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# RENEWAL THEORY WITH INDEX ZERO

KENNETH S. ALEXANDER AND QUENTIN BERGER

ABSTRACT. We study renewals  $\tau$  with index 0: the inter-arrival distribution is  $\mathbf{P}(\tau_1 = n) = \varphi(n)n^{-1}$ , with  $\varphi(\cdot)$  slowly varying. We obtain a strong renewal theorem, that is

$$\mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \mathbf{P}(\tau_1 = n)/\mathbf{P}(\tau_1 \geq n)^2.$$

If instead we only assume regular variation of  $\mathbf{P}(n \in \tau)$  and slow variation of  $U_n := \sum_{k=0}^n \mathbf{P}(k \in \tau)$ , we obtain a similar equivalence but with  $\mathbf{P}(\tau_1 = n)$  replaced by its average over a short interval. We give an application to the local asymptotics of the distribution of the first intersection of two independent renewals. Along the way we prove a local limit theorem and a local (upward) large deviation theorem, giving the asymptotics of  $\mathbf{P}(\tau_k = n)$  when  $n$  is at least the typical length of  $\tau_k$ . We further derive downward moderate and large deviations estimates, that is, the asymptotics of  $\mathbf{P}(\tau_k \leq n)$  when  $n$  is much smaller than the typical length of  $\tau_k$ .

## 1. INTRODUCTION

It is classical to study renewal processes  $\tau = \{0 = \tau_0, \tau_1, \tau_2, \dots\}$ , and in particular the relation between the renewal mass function  $\mathbf{P}(n \in \tau)$  and the inter-arrival distribution  $\mathbf{P}(\tau_1 = n)$ . We assume the inter-arrival distribution  $\mathbf{P}(\tau_1 = n)$  is regularly varying: there exists a positive slowly varying function  $\varphi(\cdot)$  and  $\alpha \geq 0$  such that

$$(1.1) \quad \mathbf{P}(\tau_1 = n) = \varphi(n) n^{-(1+\alpha)}.$$

In particular the process is aperiodic. The one case that has not received much attention (under the general assumption (1.1)) is  $\alpha = 0$ , in which  $\tau_1$  has no moments and is not in the domain of attraction of a stable law, and that is our focus here. Tauberian theorems are of less use here than in other cases, so our methods are primarily probabilistic. An example with  $\alpha = 0$  is the return times of symmetric simple random walk (SSRW) on  $\mathbb{Z}^2$ ,  $\tau = \{n, S_{2n} = 0\}$ , for which  $\mathbf{P}(\tau_1 = n) \stackrel{n \rightarrow \infty}{\sim} \pi/n(\log n)^2$ , from [18, Thm. 4].

The limiting distributions of  $\tau_n$  and related quantities in the  $\alpha = 0$  case have been studied in [8], [16], [19], [21] or [22]. An important theorem (Theorem 4.1 in [8]) states that, if we denote  $r(n) := \mathbf{P}(\tau_1 > n)$ , then provided that  $r(n)$  is slowly varying we have that for any  $y > 0$

$$(1.2) \quad \mathbf{P}(nr(\tau_n) < y) \rightarrow 1 - e^{-y} \quad \text{as } n \rightarrow +\infty.$$

However, the references mentioned above do not give renewal results nor local limit or large deviations estimates in the  $\alpha = 0$  case.

**1.1. Renewal theorems.** The assumption (1.1) is very natural: beyond the dimension-2 case, it includes the case  $\tau = \{n, S_{2n} = 0\}$ , where  $(S_n)_{n \geq 0}$  is SSRW on  $\mathbb{Z}^d$  for any  $d$ . One has  $\alpha = 1/2$  and  $\varphi(n) \stackrel{n \rightarrow \infty}{\sim} (4\pi)^{-1/2}$  for  $d = 1$  (see e.g. [13, Ch. III]); and  $\alpha = \frac{d}{2} - 1$ ,  $\varphi(n) \stackrel{n \rightarrow \infty}{\sim} c_d$  for  $d \geq 3$  (see [11, Thm. 4]). Equation (1.1) also includes the case  $\tau = \{n, S_n = 0\}$  where

$(S_n)_{n \geq 0}$  is an aperiodic random walk in the domain of attraction of a symmetric stable law, see [20, Thm. 8].

The asymptotics of the renewal function  $\mathbf{P}(n \in \tau)$  for  $\alpha > 0$  have been widely studied in the literature, including [9], [12], [14], [23]. We recall briefly the results, under assumption (1.1).

First, when  $\tau$  is transient and (1.1) holds, we have

$$(1.3) \quad \mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \frac{\mathbf{P}(\tau_1 = n)}{\mathbf{P}(\tau_1 = +\infty)^2}.$$

This is a consequence of Theorem 1 in [4], and is also proven in [15, App. A.5] with elementary methods.

If  $\tau$  is recurrent, then

- if  $\mathbf{E}[\tau_1] < +\infty$ , then the classical Renewal Theorem (see e.g. [2]) gives that

$$(1.4) \quad \lim_{n \rightarrow \infty} \mathbf{P}(n \in \tau) = \frac{1}{\mathbf{E}[\tau_1]};$$

- if  $\alpha = 1$  in (1.1), and  $\mathbf{E}[\tau_1] = +\infty$ , Erickson [12, Eq. (2.4)] proved that

$$(1.5) \quad \mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \frac{1}{\mathbf{E}[\tau_1 \wedge n]};$$

- if  $\alpha \in (0, 1)$  in (1.1), Doney [9, Thm. B] proved that

$$(1.6) \quad \mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \frac{\alpha \sin(\pi\alpha)}{\pi} n^{-(1-\alpha)} \varphi(n)^{-1}.$$

The condition (1.1) is not best possible for the validity of these *strong renewal theorems* with infinite mean. Assume simply that  $\mathbf{P}(\tau_1 > n) \stackrel{n \rightarrow \infty}{\sim} \alpha^{-1} \varphi(n) n^{-\alpha}$  with  $\alpha \in (0, 1]$  (and  $\mathbf{E}[\tau_1] = +\infty$  if  $\alpha = 1$ ), so that  $\tau_1$  is in the domain of attraction of a stable law with index  $\alpha$ . Garsia and Lamperti [14] showed that (1.6) holds whenever  $\alpha \in (\frac{1}{2}, 1)$ , and Erickson proved (1.5) in the case  $\alpha = 1$ . When  $\alpha \in (0, \frac{1}{2}]$ , some additional conditions on the distribution of  $\tau_1$  are necessary for (1.6) to be valid, and sufficient ones were given in [6], [7], [9], [23]. It is only recently that a complete necessary and sufficient condition for the *strong renewal theorem* (1.6) was proven in simultaneous papers by Caravenna [3] and Doney [10].

Throughout the paper,  $c_1, c_2, \dots$  are constants depending only on the distribution of  $\tau_1$ . Also, we treat certain large quantities at times as if they were integers, simply to avoid the clutter of integer-part notation; in all cases these can be treated as if the integer-part notation were in use.

Our first result is a local limit and local large deviation theorem, proved in Section 2, in the case of a recurrent  $\tau$ . Define  $r_n := r(n) := \mathbf{P}(\tau_1 > n)$ , which in the  $\alpha = 0$  case is slowly varying and satisfies (see [5, Proposition 1.5.9a])

$$(1.7) \quad \varphi(n) = o(r_n) \quad \text{as } n \rightarrow \infty.$$

In particular we have  $\varphi(n) \rightarrow 0$ .

**Theorem 1.1.** *If  $\tau$  is recurrent and (1.1) holds with  $\alpha = 0$ , then uniformly for  $k$  such that  $k\varphi(n) \rightarrow 0$ , we have*

$$(1.8) \quad \mathbf{P}(\tau_k = n) \stackrel{n \rightarrow \infty}{\sim} k \mathbf{P}(\tau_1 = n) (1 - r_n)^k.$$

Further, there exists a constant  $c_1 > 0$  such that for  $n$  sufficiently large and all  $1 \leq k \leq n$ ,

$$(1.9) \quad \mathbf{P}(\tau_k = n) \leq c_1 k \mathbf{P}(\tau_1 = n) (1 - r_n)^k.$$

Note that, as soon as  $k \gg r_n^{-1}$ , we have  $\mathbf{P}(\tau_k \leq n) \leq (1 - r_n)^k \rightarrow 0$ , and  $n$  is therefore much smaller than the typical size of  $\tau_k$ . By (1.7),  $k \gg r_n^{-1}$  is consistent with the hypothesis  $k \ll 1/\varphi(n)$ . Equation (1.8) therefore includes  $n$  up to a size much larger than the typical size of  $\tau_k$ , which is not true for  $\alpha > 0$  (simply because of the local limit theorem). Heuristically, (1.8) says that even for much smaller-than-usual  $n$ , when  $\tau_k = n$  it is because there was a single gap of length very close to  $n$ , among the first  $k$  gaps  $\tau_j - \tau_{j-1}$ ; this is unique to  $\alpha = 0$ .

Our strong renewal theorem is an easy consequence of Theorem 1.1, as follows. Let  $\theta_n$  satisfy  $r_n^{-1} \ll \theta_n \ll \varphi(n)^{-1}$ . We write  $\mathbf{P}(n \in \tau) = \sum_{k=1}^n \mathbf{P}(\tau_k = n)$ , and decompose it according to whether  $k$  is smaller or larger than  $\theta_n$ . Thanks to (1.8), by our choice of  $\theta_n$  we have

$$\sum_{k \leq \theta_n} \mathbf{P}(\tau_k = n) \stackrel{n \rightarrow \infty}{\sim} \sum_{k \leq \theta_n} k \mathbf{P}(\tau_1 = n) (1 - r_n)^k \stackrel{n \rightarrow \infty}{\sim} r_n^{-2} \mathbf{P}(\tau_1 = n).$$

For the rest of the sum, we use (1.9) together with  $\theta_n \gg r_n^{-1}$ , to get that, for  $n \geq n_0$

$$\sum_{k > \theta_n} \mathbf{P}(\tau_k = n) \leq c_1 \sum_{k > \theta_n} k \mathbf{P}(\tau_1 = n) (1 - r_n)^k = o(1) r_n^{-2} \mathbf{P}(\tau_1 = n) \quad \text{as } n \rightarrow +\infty.$$

We therefore obtain the following.

**Theorem 1.2.** *If  $\tau$  is recurrent and (1.1) holds with  $\alpha = 0$ , we have*

$$(1.10) \quad \mathbf{P}(n \in \tau) \sim \frac{\mathbf{P}(\tau_1 = n)}{\mathbf{P}(\tau_1 > n)^2} \quad \text{as } n \rightarrow +\infty.$$

If we combine (1.3) and Theorem 1.2, we obtain the following statement: if (1.1) holds with  $\alpha \geq 0$ , and  $\mathbf{P}(\tau_1 > n)$  is slowly varying (that is, either  $\tau$  is transient, or  $\tau$  is recurrent with  $\alpha = 0$ ), then (1.10) holds.

The heuristic behind (1.10) may be seen by restating it as  $\mathbf{P}(\tau_1 = n \mid n \in \tau) \sim \mathbf{P}(\tau_1 > n)^2$ . This says that given  $n \in \tau$ , in order to have  $\tau_1 = n$  (i.e. no renewals between 0 and  $n$ ), the trajectory mainly needs to “escape” without renewals at each end, and these two escapes are approximately independent, each with probability near  $\mathbf{P}(\tau_1 > n)$ . This independence in the recurrent case is unique to  $\alpha = 0$ , since in that case the only renewals that typically occur given  $n \in \tau$  are very close to 0 and  $n$ .

**1.2. Large and moderate deviations.** Theorem 1.1 may be viewed as both a local limit theorem and a local large deviation theorem for the case  $\alpha = 0$ , covering upward deviations (in the sense that  $n$  is much larger than the typical size of  $\tau_k$ ) and downward deviations that are not too great. As a complement we now consider estimates for downward deviations of the form  $\mathbf{P}(\tau_k \leq n)$  for  $n$  much smaller than the typical size of  $\tau_k$ , that is  $kr_n \rightarrow \infty$ .

Let  $\varphi^*$  denote a slowly varying function conjugate to  $\varphi$ , that is, such that  $x \mapsto x\varphi^*(x)$  is an asymptotic inverse of  $y \mapsto y\varphi(y)$ , see [5, §1.5.7] for more. For most common slowly varying functions  $\varphi$  one has  $\varphi^* \sim 1/\varphi$ , but this is not true if  $\varphi$  is “barely slowly varying,” for example  $\varphi(n) = n^{1/\log \log n}$ . We will prove the following in Section 1.3.

**Theorem 1.3.** *Suppose  $\tau$  is recurrent and (1.1) holds with  $\alpha = 0$ . Let  $n \geq k$ .*

(i) Given  $M > 0$  there exists  $a_M$ , with  $a_M \rightarrow 1$  as  $M \rightarrow 0$ , such that if  $n$  is large and  $k\varphi(n) \leq M$ , then

$$(1.11) \quad a_M(1 - r_n)^k \leq \mathbf{P}(\tau_k \leq n) \leq (1 - r_n)^k.$$

(ii) If  $k, n \rightarrow +\infty$  with  $k\varphi(n) \rightarrow +\infty$  and  $n/k \rightarrow +\infty$ , then we have

$$(1.12) \quad \mathbf{P}(\tau_k \leq n) = \exp \left\{ -(1 + o(1)) k r \left( \frac{n}{k} \varphi^* \left( \frac{n}{k} \right) \right) \right\}.$$

(iii) For  $n = bk$  with  $b \geq 1$ , the limit  $-I(b) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(\tau_k \leq bk)$  exists, and it is finite if  $b \geq \min\{j : \mathbf{P}(\tau_1 = j) > 0\}$ . Moreover, it satisfies

$$I(b) \sim r(b\varphi^*(b)) \quad \text{as } b \rightarrow +\infty.$$

This theorem extends the result (1.2) of Darling [8] to the case  $y \rightarrow +\infty$  as  $n \rightarrow +\infty$ . In particular, (i) allows to recover (1.2) by taking  $k = y/r_n$  (since  $\{\tau_k \leq n\} = \{r(\tau_k) \geq y/k\}$ ), and moreover extends it to  $\mathbf{P}(kr(\tau_k) \geq y) \sim e^{-y}$  as  $k \rightarrow \infty$ , uniformly for  $y \ll r_n/\varphi(n)$  (we recall (1.7)).

**1.3. Reverse renewal theorems.** Though (1.1) is very natural, verifying that it holds is often tedious, for example if  $\tau = \{n, S_n = 0\}$ , with  $(S_n)_{n \geq 0}$  an aperiodic random walk in the domain of attraction of a symmetric stable distribution, see [18]. But in that case, a local limit theorem (see [17, § 50]) easily gives the asymptotic behavior of  $\mathbf{P}(S_n = 0) = \mathbf{P}(n \in \tau)$ . Therefore, one would like to get a general result to infer from  $\mathbf{P}(n \in \tau)$  something about the behavior of  $\mathbf{P}(\tau_1 = n)$ . We call such a result a *reverse renewal theorem*. An additional application of such theorems is given in Section 1.4.

In general, it is not true that regular variation of  $\mathbf{P}(n \in \tau)$  implies regular variation of  $\mathbf{P}(\tau_1 = n)$ , an example being given in Section 4.3. But the average of the values  $\mathbf{P}(\tau_1 = n)$  over a relatively short interval may be better behaved. In fact we can obtain a reverse renewal theorem corresponding to (1.3) and Theorem 1.2 in the  $\alpha = 0$  case, as follows.

Define

$$U_n := \sum_{k=0}^n \mathbf{P}(k \in \tau), \quad U_\infty := \mathbf{E}[|\tau|] = \sum_{k=0}^{\infty} \mathbf{P}(k \in \tau) \quad \left( = \frac{1}{\mathbf{P}(\tau_1 = \infty)} \text{ if } U_\infty < \infty \right),$$

and note that

$$(1.13) \quad \text{if } U_n \text{ is slowly varying, then } U_n \stackrel{n \rightarrow \infty}{\sim} \mathbf{P}(\tau_1 > n)^{-1}.$$

This is trivial if  $\tau$  is transient:  $|\tau|$  is then a geometric random variable, and  $U_n$  converges to  $\mathbf{E}[|\tau|] = \mathbf{P}(\tau_1 = +\infty)^{-1}$ . In the recurrent case, we refer to Theorem 8.7.3 in [5]; the proof uses standard properties of convolution of Laplace transforms. Note that in the following we do not assume (1.1).

**Theorem 1.4.** *Assume that  $\mathbf{P}(n \in \tau)$  is regularly varying and  $U_n$  is slowly varying. Then there exist  $\epsilon_n \rightarrow 0$  such that*

$$(1.14) \quad \frac{1}{\epsilon_n n} \sum_{(1-\epsilon_n)n < k \leq n} \mathbf{P}(\tau_1 = k) \stackrel{n \rightarrow \infty}{\sim} \mathbf{P}(\tau_1 > n)^2 \mathbf{P}(n \in \tau).$$

*If also  $\mathbf{P}(\tau_1 = n)$  is regularly varying, then*

$$(1.15) \quad \mathbf{P}(\tau_1 = n) \stackrel{n \rightarrow \infty}{\sim} \mathbf{P}(\tau_1 > n)^2 \mathbf{P}(n \in \tau).$$

This theorem applies in the recurrent case when  $\mathbf{P}(n \in \tau)$  is regularly varying with index  $-1$ , and in the case of a transient renewal  $\tau$ . When  $\tau$  is transient, we are able to prove the following stronger statement.

**Theorem 1.5.** *If  $\mathbf{P}(n \in \tau)$  is regularly varying and  $\tau$  is transient, then*

$$\mathbf{P}(\tau_1 = n) \stackrel{n \rightarrow \infty}{\sim} \mathbf{P}(\tau_1 = \infty)^2 \mathbf{P}(n \in \tau).$$

This theorem was proved in [11] in the case where  $\tau_1, \tau_2, \dots$  are the return times to the origin of a transient aperiodic random walk, but we prove it here in general.

Section 4.1 is devoted to the proof of Theorem 1.5, and Section 4.2 to the proof of Theorem 1.4 (where only the recurrent case has to be considered, with Theorem 1.5 proved.) Finally, in Section 4.3, we give an example where  $\mathbf{P}(\tau_1 = n)$  is not regularly varying but  $\mathbf{P}(n \in \tau)$  is, and  $U_n$  is slowly varying. This shows that (1.15) cannot hold in the general case of a recurrent renewal, and our Theorem 1.4 is in that sense optimal.

In general, Theorem 1.4 reduces the problem of proving (1.15) to showing that  $\mathbf{P}(\tau_1 = k)$  is approximately constant over the interval  $((1 - \epsilon_n)n, n]$ .

**1.4. Application of reverse renewal theorems: the intersection of two independent renewals.** Let  $\tau$  and  $\sigma$  be independent renewal processes with inter-arrival distributions satisfying

$$(1.16) \quad \mathbf{P}(\tau_1 = n) = \varphi(n)n^{-(1+\alpha)}, \quad \mathbf{P}(\sigma_1 = n) = \tilde{\varphi}(n)n^{-(1+\tilde{\alpha})}$$

for some  $\alpha, \tilde{\alpha} \geq 0$  and slowly varying functions  $\varphi(\cdot), \tilde{\varphi}(\cdot)$ . We assume  $\alpha \leq \tilde{\alpha}$ .

We denote the intersection  $\rho := \tau \cap \sigma$ , which is a renewal process with renewal mass function and renewal function

$$\mathbf{P}(n \in \rho) = \mathbf{P}(n \in \tau)\mathbf{P}(n \in \sigma), \quad U_n^* = \sum_{k=0}^n \mathbf{P}(k \in \rho).$$

These are regularly varying, and their asymptotic behavior is thus known from the results for  $\sigma, \tau$  in Section 1.1. In [1] our reverse renewal theorems, 1.4 and 1.5, are applied to help establish the following. If  $\rho$  is transient (i.e.  $U_\infty^* < \infty$ ) then

$$\mathbf{P}(\rho_1 = n) \stackrel{n \rightarrow \infty}{\sim} (U_\infty^*)^{-2} \mathbf{P}(n \in \tau)\mathbf{P}(n \in \sigma).$$

If  $\rho$  is recurrent and either (i)  $\alpha, \tilde{\alpha} \in (0, 1)$  with  $\alpha + \tilde{\alpha} = 1$ , or (ii)  $\alpha = 0, \tilde{\alpha} \geq 1$ , then  $U_n^*$  is slowly varying, and

$$(1.17) \quad \mathbf{P}(\rho_1 = n) \stackrel{n \rightarrow \infty}{\sim} (U_n^*)^{-2} \mathbf{P}(n \in \tau)\mathbf{P}(n \in \sigma) \stackrel{n \rightarrow \infty}{\sim} \frac{\psi^*(n)}{n}$$

for some (asymptotically known) slowly varying  $\psi^*$ . In [1], general  $0 \leq \alpha \leq \tilde{\alpha}$  are covered, and without Theorems 1.4 and 1.5 here, the cases (i) and (ii) would have to be excluded. The key step to get from (1.14) for  $\rho$  to (1.17) is to show that, due to the regularity (1.16) in  $\sigma$  and  $\tau$ ,  $\mathbf{P}(\rho_1 = k)$  is approximately constant over short intervals, so that the left side of (1.14) (for  $\rho$ ) is asymptotic to  $\mathbf{P}(\rho_1 = n)$ .

## 2. PROOF OF THEOREM 1.1

We first prove (1.8), and turn to (1.9) as a second step. We introduce some notations: let

$$G_i := \tau_i - \tau_{i-1} \quad \text{and} \quad M_k := \max_{1 \leq i \leq k} G_i.$$

We also let  $\widehat{G}_1^{(m)}, \dots, \widehat{G}_k^{(m)}$  be i.i.d. with distribution  $\mathbf{P}(\tau_1 \in \cdot \mid \tau_1 \leq m)$ .

**2.1. Proof of the local limit and local large deviation result (1.8).** The proof is divided into three steps, in which we control several contributions to  $\mathbf{P}(\tau_k = n)$ .

- Step 1. Contribution of the case of only one jump larger than  $(1 - \epsilon)n$ , all the other ones being (necessarily) smaller than  $n/2$ . This gives the right order in Theorem 1.1 when  $k \ll \varphi(n)^{-1}$ ;
- Step 2. Contribution of the case when all jumps are smaller than  $n/2$ : it is negligible, so there must be one jump larger than  $n/2$  (and there can be only one such jump);
- Step 3. Contribution of the case when there is one jump larger than  $n/2$ , but smaller than  $(1 - \epsilon)n$ . This is also negligible.

**Step 1:** We show that, for any fixed  $\epsilon > 0$ , and provided that  $k\varphi(n) \xrightarrow{n \rightarrow \infty} 0$ ,

$$(2.1) \quad \mathbf{P}\left(\tau_k = n, M_k > (1 - \epsilon)n\right) = (1 + O(\epsilon)) k \mathbf{P}(\tau_1 = n) (1 - r_n)^k, \quad \text{as } n \rightarrow \infty.$$

We have

$$(2.2) \quad \mathbf{P}\left(\tau_k = n, M_k > (1 - \epsilon)n\right) = k(1 - r_n)^{k-1} \sum_{m=1}^{\epsilon n} \mathbf{P}\left(\sum_{i=1}^{k-1} \widehat{G}_i^{(n)} = m\right) \mathbf{P}(\tau_1 = n - m).$$

This gives the upper bound

$$(2.3) \quad \begin{aligned} \mathbf{P}\left(\tau_k = n, M_k > (1 - \epsilon)n\right) &\leq k(1 - r_n)^{k-1} \max_{(1-\epsilon)n \leq j \leq n} \mathbf{P}(\tau_1 = j) \\ &\leq (1 + 2\epsilon)k(1 - r_n)^k \mathbf{P}(\tau_1 = n), \end{aligned}$$

provided that  $n$  is large enough.

In the other direction, (2.2) gives

$$\mathbf{P}\left(\tau_k = n, M_k > (1 - \epsilon)n\right) \geq k(1 - r_n)^{k-1} \mathbf{P}\left(\sum_{i=1}^{k-1} \widehat{G}_i^{(n)} \leq \epsilon n\right) \min_{(1-\epsilon)n \leq j \leq n} \mathbf{P}(\tau_1 = j).$$

Then, using that  $\mathbf{E}[\widehat{G}_1^{(n)}] = (1 - r_n)^{-1} \sum_{x=1}^n \varphi(x) \xrightarrow{n \rightarrow \infty} n\varphi(n)$ , we have that for  $n$  large enough

$$\mathbf{P}\left(\sum_{i=1}^{k-1} \widehat{G}_i^{(n)} \leq \epsilon n\right) \geq 1 - \frac{\mathbf{E}[\widehat{G}_1^{(n)}]}{\epsilon n} \geq 1 - \frac{2(k-1)\varphi(n)}{\epsilon}.$$

Therefore, since  $k\varphi(n) \rightarrow 0$ , we end up with

$$(2.4) \quad \mathbf{P}\left(\tau_k = n, M_k > (1 - \epsilon)n\right) \geq (1 - 2\epsilon)k(1 - r_n)^k \mathbf{P}(\tau_1 = n).$$

provided that  $n$  is large enough.

**Step 2:** We want to show that the main contribution to  $\mathbf{P}(\tau_k = n)$  comes when  $M_k \geq n/2$ . We prove that there exists a constant  $c_2 > 0$  such that, if  $k\varphi(n)$  is small enough,

$$(2.5) \quad \mathbf{P}(\tau_k = n, M_k \leq n/2) \leq c_2 k^2 \varphi(n) \mathbf{P}(\tau_1 = n) (1 - r_n)^k,$$

which is negligible compared to (2.1) when  $k\varphi(n) \rightarrow 0$ . It is sufficient to show that, if  $k\varphi(n)$  is small enough,

$$(2.6) \quad \mathbf{P}\left(\sum_{i=1}^k \widehat{G}_i^{(n)} = n; \widehat{G}_i^{(n)} \leq n/2 \text{ for all } i \leq k\right) \leq c_2 k^2 \varphi(n) \mathbf{P}(\tau_1 = n)$$

To prove this, we rely on the following lemma.

**Lemma 2.1.** *Suppose (1.1) holds with  $\alpha = 0$ . There exist constants  $c_3, c_4 > 0$  such that for  $n$  large, for all  $1 \leq m \leq n$  and  $k \geq 0$ ,*

$$(2.7) \quad \mathbf{P}\left(\sum_{i=1}^k \widehat{G}_i^{(m)} \geq n/2\right) \leq \left(\frac{c_3 k m \varphi(m)}{n}\right)^{\frac{n}{2m}} \leq \left(c_4 k \varphi(n)\right)^{\frac{n}{2m}}.$$

**Proof** The second inequality is a consequence of the fact that  $m\varphi(m)$  is asymptotically increasing, so we prove the first inequality.

For any  $\lambda > 0$  we have

$$(2.8) \quad \mathbf{P}\left(\sum_{i=1}^k \widehat{G}_i^{(m)} \geq n/2\right) \leq e^{-\lambda n/2} \mathbf{E}\left[e^{\lambda \widehat{G}_1^{(m)}}\right]^k.$$

There exists a constant  $c_5$  such that for any  $j \geq 1$

$$(2.9) \quad \mathbf{E}\left[(\widehat{G}_1^{(m)})^j\right] \leq m^{j-1} \mathbf{E}[\tau_1 \mid \tau_1 \leq m] \leq c_5 m^j \varphi(m).$$

Hence, for any  $\lambda > 0$ , we have

$$\mathbf{E}\left[e^{\lambda \widehat{G}_1^{(m)}}\right] \leq 1 + c_5 \varphi(m) (e^{m\lambda} - 1).$$

Now, let us define  $\lambda$  by

$$c_5 \varphi(m) (e^{m\lambda} - 1) = \frac{n}{km},$$

so that

$$\mathbf{E}\left[e^{\lambda \widehat{G}_1^{(m)}}\right]^k \leq e^{n/m}$$

and

$$e^{-\lambda n/2} = \left(1 + \frac{n}{c_5 m k \varphi(m)}\right)^{-n/2m} \leq \left(\frac{c_5 k m \varphi(m)}{n}\right)^{n/2m}.$$

Therefore, (2.8) yields

$$(2.10) \quad \mathbf{P}\left(\sum_{i=1}^k \widehat{G}_i^{(m)} \geq n/2\right) \leq \left(\frac{c_5 k m \varphi(m)}{n}\right)^{n/2m} e^{n/m} \leq \left(\frac{c_5 e^2 k m \varphi(m)}{n}\right)^{\frac{n}{2m}}.$$

□

To control the probability on the left in (2.6), we decompose it according to the value of the largest  $\widehat{G}_i^{(n)}$ . Let us denote  $m_s := 2^{-s}n$  and  $J_s = (m_{s+1}, m_s]$ . We have

$$\begin{aligned}
& \mathbf{P}\left(\sum_{i=1}^k \widehat{G}_i^{(n)} = n; \widehat{G}_i^{(n)} \leq n/2 \text{ for all } i \leq k\right) \\
&= \sum_{n/k \leq m \leq n/2} k \mathbf{P}\left(\widehat{G}_1^{(n)} = m, \widehat{G}_i^{(n)} \leq m \text{ for all } 2 \leq i \leq k, \sum_{i=2}^k \widehat{G}_i^{(n)} = n - m\right) \\
&\leq \sum_{1 \leq s \leq \log_2 k} \sum_{m \in J_s} k \left(\frac{1-r_m}{1-r_n}\right)^{k-1} \mathbf{P}\left(\widehat{G}_1^{(n)} = m\right) \mathbf{P}\left(\sum_{i=2}^k \widehat{G}_i^{(m)} = n - m\right) \\
&\leq 2k \sum_{1 \leq s \leq \log_2 k} \sum_{m \in J_s} \frac{\varphi(m)}{m} \mathbf{P}\left(\sum_{i=1}^{k-1} \widehat{G}_i^{(m)} = n - m\right) \\
(2.11) \quad &\leq 2c_6 k \sum_{1 \leq s \leq \log_2 k} \frac{\varphi(m_{s+1})}{m_{s+1}} \mathbf{P}\left(\sum_{i=1}^{k-1} \widehat{G}_i^{(m_s)} \geq \frac{n}{2}\right),
\end{aligned}$$

where in the last inequality we used that there exists  $c_6$  such that for sufficiently large  $m_s$  and all  $m \in J_s$ ,  $\varphi(m) \leq c_6 \varphi(m_{s+1})$ . Since  $n/k \gg n\varphi(n) \rightarrow \infty$ , all values  $m_s$  in (2.11) are sufficiently large in this sense, when  $n$  is large.

Since  $\varphi$  is slowly varying, given  $a \leq 1$  we have  $\varphi(an)/\varphi(n) \leq 1/a$  for  $n$  large. With (2.11) and Lemma 2.1 this shows that

$$\begin{aligned}
(2.12) \quad \mathbf{P}\left(\sum_{i=1}^k \widehat{G}_i^{(n)} = n; \widehat{G}_i^{(n)} \leq n/2 \text{ for all } i \leq k\right) &\leq 2c_6 k \sum_{s \geq 1} \frac{2^{s+1} \varphi(n)}{2^{-(s+1)} n} \left(c_4 k \varphi(n)\right)^{\frac{n}{2m_s}} \\
&\leq 8c_6 k \frac{\varphi(n)}{n} \sum_{s \geq 1} 4^s \left(c_4 k \varphi(n)\right)^{2^{s-1}} \\
&\leq c_2 k^2 \varphi(n) \mathbf{P}(\tau_1 = n),
\end{aligned}$$

where we used in the last inequality that  $k\varphi(n)$  is small. Hence, (2.6) is proven, and so is (2.5).

**Step 3:** We show that the main contribution to  $\mathbf{P}(\tau_k = n)$  comes when not only  $M_k \geq n/2$ , but when  $M_k \geq (1 - \epsilon)n$ : we prove that for  $n$  large enough,

$$(2.13) \quad \mathbf{P}\left(\tau_k = n, n/2 < M_k \leq (1 - \epsilon)n\right) \leq \frac{6}{\epsilon} k^2 \varphi(n) \mathbf{P}(\tau_1 = n) (1 - r_n)^k.$$

Indeed, we have that

$$\mathbf{P}\left(\tau_k = n, n/2 < M_k \leq (1 - \epsilon)n\right) \leq k (1 - r_n)^{k-1} \max_{n/2 \leq j \leq n} \mathbf{P}(\tau_1 = j) \mathbf{P}\left(\sum_{i=1}^{k-1} \widehat{G}_i^{(n)} \geq \epsilon n\right).$$

Then, we use that  $\max_{n/2 \leq j \leq n} \mathbf{P}(\tau_1 = j) \leq 3\mathbf{P}(\tau_1 = n)$  provided that  $n$  is large enough, together with Markov's inequality and the fact that  $\mathbf{E}[\widehat{G}_1^{(n)}] \leq 2n\varphi(n)$  when  $n$  is large enough. This yields (2.13).

Combining (2.3)-(2.4) with (2.5) and (2.13), since  $\epsilon$  is arbitrary we get that, uniformly for  $k$  such that  $k\varphi(n) \rightarrow 0$ , (1.8) holds.  $\square$

**2.2. Proof of the uniform bound (1.9).** To prove the uniform bound, we rely on Lemma 2.1, and we decompose the probability according to the value of  $M_k$ .

Let  $n \geq n_0$  and define

$$\ell_n = \min\{\ell : 2^\ell \geq n\}, \quad \ell_{n,k} := \max\{\ell : c_3 k 2^\ell \varphi(2^\ell) \leq \frac{1}{2}n\},$$

where  $c_3$  is the constant from Lemma 2.1.

Then for some (large)  $\ell_0$ , there exists a constant  $c_7 > 0$  such that for all  $\ell_0 < \ell \leq \ell_n - 2$ ,

$$\begin{aligned} \mathbf{P}(\tau_k = n, M_k \in (2^{\ell-1}, 2^\ell]) &\leq k \mathbf{P}\left(G_1 \in (2^{\ell-1}, 2^\ell], \max_{2 \leq i \leq k} G_i \leq 2^\ell, \tau_k = n\right) \\ &\leq k(1 - r_{2^\ell})^{k-1} \max_{m \in (2^{\ell-1}, 2^\ell]} \mathbf{P}(\tau_1 = m) \mathbf{P}\left(\sum_{i=1}^{k-1} \widehat{G}_i^{(2^\ell)} \in (n - 2^\ell, n]\right) \\ (2.14) \quad &\leq c_7 k (1 - r_{2^\ell})^{k-1} \frac{\varphi(2^\ell)}{2^\ell} \mathbf{P}\left(\sum_{i=1}^{k-1} \widehat{G}_i^{(2^\ell)} > \frac{n}{2}\right). \end{aligned}$$

We now have 4 cases according to the value of  $\ell$ .

**Case 1.** For  $\ell_0 \vee \ell_{n,k} < \ell \leq \ell_n - 2$  we bound the last probability in (2.14) by 1, and observe that provided  $\ell_0$  is large enough,  $r_{2^\ell} - r_n \geq \frac{1}{2}\varphi(2^\ell)$ , which leads to

$$\begin{aligned} \mathbf{P}(\tau_k = n, 2^{\ell_0 \vee \ell_{n,k}} < M_k \leq 2^{\ell_n - 2}) &\leq 2c_7 k (1 - r_n)^k \sum_{\ell=\ell_0 \vee \ell_{n,k}+1}^{\ell_n-2} \frac{\varphi(2^\ell)}{2^\ell} \left(1 - \frac{r_{2^\ell} - r_n}{1 - r_n}\right)^k \\ &\leq 2c_7 k (1 - r_n)^k \sum_{\ell=\ell_0 \vee \ell_{n,k}+1}^{\ell_n-2} \frac{\varphi(2^\ell)}{2^\ell} e^{-k\varphi(2^\ell)/4} \\ (2.15) \quad &\leq 2c_7 k (1 - r_n)^k \frac{\varphi(n)}{n} \sum_{\ell=1}^{\ell_n-2} \frac{n \varphi(2^\ell)}{2^\ell \varphi(n)} e^{-n/8c_3 2^\ell}, \end{aligned}$$

where we used that  $2^\ell \varphi(2^\ell)$  is asymptotically increasing in  $\ell$ . We obtain easily that the last sum remains bounded as  $n \rightarrow \infty$ . In the end, we have a constant  $c_8 > 0$  such that for  $n \geq n_0$

$$(2.16) \quad \mathbf{P}(\tau_k = n, 2^{\ell_0 \vee \ell_{n,k}} < M_k \leq 2^{\ell_n - 2}) \leq c_8 k (1 - r_n)^k \mathbf{P}(\tau_1 = n).$$

**Case 2.** To handle  $\ell = \ell_n - 1, \ell_n$  we have analogously to (2.14), for  $n \geq n_0$

$$\begin{aligned} \mathbf{P}(\tau_k = n, M_k > 2^{\ell_n - 2}) &\leq k(1 - r_n)^{k-1} \max_{m \in (2^{\ell_n - 2}, 2^{\ell_n}]} \mathbf{P}(\tau_1 = m) \\ (2.17) \quad &\leq c_9 k (1 - r_n)^k \mathbf{P}(\tau_1 = n). \end{aligned}$$

**Case 3.** We now deal with  $\ell_0 < \ell \leq \ell_{n,k}$ . We bound the last probability in (2.14) using Lemma 2.1. We obtain, analogously to (2.15)

$$\begin{aligned}
\mathbf{P}(\tau_k = n, 2^{\ell_0} < M_k \leq 2^{\ell_{n,k}}) &\leq 2c_7 k \sum_{\ell=\ell_0+1}^{\ell_{n,k} \wedge \ell_n} (1-r_{2^\ell})^k \frac{\varphi(2^\ell)}{2^\ell} \left( \frac{c_3 k 2^\ell \varphi(2^\ell)}{n} \right)^{n/2^{\ell+1}} \\
&\leq 2c_7 k (1-r_n)^k \sum_{\ell=\ell_0+1}^{\ell_n} \frac{\varphi(2^\ell)}{2^\ell} \left( \frac{1}{2} \right)^{n/2^{\ell+1}} \\
&\leq c_{10} k (1-r_n)^k \frac{\varphi(2^{\ell_n+1})}{2^{\ell_n+1}} \left( \frac{1}{2} \right)^{n/2^{\ell_n+1}} \\
&\leq c_{11} k (1-r_n)^k \frac{\varphi(n)}{n} \\
(2.18) \qquad &= c_{11} k (1-r_n)^k \mathbf{P}(\tau_1 = n).
\end{aligned}$$

Here the third inequality uses the fact that  $n/2^{\ell_n+1} \geq 1/4$ , and consequently the sum in the second line of (2.18) is of the same order as the  $\ell = \ell_n$  term.

**Case 4.** Finally to handle  $\ell \leq \ell_0$  we have, using Lemma 2.1 and writing  $m_0 := 2^{\ell_0}$

$$\begin{aligned}
\mathbf{P}(\tau_k = n, M_k \leq 2^{\ell_0}) &\leq (1-r_{m_0})^k \mathbf{P} \left( \sum_{i=1}^k \widehat{G}_i^{(m_0)} = n \right) \\
&\leq (1-r_n)^k \left( \frac{1-r_{m_0}}{1-r_n} \right)^k \left( \min \left\{ \frac{c_3 m_0 \varphi(m_0) k}{n}, 1 \right\} \right)^{n/m_0} \\
(2.19) \qquad &\leq (1-r_n)^k e^{-c_{12} k} \left( \min \left\{ \frac{c_{13} k}{n}, 1 \right\} \right)^{n/m_0}.
\end{aligned}$$

Considering separately the cases  $k \leq n/2c_{13}$  and  $n/2c_{13} < k \leq n$ , we conclude that there is some  $c_{14} > 0$  such that for  $n$  large,

$$(2.20) \qquad \mathbf{P}(\tau_k = n, M_k \leq 2^{\ell_0}) \leq (1-r_n)^k e^{-c_{14} n} \leq c_{15} k (1-r_n)^k \mathbf{P}(\tau_1 = n).$$

Collecting (2.16),(2.17),(2.18) and (2.20) concludes the proof of (1.9).  $\square$

### 3. LARGE DEVIATIONS: PROOF OF THEOREM 1.3

Recall that  $G_i = \tau_i - \tau_{i-1}$ , and  $\widehat{G}_1^{(m)}, \widehat{G}_2^{(m)}, \dots$  are i.i.d. with distribution  $\mathbf{P}(\tau_1 \in \cdot \mid \tau_1 \leq m)$ .

**Proof of (i).** The second inequality is trivial, so we prove the first. Suppose  $k\varphi(n) \leq M$ . Given  $0 < \epsilon < 1$ ,

$$r_{\epsilon n} - r_n \sim \varphi(n) \log \frac{1}{\epsilon} \quad \text{as } n \rightarrow +\infty,$$

so for large  $n$ ,

$$(3.1) \qquad \mathbf{P} \left( \max_{i \leq k} \widehat{G}_i^{(n)} \leq \epsilon n \right) = \left( 1 - \frac{r_{\epsilon n} - r_n}{1 - r_n} \right)^k \geq \exp \left( -2k\varphi(n) \log \frac{1}{\epsilon} \right) \geq \epsilon^{2M}.$$

On the other hand, since  $\mathbf{E}[\widehat{G}_1^{(m)}] \stackrel{m \rightarrow \infty}{\sim} m\varphi(m)$ , given  $\epsilon > 0$  we have for  $n$  large enough

$$(3.2) \quad \mathbf{P} \left( \sum_{i=1}^k \widehat{G}_i^{(\epsilon n)} \leq n \right) \geq 1 - \frac{1}{n} k \mathbf{E} \left( \widehat{G}_1^{(\epsilon n)} \right) \geq 1 - 2\epsilon k \varphi(n) \geq 1 - 2\epsilon M.$$

If  $M \leq 1/3$ , we apply (3.2) with  $\epsilon = 1$ :

$$(3.3) \quad \mathbf{P}(\tau_k \leq n) \geq (1 - r_n)^k \mathbf{P} \left( \sum_{i=1}^n \widehat{G}_i^{(n)} \leq n \right) \geq (1 - r_n)^k (1 - 2M).$$

If  $M > 1/3$ , we take  $\epsilon = 1/4M$ , and combining (3.1) with (3.2), we obtain for  $n$  large enough

$$\mathbf{P}(\tau_k \leq n) \geq (1 - r_n)^k \mathbf{P} \left( \max_{i \leq k} \widehat{G}_i^{(n)} \leq \epsilon n \right) \mathbf{P} \left( \sum_{i=1}^k \widehat{G}_i^{(\epsilon n)} \leq n \right) \geq \frac{1}{2} \left( \frac{1}{4M} \right)^{2M} (1 - r_n)^k.$$

**Proof of (ii).** Define, for any  $\lambda > 0$ ,

$$\nu(\lambda) := 1 - \mathbf{E} \left( e^{-\lambda \tau_1} \right),$$

so  $-\log(1 - \nu(\cdot))$  is non-decreasing and strictly concave. Moreover, it is standard to obtain that

$$(3.4) \quad \nu(\lambda) \sim r \left( \frac{1}{\lambda} \right) \rightarrow 0, \quad \text{and} \quad \nu'(\lambda) \sim \frac{1}{\lambda} \varphi \left( \frac{1}{\lambda} \right) \rightarrow +\infty \quad \text{as } \lambda \searrow 0.$$

We may view (1.12) as a combination of an upper and a lower bound, which we now prove.

*Upper bound in (1.12).* Define

$$f_n(\lambda) := -n\lambda - k \log(1 - \nu(\lambda));$$

note the notation suppresses the dependence on  $k$ . We will use the standard exponential bound

$$(3.5) \quad \mathbf{P}(\tau_k \leq n) = \mathbf{P} \left( e^{-\lambda \tau_k} \geq e^{-\lambda n} \right) \leq e^{\lambda n} (1 - \nu(\lambda))^k = e^{-f_n(\lambda)} \quad \text{for all } \lambda > 0.$$

Now, we define  $\lambda_n > 0$  by  $f'_n(\lambda_n) = 0$ , or equivalently,

$$(3.6) \quad \frac{\nu'(\lambda_n)}{1 - \nu(\lambda_n)} = \frac{n}{k},$$

so that  $f_n$  achieves its (positive) supremum at  $\lambda_n$ . Then  $\lambda_n \rightarrow 0$ , since  $n/k \rightarrow +\infty$ . Therefore, thanks to (3.4), we get that

$$(3.7) \quad \frac{n}{k} \stackrel{n \rightarrow \infty}{\sim} \nu'(\lambda_n) \stackrel{n \rightarrow \infty}{\sim} \frac{1}{\lambda_n} \varphi \left( \frac{1}{\lambda_n} \right),$$

which is equivalent to

$$(3.8) \quad \frac{1}{\lambda_n} \stackrel{n \rightarrow \infty}{\sim} \frac{n}{k} \varphi^* \left( \frac{n}{k} \right).$$

Then, (3.4) gives that  $\nu(\lambda_n) \stackrel{n \rightarrow \infty}{\sim} r(1/\lambda_n) \gg \varphi(1/\lambda_n)$ , which with (3.7) shows that  $n\lambda_n \ll k\nu(\lambda_n)$ . In the end, we get

$$(3.9) \quad f_n(\lambda_n) = (1 + o(1))k\nu(\lambda_n) \stackrel{n \rightarrow \infty}{\sim} k r \left( \frac{1}{\lambda_n} \right) \stackrel{n \rightarrow \infty}{\sim} k r \left( \frac{n}{k} \varphi^* \left( \frac{n}{k} \right) \right).$$

With (3.5) this lets us conclude

$$(3.10) \quad \mathbf{P}(\tau_k \leq n) \leq \exp \left[ -(1 + o(1))k r \left( \frac{n}{k} \varphi^* \left( \frac{n}{k} \right) \right) \right].$$

*Lower bound in (1.12)* As is standard, we will obtain a corresponding lower bound using a tilted distribution. Let  $\epsilon > 0$ , and let  $\tilde{\lambda}_n$  satisfy (analogously to (3.6))

$$(3.11) \quad \frac{\nu'(\tilde{\lambda}_n)}{1 - \nu(\tilde{\lambda}_n)} = (1 - \epsilon) \frac{n}{k}.$$

Then, let  $\tilde{\mathbf{P}}, \tilde{\mathbf{E}}, \tilde{\mathbf{V}}\text{ar}$  denote the probability, expectation and variance with respect to the tilted distribution of the i.i.d. sequence  $(G_1, G_2, \dots)$  given by

$$\tilde{\mathbf{P}}(G_1 \in \cdot) = \frac{\mathbf{E} \left( e^{-\tilde{\lambda}_n \tau_1} \mathbf{1}_{\{\tau_1 \in \cdot\}} \right)}{\mathbf{E}(e^{-\tilde{\lambda}_n \tau_1})}.$$

We estimate

$$(3.12) \quad \begin{aligned} \mathbf{P}(\tau_k \leq n) &\geq \frac{\mathbf{E}(e^{-\tilde{\lambda}_n \tau_k})}{e^{-(1-2\epsilon)n\tilde{\lambda}_n}} \frac{\mathbf{E} \left( e^{-\tilde{\lambda}_n \tau_k} \mathbf{1}_{\{\tau_k \in ((1-2\epsilon)n, n)\}} \right)}{\mathbf{E}(e^{-\tilde{\lambda}_n \tau_k})} \\ &\geq \exp \left( (1 - 2\epsilon)n\tilde{\lambda}_n + k \log(1 - \nu(\tilde{\lambda}_n)) \right) \tilde{\mathbf{P}}(\tau_k \in ((1 - 2\epsilon)n, n)). \end{aligned}$$

Note that (3.8) translates here as

$$\frac{1}{\tilde{\lambda}_n} \stackrel{n \rightarrow \infty}{\sim} (1 - \epsilon) \frac{n}{k} \varphi^* \left( \frac{n}{k} \right),$$

so that  $\nu(\tilde{\lambda}_n) \stackrel{n \rightarrow \infty}{\sim} r(1/\tilde{\lambda}_n) \stackrel{n \rightarrow \infty}{\sim} \nu(\lambda_n)$ . As in (3.9), we get that

$$(3.13) \quad \mathbf{P}(\tau_k \leq n) \geq \exp \left[ -(1 + o(1))k r \left( \frac{n}{k} \varphi^* \left( \frac{n}{k} \right) \right) \right] \times \tilde{\mathbf{P}}(\tau_k \in ((1 - 2\epsilon)n, n)),$$

and it only remains to show that the last probability converges to 1 as  $n \rightarrow +\infty$ .

It is standard that

$$(3.14) \quad \tilde{\mathbf{E}}(G_1) = \frac{\nu'(\tilde{\lambda}_n)}{1 - \nu(\tilde{\lambda}_n)} = (1 - \epsilon) \frac{n}{k},$$

so we only need to show that  $\tilde{\mathbf{V}}\text{ar}(G_1) = o(n^2/k)$ . In fact, we have

$$(3.15) \quad \tilde{\mathbf{E}}[(G_1)^2] = \frac{1}{1 - \nu(\tilde{\lambda}_n)} \sum_{j=1}^{\infty} j \varphi(j) e^{-\tilde{\lambda}_n j} \stackrel{n \rightarrow \infty}{\sim} \frac{1}{(\tilde{\lambda}_n)^2} \varphi \left( \frac{1}{\tilde{\lambda}_n} \right) \stackrel{n \rightarrow \infty}{\sim} \frac{1}{\tilde{\lambda}_n} (1 - \epsilon) \frac{n}{k},$$

where the last equivalence is a slight variant of (3.7). Since  $k\varphi(n) \rightarrow \infty$ , by a similar variant of (3.8) we have

$$\frac{1}{\tilde{\lambda}_n} \varphi \left( \frac{1}{\tilde{\lambda}_n} \right) \sim (1 - \epsilon) \frac{n}{k} \ll n\varphi(n),$$

and therefore  $\tilde{\lambda}_n^{-1} = o(n)$ . With (3.15) this shows that indeed  $\tilde{\mathbf{V}}\text{ar}(G_1) = o(n^2/k)$ .

**Proof of (iii).** The existence of  $I(b)$  is standard, and its asymptotics as  $b \rightarrow \infty$  simply follow from (ii).

## 4. REVERSE RENEWAL THEOREMS

4.1. **Transient case, proof of Theorem 1.5.** Denote  $p_\infty := \mathbf{P}(\tau_1 = +\infty) > 0$ . We fix  $\epsilon > 0$ , and  $A$  large enough so  $\mathbf{P}(\tau_1 > A) \in [p_\infty, p_\infty + \epsilon]$ , and hence  $\mathbf{P}(A < \tau_1 < +\infty) \leq \epsilon$ . We then define the events

$$\mathcal{A}_1 = \{\tau \cap (0, A] = \emptyset\} \quad \text{and} \quad \mathcal{A}_2 = \{\tau \cap [n - A, n) = \emptyset\}.$$

We claim that if  $n$  is large enough,

$$(4.1) \quad \begin{aligned} (1 - \epsilon)\mathbf{P}(\tau_1 \leq A) &\leq \mathbf{P}(\mathcal{A}_1^c | n \in \tau) \leq (1 + \epsilon)\mathbf{P}(\tau_1 \leq A), \\ (1 - \epsilon)\mathbf{P}(\tau_1 \leq A) &\leq \mathbf{P}(\mathcal{A}_2^c | n \in \tau) \leq (1 + \epsilon)\mathbf{P}(\tau_1 \leq A), \\ (1 - \epsilon)\mathbf{P}(\tau_1 \leq A)^2 &\leq \mathbf{P}(\mathcal{A}_1^c \cap \mathcal{A}_2^c | n \in \tau) \leq (1 + \epsilon)\mathbf{P}(\tau_1 \leq A)^2. \end{aligned}$$

Indeed, we can write

$$(4.2) \quad \mathbf{P}(\mathcal{A}_1^c \cap \mathcal{A}_2^c | n \in \tau) = \sum_{i=1}^A \sum_{j=1}^A \mathbf{P}(\tau_1 = i) \mathbf{P}(\tau_1 = j) \frac{\mathbf{P}(n - i - j \in \tau)}{\mathbf{P}(n \in \tau)}.$$

Since  $\mathbf{P}(n \in \tau)$  is regularly varying, for large  $n$ , the last ratio is close to 1 uniformly in  $i, j \leq A$ , and the third line in (4.1) follows. The first two lines are proved similarly.

It follows from (4.1) that

$$(4.3) \quad \begin{aligned} \mathbf{P}(\mathcal{A}_1 \cap \mathcal{A}_2 | n \in \tau) &= 1 - \mathbf{P}(\mathcal{A}_1^c | n \in \tau) - \mathbf{P}(\mathcal{A}_2^c | n \in \tau) + \mathbf{P}(\mathcal{A}_1^c \cap \mathcal{A}_2^c | n \in \tau) \\ &\leq 1 - 2\mathbf{P}(\tau_1 \leq A) + \mathbf{P}(\tau_1 \leq A)^2 + 3\epsilon \\ &\leq \mathbf{P}(\tau_1 > A)^2 + 3\epsilon \\ &\leq (p_\infty + \epsilon)^2 + 3\epsilon. \end{aligned}$$

Therefore for large  $n$ ,

$$(4.4) \quad \mathbf{P}(\tau_1 = n) \leq \mathbf{P}(\mathcal{A}_1, \mathcal{A}_2, n \in \tau) \leq ((p_\infty + \epsilon)^2 + 3\epsilon) \mathbf{P}(n \in \tau).$$

Similarly to (4.3),  $\mathbf{P}(\mathcal{A}_1 \cap \mathcal{A}_2 | n \in \tau) \geq p_\infty^2 - 3\epsilon$  and hence

$$(4.5) \quad \mathbf{P}(\mathcal{A}_1, \mathcal{A}_2, n \in \tau) \geq (p_\infty^2 - 3\epsilon) \mathbf{P}(n \in \tau).$$

To turn this into a lower bound on  $\mathbf{P}(\tau_1 = n)$ , we show that conditionally on  $\{\mathcal{A}_1, \mathcal{A}_2, n \in \tau\}$ , it is very likely that  $\tau_1 = n$ . More precisely, we claim that there exists  $c_{16}$  such that, for  $n$  large,

$$(4.6) \quad \mathbf{P}(\tau_1 \neq n, \mathcal{A}_1, \mathcal{A}_2, n \in \tau) \leq c_{16}\epsilon \mathbf{P}(n \in \tau).$$

With (4.5), this shows that

$$(4.7) \quad \mathbf{P}(\tau_1 = n) = \mathbf{P}(\tau_1 = n, \mathcal{A}_1, \mathcal{A}_2, n \in \tau) \geq (p_\infty^2 - 3\epsilon - c_{16}\epsilon) \mathbf{P}(n \in \tau).$$

Since  $\epsilon$  is arbitrary, (4.4) and (4.7) complete the proof of Theorem 1.5.

To prove (4.6), we write

$$(4.8) \quad \begin{aligned} \mathbf{P}(\tau_1 \neq n, \mathcal{A}_1, \mathcal{A}_2, n \in \tau) &\leq \sum_{i=A+1}^{n/2} \mathbf{P}(\tau_1 = i) \mathbf{P}(n - i \in \tau) \\ &+ \sum_{i=n/2+1}^{n-A-1} \sum_{j=A+1}^{n-i} \mathbf{P}(\tau_1 = i) \mathbf{P}(\tau_1 = j) \mathbf{P}(n - i - j \in \tau). \end{aligned}$$

For the first sum in (4.8), since  $\mathbf{P}(n \in \tau)$  is regularly varying, there is a constant  $c_{17}$  such that, provided that  $n$  is large,  $\mathbf{P}(n - i \in \tau) \leq c_{17} \mathbf{P}(n \in \tau)$  for every  $i \leq n/2$ . Hence

$$(4.9) \quad \sum_{i=A+1}^{n/2} \mathbf{P}(\tau_1 = i) \mathbf{P}(n - i \in \tau) \leq c_{17} \mathbf{P}(n \in \tau) \mathbf{P}(A < \tau_1 < +\infty) \leq c_{17} \epsilon \mathbf{P}(n \in \tau).$$

For the second sum in (4.8), we use that  $\mathbf{P}(\tau_1 = i) \leq \mathbf{P}(i \in \tau) \leq c_{17} \mathbf{P}(n \in \tau)$  for  $n$  large enough, since  $i \in (n/2, n)$ . Therefore,

$$(4.10) \quad \begin{aligned} &\sum_{i=n/2+1}^{n-A-1} \sum_{j=A+1}^{n-i} \mathbf{P}(\tau_1 = i) \mathbf{P}(\tau_1 = j) \mathbf{P}(n - i - j \in \tau) \\ &\leq c_{17} \mathbf{P}(n \in \tau) \mathbf{P}(A < \tau_1 < +\infty) \times \sum_{k=0}^{+\infty} \mathbf{P}(k \in \tau) \leq \frac{c_{17}}{p_\infty} \epsilon \mathbf{P}(n \in \tau), \end{aligned}$$

and the proof of (4.6) is complete.

**4.2. Recurrent case, proof of Theorem 1.4.** We assume now that  $\tau$  is recurrent, so that  $U_n$  grows to infinity as a slowly varying function. We can rewrite (1.14) as

$$(4.11) \quad \mathbf{P}(\tau_1 \in ((1 - \epsilon_n)n, n]) = r_{(1-\epsilon_n)n} - r_n \stackrel{n \rightarrow \infty}{\sim} r_n^2 \epsilon_n n \mathbf{P}(n \in \tau).$$

Note that  $r_n$  is slowly varying by (1.13), so if  $\epsilon_n \rightarrow 0$  slowly enough then  $r_{\epsilon_n n} \sim r_n$ . We prove separately an upper and lower bound for the probability on the left in (4.11). The upper bound is provided by the following lemma, which will also be useful in the proof of the lower bound.

**Lemma 4.1.** *Assume that  $\mathbf{P}(n \in \tau)$  is regularly varying and  $U_n$  is slowly varying. Suppose  $\epsilon_n \in (0, 1)$  satisfies  $r_{\epsilon_n n} \stackrel{n \rightarrow \infty}{\sim} r_n$ . Then*

$$(4.12) \quad \mathbf{P}(\tau_1 \in ((1 - \epsilon_n)n, n]) = r_{(1-\epsilon_n)n} - r_n \leq (1 + o(1)) r_n^2 \epsilon_n n \mathbf{P}(n \in \tau).$$

**Proof** The idea is to obtain a lower bound on  $\epsilon_n n \mathbf{P}(n \in \tau) \sim U_n - U_{(1-\epsilon_n)n}$ , the mean number of renewals in  $((1 - \epsilon_n)n, n]$ , by considering trajectories in which the first “big” gap has size in  $((1 - \epsilon_n)n, n]$ , and lands in the interval  $((1 - \epsilon_n)n, (1 + \delta_n)n]$ , see (4.15).

Since  $\{r_k\}$  is slowly varying, we can choose such  $\delta_n = o(\epsilon_n)$  with  $r_{\delta_n n} \stackrel{n \rightarrow \infty}{\sim} r_n$ . Define

$$(4.13) \quad I_n := \min\{i \geq 1 : \tau_i - \tau_{i-1} > \delta_n n\}, \quad \text{and} \quad L_n := \tau_{I_n} - \tau_{I_n-1},$$

the index and length of the first “big” gap. Let  $N_n$  be the number of renewals in  $((1 - \epsilon_n)n, (1 + 2\delta_n)n]$ , and let  $T_n$  be the location of the first renewal in the smaller interval  $((1 - \epsilon_n)n, (1 + \delta_n)n]$ , when one exists, otherwise  $T_n = \infty$ . Then

$$\mathbf{E}[N_n \mid T_n = t] \geq U_{\delta_n n} \quad \text{for all } t \in ((1 - \epsilon_n)n, (1 + \delta_n)n],$$

so

$$(4.14) \quad \mathbf{E} \left[ N_n \mid T_n \in ((1 - \epsilon_n)n, (1 + \delta_n)n] \right] \geq U_{\delta_n n}.$$

By independence, we therefore get that

$$(4.15) \quad \mathbf{P} \left( T_n \in ((1 - \epsilon_n)n, (1 + \delta_n)n] \right) \geq \mathbf{P} (\tau_{I_n-1} \leq \delta_n n) \mathbf{P} (L_n \in ((1 - \epsilon_n)n, n]).$$

We claim that

$$(4.16) \quad \mathbf{P} (\tau_{I_n-1} \leq \delta_n n) \rightarrow 1 \quad \text{as } n \rightarrow \infty,$$

so that

$$(4.17) \quad \mathbf{P} \left( T_n \in ((1 - \epsilon_n)n, (1 + \delta_n)n] \right) \geq (1 - o(1)) \mathbf{P} (\tau_1 \in ((1 - \epsilon_n)n, n] \mid \tau_1 > \delta_n n).$$

Together with (4.14), this shows that

$$(4.18) \quad U_{(1+2\delta_n)n} - U_{(1-\epsilon_n)n} = \mathbf{E}(N_n) \geq (1 - o(1)) \frac{r_{(1-\epsilon_n)n} - r_n}{r_{\delta_n n}} U_{\delta_n n} = (1 - o(1)) \frac{r_{(1-\epsilon_n)n} - r_n}{r_n^2},$$

where we used (1.13). Since  $\mathbf{P}(k \in \tau)$  is regularly varying and  $\delta_n \ll \epsilon_n$ , we also have

$$(4.19) \quad U_{(1+2\delta_n)n} - U_{(1-\epsilon_n)n} \stackrel{n \rightarrow \infty}{\sim} (\epsilon_n + 2\delta_n)n \mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \epsilon_n n \mathbf{P}(n \in \tau).$$

Together with (4.18), this completes the proof of the lemma.

We now prove (4.16). Notice that

$$(4.20) \quad \mathbf{E} [\tau_1 \mathbf{1}_{\{\tau_1 \leq \delta_n n\}}] = \sum_{0 \leq j < \delta_n n} (r_j - r_{\delta_n n}) \ll \sum_{0 \leq j < \delta_n n} r_j \stackrel{n \rightarrow \infty}{\sim} \delta_n n r_n.$$

Therefore we can choose  $D_n \rightarrow \infty$  satisfying  $D_n \mathbf{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq \delta_n n\}}] = o(\delta_n n r_n)$ . Then,

$$(4.21) \quad \mathbf{P} \left( I_n \leq \frac{D_n}{r_n} \right) = 1 - (1 - r_{\delta_n n})^{D_n/r_n} \xrightarrow{n \rightarrow \infty} 1.$$

Also, for all  $j \geq 1$ ,  $\mathbf{E}[\tau_{I_n-1} \mid I_n = j] = (j - 1) \mathbf{E}[\tau_1 \mid \tau_1 \leq \delta_n n]$ , so that

$$(4.22) \quad \mathbf{E} \left[ \tau_{I_n-1} \mid I_n \leq \frac{D_n}{r_n} \right] \leq \frac{D_n}{r_n} (1 - r_{\delta_n n})^{-1} \mathbf{E} [\tau_1 \mathbf{1}_{\{\tau_1 \leq \delta_n n\}}] = o(\delta_n n).$$

With (4.21), it shows that

$$(4.23) \quad \mathbf{P} (\tau_{I_n-1} \leq \delta_n n) \geq \mathbf{P} \left( \tau_{I_n-1} \leq \delta_n n \mid I_n \leq \frac{D_n}{r_n} \right) \mathbf{P} \left( I_n \leq \frac{D_n}{r_n} \right) \rightarrow 1 \quad \text{as } n \rightarrow +\infty.$$

□

For the rest of the section, we let  $\delta_n, I_n, L_n$  be as in the proof of Lemma 4.1.

We now complete the proof of Theorem 1.4 by proving that the upper bound in Lemma 4.1 is sharp, essentially by showing that the lower bound (4.17) is sharp. More precisely, denoting  $J_n$  the numbers of renewals in  $((1 - \epsilon_n + 2\delta_n)n, n]$ , we claim that

$$(4.24) \quad \mathbf{P}(J_n \geq 1) \leq (1 + o(1)) \frac{r_{(1-\epsilon_n)n} - r_n}{r_n}.$$

Since  $\mathbf{E}[J_n \mid J_n \geq 1] \leq U_{\epsilon_n n} \leq U_n$ , together with (1.13) this yields

$$U_n - U_{(1-\epsilon_n+2\delta_n)n} = \mathbf{E}[J_n] \leq \mathbf{P}(J_n \geq 1)U_n \leq (1 + o(1)) \frac{r_{(1-\epsilon_n)n} - r_n}{r_n^2}.$$

Analogously to (4.19) we have  $U_n - U_{(1-\epsilon_n+2\delta_n)n} \stackrel{n \rightarrow \infty}{\sim} \epsilon_n n \mathbf{P}(n \in \tau)$  since  $\delta_n = o(\epsilon_n)$ , and this gives (4.12) with the inequality reversed. We are therefore left with proving (4.24).

First, we deal with the main contribution to the event  $\{J_n \geq 1\}$ :

$$\begin{aligned} \mathbf{P}(J_n \geq 1, \tau_{I_{n-1}} \leq \delta_n n, \tau_{I_n} \in ((1-\epsilon_n+\delta_n)n, n]) &\leq \mathbf{P}(\tau_{I_{n-1}} \leq \delta_n n, L_n \in ((1-\epsilon_n)n, n]) \\ &\leq \frac{r_{(1-\epsilon_n)n} - r_n}{r_{\delta_n n}} \\ (4.25) \qquad \qquad \qquad &\leq (1 + o(1)) r_n \epsilon_n n \mathbf{P}(n \in \tau), \end{aligned}$$

where we used Lemma 4.1 and  $r_{\delta_n n} \sim r_n$  for the last inequality.

Then we need to show that other ways of achieving  $J_n \geq 1$  have a negligible probability. We claim that

$$(4.26) \quad \mathbf{P}(J_n \geq 1, \tau_{I_{n-1}} \leq \frac{n}{2}, \tau_{I_n} \leq (1-\epsilon_n+\delta_n)n) = o(1) r_n \epsilon_n n \mathbf{P}(n \in \tau),$$

$$(4.27) \quad \mathbf{P}(J_n \geq 1, \delta_n n < \tau_{I_{n-1}} \leq \frac{n}{2}, \tau_{I_n} \in ((1-\epsilon_n+\delta_n)n, n]) = o(1) r_n \epsilon_n n \mathbf{P}(n \in \tau),$$

and

$$(4.28) \quad \mathbf{P}(\tau_{I_{n-1}} > \frac{n}{2}) = o(1) r_n \epsilon_n n \mathbf{P}(n \in \tau).$$

In (4.26) the big gap ends early (relative to (4.25)), in (4.27) it starts late, and in (4.28) it starts very late. All together, (4.25)–(4.28) prove (4.24).

**Proof of (4.26).** Let us fix  $n \geq 1$  and let

$$Q_1 = (n - 2\epsilon_n n, (1 - \epsilon_n + \delta_n)n], \quad Q_q = (n - 2^q \epsilon_n n, n - 2^{q-1} \epsilon_n n] \text{ for } q \geq 2,$$

and let  $q_n = \min\{q : 2^q \epsilon_n \geq 1\}$ . For any fixed  $j \leq n/2$  the position of  $\tau_{I_{n-1}}$ , we decompose according to the interval  $Q_q$  containing  $\tau_{I_n}$ :

$$\begin{aligned} \mathbf{P}(J_n \geq 1, \tau_{I_n} \leq (1 - \epsilon_n + \delta_n)n \mid \tau_{I_{n-1}} = j) &= \sum_{q=1}^{q_n} \mathbf{P}(J_n \geq 1, \tau_{I_n} \in Q_q \mid \tau_{I_{n-1}} = j) \\ (4.29) \qquad \qquad \qquad &\leq \sum_{q=1}^{q_n} \mathbf{P}(\tau_1 \in Q_q - j \mid \tau_1 \geq \delta_n n) \max_{m \in n - Q_q} \mathbf{P}(|\tau \cap (m - (\epsilon_n - 2\delta_n)n, m]| \geq 1). \end{aligned}$$

Here the big gap corresponds to  $[j, n - m]$ .

First, we control the last probability in (4.29). Note that for any given interval  $(a, b] \subset (0, \infty]$ , conditioning on the location of the first renewal (if any) in  $(a, b]$ , we get for any  $h \geq 1$

$$(4.30) \quad \mathbf{P}(|\tau \cap (a, b]| \geq 1) U_h \leq U_{b+h} - U_a.$$

For any  $q \geq 2$  and  $m \in n - Q_q$  we have  $m \in (2^{q-1} \epsilon_n n, 2^q \epsilon_n n]$ , and in particular  $m \geq 2\epsilon_n n$ . Therefore, applying (4.30) with  $h = \delta_n n$ , there exists a constant  $c_{18}$  such that

$$(4.31) \quad \mathbf{P}(|\tau \cap (m - (\epsilon_n - 2\delta_n)n, m]| \geq 1) \leq \frac{U_{m+\delta_n n} - U_{m-\epsilon_n n}}{U_{\delta_n n}} \leq c_{18} r_n \epsilon_n n \mathbf{P}(2^q \epsilon_n n \in \tau).$$

For the last inequality we used regular variation of  $\mathbf{P}(k \in \tau)$ , and (1.13).

For  $q = 1$  and  $m \in n - Q_q$ , we have  $m \in ((\epsilon_n - \delta_n)n, 2\epsilon_n)$ . Hence, applying again (4.30) with  $h = \delta_n n$ , we get

$$(4.32) \quad \mathbf{P}(|\tau \cap (m - (\epsilon_n - 2\delta_n)n, m]| \geq 1) \leq \mathbf{P}(|\tau \cap (\delta_n n, 2\epsilon_n n]| \geq 1) \leq \frac{U_{(\epsilon_n + \delta_n)n} - U_{\delta_n n}}{U_{\delta_n n}} \xrightarrow{n \rightarrow \infty} 0,$$

where we used that  $U_{(\epsilon_n + \delta_n)n} \stackrel{n \rightarrow \infty}{\sim} U_n \stackrel{n \rightarrow \infty}{\sim} U_{\delta_n n}$  for the last convergence, which is uniform in  $m \in n - Q_q$ .

We now control  $\mathbf{P}(\tau_1 \in Q_q - j \mid \tau_1 \geq \delta_n n)$ . For  $1 \leq q \leq q_n - 3$ , we have  $2^q \epsilon_n n < n/4$  so  $n - 2^q \epsilon_n n - j > n/4$ . Hence, using Lemma 4.1 (with an interval of length  $2^{q-1} \epsilon_n n$ , which satisfies the hypotheses since  $r_n \leq r_{2^{q-1} \epsilon_n} \leq r_{\epsilon_n n} \stackrel{n \rightarrow \infty}{\sim} r_n$ ),

$$\mathbf{P}(\tau_1 \in Q_q - j) \leq (1 + o(1)) r_n^2 2^{q-1} \epsilon_n n \max_{n/4 < x \leq n} \mathbf{P}(x \in \tau) \leq c_{19} r_n^2 2^{q-1} \epsilon_n n \mathbf{P}(n \in \tau),$$

and therefore, there is some constant  $c_{20}$  such that

$$(4.33) \quad \mathbf{P}(\tau_1 \in Q_q - j \mid \tau_1 \geq \delta_n n) \leq c_{20} r_n 2^{q-1} \epsilon_n n \mathbf{P}(n \in \tau).$$

For  $q = q_n - 2, q_n - 1, q_n$  we have

$$(4.34) \quad \mathbf{P}(\tau_1 \in Q_q - j \mid \tau_1 \geq \delta_n n) \leq \frac{r_{\delta_n n} - r_n}{r_{\delta_n n}} \xrightarrow{n \rightarrow \infty} 0,$$

and, similarly to (4.31), there is a constant  $c_{21}$  such that, for  $m \in n - Q_q$ ,

$$(4.35) \quad \mathbf{P}(|\tau \cap (m - \epsilon_n n, m]| \geq 1) \leq c_{21} r_n \epsilon_n n \mathbf{P}(n \in \tau).$$

Combining (4.31)–(4.35) with (4.29), we finally obtain

$$(4.36) \quad \begin{aligned} & \mathbf{P}(J_n \geq 1, \tau_{I_n} \leq (1 - \epsilon_n - \delta_n)n \mid \tau_{I_{n-1}} = j) \\ & \leq c_{22} r_n \epsilon_n n \mathbf{P}(n \in \tau) \left( o(1) + r_n \sum_{q=2}^{q_n-3} 2^{q-1} \epsilon_n n \mathbf{P}(2^q \epsilon_n n \in \tau) \right). \end{aligned}$$

Now there is a constant  $c_{23}$  such that

$$\sum_{q=2}^{q_n-3} 2^{q-1} \epsilon_n n \mathbf{P}(2^q \epsilon_n n \in \tau) \leq c_{23} \sum_{q=1}^{q_n-3} \sum_{x \in n - Q_q} \mathbf{P}(x \in \tau) \leq c_{23} (U_n - U_{\epsilon_n n}).$$

Therefore, since  $U_n - U_{\epsilon_n n} = o(U_n) = o(r_n^{-1})$ , we get that

$$\mathbf{P}(J_n \geq 1, \tau_{I_n} \leq (1 - \epsilon_n - \delta_n)n \mid \tau_{I_{n-1}} = j) = o(1) r_n \epsilon_n n \mathbf{P}(n \in \tau),$$

uniformly for  $j \leq n/2$ , and (4.26) follows.

**Proof of (4.27).** For all  $j \in (\delta_n n, n/2]$ , by Lemma 4.1 we have

$$(4.37) \quad \begin{aligned} & \mathbf{P}(\tau_{I_n} \in ((1 - \epsilon_n + \delta_n)n, n] \mid \tau_{I_{n-1}} = j) = \mathbf{P}(\tau_1 \in ((1 - \epsilon_n + \delta_n)n - j, n - j] \mid \tau_1 > \delta_n n) \\ & \leq (1 + o(1)) r_n (\epsilon_n - \delta_n) n \mathbf{P}(n - j \in \tau) \leq c_{24} r_n \epsilon_n n \mathbf{P}(n \in \tau). \end{aligned}$$

Therefore, we get that

$$\mathbf{P}(\tau_{I_n} \in ((1 - \epsilon_n - \delta_n)n, n] \mid \delta_n n < \tau_{I_{n-1}} \leq \frac{n}{2}) \leq c_{24} r_n \epsilon_n n \mathbf{P}(n \in \tau),$$

and since  $\mathbf{P}(\tau_{I_{n-1}} > \delta_n n) \xrightarrow{n \rightarrow \infty} 0$  (see (4.23)), (4.27) follows.

**Proof of (4.28).** Since  $\mathbf{P}(n \in \tau)$  is regularly varying and  $U_n$  is slowly varying with  $U_n \rightarrow \infty$ ,  $\mathbf{P}(n \in \tau)$  must have index of regular variation  $-1$ . Hence by [5, Proposition 1.5.9a] we have  $n\mathbf{P}(n \in \tau) \ll U_n$ . Therefore

$$(4.38) \quad \widehat{\varphi}(n) := r_n^2 n\mathbf{P}(n \in \tau)$$

is a slowly varying function which by (1.13) satisfies

$$(4.39) \quad \widehat{\varphi}(n) \ll r_n,$$

and (4.28) is equivalent to proving

$$(4.40) \quad \mathbf{P}(\tau_{I_n-1} > n/2) = o(1) \epsilon_n \frac{\widehat{\varphi}(n)}{r_n}.$$

Note that if we knew (1.1) held, necessarily with  $\alpha = 0$ , we could conclude from Theorem 1.2 that  $\widehat{\varphi}(n) \sim \varphi(n)$ , so  $\widehat{\varphi}$  may be viewed as a surrogate for  $\varphi$  in the absence of (1.1). Decomposing over the value of  $I_n - 1$ , we get, recalling  $\widehat{G}_i^{(m)}$  from Section 2,

$$(4.41) \quad \mathbf{P}(\tau_{I_n-1} > n/2) = \sum_{k=1}^{+\infty} \mathbf{P}(I_n-1 = k, \tau_k > n/2) = \sum_{k=1}^{+\infty} r_{\delta_n n} (1-r_{\delta_n n})^{k-1} \mathbf{P}\left(\sum_{i=1}^k \widehat{G}_i^{(\delta_n n)} > n/2\right).$$

We now need an analogue of Lemma 2.1.

**Lemma 4.2.** *Assume that  $\mathbf{P}(n \in \tau)$  is regularly varying with index  $-1$  and  $U_n$  is slowly varying. There exist  $m_0, c_{25} > 0$  such that for all  $m_0 \leq m \leq n$  and  $k \geq 1$ ,*

$$(4.42) \quad \mathbf{P}\left(\sum_{i=1}^k \widehat{G}_i^{(m)} \geq \frac{n}{2}\right) \leq \left(\frac{c_{25} k m \widehat{\varphi}(m)}{n}\right)^{\frac{n}{2m}}.$$

**Proof** The proof follows the same lines as that of Lemma 2.1. The only modification needed is in (2.9): the computation of  $\mathbf{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq m\}}]$  required knowledge on  $\mathbf{P}(\tau_1 = n)$ . Therefore it is sufficient to show that there exists some constant  $c_{26}$  such that

$$(4.43) \quad \mathbf{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq m\}}] \leq c_{26} m \widehat{\varphi}(m).$$

Indeed, thanks to Lemma 4.1, we get that there exist  $\ell_0 > 0$  and a constant  $c_{27}$  such that, for any  $\ell \geq \ell_0$

$$\mathbf{P}(\tau_1 \in (2^{\ell-1}, 2^\ell]) \leq c_{27} (r_{2^\ell})^2 2^{\ell-1} \mathbf{P}(2^\ell \in \tau) \leq c_{27} \widehat{\varphi}(2^\ell).$$

Therefore for any  $m \geq \ell_0$ ,

$$\begin{aligned} \mathbf{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq m\}}] &\leq 2^{\ell_0} + \sum_{\ell=\ell_0+1}^{\lceil \log_2(m) \rceil} 2^\ell \mathbf{P}(\tau_1 \in (2^{\ell-1}, 2^\ell]) \\ &\leq 2^{\ell_0} + c_{27} \sum_{\ell=1}^{\lceil \log_2(m) \rceil} 2^\ell \widehat{\varphi}(2^\ell) \\ &\leq 2^{\ell_0} + c_{28} \sum_{x=1}^m \widehat{\varphi}(x), \end{aligned}$$

and (4.43) is proved.  $\square$

Going back to (4.41), we can apply Lemma 4.2 with  $m = \delta_n n$ , and note that  $2r_n \geq r_{\delta_n n} \geq r_n$  for large  $n$ , to obtain

$$(4.44) \quad \mathbf{P}(\tau_{I_{n-1}} > n/2) \leq 3r_n (c_{25} \delta_n \widehat{\varphi}(\delta_n n))^{1/2\delta_n} \sum_{k=1}^{+\infty} e^{-kr_n} k^{1/2\delta_n}.$$

The sum here is readily approximated by a gamma function, and an application of Stirling's formula then yields

$$\sum_{k=1}^{+\infty} e^{-kr_n} k^{1/2\delta_n} \leq \frac{c_{29}}{\delta_n^{1/2} r_n} \left( c_{30} \frac{\widehat{\varphi}(\delta_n n)}{\delta_n r_n} \right)^{1/2\delta_n},$$

so that for large  $n$ ,

$$(4.45) \quad \mathbf{P}(\tau_{I_{n-1}} > n/2) \leq \frac{c_{29}}{\delta_n^{1/2}} \left( c_{31} \frac{\widehat{\varphi}(\delta_n n)}{r_n} \right)^{1/2\delta_n} \leq \delta_n^3 \left( c_{32} \frac{\widehat{\varphi}(\delta_n n)}{r_n} \right)^{1/2\delta_n} \leq \delta_n \left( \frac{\delta_n \widehat{\varphi}(\delta_n n)}{r_n} \right)^2.$$

Since  $\widehat{\varphi}$  is slowly varying, for  $n$  large enough we have  $\delta_n \widehat{\varphi}(\delta_n n) \leq \widehat{\varphi}(n)$ . Since we chose  $\delta_n = o(\epsilon_n)$ , we therefore obtain

$$(4.46) \quad \mathbf{P}(\tau_{I_{n-1}} > n/2) \leq \epsilon_n \left( \frac{\widehat{\varphi}(n)}{r_n} \right)^2,$$

which with (4.39) completes the proof of (4.40), hence also of (4.28), (4.24) and finally (4.11), or equivalently (1.14).

Equation (1.15) is an immediate consequence of (1.14).  $\square$

**4.3. Why not expect a stronger reverse renewal theorem?** We give here an example of a distribution for a recurrent renewal  $\tau$  where  $\mathbf{P}(n \in \tau)$  is regularly varying (with index of regular variation  $-1$ ) but  $\mathbf{P}(\tau_1 = n)$  is not. This shows that (1.15) is not true in general under the assumptions used to obtain (1.14).

Let  $\sigma$  be a recurrent renewal with inter-arrival distribution of form

$$(4.47) \quad \mathbf{P}(\sigma_1 = n) = \varphi(n) n^{-1}.$$

Now, let  $\tau_1$  be  $2\sigma_1$  or  $1$ , with probability  $1/2$  each:

$$\mathbf{P}(\tau_1 = 1) = \frac{1}{2}, \quad \mathbf{P}(\tau_1 = 2m) = \frac{1}{2} \mathbf{P}(\sigma_1 = m), \quad \mathbf{P}(\tau_1 = 2m - 1) = 0 \quad \text{for } m \geq 1.$$

Note that  $r_n := \mathbf{P}(\tau_1 > n) \stackrel{n \rightarrow \infty}{\sim} \frac{1}{2} \mathbf{P}(\sigma_1 > n)$ . Then  $\mathbf{P}(\tau_1 = n)$  is not regularly varying, but we show that the gaps of length 1 have a smoothing effect, and make  $\mathbf{P}(n \in \tau)$  regularly varying. More precisely, we prove

$$(4.48) \quad \mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \frac{\varphi(n)}{2r_n^2 n}.$$

**Proof of (4.48)** We choose  $\theta_n, \lambda_n$  satisfying

$$r_n^{-1} \ll \theta_n \ll \varphi(n)^{-1} \quad \text{and} \quad 1 \ll \lambda_n \ll r_n^{-1/2},$$

and decompose  $\mathbf{P}(n \in \tau)$  into three sums:

$$(4.49) \quad \mathbf{P}(n \in \tau) = \sum_{k \leq (\lambda_n r_n)^{-1}} \mathbf{P}(\tau_k = n) + \sum_{(\lambda_n r_n)^{-1} < k \leq \theta_n} \mathbf{P}(\tau_k = n) + \sum_{k > \theta_n} \mathbf{P}(\tau_k = n).$$

We will show that the main contribution comes from the middle sum, see (4.55), the first and last sum being negligible, see (4.56)-(4.57).

**Middle sum.** We introduce  $X_k$  the number of gaps of length 1 in the first  $k$  gaps of  $\tau$ . For  $(\lambda_n r_n)^{-1} < k \leq \theta_n$ , note that  $\lambda_n \leq k\lambda_n^2 r_n \ll k$  and  $k \ll n$ , and write

$$(4.50) \quad \begin{aligned} \mathbf{P}(\tau_k = n) &= \mathbf{P}\left(X_k - \frac{k}{2} \in (-k\lambda_n r_n^{1/2}, k\lambda_n r_n^{1/2}); \tau_k = n\right) \\ &\quad + \mathbf{P}\left(|X_k - \frac{k}{2}| \geq k\lambda_n r_n^{1/2}; \tau_k = n\right). \end{aligned}$$

The last probability is small. Indeed, there is a constant  $c_{33}$  such that

$$\mathbf{P}\left(|X_k - \frac{k}{2}| \geq k\lambda_n r_n^{1/2}\right) \leq e^{-c_{33}\lambda_n^2 r_n k} \quad \text{for all } k \geq 1;$$

conditioning on  $X_k$  we therefore get that

$$(4.51) \quad \mathbf{P}\left(|X_k - \frac{k}{2}| \geq k\lambda_n r_n^{1/2}; \tau_k = n\right) \leq e^{-c_{33}\lambda_n^2 r_n k} \sup_{1 \leq j \leq k} \sup_{\frac{n-k}{2} \leq m \leq \frac{n}{2}} \mathbf{P}(\sigma_j = m).$$

Here the sups are over all possible values of  $j = k - X_k$  and  $m = (n - X_k)/2$ . Applying (1.9) we see that for  $n$  large, for all  $m \geq (n - k)/2 \geq n/4$  and  $j \leq k$ , we have

$$\mathbf{P}(\sigma_j = m) \leq c_{34} k \mathbf{P}(\sigma_1 = n).$$

Since  $e^{-c_{33}\lambda_n^2 r_n k} = o(1) e^{-kr_n}$  as  $n \rightarrow \infty$ , uniformly in middle-sum values of  $k$ , we get that

$$(4.52) \quad \mathbf{P}\left(|X_k - \frac{k}{2}| \geq k\lambda_n r_n^{1/2}; \tau_k = n\right) = o(1) k e^{-kr_n} \frac{\varphi(n)}{n} \quad \text{as } n \rightarrow \infty,$$

with the  $o(1)$  uniform over middle-sum values of  $k$ .

For the first probability on the right in (4.50), we use (1.8). Uniformly for  $j$  in the interval  $\frac{k}{2} + (-k\lambda_n r_n^{1/2}, k\lambda_n r_n^{1/2})$  with  $j \equiv n - k \pmod{2}$ , and for middle-sum values of  $k$  (which satisfy  $k\varphi(n) \xrightarrow{n \rightarrow \infty} 0$  and  $k \rightarrow +\infty$ ), we have  $j \sim k/2$  and  $k \ll n$ , so

$$(4.53) \quad \begin{aligned} \mathbf{P}\left(\sigma_j = \frac{n - k + j}{2}\right) &= (1 + o(1)) k (1 - 2r_{(n-k+j)/2})^j \frac{\varphi(n/2)}{n/2} \\ &= (1 + o(1)) 2k e^{-kr_n(1+o(1))} \frac{\varphi(n)}{n}, \end{aligned}$$

since  $\mathbf{P}(\sigma_1 > n) \xrightarrow{n \rightarrow \infty} 2r_n$ . Therefore, since

$$\mathbf{P}\left(X_k - \frac{k}{2} \in (-k\lambda_n r_n^{1/2}, k\lambda_n r_n^{1/2}); X_k \equiv n \pmod{2}\right) \rightarrow \frac{1}{2} \quad \text{as } n \rightarrow \infty,$$

conditioning again on  $X_k$  we get that for middle-sum values of  $k$ ,

$$(4.54) \quad \mathbf{P}\left(X_k - \frac{k}{2} \in (-k\lambda_n r_n^{1/2}, k\lambda_n r_n^{1/2}); \tau_k = n\right) = (1 + o(1)) k e^{-kr_n(1+o(1))} \frac{\varphi(n)}{n},$$

with the  $o(1)$  uniform over middle-sum values of  $k$ .

Summing (4.52) and (4.54), we obtain straightforwardly that

$$(4.55) \quad \sum_{(\lambda_n r_n)^{-1} < k \leq \theta_n} \mathbf{P}(\tau_k = n) = \sum_{(\lambda_n r_n)^{-1} < k \leq \theta_n} (1 + o(1)) k e^{-kr_n(1+o(1))} \frac{\varphi(n)}{n} = (1 + o(1)) \frac{\varphi(n)}{2r_n^2 n}.$$

We are therefore left with showing that the two other sums in (4.49) are negligible.

**First sum.** Using (1.9) and (4.47) and conditioning once more on  $X_k$ , we get that there exists a constant  $c_{35}$  such that for  $n$  large enough, for any  $k \leq (\lambda_n r_n)^{-1}$ ,

$$\mathbf{P}(\tau_k = n) \leq \sup_{1 \leq j \leq k} \sup_{\frac{n-k}{2} \leq m \leq \frac{n}{2}} \mathbf{P}(\sigma_j = m) \leq c_{35} k \frac{\varphi(n)}{n},$$

which gives

$$(4.56) \quad \sum_{k \leq (\lambda_n r_n)^{-1}} \mathbf{P}(\tau_k = n) \leq \frac{c_{35}}{(\lambda_n r_n)^2} \frac{\varphi(n)}{n} = o(1) \frac{\varphi(n)}{r_n^2 n}.$$

**Last sum.** Similarly to (4.52)-(4.54) but using (1.9) in place of (1.8), we obtain that there exists  $c_{36}$  such that for all  $\theta_n < k \leq n/2$ ,

$$\mathbf{P}(\tau_k = n) \leq c_{36} k e^{-kr_n(1+o(1))} \frac{\varphi(n)}{n} \leq k e^{-kr_n/2} \frac{\varphi(n)}{n},$$

the last inequality being valid for  $n$  large, since  $kr_n \rightarrow +\infty$ .

For  $k \in (n/2, n]$ , we use that

$$\mathbf{P}(\tau_k = n) \leq (1 - r_n)^k \leq e^{-nr_n/2}.$$

Since  $\theta_n \gg r_n^{-1}$ , we therefore obtain that

$$(4.57) \quad \sum_{k > \theta_n} \mathbf{P}(\tau_k = n) \leq \sum_{k > \theta_n} k e^{-kr_n/2} \frac{\varphi(n)}{n} + n e^{-nr_n/2} = o(1) \frac{\varphi(n)}{r_n^2 n}.$$

This completes the proof of (4.48). □

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