\, q^n + b\sum_{n\geqslant 0} a_2(n)\, j^nq^n \\ &\quad + c\sum_{n\geqslant 0} a_2(n)\, j^{2n}q^n \\ &= (a+b+c)\sum_{m\geqslant 0} a_2(3m)\, q^{3m} \\ &\quad + (a+jb+j^2c)\sum_{m\geqslant 0} a_2(3m+1)\, q^{3m+1} \\ &\quad + (a+j^2b+jc)\sum_{m\geqslant 0} a_2(3m+2)\, q^{3m+2}. \end{aligned}$$

It follows from (1.7) that

$$\begin{aligned} ag(q) + bg(jq) + cg(j^2q) &= (a+b+c)\sum_{m\geqslant 0} a_6(3m)\, q^{3m} \\ &\quad + 4(a+jb+j^2c)\sum_{m\geqslant 0} a_6(3m+1)\, q^{3m+1} \\ &\quad - 2(a+j^2b+jc)\sum_{m\geqslant 0} a_6(3m+2)\, q^{3m+2}. \end{aligned}$$

The right-hand side is equal to $f(q)$ since $a+b+c = 1$, $a+jb+j^2c = 1/4$, and $a+j^2b+jc = -1/2$. Q.e.d.

References

[1] L. E. Dickson, *Introduction to the theory of numbers*, The University of Chicago Press, Chicago, Illinois, Sixth impression, 1946.

[2] N. J. Fine, *Basic hypergeometric series and applications*, Mathematical Surveys and Monographs, 27, Amer. Math. Soc., Providence, RI, 1988.

[3] C. Kassel, C. Reutenauer, *On the zeta function of a punctual Hilbert scheme of the two-dimensional torus*, arXiv:1505.07229.

[4] G. Köhler, *Eta products and theta series identities* Springer Monographs in Mathematics. Springer, Heidelberg, 2011.

[5] *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://oeis.org.

Christian Kassel: Institut de Recherche Mathématique Avancée, CNRS & Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg, France

E-mail address: kassel@math.unistra.fr

URL: www-irma.u-strasbg.fr/~kassel/

Christophe Reutenauer: Mathématiques, Université du Québec à Montréal, Montréal, CP 8888, succ. Centre Ville, Canada H3C 3P8

E-mail address: reutenauer.christophe@uqam.ca

URL: www.lacim.uqam.ca/~christo/