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LIMIT VALUE OF DYNAMIC ZERO-SUM GAMES WITH VANISHING STAGE DURATION

SYLVAIN SORIN

Abstract. We consider two person zero-sum games where the players control, at discrete times $\{t_n\}$ induced by a partition Π of $\mathbb{R}^+$, a continuous time Markov state process. We prove that the limit of the values v_Π exist as the mesh of Π goes to 0. The analysis covers the cases of :
1) stochastic games (where both players know the state)
2) symmetric no information.
The proof is by reduction to a deterministic differential game.

1. Introduction

Repeated interactions in a stationary environment have been traditionally represented by dynamic games played in stages. An alternative approach is to consider a continuous time process on which the players act at discrete times. In the first case the expected number of interactions increases as the weight θ_n of each stage n goes to zero. In the second case the number of interactions increases when the duration δ_n of each time interval n vanishes.

In a repeated game framework one can normalize the model using the evaluation of the stages, so that stage n is associated to time $t_n = \sum_{j=1}^{n-1} \theta_j$, and then consider the game played on $[0,1]$ where time t corresponds to the fraction t of the total duration. Each evaluation θ (in the original repeated game) thus induces a partition Π_θ of $[0,1]$ with vanishing mesh corresponding to vanishing stage weight. Tools adapted from continuous time models can be used to obtain convergence results, given an ordered set of evaluations, for the corresponding family of values v_θ, see e.g. for different classes of games, Sorin [32], [33], [34], Vieille [40], Laraki [23], Cardaliaguet, Laraki and Sorin [7].

In the alternative approach considered here, there is a given evaluation $\mathbf{k}$ on $\mathbb{R}^+$ and one consider a sequence of partitions $\Pi(m)$ of $\mathbb{R}^+$ with vanishing mesh corresponding to vanishing stage duration and the associated sequence of values.

In both cases, for each given partition the value function exists at the times defined by the partition and the stationarity of the model allows to write a recursive formula (RF). Then one extends the value function to $[0,1]$ (resp. $\mathbb{R}^+$) by linearity and one considers the family of values as the mesh of the partition goes to 0. The next two steps in the proof of convergence of the family of values consist in defining a PDE (Main Equation ME) and proving :
1) that any accumulation point of the family is a viscosity solution of (ME)
2) that (ME) has a unique viscosity solution.

Altogether the tools are quite similar to those used in differential games, however in the current framework the state is basically a random variable and the players use mixed strategies.

Section 2 describes the model. Section 3 is devoted to the framework where both players observe the state variable. Section 4 deals with the situation where the state is unknown but the moves are observed. In both cases the analysis is done by reduction to a differential game. Section 5 presents the main results concerning differential games that are used in the paper.

2. Smooth continuous time games and discretization

2.1. Discretization of a continuous time process and associated game.

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Consider a time homogeneous state process Z_t, defined on $\mathbb{R}^+ = [0, +\infty)$, with values in a state space Ω and an evaluation given by a probability density $\mathbf{k}(t)$ on $\mathbb{R}^+$.
Each partition $\Pi = \{t_1 = 0, t_2, ..., t_n, ...\}$ of $\mathbb{R}^+$ induces a discrete time game as follows. The time interval $L_n = [t_n, t_{n+1}[$ corresponds to stage n and has duration δ_n. The law of Z_t on L_n is determined by its value at time t_n, $\hat{Z}_n = Z_{t_n}$ and the actions $(i_n, j_n) \in I \times J$ chosen by the players at time t_n, that last for stage n. The payoff at time t in stage n ($t \in L_n$) is defined trough a map $\mathbf{g}$ from $\Omega \times I \times J$ to $\mathbb{R}$:

$$\mathbf{g}_\Pi(t) = g(Z_t, i_n, j_n)$$

(An alternative choice leading to the same asymptotic results would be $g_\Pi(t) = \mathbf{g}(\hat{Z}_n, i_n, j_n)$).
The evaluation along a play is:

$$\gamma_{\Pi,\mathbf{k}} = \int_0^{+\infty} \mathbf{g}_\Pi(t)\mathbf{k}(dt)$$

and the corresponding value function is $v_{\Pi,\mathbf{k}}$.
One will study the asymptotics of the family $\{v_{\Pi,\mathbf{k}}\}$ as the mesh $\delta = \sup \delta_n$ of the partition Π vanishes.

2.2. **Markov process.**

From now on we consider the case where $Z_t, t \in \mathbb{R}^+$ follows a continuous time Markov process: it is specified by a transition rate $\mathbf{q}$ that belongs to the set $\mathcal{M}$ of real bounded maps on $I \times J \times \Omega \times \Omega$ with:

$$\mathbf{q}(i,j)[\omega, \omega'] \geq 0, \quad \text{if } \omega' \neq \omega, \quad \text{and} \sum_{\omega' \in \Omega} \mathbf{q}(i,j)[\omega, \omega'] = 0, \quad \forall i, j, \omega.$$

Let $\mathsf{P}^h(i,j), h \in \mathbb{R}^+$ be the continuous time Markov chain on Ω generated by the kernel $\mathbf{q}(i,j)$:

$$\dot{\mathsf{P}}^h(i,j) = \mathsf{P}^h(i,j)\,\mathbf{q}(i,j) = \mathbf{q}(i,j)\ \mathsf{P}^h(i,j)$$

and for $t \geq 0$:

$$\mathsf{P}^{t+h}(i,j) = \mathsf{P}^t(i,j)\, e^{h\ \mathbf{q}(i,j)}.$$

In particular, one has:

$$\begin{aligned} \mathsf{P}^h(i,j)[z,z'] &= Prob\,(Z_{t+h} = z' | Z_t = z), \qquad \forall t \geq 0 \\ &= \mathbf{1}_{\{z\}}(z') + h\,\mathbf{q}(i,j)[z,z'] + o(h) \end{aligned}$$

2.3. **Hypotheses.**

One assume from now on:
the state space Ω is finite,
the evaluation $\mathbf{k}$ is Lipschitz continuous on $\mathbb{R}^+$.
the action sets I, J are compact metric spaces,
the payoff $\mathbf{g}$ and the transition $\mathbf{q}$ are continuous on $I \times J$.

2.4. **Notations.**

If μ is a bounded measurable function defined on $I \times J$ with values in a convex set, $\mu(x,y)$ denotes its multilinear extension to $X \times Y$, with $X = \Delta(I)$ (resp. $Y = \Delta(J)$), set of regular Borel probabilities on I (resp. J). (This applies in particular to $\mathbf{g}$ and $\mathbf{q}$).
For $\zeta \in \Delta(\Omega)$ and $\mu \in \mathbb{R}^{\Omega^2}$ we define :

$$\zeta * \mu\ (z) = \sum_{\omega \in \Omega} \zeta(\omega)\mu[\omega, z].$$

(When g is a map from Ω to itself and $\mu[\omega, z] = \mathbf{1}_{\{g(\omega)=z\}}$, $\zeta * g$ is the usual image measure).
In particular, if $\zeta_t \in \Delta(\Omega)$ is the law of Z_t one has, if (i,j) is played on $[t, t+h]$

$$\zeta_{t+h} = \zeta_t * \mathsf{P}^h(i,j)$$

and

$$\dot{\zeta}_t = \zeta_t * \mathbf{q}(i,j).$$

Similarly we use the following notation for a transition probability or a transition rate μ operating on a real function f on Ω:

$$\mu[z,.]\circ f(\cdot)=\sum_{z'}\mu[z,z']f(z')=\mu\circ f\,[z].$$

3. State controlled and publicly observed

This section is devoted to the case were the process Z_t is controlled by both players and observed by both (there is no assumptions on the signals on the actions). A stage n (time t_n) both players know Z_{t_n}. This corresponds to a stochastic game G in continuous time analyzed trough a time discretization along Π, G_Π.
Previous related papers to stochastic games in continuous time include Zachrisson [42], Tanaka and Wakuta [39], Guo and Hernandez-Lerma [19], [20], Neyman [25].
The approach via time discretization is related to similar procedures in differential games, see Section 5, Fleming [14], [15], [16], Scarf [30] and Neyman [26].

3.1. **General case.**

Consider a general evaluation $\mathbf{k}$. Since $\mathbf{k}$ is fixed during the analysis we will write v_Π for $v_{\Pi,\mathbf{k}}$, defined on $\mathbb{R}^+\times\Omega$.

3.1.1. *Recursive formula.*

The hypothesis on the data implies that v_Π exists, see e.g. [24], Chapters IV and VII, or [27], and in the current framework the recursive formula takes the following form:

Proposition 3.1.
The game G_Π has a value v_Π satisfying the recursive equation:

$$\begin{aligned} v_\Pi(t_n,Z_{t_n}) &= \mathtt{val}_{X\times Y}\,\mathsf{E}_{z,x,y}[\int_{t_n}^{t_{n+1}}\mathbf{g}(Z_s,i,j)\mathbf{k}(s)ds+v_\Pi(t_{n+1},Z_{t_{n+1}})]\\ &= \mathtt{val}_{X\times Y}\,[\mathsf{E}_{z,x,y}(\int_{t_n}^{t_{n+1}}\mathbf{g}(Z_s,i,j)\mathbf{k}(s)ds)+\mathsf{P}^{\delta_n}(x,y)[Z_{t_n},.]\circ v_\Pi(t_{n+1},.)] \qquad (1)\end{aligned}$$

Proof
This is the basic recursive formula for the stochastic game with state space Ω, action sets I and J and transition kernel $\mathsf{P}^{\delta_n}(i,j)$, going back to Shapley [31]. ∎

Recall that the value $v_\Pi(.,z)$ is defined at times $t_n\in\Pi$ and extended by linearity to $\mathbb{R}^+$.

3.1.2. *Main equation.*

The first property is standard in this framework.

Proposition 3.2.
The family of values $\{v_\Pi\}$ is uniformly Lipschitz w.r.t. $t\in\mathbb{R}^+$.

Denote thus by $\mathbf{V}$ the (non empty) set of accumulation points of the family $\{v_\Pi\}_\Pi$ (for the uniform convergence on compact subsets of $\mathbb{R}^+\times\Omega$) as the mesh δ vanishes.

Definition 3.1. *A continuous real function u on $\mathbb{R}^+\times\Omega$ is a viscosity solution of:*

$$0=\frac{d}{dt}u(t,z)+\mathtt{val}_{X\times Y}\{\mathbf{g}(z,x,y)\mathbf{k}(t)+\mathbf{q}(x,y)[z,.]\circ u(t,\cdot)\}, \qquad (2)$$

if for any real function ψ, $\mathcal{C}^1$ on $\mathbb{R}^+\times\Omega$ with $u-\psi$ having a strict maximum at $(\bar t,\bar z)\in\mathbb{R}^+\times\Omega$:

$$0\le\frac{d}{dt}\psi(\bar t,\bar z)+\mathtt{val}_{X\times Y}\{\mathbf{g}(\bar z,x,y)\mathbf{k}(\bar t)+\mathbf{q}(x,y)[\bar z,.]\circ\psi(\bar t,\cdot)\}$$

and the dual condition.

Proposition 3.3.
Any $u\in\mathbf{V}$ is a viscosity solution of (2).

Proof
Let $\psi(t,z)$ be a $\mathcal{C}^1$ test function such that $u-\psi$ has a strict maximum at $(\bar{t},\bar{z})$. Consider a sequence $V_m = v_{\Pi(m)}$ converging uniformy locally to u as $m\to\infty$ and let $(t^*(m),z(m))$ be a minimizing sequence for $\{(\psi - V_m)(t,z), t\in\Pi_m\}$. In particular $(t^*(m),z(m))$ converges to $(\bar{t},\bar{z})$ as $m\to\infty$. Given x^*_m optimal for $V_m(t^*(m),z(m))$ in (1), one obtains, with $t^*(m)=t_n\in\Pi_m$:

$$V_m(t_n,z(m)) \le \mathsf{E}_{z(m),x^*_m,y}[\int_{t_{n-1}}^{t_n} \mathbf{g}(Z_s,i,j)\mathbf{k}(s)ds] + \mathsf{P}^{\delta_n}(x^*_m,y)[z(m),.]\circ V_m(t_{n+1},.), \qquad \forall y\in Y,$$

so that by the choice of $(t^*(m),z(m))$:

$$\begin{aligned}\psi(t_n,z(m)) &\le \mathsf{E}_{z(m),x^*_m,y}[\int_{t_{n-1}}^{t_n} \mathbf{g}(Z_s,i,j)\mathbf{k}(s)ds] + \mathsf{P}^{\delta_n}(x^*_m,y)[z(m),.]\circ\psi(t_{n+1},.)]\\ &\le \delta_n\mathbf{k}(t_n)\,\mathbf{g}(z(m),x^*_m,y) + \psi(t_{n+1},z(m)) + \delta_n\,\mathbf{q}(x^*_m,y)[z(m),.]\circ\psi(t_{n+1},.) + o(\delta_n).\end{aligned}$$

This implies:

$$0 \le \delta_n\mathbf{k}(t_n)\,\mathbf{g}(z(m),x^*_m,y) + \delta_n\,\frac{d}{dt}\psi(t_n,z(m)) + \delta_n\,\mathbf{q}(x^*_m,y)[z(m),.]\circ\psi(t_{n+1},.) + o(\delta_n)$$

hence dividing by δ_n and taking the limit as $m\to\infty$ one obtains, for some accumulation point x^* in the compact set $\Delta(I)$:

$$0 \le \mathbf{k}(\bar{t})\,\mathbf{g}(\bar{z},x^*,y) + \frac{d}{dt}\psi(\bar{t},\bar{z}) + \mathbf{q}(x^*,y)[\bar{z},.]\circ\psi(\bar{t},.), \qquad \forall y\in Y \tag{3}$$

so that:

$$0 \le \frac{d}{dt}\psi(\bar{t},\bar{z}) + \mathtt{val}_{X\times Y}[\,\mathbf{g}(\bar{z},x,y)\,\mathbf{k}(\bar{t}) + \mathbf{q}(x,y)[\bar{z},.]\circ\psi(\bar{t},.)].$$

■

3.1.3. *Convergence.*
A first proof of the convergence of the family $\{v_\Pi\}_\Pi$ would follow from the property:
(P) Equation (2) has a unique viscosity solution.

An alternative approach is to relate the game to a differential game on an extended state space $\Delta(\Omega)$. Define V_Π on $\mathbb{R}^+\times\Delta(\Omega)$ as the expectation of v_Π, namely:

$$V_\Pi(t,\zeta) = \langle\zeta, v_\Pi(t,.)\rangle = \sum_{\omega\in\Omega}\zeta(\omega)v_\Pi(t,\omega)$$

and denote $\mathbf{X}=X^\Omega$ and $\mathbf{Y}=Y^\Omega$.

Proposition 3.4.
V_Π satisfies:

$$V_\Pi(t_n,\zeta_{t_n}) = \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\,[\sum_\omega \zeta_{t_n}(\omega)\mathsf{E}_{\omega,\mathbf{x}(\omega),\mathbf{y}(\omega)}(\int_{t_n}^{t_{n+1}} \mathbf{g}(Z_s,i,j)\rangle\mathbf{k}(s)ds) + V_\Pi(t_{n+1},\zeta_{t_{n+1}})] \tag{4}$$

where $\zeta_{t_{n+1}}(z) = \sum_\omega \zeta_{t_n}(\omega)\mathsf{P}^{\delta_n}(\mathbf{x}(\omega),\mathbf{y}(\omega))(\omega,z)$.

Proof
(4) follows from (1), the definition of V_Π and the formula expressing $\zeta_{t_{n+1}}$. By independence the optimization in X at each ω can be replaced by optimization in $\mathbf{X}$ and one uses the linearity in the transition. ■

Equation (1) corresponds to the usual approach following the trajectory of the process. Equation (4) expresses the dynamics of the law ζ of the process, where the players act differently at different states ω.

3.1.4. *Related differential game.*
We will prove that the recursive equation (4) is satisfied by the value of the time discretization along Π of the mixed extension of a deterministic differential game $\mathcal{G}$ (see Section 5) on $\mathbb{R}^+$, defined as follows:
1) the state space is $\Delta(\Omega)$,
2) the action sets are $\mathbf{I} = I^\Omega$ and $\mathbf{J} = J^\Omega$,
3) the dynamics on $\Delta(\Omega) \times \mathbb{R}^+$ is:

$$\dot{\zeta}_t = f(\zeta_t, \mathbf{i}, \mathbf{j})$$

with

$$f(\zeta, \mathbf{i}, \mathbf{j})(z) = \sum_{\omega \in \Omega} \zeta_t(\omega) \mathbf{q}(\mathbf{i}(\omega), \mathbf{j}(\omega))[\omega, z].$$

4) the flow payoff function is given by:

$$\langle \zeta, \mathbf{g}(., \mathbf{i}(.), \mathbf{j}(.)) \rangle = \sum_{\omega \in \Omega} \zeta(\omega) \mathbf{g}(\omega, \mathbf{i}(\omega), \mathbf{j}(\omega)).$$

5) the global outcome is:

$$\int_0^{+\infty} \gamma_t \, \mathbf{k}(t) dt$$

where γ_t is the payoff at time t.
In $\mathcal{G}_\Pi$ the state is deterministic and at each time t_n the players know ζ_{t_n} and choose $\mathbf{i}_n$ (resp. $\mathbf{j}_n$).
Consider now the mixed extension $\mathcal{G}_\Pi^{II}$ (Section 5) and let $\mathcal{V}_\Pi(t, \zeta)$ be the associated value.

Proposition 3.5.
The value $\mathcal{V}_\Pi(t, \zeta)$ satisfies the recursive equation (4).

Proof
The mixed action set for player 1 is $\tilde{\mathbf{X}}$ but due to the separability in ω one can work with $\mathbf{X}$. Then it is easy to see that equation (32) corresponds to (4). ∎

The analysis in section 5 thus implies that :
- any accumulation point U of the sequence $\mathcal{V}_\Pi$ is a viscosity solution of

$$0 = \frac{d}{dt} U(t, \zeta) + \mathtt{val}_{\mathbf{X} \times \mathbf{Y}} \left[\langle \zeta, \mathbf{g}(., \mathbf{x}(.), \mathbf{y}(.)) \rangle \mathbf{k}(t) + \langle f(\zeta, \mathbf{x}, \mathbf{y}), \nabla U(t, \zeta) \rangle \right] \tag{5}$$

- Equation (5) has a unique viscosity solution.

In particular let $\mathsf{U}(t, \zeta) = \langle \zeta, u(t, .) \rangle = \sum_\omega \zeta(\omega) u(t, \omega)$ where $u \in \mathbf{V}$.

Proposition 3.6.
$\mathsf{U}(t, \zeta)$ *is the viscosity solution of (5).*

Proof
Follows from the fact that V_Π and $\mathcal{V}_\Pi$ satisfy the same recursive formula, hence U is an acumulation point of the sequence $\mathcal{V}_\Pi$. ∎

This leads to the convergence property.

Corollary 3.1. *Both families V_Π and v_Π converge to some V and v with*

$$V(t, \zeta) = \sum_\omega \zeta(\omega) v(t, \omega).$$

V is the viscosity solution of (5).
v is the viscosity solution of (2).

Proof
One has $\nabla\mathcal{V}(t,\zeta)=\{v(t,.)\}$ hence:

$$\begin{aligned} 0 &= \langle\zeta,\frac{d}{dt}v(t,.)\rangle+\mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\,[\langle\zeta,\mathbf{g}(.,\mathbf{x}(.),\mathbf{y}(.))\rangle\mathbf{k}(t)+\sum_z[\sum_\omega\zeta(\omega)\mathbf{q}(\mathbf{x}(\omega),\mathbf{y}(\omega))[\omega,z]v(t,z)] \\ &= \langle\zeta,\frac{d}{dt}v(t,.)\rangle+\mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\,[\langle\zeta,\mathbf{g}(.,\mathbf{x}(.),\mathbf{y}(.))\rangle\mathbf{k}(t)+\sum_\omega\zeta(\omega)[\sum_z\mathbf{q}(\mathbf{x}(\omega),\mathbf{y}(\omega))[\omega,z]v(t,z)] \end{aligned}$$

This gives:

$$0=\langle\zeta,\frac{d}{dt}v(t,.)+\mathtt{val}_{X\times Y}[\mathbf{g}(.,x,y)\mathbf{k}(t)+\mathbf{q}(x,y)[.,\bullet]\circ v(t,\bullet)]\rangle$$

which is equivalent to:

$$0=\frac{d}{dt}v(t,z)+\mathtt{val}_{X\times Y}[\mathbf{g}(z,x,y)\mathbf{k}(t)+\mathbf{q}(x,y)[z,.]\circ v(t,\cdot)].$$

and this is (2).

■

3.2. **Stationary case.**
We consider the case $\mathbf{k}(t)=\rho e^{-\rho t}$ and again the game along the partition Π.

3.2.1. *Recursive formula.*
The general recursive formula (1) takes now the following form:

Proposition 3.7.

$$\begin{aligned} v_{\Pi,\rho}(t_n,Z_{t_n}) &= \mathtt{val}_{X\times Y}\,\mathsf{E}_{z,x,y}[\int_{t_n}^{t_{n+1}}\mathbf{g}(Z_s,i,j)\rho e^{-\rho}ds+v_{\Pi,\rho}(t_{n+1},Z_{t_{n+1}})] \\ &= \mathtt{val}_{X\times Y}\,[\mathsf{E}_{z,x,y}(\int_{t_n}^{t_{n+1}}\mathbf{g}(Z_s,i,j)\rho e^{-\rho}ds)+\mathsf{P}^{\delta_n}(x,y)[Z_{t_n},.]\circ v_{\Pi,\rho}(t_{n+1},.)] \end{aligned} \tag{6}$$

and if Π is uniform, $v_{\Pi,\rho}(t,z)=e^{-\rho t}\nu_{\delta,\rho}(z)$ with:

$$\nu_{\delta,\rho}(Z_0)=\mathtt{val}_{X\times Y}\,[\mathsf{E}_{z,x,y}(\int_0^\delta\mathbf{g}(Z_s,x,y)\rho e^{-\rho}ds)+e^{-\rho\delta}\,\mathsf{P}^\delta(x,y)[Z_0,.]\circ\nu_{\delta,\rho}(.)] \tag{7}$$

3.2.2. *Main equation.*
The next result is standard, see e.g. Neyman [26], Prieto-Rumeau and Hernandez-Lerma [28], p. 235. We provide a short proof for convenience.

Proposition 3.8.
1) For any $R\in\mathcal{M}$ and any $\rho\in(0,1]$ the equation, with variable φ from Ω to $\mathbb{R}$:

$$\rho\,\varphi(z)=\mathtt{val}_{X\times Y}[\rho\,\mathbf{g}(z,x,y)+R(x,y)[z,.]\circ\ \varphi(.)] \tag{8}$$

has a unique solution, denoted W_ρ.
2) For any $\delta\in(0,1]$ such that $\|\delta R/(1-\delta\rho)\|\le 1$ the solution of (8) is the value of the repeated stochastic game with payoff g, transition $P=I+\delta R/(1-\delta\rho)$ and discounted factor $\delta\rho$.

Proof
Recall from Shapley [31], that the value $W_{\rho\delta}$ of a repeated stochastic game with payoff $\mathbf{g}$ and discounted factor $\delta\rho$ satisfies:

$$W_{\rho\delta}(z)=\mathtt{val}_{X\times Y}[\delta\rho\,\mathbf{g}(z,x,y)+(1-\delta\rho)\mathsf{E}_{z,x,y}\{W_{\rho\delta}(.)\}]. \tag{9}$$

Assume the transition to be of the form $P=I+\delta q$ with $q\in\mathcal{M}$. One obtains:

$$W_{\rho\delta}(z)=\mathtt{val}_{X\times Y}[\delta\rho\,\mathbf{g}(z,x,y)+(1-\delta\rho)\{W_{\rho\delta}(z)+\delta\,q(x,y)[z,.]\circ W_{\rho\delta}(.)\}] \tag{10}$$

which gives:

$$\delta\rho\,W_{\rho\delta}(z)=\mathtt{val}_{X\times Y}[\delta\rho\,\mathbf{g}(z,x,y)+\delta(1-\delta\rho)\,q(x,y)[z,.]\circ W_{\rho\delta}(.)] \tag{11}$$

so that:

$$\rho\, W_{\rho\delta}(z) = \mathtt{val}_{X\times Y}[\rho\, \mathbf{g}(z,x,y) + (1-\delta\rho)\, q(x,y)[z,.] \circ W_{\rho\delta}(.)]. \tag{12}$$

Hence with $q = R/(1-\delta\rho)$ one obtains:

$$\rho\, W_{\rho\delta}(z) = \mathtt{val}_{X\times Y}[\rho\, \mathbf{g}(z,x,y) + R(x,y)[z,.] \circ W_{\rho\delta}(.)]. \tag{13}$$

■

3.2.3. *Convergence.*
Again the following result can be found in Neyman [26], Theorem 1, see also Guo Hernandez-Lerma [19, 20].

Proposition 3.9.
As the mesh δ of the partition Π goes to 0, $v_{\Pi,\rho}$ converges to the solution W_ρ of (8) with $R = \mathbf{q}$:

$$\rho\, W_\rho(z) = \mathtt{val}_{X\times Y}[\rho\, g(z,x,y) + \mathbf{q}(x,y)[z,.] \circ\ W_\rho(.)] \tag{14}$$

<u>Proof</u>
Consider the strategy σ of Player 1 in G_Π defined as follows: at state z, use an optimal strategy $x \in X = \Delta(I)$, for $W_\rho(z)$ given by (14). Let us evaluate, given τ, strategy of Player 2, the following amount:

$$A_1 = \mathsf{E}_{\sigma,\tau}[\int_{t_1}^{t_2} g_\Pi(s)\rho e^{-\rho s} ds + e^{-\rho t_1} W_\rho(Z_{t_1})].$$

Let x_1 the mixed move of Player 1 at stage one given $Z_0 = \hat{Z}_1$. Then if y_1 is induced by τ, there exists a constant L such that:

$$\begin{aligned} A_1 &\geq \delta_1\rho\, g(\hat{Z}_1, x_1, y_1) + (1-\delta_1\rho)[W_\rho(\hat{Z}_1) + \delta_1\, \mathbf{q}(x_1,y_1)[\hat{Z}_1,.] \circ W_\rho(.)] - \delta_1 L\delta \\ &\geq \delta_1\rho\, g(\hat{Z}_1, x_1, y_1) - \delta_1\rho\, W_\rho(\hat{Z}_1) + \delta_1\, \mathbf{q}(x_1,y_1)[\hat{Z}_1,.] \circ W_\rho(.) + W_\rho(\hat{Z}_1) - 2\delta_1 L\delta \\ &\geq W_\rho(\hat{Z}_1) - 2\delta_1 L\delta. \end{aligned}$$

Similarly let:

$$A_n = \mathsf{E}_{\sigma,\tau}[\int_{t_n}^{t_{n+1}} g_\Pi(s)\rho e^{-\rho s} ds ds + e^{-\rho t_n} W_\rho(\hat{Z}_{n+1}) | h_n]$$

where $h_n = (\hat{Z}_1, i_1, j_1, \cdots, i_{n-1}, j_{n-1}, \hat{Z}_n)$.
Then, with obvious notations:

$$\begin{aligned} A_n &\geq e^{-\rho t_{n-1}}[\delta_n\rho\, g(\hat{Z}_n, x_n, y_n) + (1-\delta_n\rho)[W_\rho(\hat{Z}_n) + \delta_n\, \mathbf{q}(x_n,y_n)[\hat{Z}_n,.] \circ W_\rho(.) - \delta_n L\delta] \\ &\geq e^{-\rho t_{n-1}}[\delta_n\rho\, g(\hat{Z}_n, x_n, y_n) - \delta_n\rho\, W_\rho(\hat{Z}_n) + \delta_n\, \mathbf{q}(x_n,y_n)[\hat{Z}_n,.] \circ W_\rho(.) + W_\rho(\hat{Z}_n) - 2\delta_n L\delta] \\ &\geq e^{-\rho t_{n-1}}[W_\rho(\hat{Z}_n) - 2\delta_n L\delta]. \end{aligned}$$

Taking the sum and the expectation, one obtains that the payoff induced by (σ,τ) in G_Π satisfies:

$$\mathsf{E}_{\sigma,\tau}[\int_0^{+\infty} g_\Pi(s)k(s)ds] \geq W_\rho(\hat{Z}_1) - 2(\sum_n \delta_n e^{-\rho t_{n-1}})L\delta$$

and $(\sum_n \delta_n e^{-\rho t_{n-1}})L\delta \to 0$ as $\delta \to 0$. ■

Comments:

The proof in Neyman [26] is done, in the finite case, for a uniform partition but shows the robustness with respect to the parameters (converging family of games).

This procedure of proof is reminiscent of the "direct approach" introduced by Isaacs [21]. To show convergence of the family of values of the discretizations v_Π: *i)* one identifies a tentative limit value v and a recursive formula $RF(v)$ and *ii)* one shows that to play in the discretized game G_Π an optimal strategy in $RF(v)$ gives an amount close to v for δ small enough.

For an alternative approach and proof, based on properties of the Shapley operator, see Sorin and Vigeral [37].

Remark that if $\mathbf{k}(t) = \rho e^{-\rho t}$, $v(t,z) = e^{-\rho t}\nu(z)$ satisfies (2) iff $\nu(z)$ satisfies (14).

4. State controlled and not observed

This section studies the game G where the process Z_t is controlled by both players but not observed. However the past actions are known: this defines a symmetric framework were the new state variable is the law of Z_t, $\zeta_t \in \Delta(\Omega)$. Even in the stationary case there is no explicit smooth solution to the main equation hence a direct approach for proving convergence, as in the previous Section 3.2, is not feasible.

Here also the analysis will be trough the connection to a differential game $\overline{\mathcal{G}}$ on $\Delta(\Omega)$ but different from the previous one $\mathcal{G}$, introduced in Section 3.

Given a partition Π denote by G_Π the associated game and again, since $\mathbf{k}$ is fixed during the analysis we will write V_Π for its value $\mathsf{V}_{\Pi,\mathbf{k}}$ defined on $\mathbb{R}^+ \times \Delta(\Omega)$.
Recall that given the initial law ζ_{t_n} and the actions $(i_{t_n}, j_{t_n}) = (i,j)$ one has:

$$\zeta^{ij}_{t_{n+1}} = \zeta_{t_n+\delta_n} = \zeta_{t_n} * \mathsf{P}^{\delta_n}(i,j) \tag{15}$$

and that this parameter is known by both players.
Extend $\mathbf{g}(.,x,y)$ from Ω to $\Delta(\Omega)$ by linearity: $\mathbf{g}(\zeta,x,y) = \sum \zeta(z)\mathbf{g}(z,x,y)$.

4.1. **Recursive formula.**

In this framework the recursive structure leads to:

Proposition 4.1.
The value V_Π satisfies the following recursive formula:

$$\mathsf{V}_\Pi(t_n,\zeta_{t_n}) \quad = \quad \mathtt{val}_{X\times Y}\, \mathsf{E}_{\zeta,x,y}[\int_{t_n}^{t_{n+1}} \mathbf{g}(\zeta_s,i,j)\mathbf{k}(s)ds + \mathsf{V}_\Pi(t_{n+1},\zeta^{ij}_{t_{n+1}})] \tag{16}$$

<u>Proof</u>
Standard, since G_Π is basically a stochastic game with parameter ζ. ∎

4.2. **Main equation.**

Consider the differential game $\overline{\mathcal{G}}$ on $\Delta(\Omega)$ with actions sets I and J, dynamics on $\Delta(\Omega)\times\mathbb{R}^+$ given by:

$$\dot{\zeta}_t = \zeta_t * \mathbf{q}(i,j),$$

current payoff $\mathbf{g}(\zeta,i,j)$ and evaluation $\mathbf{k}$.

As in Section 5, consider the discretized mixed extension $\overline{\mathcal{G}}^{II}_\Pi$ to $X\times Y$ and let $\overline{\mathcal{V}}_\Pi$ be its value.

Proposition 4.2.
$\overline{\mathcal{V}}_\Pi$ satisfies (16).

<u>Proof</u>
$\overline{\mathcal{V}}_\Pi$ satisfies (32) which is, using (15), equivalent to (16). ∎

The analysis in Section 5, Proposition 5.12 thus implies:

Proposition 4.3.
The family of values V_Π converge to V unique viscosity solution of :

$$0 = \frac{d}{dt}u(t,\zeta) + \mathtt{val}_{X\times Y}[\mathbf{g}(\zeta,x,y)\mathbf{k}(t) + \langle \zeta * \mathbf{q}(x,y), \nabla u(t,\zeta)]. \tag{17}$$

4.3. **Stationary case.**

Assume $\mathbf{k}(t) = \rho e^{-\rho t}$.
In this case one has $\mathsf{V}(\zeta,t) = e^{-\rho t}\mathsf{v}(\zeta)$ hence (17) becomes

$$\rho\mathsf{v}(\zeta) = \mathtt{val}_{X\times Y}[\rho\, \mathbf{g}(\zeta,x,y) + \langle \zeta * \mathbf{q}(x,y), \nabla\mathsf{v}(\zeta)\rangle] \tag{18}$$

4.4. **Comments.**
A differential game similar to $\overline{\mathcal{G}}$ where the state space is the set of probabilities on some set Ω has been studied in full generality by Cardaliaguet and Quincampoix [8], see also As Soulaimani [1].
Equation (18) is satisfied by the value of the Non-Revealing game in the framework analyzed by Cardaliaguet, Rainer, Rosenberg and Vieille [9] see Section 6.

5. Discretization and mixed extension of differential games

We study here the value of a continuous time game by introducing a time discretization Π and analyzing the limit behavior of the associated family of values v_Π as the mesh of the partition vanishes. This approach was initiated in Fleming [14], [15],[16], and developped in Friedman [17], [18], Eliott and Kalton [12].

A differential game γ is defined trough the following components: $Z \subset \mathbb{R}^n$ is the state space, I and J are the action sets of player 1 (maximizer) and 2, f from $Z \times I \times J$ to $\mathbb{R}^n$ is the dynamics kernel, g from $Z \times I \times J$ to $\mathbb{R}$ is the payoff-flow function and k from $\mathbb{R}^+$ to $\mathbb{R}^+$ determines the evaluation.
Formally the dynamics is defined on $[0, +\infty) \times Z$ by :

$$\dot{z}_t = f(z_t, i_t, j_t) \tag{19}$$

and the total payoff is :

$$\int_0^{+\infty} g(z_s, i_s, j_s)\, k(s)\, ds.$$

We assume:
I and J metric compact sets,
f and g continuous and uniformly Lipschitz in z,
g bounded,
k Lipschitz with $\int_0^{+\infty} k(s)ds = 1$.
$\Phi^h(z; i, j)$ denote the value at time $t+h$ of the solution of (19) starting at time t from z and with play $\{i_s = i, j_s = j\}$ on $[t, t+h]$.
To define the strategies we have to specify the information: we assume that the players know the initial state z_0, and at time t the previous play $\{i_s, j_s; 0 \le s < t\}$ hence the trajectory of the state $\{z_s; 0 \le s \le t\}$.
The analysis below will show that Markov strategies (i.e. depending only, at time t, on t and z_t) will suffice.

5.1. **Deterministic analysis.**
Let $\Pi = (\{t_n\}, n = 1, ...)$ be a partition of $[0, +\infty)$ with $t_1 = 0, \delta_n = t_{n+1} - t_n$ and $\delta = \sup \delta_n$. We consider the associated discrete time game γ_Π where on each interval $[t_n, t_{n+1})$ players use constant moves (i_n, j_n) in $I \times J$. This defines the dynamics on the state. At time t_{n+1}, (i_n, j_n) is announced and the corresponding value of the state, $z_{t_{n+1}} = \Phi^{\delta_n}(z_{t_n}; i_n, j_n)$ is known.
The associated maxmin w_Π^- satisfies the recursive formula:

$$w_\Pi^-(t_n, z_{t_n}) = \sup_I \inf_J [\int_{t_n}^{t_{n+1}} g(z_s, i, j)k(s)ds + w_\Pi^-(t_{n+1}, z_{t_{n+1}})] \tag{20}$$

The fonction $w_\Pi^-(., z)$ is extended by linearity to $[0, +\infty)$ and note that:

$$\forall \varepsilon > 0, \exists T, \text{ such that } t \ge T \text{ implies } |w_\Pi^-(t, .)| \le \varepsilon \tag{21}$$

and that all "value" functions that we will consider here will satisfy this property.

The next four results follow from the analysis in Evans and Souganidis [13], see also Barron, Evans and Jensen [3], Souganidis [38] and the presentation in Bardi and Capuzzo-Dolcetta [2], Chapter VII, Section 3.2.

Proposition 5.1.
The family $\{w^-_\Pi(t,z)\}$ is uniformly equicontinuous in both variables.

Hence the set U of accumulation points of the family $\{w^-_\Pi\}$ (for the uniform convergence on compact subsets of $\mathbb{R}^+ \times Z$), as the mesh δ of Π goes to zero, is non empty.

We first introduce the notion of viscosity solution, see Crandall and Lions [10].

Definition 5.1. *Given an Hamiltonian H from $\mathbb{R}^+ \times Z \times \mathbb{R}^n$ to $\mathbb{R}$, a continuous real function u on $\mathbb{R}^+ \times Z$ is a viscosity solution of:*

$$0 = \frac{d}{dt}u(t,z) + H(t,z,\nabla u(t,z)) \tag{22}$$

if for any real function ψ, $\mathcal{C}^1$ on $\mathbb{R}^+ \times Z$ with $u - \psi$ having a strict maximum at $(\bar{t},\bar{z}) \in \mathbb{R}^+ \times Z$:

$$0 \leq \frac{d}{dt}\psi(\bar{t},\bar{z}) + H(t,z,\nabla\psi(t,z))$$

and the dual condition holds.

We can now introduce the Hamilton-Jacobi-Isaacs (HJI) equation that follows from (20), corresponding to the Hamiltonian:

$$h^-(t,z,p) = \sup_I \inf_J [g(z,i,j)k(t) + \langle f(z,i,j),p\rangle]. \tag{23}$$

Proposition 5.2.
Any accumulation point $u \in U$ is a viscosity solution of:

$$0 = \frac{d}{dt}u(t,z) + \sup_I \inf_J [g(z,i,j)k(t) + \langle f(z,i,j),\nabla u(t,z)\rangle]. \tag{24}$$

Note that in the discounted case, $k(t) = \lambda e^{-\lambda t}$, with the change of variable $u(t,z) = e^{-\lambda t}\phi(z)$, one obtains:

$$\lambda\phi(z) = \sup_I \inf_J [\lambda g(z,i,j) + \langle f(z,i,j),\nabla\phi(z)\rangle]. \tag{25}$$

The main property is the following:

Proposition 5.3.
Equation (24) has a unique viscosity solution.

Recall that this notion and this result are due to Crandall and Lions [10], for more properties see Crandall, Ishii and Lions [11].

The uniqueness of accumulation point implies:

Corollary 5.1.
The family $\{w^-_\Pi\}$ converges to some w^-.

An alternative approach is the consider the game γ in normal form on $\mathbb{R}^+$. Let w^-_∞ be the maxmin (lower value) of the continuous time differential game played using non anticipative strategies with delay. Then from Evans and Souganidis [13], extended in Cardaliaguet [6], Chapter 3, one obtains:

Proposition 5.4.
1) w^-_∞ is a viscosity solution of (24).
2) Hence:

$$w^-_\infty = w^-.$$

Obviously similar properties hold for the minmax w^+_Π and w^+_∞.

Finally define Isaacs's condition on $I \times J$ by :

(26)
$$\sup_I \inf_J [g(z,i,j)\ k(t) + \langle f(z,i,j), p\rangle] = \inf_J \sup_I [g(z,i,j)\ k(t) + \langle f(z,i,j), p\rangle],\ \forall t \in \mathbb{R}^+, \forall z \in Z, \forall p \in \mathbb{R}^n,$$

which, with the notation (23), corresponds to :

$$h^-(t,z,p) = h^+(t,z,p).$$

Proposition 5.5.
Assume condition (26).
Then the limit value exists, in the sense that:

$$w^- = w^+ (= w^-_\infty = w^+_\infty)$$

Note that the same analysis holds if the players use strategies that depend only at time t_n on t_n and z_{t_n}.

5.2. **Mixed extension.**
We define two mixed extensions of γ as follows: for each partition Π we introduce two discrete time games associated to γ_Π and played on $X = \Delta(I)$ and $Y = \Delta(Y)$ (set of probabilities on I and J respectively). We will then prove that their asymptotic properties coincide.

5.2.1. *Deterministic actions.*
The first game Γ^I is defined as in subsection 5.1 were X and Y are now the sets of actions (this corresponds to "relaxed controls") replacing I and J.
The main point is that the dynamics f (hence the flow) and the payoff g are extended to $X \times Y$ by taking the expectation w.r.t. x and y:

$$f(z,x,y) = \int_{I\times J} f(z,i,j)x(di)y(dj)$$

$$\dot{z}_t = f(z_t, x_t, y_t) \tag{27}$$

$$g(z,x,y) = \int_{I\times J} g(z,i,j)x(di)y(dj).$$

Γ^I_Π is the associated discrete time game where on each interval $[t_n, t_{n+1})$ players use constant actions (x_n, y_n) in $X \times Y$. This defines the dynamics: $\bar{\Phi}^h(z;x,y)$ denotes the value at time $t+h$ of the solution of (27) starting at time t from z and with play $\{x_s = x, y_s = y\}$ on $[t, t+h]$. Note that $\bar{\Phi}^h(z;x,y)$ is *not* the bilinear extension of $\Phi^h(z;i,j)$. At time t_{n+1}, (x_n, y_n) is announced and the current value of the state, $z_{t_{n+1}} = \bar{\Phi}^{\delta_n}(z_{t_n}; x_n, y_n)$ is known.
The maxmin W^-_Π satisfies the recursive formula:

$$W^-_\Pi(t_n, z_{t_n}) = \sup_X \inf_Y [\int_{t_n}^{t_{n+1}} g(z_s, x, y)k(s)ds + W^-_\Pi(t_{n+1}, z_{t_{n+1}})].$$

The analysis of the previous paragraph applies, leading to:

Proposition 5.6.
The family $\{W^-_\Pi(t,z)\}$ is uniformly equicontinuous in both variables.

The HJI equation corresponds here to the Hamiltonian:

$$H^-(t,z,p) = \sup_X \inf_Y [g(z,x,y)k(t) + \langle f(z,x,y), p\rangle]. \tag{28}$$

Proposition 5.7.
1) Any accumulation point of the family $\{W^-_\Pi\}$, as the mesh δ of Π goes to zero, is a viscosity solution of:

$$0 = \frac{d}{dt}W^-(t,z) + \sup_X \inf_Y [g(z,x,y)k(t) + \langle f(z,x,y), \nabla W^-(t,z)\rangle] \tag{29}$$

2) The family $\{W^-_\Pi\}$ converges to W^-, unique viscosity solution of (29).

Finally let W^-_∞ be the maxmin of the differential game Γ^I played (on $X \times Y$) using non anticipative strategies with delay. Then:

Proposition 5.8.
1) W^-_∞ is a viscosity solution of (29).
2)

$$W^-_\infty = W^-.$$

As above, similar properties hold for W^+_Π and W^+_∞.

Due to the bilinear extension, Isaacs's condition on $X \times Y$ which is, with the notation (28):

$$H^-(t,z,p) = H^+(t,z,p) \quad \forall t \in \mathbb{R}^+, \forall z \in Z, \forall p \in \mathbb{R}^n, \tag{30}$$

always holds. Thus one obtains:

Proposition 5.9.
The limit value W exists:

$$W = W^- = W^+,$$

and is also the value of the differential game played on $X \times Y$.
It is the unique viscosity solution of :

$$0 = \frac{d}{dt}W(t,z) + \mathtt{val}_{X\times Y}[g(z,x,y)k(t) + \langle f(z,x,y), \nabla W(t,z)\rangle]. \tag{31}$$

5.2.2. *Random actions.*
We define now another game Γ^{II}_Π where the actions $(i_n, j_n) \in I \times J$ are chosen at random at time t_n according to $x_n \in X$ and $y_n \in Y$, then constant on $[t_n, t_{n+1})$ and announced at time t_{n+1}. The new state is thus, if $(i_n, j_n) = (i,j)$, $z^{ij}_{t_{n+1}} = \Phi^{\delta_n}(z_{t_n}; i, j)$.
It is clear, see e.g. [24] Chapter 4, that the next dynamic programming property holds:

Proposition 5.10. *The game Γ^{II}_Π has a value W_Π, which satisfies the recursive formula:*

$$\mathsf{W}_\Pi(t_n, z_{t_n}) = \mathtt{val}_{X\times Y}\mathsf{E}_{x,y}[\int_{t_n}^{t_{n+1}} g(z_s, i, j)k(s)ds + \mathsf{W}_\Pi(t_{n+1}, z^{ij}_{t_{n+1}})] \tag{32}$$

and given the hypothesis one obtains as above:

Proposition 5.11.
The family $\{\mathsf{W}_\Pi(t,z)\}$ is equicontinuous in both variables.

Moreover one has:

Proposition 5.12.
1) Any accumulation point U of the family $\{\mathsf{W}_\Pi\}$, as the mesh δ of Π goes to zero, is a viscosity solution of (31).
2) The family $\{\mathsf{W}_\Pi\}$ converges to W, unique solution of (31).

Proof
1) Let $\psi(t,z)$ be a $\mathcal{C}^1$ test function such that $\mathsf{U}-\psi$ has a strict maximum at $(\bar{t},\bar{z})$. Consider a sequence $W_m = \mathsf{W}_{\Pi(m)}$ converging uniformy locally to $/U$ as $m\to\infty$ and let $(t^*(m),z(m))$ be a minimizing sequence for $(\psi - W_m)(t,z), t\in\Pi(m)$. In particular $(t^*(m),z(m))$ converges to $(\bar{t},\bar{z})$ as $m\to\infty$. Given $x^*(m)$ optimal in (32) one has with $t^*(m)=t_n\in\Pi(m)$:

$$W_m(t_n,z(m)) \le \mathsf{E}_{x^*(m),y}[\int_{t_n}^{t_{n+1}} g(z_s,i,j)k(s)ds + W_m(t_{n+1},z^{ij}_{t_{n+1}})], \qquad \forall y\in Y$$

so that by the choice of $(t^*(m),z(m))$:

$$\begin{aligned}\psi(t_n,z(m)) &\le \mathsf{E}_{z,x^*(m),y}[\int_{t_n}^{t_{n+1}} g(z_s,i,j)k(s)ds + \psi(t_{n+1},z^{ij}_{t_n+1})]\\ &\le \delta_n k(t_n)\ g(z(m),x^*(m),y) + \psi(t_{n+1},z(m))\\ &\quad + \delta_n\ \mathsf{E}_{x^*(m)\,y}\,\langle f(z(m),i,j),\nabla\psi(t_{n+1},z(m))\rangle + o(\delta_n).\end{aligned}$$

This implies:

$$0 \le \delta_n\ \frac{d}{dt}\psi(t_n,z(m)) + \delta_n\ k(t_n)\ g(z(m),x^*(m),y) + \delta_n\ \mathsf{E}_{x^*(m)\,y}\langle f(z(m),i,j),\nabla\psi(t_n,z(m))\rangle + o(\delta_n)$$

hence dividing by δ_n and taking the limit as $m\to\infty$ one obtains, for some accumulation point $x^*\in\Delta(I)$:

$$0 \le \frac{d}{dt}\psi(\bar{t},\bar{z}) + k(\bar{t})\ g(\bar{z},x^*,y) + \mathsf{E}_{x^*\,y}\langle f(\bar{z},i,j),\nabla\psi(\bar{t},\bar{z})\rangle, \qquad \forall y\in Y. \tag{33}$$

Thus U is a viscosity solution of :

$$0 = \frac{d}{dt}u(t,z) + \mathtt{val}_{X\times Y}\int_{I\times J}[g(z,i,j)k(t) + \langle f(z,i,j),\nabla u(t,z)\rangle]x(di)y(dj)$$

which by linearity, reduces to (31).
2) The proof of uniqueness follows from Proposition 5.9. ∎

Note again that the same analysis holds if the players use strategies that depend only at time t_n on t_n and z_{t_n}.

5.2.3. *Comments.*

Both games lead to the same limit PDE (31) but with different sequences of approximations: In the first case (Γ^I), the evolution is deterministic and the state (or (x,y)) is announced.
In the second case (Γ^{II}), the evolution is random and the state (or the actions) are announced (the knowledge of (x,y) would not be enough).

The fact that both games have same limit value is a justification for playing distribution or mixed actions as pure actions in continuous time and for assuming that the distributions are observed, see Neyman [25].

Remark also that the same analysis holds if f and g depend in addition continuously on t.

A related study of differential games with mixed actions, but concerned with the analysis trough strategies can be found in [4], [5], [22].

The advantage of working with discretization is to have a well defined and simple set of strategies hence the recursive formula is immediate to check for the associated maxmin or minmax $W^{\pm}_{\Pi}$. On the other hand the main equation (HJI) is satisfied by accumulation points.
The use of mixed actions in extensions of type II allows to have values in the associated game.

6. Concluding comments and extensions

This research is part of an analysis of asymptotic properties of dynamic games through their recursive structure : operator approach [31], [29].

Recall that the analysis in terms of repeated games may lead to non convergence, in the framework of Section 3 with compact action spaces, see Vigeral [41], or in the framework of

Section 4 even with finite action spaces, see Ziliotto [43] (for an overview of similar phenomena see Sorin and Vigeral [36]).

The approach in terms of vanishing duration of a continuous time process allows, via the extension of the state space from Ω to $\Delta(\Omega)$ to obtain smooth transition and nice limit behavior as δ vanishes.

A similar procedure has been analyzed by Neyman [26], in the finite case, for more general classes of approximating games and developed in Sorin and Vigeral [37].

The case of private information on the state variable has been treated by Cardaliaguet, Rainer, Rosenberg and Vieille [9] in the stationary finite framework: the viscosity solution corresponding to (ME) involves a geometric aspect due to the revelation of information that makes the analysis much more difficult. The (ME) obtained here in Section 3 corresponds to the Non Revealing value that players can obtained without using their private information.

Let us finally mention three directions of research:
the study of the general symmetric case i.e. a framework between Section 3 and Section 4 where the players receive partially revealing symmetric signals on the state, [35],
the asymptotic properties when both the evaluation tends to $+\infty$ and the mesh goes to 0: in the stationary case this means both ρ and δ vanishes. In the framework of Section 3, with finite actions spaces this was done by Neyman [26] using the algebraic property of equation (14),
the construction of optimal strategies based at time t on the current state z_t and the instantaneous discount rate $\mathbf{k}(t)/\sum_t^{+\infty}\mathbf{k}(s)ds$.

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