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Kenneth S. Alexander, Quentin Berger

LOCAL ASYMPTOTICS FOR THE FIRST INTERSECTION OF TWO INDEPENDENT RENEWALS

KENNETH S. ALEXANDER AND QUENTIN BERGER

Abstract. We study the intersection of two independent renewal processes, $\rho = \tau \cap \sigma$. Assuming that $\mathbf{P}(\tau_1 = n) = \varphi(n)\, n^{-(1+\alpha)}$ and $\mathbf{P}(\sigma_1 = n) = \widetilde{\varphi}(n)\, n^{-(1+\widetilde{\alpha})}$ for some $\alpha, \widetilde{\alpha} \geqslant 0$ and some slowly varying $\varphi, \widetilde{\varphi}$, we give the asymptotic behavior first of $\mathbf{P}(\rho_1 > n)$ (which is straightforward except in the case of $\min(\alpha, \widetilde{\alpha}) = 1$) and then of $\mathbf{P}(\rho_1 = n)$. The result may be viewed as a kind of reverse renewal theorem, as we determine probabilities $\mathbf{P}(\rho_1 = n)$ while knowing asymptotically the renewal mass function $\mathbf{P}(n \in \rho) = \mathbf{P}(n \in \tau)\mathbf{P}(n \in \sigma)$. Our results can be used to bound coupling-related quantities, specifically the increments $|\mathbf{P}(n \in \tau) - \mathbf{P}(n-1 \in \tau)|$ of the renewal mass function.

1. Intersection of two independent renewals

We consider two independent (discrete) renewal processes τ and σ, whose law are denoted respectively $\mathbf{P}_\tau$ and $\mathbf{P}_\sigma$, and the renewal process of intersections, $\rho = \tau \cap \sigma$. We denote $\mathbf{P} = \mathbf{P}_\tau \otimes \mathbf{P}_\sigma$.

The process ρ appears in various contexts. In pinning models, for example, it may appear directly in the definition of the model (as in [1], where σ represents sites with nonzero disorder values, and τ corresponds to the polymer being pinned) or it appears in the computation of the variance of the partition function via a replica method (see for example [20]), and is central in deciding whether disorder is relevant or irrelevant in these models, cf. [3].

When τ and σ have the same inter-arrival distribution, ρ_1 is related to the coupling time of τ and σ, if we allow τ and σ to start at different points. In particular, in the case $\mu := \mathbf{E}[\tau_1] < +\infty$, the coupling time ρ_1 has been used to study the rate of convergence in the renewal theorem, see [16, 17], using that

$$|\mathbf{P}(n \in \tau) - \mathbf{P}(n \in \sigma)| \leqslant \mathbf{E}\big[|\mathbf{1}_{\{n\in\tau\}} - \mathbf{1}_{\{n\in\sigma\}}|\mathbf{1}_{\{\rho_1 > n\}}\big] \leqslant \mathbf{P}(\rho_1 > n)\,.$$

Hence, if σ is *delayed* by a random X having the waiting time distribution ν of the renewal process (and denoting $\mathbf{P}_\nu$ the delayed law of σ), we have that $\mathbf{P}_\nu(n \in \sigma) = \frac{1}{\mu}$ for all n, and so $\mathbf{P}_\tau \otimes \mathbf{P}_\nu(\rho_1 > n)$ gives the rate of convergence in the renewal theorem. This question has also been studied via a more analytic method in [18, 13]. Denoting $u_n := \mathbf{P}(n \in \tau)$ the renewal mass function ot τ, Rogozin [18] proved that $u_n - \frac{1}{\mu} \sim \frac{1}{\mu^2} \sum_{k>n}^{+\infty} \mathbf{P}(\tau_1 > k)$ as $n \to \infty$.

In this paper, we consider only the non-delayed case, with a brief exception to study $|u_n - u_{n-1}|$, see Theorem 1.6.

1.1. **Setting of the paper.** We assume that there exist $\alpha, \widetilde{\alpha} \geqslant 0$ and slowly varying functions $\varphi, \widetilde{\varphi}$ such that

$$\mathbf{P}(\tau_1 = n) = \varphi(n)\, n^{-(1+\alpha)}\,, \qquad \mathbf{P}(\sigma_1 = n) = \widetilde{\varphi}(n)\, n^{-(1+\widetilde{\alpha})}\,. \tag{1.1}$$

(As mentioned above, τ and σ are *non-delayed*, if not specified otherwise.) With no loss of generality, we assume that $\alpha \leqslant \widetilde{\alpha}$. We define $\mu_n := \mathbf{E}[\tau_1 \wedge n]$ and $\widetilde{\mu}_n := \mathbf{E}[\sigma_1 \wedge n]$ the truncated means, and also $\mathbf{E}[\tau_1] = \mu = \lim_{n\to\infty} \mu_n \leqslant \infty$, and similarly $\widetilde{\mu} = \lim_{n\to\infty} \widetilde{\mu}_n$.

The assumption (1.1) is very natural, and is widely used in the literature (for example, once again in pinning models). It covers in particular the case of the return times $\tau = \{n\,,\, S_{2n} = 0\}$, where $(S_n)_{n\geqslant 0}$ is the simple symmetric nearest-neighbor random walk on $\mathbb{Z}^d$ (see e.g. [8, Ch. III] for $d = 1$, [14, Thm. 4] for $d = 2$ and [6, Thm. 4] for $d = 3$), or the case $\tau = \{n\,, S_n = 0\}$ where $(S_n)_{n\geqslant 0}$ an aperiodic random walk in the domain of attraction of a symmetric stable law, see [15, Thm. 8].

In Section 2, we recall the strong renewal theorems for τ and σ under assumption (1.1) (from [2, 5, 7] in the recurrent case, [11, App. A.5] in the transient case), as well as newer reverse renewal theorems (from [2]). We collect the results when τ is recurrent in the following table, denoting $r_n := \mathbf{P}(\tau_1 > n)$, and we refer to (2.1) for the transient case.

	$\alpha \geqslant 1$	$\alpha \in (0,1)$	$\alpha = 0$
$\mathbf{P}(n\in\tau) \overset{n\to\infty}{\sim}$	$(\mu_n)^{-1}$	$\frac{\alpha\sin(\pi\alpha)}{\pi}\, n^{-(1-\alpha)}\varphi(n)^{-1}$	$\frac{\varphi(n)}{n\, r_n^2}$

TABLE 1. Asymptotics of the renewal mass function if τ is recurrent, and has inter-arrival distribution $\mathbf{P}(\tau_1 = n) = \varphi(n)n^{-(1+\alpha)}$ with $\alpha \geqslant 0$.

From Table 1 and (2.1), the renewal mass function of ρ satisfies

$$\mathbf{P}(n\in\rho) = \mathbf{P}(n\in\tau)\mathbf{P}(n\in\sigma) = \psi^*(n)\, n^{-\theta^*} \tag{1.2}$$

for some $\theta^* \geqslant 0$ and slowly varying function $\psi^*(n)$. For example, if both τ and σ are recurrent we have

$$\theta^* = 2 - \alpha\wedge 1 - \widetilde{\alpha}\wedge 1; \tag{1.3}$$

if also $\alpha, \widetilde{\alpha} \in (0,1)$, then ψ^* is a constant multiple of $1/\varphi\widetilde{\varphi}$. If instead both τ and σ are transient then $\theta^* = 2 + \alpha + \widetilde{\alpha}$. Note that ρ is transient for $\theta^* > 1$ and recurrent for $\theta^* < 1$. Recalling that $\alpha \leqslant \widetilde{\alpha}$, if we define

$$\alpha^* = \begin{cases} \alpha & \text{if } \rho \text{ is recurrent and } \alpha \geqslant 1,\\ 1-\theta^* & \text{if } \rho \text{ is recurrent and } \alpha < 1,\\ \theta^* - 1 & \text{if } \rho \text{ is transient,} \end{cases} \tag{1.4}$$

then, based on Theorem 2.1 in the transient case and Table 1 in the recurrent case, we expect $\mathbf{P}(\rho_1 = n)$ to be expressed as $n^{-(1+\alpha^*)}$ multiplied by a slowly varying function.

Observe that the renewal function of ρ, defined as

$$U_n^* := \sum_{k=0}^{n} \mathbf{P}(n \in \rho),$$

is always regularly varying, with exponent $\alpha^* = 1 - \theta^*$ in the recurrent case and 0 in the transient case.

Our goal is to derive from (1.1) the local asymptotics of the inter-arrival distribution, that is, the asymptotics of $\mathbf{P}(\rho_1 = n)$. For general renewal processes ρ these asymptotics should not be uniquely determined by the asymptotic behavior of the renewal mass function (1.2) (which is known is our case), but the extra structure given by $\rho = \tau \cap \sigma$ under (1.1) makes such determination possible.

Remark 1.1. For ρ to be recurrent, it is necessary that both τ and σ are recurrent, so (1.3) holds. It follows from Table 1 that ρ is recurrent if and only if one of the following also holds:

(i) $\alpha + \widetilde{\alpha} > 1$,

(ii) $\alpha, \widetilde{\alpha} \in (0,1), \alpha + \widetilde{\alpha} = 1$ and $\sum_{n \geqslant 1} \frac{1}{n\varphi(n)\widetilde{\varphi}(n)} = +\infty$,

(iii) $\alpha = 0, \widetilde{\alpha} = 1$ and $\sum_{n \geqslant 1} \frac{\varphi(n)}{n\, r_n^2 \widetilde{\mu}_n} = +\infty$.

1.2. Main results.

Case of transient ρ. Since $\mathbf{P}(n \in \rho)$ is summable (with sum $\mathbf{E}(|\rho|)$), we must have $\theta^* \geqslant 1$. Here the following is immediate from ([2], Theorem 1.4), given below as Theorem 2.1.

Theorem 1.2. *Assume* (1.1)*, and suppose that ρ is transient. Then*

$$\mathbf{P}(\rho_1 = n) \overset{n\to\infty}{\sim} \frac{1}{\mathbf{E}(|\rho|)^2} \mathbf{P}(n \in \tau)\mathbf{P}(n \in \sigma)\,.$$

Case of recurrent ρ. Here τ and σ must be recurrent, so (1.3) holds with $\theta^* \in [0,1]$, and $\alpha^* = 1 - \theta^*$ if $\alpha \leqslant 1$, $\alpha^* = \alpha$ if $\alpha \geqslant 1$.

Theorem 1.3. *Assume* (1.1)*, and suppose that ρ is recurrent. Then for α^* from* (1.4) *(with θ^* defined as in* (1.3)*), the following hold.*

(i) If $\alpha^* \in (0,1)$ then

$$\mathbf{P}(\rho_1 > n) \overset{n\to\infty}{\sim} \frac{\sin(\pi\alpha^*)}{\pi} \psi^*(n)^{-1}\, n^{-\alpha^*}\,. \tag{1.5}$$

(ii) If $\alpha^* = 0$ then

$$\mathbf{P}(\rho_1 > n) \overset{n\to\infty}{\sim} \left(\sum_{j=1}^{n} \frac{\psi^*(j)}{j}\right)^{-1}, \tag{1.6}$$

which is slowly varying.

(iii) If $\alpha^* \geqslant 1$ then

$$\mathbf{P}(\rho_1 > n) \overset{n\to\infty}{\sim} \widetilde{\mu}_n \mathbf{P}(\tau_1 > n) + \mu_n \mathbf{P}(\sigma_1 > n). \tag{1.7}$$

In Theorem 1.3(iii), $\widetilde{\mu}_n, \mu_n$ are slowly varying since $\widetilde{\alpha} \geqslant \alpha \geqslant 1$ (recall (1.4)), and they may be replaced by μ or $\widetilde{\mu}$ if that mean is finite.

We will prove Theorem 1.3 in Sections 3–4. The cases (i) and (ii) are essentially immediate from known relations of form $\mathbf{P}(\rho_1 > n) \sim c/U_n^*$ and are given in Section 3. Item (iii) seems to be a new result, and is treated in Section 4 via a probabilistic method. Note that in all cases, $\mathbf{P}(\rho_1 > n)$ is regularly varying with exponent $-\alpha^*$.

To obtain the asymptotics of $\mathbf{P}(\rho_1 = n)$ from Theorem 1.3 (in the case $\alpha^* > 0$), or using the weak reverse renewal Theorem 2.2 (in the case $\alpha^* = 0$), we only need to show that $\mathbf{P}(\rho_1 = k)$ is approximately constant on an interval $[(1-\varepsilon)n, n]$ with ε small. To that end we have the following lemma, which we will prove in Section 5.

Lemma 1.4. *Assume* (1.1)*, and suppose that ρ is recurrent. Let $v_n := \mathbf{P}(\rho_1 > n)^2\mathbf{P}(n \in \rho)$. Then for every $\delta > 0$, there exists some $\varepsilon > 0$ such that, if n is large enough we have for all $k \in (0, \varepsilon n)$*

$$(1-\delta)\mathbf{P}(\rho_1 = n-k) - \delta v_n \leqslant \mathbf{P}(\rho_1 = n) \leqslant (1+\delta)\mathbf{P}(\rho_1 = n+k) + \delta v_n. \tag{1.8}$$

We will see later that $v_n = O(\mathbf{P}(\rho_1 = n))$, so Lemma 1.4 is actually true without the δv_n terms, but we will not need this improved result.

We can now state our main theorem, which we will prove in Section 6.

Theorem 1.5. *Assume* (1.1) *with $\widetilde{\alpha} \geqslant \alpha$, and suppose that ρ is recurrent. Let α^* be as in* (1.4).

(i) If $\alpha^* \in (0,1)$ then

$$\mathbf{P}(\rho_1 = n) \overset{n\to\infty}{\sim} \frac{\alpha^* \sin(\pi\alpha^*)}{\pi}\, \psi^*(n)^{-1} n^{-(1+\alpha^*)} \,.$$

(ii) If $\alpha^* = 0$ then

$$\mathbf{P}(\rho_1 = n) \overset{n\to\infty}{\sim} \left(\sum_{k=1}^{n} \frac{\psi^*(k)}{k}\right)^{-2} \frac{\psi^*(n)}{n} \,.$$

(iii) If $\alpha^* \geqslant 1$ then

$$\mathbf{P}(\rho_1 = n) \overset{n\to\infty}{\sim} \widetilde{\mu}_n\, \mathbf{P}(\tau_1 = n) + \mu_n\, \mathbf{P}(\sigma_1 = n) \overset{n\to\infty}{\sim} \frac{\widetilde{\mu}_n \varphi(n) + \mu_n \widetilde{\varphi}(n) n^{\alpha-\widetilde{\alpha}}}{n^{1+\alpha^*}} \,.$$

As in Theorem 1.3, in (iii), $\widetilde{\mu}_n, \mu_n$ may be replaced by μ or $\widetilde{\mu}$ if that mean is finite.

We now illustrate this theorem with some subcases, using Table 1.

1. If τ, σ are recurrent with $\alpha, \widetilde{\alpha} \in (0,1)$ and $\alpha + \widetilde{\alpha} > 1$, then ρ is recurrent with $\alpha^* = \alpha + \widetilde{\alpha} - 1 \in (0,1)$ and

$$\mathbf{P}(\rho_1 = n) \overset{n\to\infty}{\sim} c_{\alpha,\widetilde{\alpha}}\, \varphi(n)\widetilde{\varphi}(n)\, n^{-(\alpha+\widetilde{\alpha})} \qquad \text{with } c_{\alpha,\widetilde{\alpha}} = \frac{\pi\alpha^* \sin(\pi\alpha^*)}{\alpha\widetilde{\alpha}\sin(\pi\alpha)\sin(\pi\widetilde{\alpha})} \,. \tag{1.9}$$

2. If τ, σ are recurrent with $\alpha, \widetilde{\alpha} \in (0,1), \alpha + \widetilde{\alpha} = 1$ and $\sum_{n=1}^{\infty} 1/n\varphi(n)\widetilde{\varphi}(n) = \infty$, then ρ is recurrent, $\alpha^* = 0$, $\psi^*(n) \sim c'_{\alpha,\widetilde{\alpha}} \varphi(n)^{-1}\widetilde{\varphi}(n)^{-1}$ with $c'_{\alpha,\widetilde{\alpha}} = \frac{\alpha\widetilde{\alpha}\sin(\pi\alpha)\sin(\pi\widetilde{\alpha})}{\pi^2}$.

Therefore,

$$\mathbf{P}(\rho_1 = n) \overset{n\to\infty}{\sim} \frac{1}{c'_{\alpha,\widetilde{\alpha}}} \left(\sum_{k=1}^{n} \frac{1}{k\varphi(k)\widetilde{\varphi}(k)} \right)^{-2} \frac{1}{n\varphi(n)\widetilde{\varphi}(n)} . \tag{1.10}$$

As a special case, suppose $\tau = \{n, S_{2n} = 0\}$, $\sigma = \{n, S'_{2n} = 0\}$ are the return times of independent symmetric simple random walks (SSRW) on $\mathbb{Z}$. Then $\alpha = \widetilde{\alpha} = 1/2$ and $\varphi(n) = \widetilde{\varphi}(n) \to \frac{1}{2\sqrt{\pi}}$ so

$$\mathbf{P}(\rho_1 = n) \overset{n\to\infty}{\sim} \frac{\pi}{n(\log n)^2}. \tag{1.11}$$

Rotating the lattice by $\pi/4$ shows that this is the same as the return time distribution for $(\mathbf{S}_n)_{n\geqslant 0}$ the SSRW on $\mathbb{Z}^2$ (the *even* return times: $\rho = \{n, \mathbf{S}_{2n} = 0\}$). Hence (1.11) is a classical result of Jain and Pruitt [14].

3. If τ is recurrent with $\alpha \in (0,1)$, and $\widetilde{\alpha} \geqslant 1$ (so $\widetilde{\mu}_n$ is slowly varying; this includes the case when $\widetilde{\mu} < +\infty$), then $\alpha^* = \alpha$ and

$$\mathbf{P}(\rho_1 = n) \overset{n\to\infty}{\sim} \widetilde{\mu}_n\, \varphi(n) n^{-(1+\alpha)} \overset{n\to\infty}{\sim} \widetilde{\mu}_n\, \mathbf{P}(\tau_1 = n) . \tag{1.12}$$

1.3. **Application to a coupling-related quantity.** We now provide an application of Theorem 1.3.

Theorem 1.6. *Let τ be a recurrent renewal process satisfying* (1.1)*, and let $u_n := \mathbf{P}(n \in \tau)$ be its renewal mass function. There exist constants $c_i > 0$ such that*

$$|u_n - u_{n-1}| \leqslant c_1\, u_n \mathbf{P}(\rho_1 > n) \leqslant \begin{cases} c_2\, n^{-1/2} \varphi(n)^{-1} \left(\sum_{j=1}^{n} \frac{1}{j\varphi(j)^2} \right)^{-1} & \text{if } \alpha = 1/2 \\ c_2\, n^{-\alpha} \varphi(n)^{-1} & \text{if } \alpha > 1/2. \end{cases} \tag{1.13}$$

Note that the right side of (1.13) is of order $\mathbf{P}(\tau_1 > n)$ when $\alpha > 1/2$. It is summable precisely when $\mu = \mathbf{E}[\tau_1] < +\infty$, and then, by Theorem 1.3(iii), (1.13) says $|u_n - u_{n-1}| \leqslant c_3\, \mathbf{P}(\tau_1 > n)$. This gives additional information compared to the known asymptotics

$$u_n - \frac{1}{\mu} \sim \frac{1}{\mu^2} \sum_{k>n} \mathbf{P}(\tau_1 > k)$$

from [18]. We can sum (1.13) to obtain $|u_n - 1/\mu| \leqslant c_3 \sum_{k>n} \mathbf{P}(\tau_1 > k)$, which is of the right order, but we cannot obtain the proper constant $1/\mu^2$.

We also mention the works of Topchii [21, 22], treating the case when τ_1 is a continuous random variable with $E[\tau_1] = \infty$ and density $f(t) \overset{t\to\infty}{\sim} \varphi(t) t^{-(1+\alpha)}$, studying the density $u(t) = \sum_{k=0}^{\infty} f^{*k}(t)\; u(t)$ of the renewal function, and also $u'(t)$. Under some additional regularity conditions on $f'(t)$, letting $m(t) := \mathbf{E}[\tau_1 \wedge t]$, it is proven that

$$u'(t) \overset{t\to\infty}{\sim} \begin{cases} \frac{\alpha(\alpha-1)\sin(\pi\alpha)}{\pi}\; \varphi(t)^{-1}\, t^{-(2-\alpha)} & \text{if } 0 < \alpha < 1, \\ \frac{1}{m(t)^2}\; \varphi(t)\, t^{-1} & \text{if } \alpha = 1 \text{ and } \mathbf{E}[\tau_1] = \infty. \end{cases}$$

This is a better estimate than its analog in the infinite-mean case in Theorem 1.6, but the techniques of [21, 22] do not appear adaptable to the discrete setting.

Proof of Theorem 1.6. The second inequality in (1.13) is a direct consequence of Theorem 1.3(iii) and Table 1, so we prove the first one. Take σ a renewal process independent from τ, with the same inter-arrival distribution, but starting from $\sigma_0 = 1$. We can couple τ and σ so that $\tau = \sigma$ on $[\rho_1, \infty)$. Then denoting the corresponding joint distribution by $\mathbf{P}_{0,1}$ we have

$$|u_n - u_{n-1}| = \left|\mathbf{E}_{0,1}[\mathbf{1}_{\{n\in\tau\}} - \mathbf{1}_{\{n\in\sigma\}}]\right| \leqslant \mathbf{P}_{0,1}(n \in \tau, \rho_1 > n) + \mathbf{P}_{0,1}(n \in \sigma, \rho_1 > n).$$

By Lemma A.1 there is a constant C_0 such that

$$\mathbf{P}_{0,1}(n \in \tau, \rho_1 > n) \leqslant \mathbf{P}_{0,1}(n \in \tau)\mathbf{P}_{0,1}(\rho_1 > n/4 \mid n \in \tau) \leqslant C_0\, u_n \mathbf{P}_{0,1}(\rho_1 > n/4),$$

and similarly for $\mathbf{P}_{0,1}(n \in \sigma, \rho_1 > n)$, since $u_{n-1} \sim u_n$. Now, fix k_0 such that $\mathbf{P}(\tau_1 = k_0 + 1)\mathbf{P}(\tau_1 = k_0) > 0$, and observe that for any $x > 0$

$$\mathbf{P}(\rho_1 > x + k_0) \geqslant \mathbf{P}(\sigma_1 = k_0 + 1)\mathbf{P}(\tau_1 = k_0)\mathbf{P}_{0,1}(\rho_1 > x).$$

Since $\mathbf{P}(\rho_1 > n)$ is regularly varying (cf. Theorem 1.3), it follows that there is a constant $c_4 > 0$ such that $\mathbf{P}_{0,1}(\rho_1 > n/4) \leqslant c_4\mathbf{P}(\rho_1 > n)$, and hence Theorem 1.6 follows. $\square$

1.4. **Organization of the rest of the paper and idea of the proof.** First of all, we recall renewal and reverse renewal theorems in Section 2, which are used throughout the paper.

Sections 3–4 are devoted to the proof of Theorem 1.3. Items (i)-(ii) are dealt with using Theorem 8.7.3 in [4], and our main contribution is the proof of item (iii). The underlying idea is that, in order to have $\{\rho_1 > n\}$ either one of τ or σ typically makes a jump of order at least n. We decompose $\mathbf{P}(\rho_1 > n)$ according to the number k of steps before τ (resp. σ) *escapes* beyond n by a jump larger than $(1-\varepsilon)n$: we find that the expected number of steps is approximately $\widetilde{\mu}_n$ (resp. μ_n), giving Theorem 1.3(iii).

Sections 5–6 contain the proof of Theorem 1.5. In Section 5, we prove Lemma 1.4 in two steps. First, we show that when $\rho_1 = n$, having only gaps of length $\leqslant \delta n$ is very unlikely ; then, given that there is, say in τ, a gap larger than δn, we can stretch it (together with associated σ intervals) by $k \ll \delta n$ at little cost: this proves that $\mathbf{P}(\rho_1 = n) \approx \mathbf{P}(\rho_1 = n + k)$. In Section 6, we conclude the proof of Theorem 1.5 by combining Lemma 1.4 with Theorem 1.3.

2. Background on renewal and reverse renewal theorems

We consider a renewal $\tau = \{\tau_0, \tau_1, \dots\}$, with $\tau_0 = 0$. The corresponding *renewal mass function* is $\mathbf{P}(n \in \tau), n \geqslant 0$.

2.1. **On renewal theorems.** In what follows we assume that the inter-arrival distribution of τ satisfies (1.1).

Transient case. If τ is transient, then (see [11, App. A.5])

$$\mathbf{P}(n \in \tau) \overset{n\to\infty}{\sim} \frac{1}{(p_\infty^\tau)^2}\mathbf{P}(\tau_1 = n)\,, \tag{2.1}$$

where $p_\infty^\tau := \mathbf{P}(\tau_1 = +\infty) \in (0,1)$.

Recurrent case. Here there are multiple subcases, as follows.

- If $\mathbf{E}[\tau_1] < +\infty$, then the classical Renewal Theorem says

$$\lim_{n\to\infty} \mathbf{P}(n \in \tau) = \frac{1}{\mathbf{E}[\tau_1]}. \tag{2.2}$$

- If $\alpha = 1$, $\mathbf{E}[\tau_1] = +\infty$, then from [7, eq. (2.4)],

$$\mathbf{P}(n \in \tau) \overset{n\to\infty}{\sim} (\mu_n)^{-1}, \tag{2.3}$$

where $\mu_n := \mathbf{E}(\tau_1 \wedge n)$ is slowly varying.

- If $\alpha \in (0,1)$ then by [5, Thm. B],

$$\mathbf{P}(n \in \tau) \overset{n\to\infty}{\sim} \frac{\alpha \sin(\pi\alpha)}{\pi} \, n^{-(1-\alpha)} \varphi(n)^{-1}. \tag{2.4}$$

(Note that there is a typo in [5, Eq. (1.8)].)

- If $\alpha = 0$, then from [2, Thm. 1.2],

$$\mathbf{P}(n \in \tau) \overset{n\to\infty}{\sim} \frac{\mathbf{P}(\tau_1 = n)}{\mathbf{P}(\tau_1 > n)^2}. \tag{2.5}$$

We recall that the results in the case of a recurrent τ are collected in Table 1.

2.2. **On reverse renewal theorems.** In the opposite direction, if in place of (1.1), one assumes that $\mathbf{P}(n \in \tau)$ is regularly varying with exponent $1-\alpha$, then for $0 \leqslant \alpha < 1$ the asymptotics of $\mathbf{P}(\tau_1 > n)$ follow from [4, Thm. 8.7.3]. It is not possible in general to deduce the asymptotics of $\mathbf{P}(\tau_1 = n)$, which need not even be regularly varying. However, in certain cases, one can recover at least some behavior of $\mathbf{P}(\tau_1 = n)$ from that of $\mathbf{P}(n \in \tau)$ when the latter is regularly varying; we call such a result a *reverse renewal theorem.* Specifically, if the renewal function

$$U_n := \sum_{k=0}^{n} \mathbf{P}(k \in \tau), \quad n \leqslant \infty,$$

is slowly varying (as happens in the case of transient τ or $\alpha = 0$), the following theorems apply.

Transient case. We write $|\tau|$ for $|\{\tau_0, \tau_1, \dots\}|$, which is geometrically distributed in the transient case, with $E(|\tau|) = 1/p_\infty^\tau$.

Theorem 2.1 (Theorem 1.4 in [2])**.** *If $\mathbf{P}(n \in \tau)$ is regularly varying and τ is transient, then*

$$\mathbf{P}(\tau_1 = n) \overset{n\to\infty}{\sim} \frac{1}{\mathbf{E}(|\tau|)^2} \mathbf{P}(n \in \tau).$$

Recurrent case. If U_n is growing to infinity as a slowly varying function, then we have only a weaker reverse renewal theorem corresponding to (2.5).

Theorem 2.2 (Theorem 1.3 in [2])**.** *If $\mathbf{P}(n \in \tau)$ is regularly varying, and if U_n is slowly varying, then there exists some $\varepsilon_n \stackrel{n\to\infty}{\to} 0$ such that*

$$\frac{1}{\varepsilon_n n} \sum_{k=(1-\varepsilon_n)n}^{n} \mathbf{P}(\tau_1 = k) \stackrel{n\to\infty}{\sim} (U_n)^{-2} \mathbf{P}(n \in \tau)\,.$$

One can therefore obtain the local asymptotics of $\mathbf{P}(\tau_1 = n)$ from this last theorem when one can show $\mathbf{P}(\tau_1 = n)$ is approximately constant over an interval of length $o(n)$, as done in Lemma 1.4.

3. Proof of Theorem 1.3(i), (ii)

In case (i) we have $U_n^* \sim \frac{1}{\alpha^*}\psi^*(n) n^{\alpha^*}$, and in case (ii) $U_n^* = \sum_{j=1}^n \frac{\psi^*(j)}{j}$ which is slowly varying. Hence by [4, Thm. 8.7.3], in case (i),

$$\mathbf{P}(\rho_1 > n) \stackrel{n\to\infty}{\sim} \frac{1}{\Gamma(1-\alpha^*)\Gamma(1+\alpha^*)} \frac{1}{U_n^*} \stackrel{n\to\infty}{\sim} \frac{\sin(\pi\alpha^*)}{\pi} \psi^*(n)^{-1}\, n^{-\alpha^*},$$

and in case (ii),

$$\mathbf{P}(\rho_1 > n) \stackrel{n\to\infty}{\sim} \frac{1}{U_n^*} = \left(\sum_{j=1}^{n} \frac{\psi^*(j)}{j} \right)^{-1}. \tag{3.1}$$

4. Proof of Theorem 1.3(iii)

For $\alpha^* \geqslant 1$ (i.e. $\alpha \geqslant 1$), we cannot extract the behavior of $\mathbf{P}(\rho_1 > n)$ directly from that of U_n^* as in Section 3, and we need a preliminary result: we prove that $\mathbf{P}(\rho_1 > n)$ is regularly varying and hence for any $\varepsilon > 0$ we have

$$\mathbf{P}(\rho_1 > \varepsilon n) = O(\mathbf{P}(\rho_1 > n)) \quad \text{as } n \to \infty. \tag{4.1}$$

In Section 4.1, we prove (4.1), with the help of [10]. In Section 4.3, we prove an upper bound for $\mathbf{P}(\rho_1 > n)$. Finally, in Section 4.4, we prove the corresponding lower bound.

4.1. **Proof of** (4.1)**.** A sequence $\{u_n\}$ is said to be in the *de Haan class* Π if there exists a slowly varying sequence ℓ_n such that for all $\lambda > 0$,

$$\frac{u_{\lfloor \lambda n \rfloor} - u_n}{\ell_n} \to \log \lambda \quad \text{as } n \to \infty.$$

We write $\mathrm{RVS}_{-\alpha}$ for the set of regularly varying sequences of index $-\alpha$. We can state the results of Frenk [10] as follows.

Proposition 4.1 ([10], main theorem and Lemma 4)**.** *Let ν be a renewal process, and denote $u_n = \mathbf{P}(n \in \nu)$. Then, we have*

$$\mathbf{P}(\nu_1 > n) \in \mathrm{RVS}_{-1} \quad \Leftrightarrow \quad u_n \in \Pi\,. \tag{4.2}$$

Moreover, for any $\alpha > 1$, denoting $m = \mathbf{E}[\nu_1] < +\infty$, we have

$$\mathbf{P}(\nu_1 > n) \in \mathrm{RVS}_{-\alpha} \quad \Leftrightarrow \quad u_n - \frac{1}{m} \in \mathrm{RVS}_{1-\alpha}\,, \tag{4.3}$$

and each implies that

$$u_n - \frac{1}{m} \overset{n\to\infty}{\sim} \frac{1}{m^2(\alpha-1)}\, n\mathbf{P}(\nu_1 > n)\,. \tag{4.4}$$

Using Proposition 4.1, we prove that $\mathbf{P}(\rho_1 > n)$ is regularly varying with exponent $-\alpha$, as follows, yielding (4.1).

If $\alpha = \widetilde{\alpha} = 1$, then Proposition 4.1 tells that the slowly varying sequences $u_n = \mathbf{P}(n \in \tau), \widetilde{u}_n = \mathbf{P}(n \in \sigma)$ are both in Π, with some corresponding slowly varying sequences $\ell_n, \widetilde{\ell}_n$. (One expects $\ell_n \sim \varphi(n)$ but we do not have or need proof of this.) Therefore, letting $L_n := \widetilde{\ell}_n u_n + \ell_n \widetilde{u}_n$, the product sequence $\mathbf{P}(n \in \rho) = u_n \widetilde{u}_n$ satisfies

$$\frac{u_{\lfloor \lambda n\rfloor}\widetilde{u}_{\lfloor \lambda n\rfloor} - u_n\widetilde{u}_n}{L_n} = \frac{u_{\lfloor \lambda n\rfloor}}{u_n}\frac{\widetilde{u}_{\lfloor \lambda n\rfloor} - \widetilde{u}_n}{\widetilde{\ell}_n}\frac{\widetilde{\ell}_n u_n}{L_n} + \frac{u_{\lfloor \lambda n\rfloor} - u_n}{\ell_n}\frac{\ell_n \widetilde{u}_n}{L_n} \overset{n\to\infty}{\to} \log\lambda \tag{4.5}$$

for all $\lambda > 0$, so the product sequence is in Π. Applying Proposition 4.1 again, we see that $\mathbf{P}(\rho_1 > n)$ is regularly varying with index -1.

If $\alpha = 1$, $\widetilde{\alpha} > 1$, then $\{u_n\}$ is in Π (with some corresponding slowly varying sequence ℓ_n), and $\widetilde{u}_n - \frac{1}{\widetilde{\mu}}$ is regularly varying with index $1 - \widetilde{\alpha}$. Hence,

$$\frac{u_{\lfloor \lambda n\rfloor}\widetilde{u}_{\lfloor \lambda n\rfloor} - u_n\widetilde{u}_n}{\mu^{-1}\ell_n} = \frac{\widetilde{u}_{\lfloor \lambda_n\rfloor}}{\mu^{-1}}\frac{u_{\lfloor \lambda n\rfloor} - u_n}{\ell_n} + u_n\frac{\widetilde{u}_{\lfloor \lambda n\rfloor} - \widetilde{u}_n}{\mu^{-1}\ell_n} \overset{n\to\infty}{\to} \log\lambda,$$

where we used that $\widetilde{u}_{\lfloor \lambda n\rfloor} - \widetilde{u}_n$ is in $\mathrm{RVS}_{1-\widetilde{\alpha}}$ so that the second term in the sum goes to 0 (since u_n/ℓ_n is regularly varying with index 0). Hence $\mathbf{P}(n \in \rho) = u_n\widetilde{u}_n$ is in Π, and applying Proposition 4.1, we get that $\mathbf{P}(\rho_1 > n)$ is regularly varying with index -1.

If $1 < \alpha \leqslant \widetilde{\alpha}$, then using Proposition 4.1, we get that

$$\begin{aligned} u_n\widetilde{u}_n - \frac{1}{\mu\widetilde{\mu}} &= \Big(\frac{1}{\mu} + \frac{1+o(1)}{\mu^2(\alpha-1)} n\mathbf{P}(\tau_1 > n)\Big)\Big(\frac{1}{\widetilde{\mu}} + \frac{1+o(1)}{\widetilde{\mu}^2(\alpha-1)} n\mathbf{P}(\sigma_1 > n)\Big) - \frac{1}{\mu\widetilde{\mu}} \\ &= \frac{1+o(1)}{\widetilde{\mu}\mu^2(\widetilde{\alpha}-1)}\, n\mathbf{P}(\tau_1 > n) + \frac{1+o(1)}{\mu\widetilde{\mu}^2(\widetilde{\alpha}-1)}\, n\mathbf{P}(\sigma_1 > n), \end{aligned} \tag{4.6}$$

and therefore $u_n\widetilde{u}_n - \frac{1}{\mu\widetilde{\mu}} \in \mathrm{RVS}_{1-\alpha}$. Applying Proposition 4.1 again, we get that $\mathbf{P}(\rho_1 > n)$ is regularly varying with index $-\alpha$, and so (4.1) is proven. Proposition 4.1 and (4.6) further give that

$$\begin{aligned} \mathbf{P}(\rho_1 > n) &= (1+o(1))\frac{1}{n}(\mu\widetilde{\mu})^2(\alpha-1)\Big(u_n\widetilde{u}_n - \frac{1}{\mu\widetilde{\mu}}\Big) \\ &= (1+o(1))\widetilde{\mu}\mathbf{P}(\tau_1 > n) + (1+o(1))\mu\frac{\alpha-1}{\widetilde{\alpha}-1}\mathbf{P}(\sigma_1 > n) \end{aligned} \tag{4.7}$$

The second term is negligible compared to the first if $\widetilde{\alpha} > \alpha > 1$, so this proves Theorem 1.3(iii) when $1 < \alpha \leqslant \widetilde{\alpha}$.

We will present the rest of our proof of Theorem 1.3 in the whole range $1 \leqslant \alpha \leqslant \widetilde{\alpha}$ even though it is now needed only for $\alpha = 1$; this adds no complexity. The advantage is that it is a more probabilistic approach, in that we use Proposition 4.1 only to get the regular variation of $\mathbf{P}(\rho_1 > n)$, and avoid using the un-probabilistic (4.4) (with $\nu = \rho$) to estimate $\mathbf{P}(\rho_1 > n)$ as in (4.7). The method also provides an interpretation of the terms $\mu_n, \widetilde{\mu}_n$ appearing in Theorem 1.3(iii).

4.2. **Some useful preliminary lemmas.** Before we prove Theorem 1.3(iii), we need two technical lemmas.

Lemma 4.2. *Let τ, σ be independent renewal processes, suppose $\rho = \tau \cap \sigma$ is recurrent with $\mathbf{E}(\sigma_1) < \infty$, and let $K := \min\{k \geqslant 1 : \tau_k \in \sigma\}$. Then $\mathbf{E}(K) = \mathbf{E}(\sigma_1)$.*

Proof Since $\mathbf{P}(n \in \rho) = \mathbf{P}(n \in \tau)\mathbf{P}(n \in \sigma)$, the renewal theorem gives

$$\mathbf{E}(\rho_1) = \mathbf{E}(\sigma_1)\mathbf{E}(\tau_1). \tag{4.8}$$

Let $K_1, K_2, \dots$ be i.i.d. copies of K and let $S_m := K_1 + \cdots + K_m$. Then τ_{S_m} has the distribution of ρ_m, so using (4.8),

$$\frac{\tau_{S_m}}{m} \to \mathbf{E}[\rho_1] = \mathbf{E}[\tau_1]\mathbf{E}[\sigma_1] \quad \text{a.s.,} \quad \text{and} \quad \frac{\tau_{S_m}}{m} = \frac{\tau_{S_m}}{S_m}\frac{S_m}{m} \to \mathbf{E}[\tau_1]\mathbf{E}[K] \quad \text{a.s.,}$$

and the lemma follows. ☐

Write $\mathbf{P}_{x,y}(\cdot)$ for $\mathbf{P}(\cdot \mid \tau_0 = x, \sigma_0 = y)$, and write $\mathbf{E}_{x,y}$ the corresponding expectation.

Lemma 4.3. *Assume* (1.1)*, and suppose ρ is recurrent and $\alpha^* > 0$ (equivalently, $\alpha + \widetilde{\alpha} > 1$.) Given $\eta > 0$, provided δ is sufficiently small we have for large n and all $0 \leqslant x \leqslant \delta n$:*

$$\mathbf{P}_{-x,0}(\rho \cap [0,n] = \emptyset) < \eta\,. \tag{4.9}$$

If also $\alpha \geqslant 1$, then the same is true with $\delta > 0$ arbitrary. The analogous results with τ, σ interchanged hold as well.

Proof Fix $x \leqslant \delta n$ and let $N := |\rho \cap [0,n]|$. Then $\mathbf{P}_{-x,0}(\rho \cap [0,n] = \emptyset) = \mathbf{P}_{-x,0}(N = 0)$ and $\mathbf{E}_{-x,0}(N \mid N \geqslant 1) \leqslant U_n^*$, so

$$\mathbf{P}_{-x,0}(N = 0) = \frac{\mathbf{E}_{-x,0}(N \mid N \geqslant 1) - \mathbf{E}_{-x,0}(N)}{\mathbf{E}_{-x,0}(N \mid N \geqslant 1)} \leqslant \frac{U_n^* - \mathbf{E}_{-x,0}(N)}{U_n^*} \tag{4.10}$$

while

$$U_n^* - \mathbf{E}_{-x,0}(N) = \sum_{j=0}^{n} \mathbf{P}(j \in \sigma)\big[\mathbf{P}(j \in \tau) - \mathbf{P}(j + x \in \tau)\big]\,. \tag{4.11}$$

Since $\mathbf{P}(j \in \tau)$ is regularly varying, given $\eta > 0$, there exists A (large) such that for $\delta > 0$, for n large we have for all $x \leqslant \delta n$ and $A\delta n \leqslant j \leqslant n$ that

$$\mathbf{P}(j \in \tau) - \mathbf{P}(j + x \in \tau) \leqslant \frac{\eta}{2}\mathbf{P}(j \in \tau)\,. \tag{4.12}$$

Since U_k^* is regularly varying, with positive index since $\alpha^* > 0$, if δ, and therefore $A\delta$, is sufficiently small then for large n we have $U_{A\delta n}^* \leqslant \frac{\eta}{2} U_n^*$. With (4.11) this gives that for large n,

$$U_n^* - \mathbf{E}_{-x,0}(N) \leqslant U_{A\delta n}^* + \frac{\eta}{2} U_n^* \leqslant \eta U_n^*. \tag{4.13}$$

With (4.10), this proves (4.9) for large n.

Now consider $\alpha \geqslant 1$, meaning $\mathbf{P}(k \in \tau)$ is slowly varying. Given $\eta > 0$, for any $\delta > 0$ we can choose A (small this time) so that $U_{A\delta n}^* \leqslant \frac{\eta}{2} U_n^*$ for large n. Inequality (4.12) holds for all $j \geqslant A\delta n$ and $x \leqslant \delta n$, for n large, so (4.13) is valid and (4.9) follows. □

4.3. **Upper bound for $\mathbf{P}(\rho_1 > n)$.** Let us fix $\varepsilon > 0$. Let us call a gap $\tau_k - \tau_{k-1}$ or $\sigma_k - \sigma_{k-1}$ *long* if it exceeds $(1-2\varepsilon)n$; the *starting* and *ending points* of such a gap are τ_{k-1}, τ_k or σ_{k-1}, σ_k. Let S be the first starting point of a long gap in τ or σ, and let T be the ending point of the gap that starts at S. (To make things well-defined, if both τ and σ have long gaps starting at S, then we take T to be the first endpoint among these two gaps.) Then

$$\begin{aligned} \mathbf{P}(\rho_1 > n) \leqslant \mathbf{P}(\rho_1 > n, \sigma \cap [\varepsilon n, (1-\varepsilon)n] \neq \emptyset, \tau \cap [\varepsilon n, (1-\varepsilon)n] \neq \emptyset) \\ + \mathbf{P}(\rho_1 \geqslant T). \end{aligned} \tag{4.14}$$

For fixed n, we let $\bar\tau_1$ have the distribution of τ_1 given $\tau_1 \leqslant (1-2\varepsilon)n$, and similarly for $\bar\sigma_1$. Let $\bar\tau$ and $\bar\sigma$ be renewal processes with gaps distributed as $\bar\tau_1$ and $\bar\sigma_1$, respectively, and let $K := \min\{k \geqslant 1 : \tau_k \in \sigma\}$ and $\bar K := \min\{k \geqslant 1 : \bar\tau_k \in \bar\sigma\}$. Then, we have

$$\begin{aligned} &\mathbf{P}(\rho_1 \geqslant T, S \in \tau) \\ &\quad = \sum_{k \geqslant 0} \mathbf{P}\bigg(K > k, \tau_i - \tau_{i-1} \leqslant (1-2\varepsilon)n \text{ for all } i \leqslant k, \tau_{k+1} - \tau_k > (1-2\varepsilon)n, \\ &\qquad\qquad \sigma_i - \sigma_{i-1} \leqslant (1-2\varepsilon)n \text{ for all } i \text{ with } \sigma_{i-1} \leqslant \tau_k \bigg) \\ &\quad \leqslant \sum_{k \geqslant 0} \mathbf{P}(\bar K > k) \mathbf{P}(\tau_1 > (1-2\varepsilon)n) \\ &\quad = \mathbf{E}[\bar K] \mathbf{P}(\tau_1 > (1-2\varepsilon)n). \end{aligned} \tag{4.15}$$

From Lemma 4.2 we have $\mathbf{E}[\bar K] = \mathbf{E}(\sigma_1 \mid \sigma_1 \leqslant (1-2\varepsilon)n) \leqslant \widetilde\mu_n$. Thus for large n we have

$$\mathbf{P}(\rho_1 \geqslant T, S \in \tau) \leqslant (1-3\varepsilon)^{-\alpha} \widetilde\mu_n \mathbf{P}(\tau_1 > n).$$

A similar computation holds for $\mathbf{P}(\rho_1 \geqslant T, S \in \sigma)$ so we have for large n:

$$\mathbf{P}(\rho_1 \geqslant T) \leqslant (1-3\varepsilon)^{-\widetilde\alpha} \left\{ \widetilde\mu_n \mathbf{P}(\tau_1 > n) + \mu_n \mathbf{P}(\sigma_1 > n) \right\}. \tag{4.16}$$

We now need a much smaller bound for the first term on the right side of (4.14). Define $U := \min \tau \cap (\varepsilon n, \infty)$ and $V := \min \sigma \cap (\varepsilon n, \infty)$. Then

$$\mathbf{P}\big(\rho_1 > n, \sigma \cap (\varepsilon n, (1-\varepsilon)n) \neq \emptyset, \tau \cap (\varepsilon n, (1-\varepsilon)n) \neq \emptyset, U < V\big)$$
$$\leqslant \sum_{u<v, u,v \in (\varepsilon n,(1-\varepsilon)n)} \mathbf{P}(\rho_1 > \varepsilon n, U = u, V = v)\mathbf{P}_{u-v,0}(\rho_1 > \varepsilon n). \tag{4.17}$$

We may now apply Lemma 4.3 for the last probability. Fix $\eta > 0$. Then, since $\widetilde{\alpha} \geqslant \alpha \geqslant 1$ for n large enough,

$$\mathbf{P}_{-x,0}(\rho_1 > \varepsilon n) < \eta \quad \text{for all } 0 \leqslant x \leqslant n. \tag{4.18}$$

Therefore, summing over u, v, the right side of (4.17) is bounded by $\eta \mathbf{P}(\rho_1 > \varepsilon n, U < V)$, and a similar bound holds when $U > V$. Hence, combining this with with (4.14) and (4.16), we get that

$$\mathbf{P}(\rho_1 > n) \leqslant (1-3\varepsilon)^{-\widetilde{\alpha}} \left\{\widetilde{\mu}_n \mathbf{P}(\tau_1 > n) + \mu_n \mathbf{P}(\sigma_1 > n)\right\} + \eta \mathbf{P}(\rho_1 > \varepsilon n)\,. \tag{4.19}$$

Now we may use (4.1) to control the last term: we finally get that, provided η is small enough, for large n,

$$\mathbf{P}(\rho_1 > n) \leqslant (1 + 4\widetilde{\alpha}\varepsilon) \left\{\widetilde{\mu}_n \mathbf{P}(\tau_1 > n) + \mu_n \mathbf{P}(\sigma_1 > n)\right\}\,. \tag{4.20}$$

4.4. **Lower bound for $\mathbf{P}(\rho_1 > n)$.** We use a modification of our earlier truncation. Fix n and, analogously to $\bar\tau, \bar\sigma$, let $\widehat\tau$ and $\widehat\sigma$ be renewal processes with gaps $\widehat\tau_i - \widehat\tau_{i-1} = (\tau_i - \tau_{i-1}) \wedge (n+1)$ and $\widehat\sigma_i - \widehat\sigma_{i-1} = (\sigma_i - \sigma_{i-1}) \wedge (n+1)$, respectively, and let $\widehat\rho = \widehat\tau \cap \widehat\sigma$ and $\widehat K := \min\{k \geqslant 1 : \widehat\tau_k \in \widehat\sigma\}$. We call a gap in $\widehat\tau$ or $\widehat\sigma$ *large* if its length is $n+1$. Let $[S_{\widehat\tau}, T_{\widehat\tau}]$ and $[S_{\widehat\sigma}, T_{\widehat\sigma}]$ be the first large gaps in $\widehat\tau$ and $\widehat\sigma$ respectively, and let $J_{\widehat\tau}$ and $J_{\widehat\sigma}$ be the number of large gaps in $\widehat\tau$ and $\widehat\sigma$ respectively before time $\widehat\rho_1^{(n)}$.

Observe that

$$\mathbf{P}(\rho_1 > n) = \mathbf{P}(\widehat\rho_1 > n) \geqslant \mathbf{P}(J_{\widehat\tau} \geqslant 1) + \mathbf{P}(J_{\widehat\sigma} \geqslant 1) - \mathbf{P}(J_{\widehat\tau} \geqslant 1, J_{\widehat\sigma} \geqslant 1). \tag{4.21}$$

We claim that

$$\mathbf{P}(J_{\widehat\tau} \geqslant 1) \geqslant (1 - o(1))\mathbf{E}[J_{\widehat\tau}] \quad \text{as } n \to \infty \tag{4.22}$$

and

$$\mathbf{P}(J_{\widehat\tau} \geqslant 1, J_{\widehat\sigma} \geqslant 1) = o\left(\mathbf{P}(J_{\widehat\tau} \geqslant 1) + \mathbf{P}(J_{\widehat\sigma} \geqslant 1)\right) \quad \text{as } n \to \infty. \tag{4.23}$$

Assuming (4.22) and (4.23), we have

$$\mathbf{P}(\rho_1 > n) \geqslant (1 - o(1))\bigg(\mathbf{E}[J_{\widehat\tau}] + \mathbf{E}[J_{\widehat\sigma}]\bigg). \tag{4.24}$$

Then using Lemma 4.2 to get $\mathbf{E}[\widehat K] = \mathbf{E}[\widehat\sigma_1] = \widetilde\mu_{n+1}$ we obtain

$$\mathbf{E}[J_{\widehat\tau}] \sum_{k \geqslant 0} \mathbf{P}\left(\tau_{k+1} - \tau_k > n, \widehat K > k\right)$$
$$= \mathbf{E}[\widehat K]\mathbf{P}(\tau_1 > n) = \widetilde\mu_{n+1}\mathbf{P}(\tau_1 > n), \tag{4.25}$$

and similarly for $\mathbf{E}[J_{\widehat\sigma}]$. With (4.24) this shows that

$$\mathbf{P}(\rho_1 > n) \geqslant (1 - o(1)) \left\{\widetilde{\mu}_n \mathbf{P}(\tau_1 > n) + \mu_n \mathbf{P}(\sigma_1 > n)\right\}\,. \tag{4.26}$$

This and (4.20) prove Theorem 1.3(iii).

It remains to prove (4.22) and (4.23). We begin with (4.23). We write

$$
\begin{aligned}
\mathbf{P}(J_{\widehat\tau} \geqslant 1, J_{\widehat\sigma} \geqslant 1) = \mathbf{P}(J_{\widehat\tau} \geqslant 1, J_{\widehat\sigma} \geqslant 1, S_{\widehat\tau} < S_{\widehat\sigma}) \\
+ \mathbf{P}(J_{\widehat\tau} \geqslant 1, J_{\widehat\sigma} \geqslant 1, S_{\widehat\tau} > S_{\widehat\sigma}),
\end{aligned}
\tag{4.27}
$$

and we control both terms separately. On the event $\{S_{\widehat\tau} < S_{\widehat\sigma}\}$, we decompose over the first $\widehat\sigma$ renewal in the interval $(S_{\widehat\tau}, T_{\widehat\tau})$, to obtain that

$$
\mathbf{P}(J_{\widehat\tau} \geqslant 1, J_{\widehat\sigma} \geqslant 1, S_{\widehat\tau} < S_{\widehat\sigma}) \leqslant \mathbf{P}(J_{\widehat\tau} \geqslant 1) \times \sup_{x\in(0,n]} \mathbf{P}_{x,0}\left(J_{\widehat\sigma} \geqslant 1\right). \tag{4.28}
$$

From Lemma 4.3 we have that for any $\eta > 0$, for n large enough, for all $1 \leqslant x \leqslant n/2$,

$$
\mathbf{P}_{x,0}\left(J_{\widehat\sigma} \geqslant 1\right) \leqslant \mathbf{P}_{x,0}\left(\rho_1 \geqslant n/2\right) \leqslant \eta\,. \tag{4.29}
$$

If $x \in (n/2, n]$, then we decompose over the first σ renewal in the interval $[x/2, x)$ if it exists, to get

$$
\mathbf{P}_{x,0}\left(J_{\widehat\sigma} \geqslant 1\right) \leqslant \mathbf{P}(\sigma \cap [x/2, x) = \emptyset) + \sup_{y\in[1,x/2]} \mathbf{P}_{y,0}\left(J_{\widehat\sigma} \geqslant 1\right). \tag{4.30}
$$

The last sup in bounded as in (4.29). For the first probability on the right, using the renewal theorem when $\alpha > 1$ and [7] when $\alpha = 1$, we get that there is a constant c_5 such that

$$
\mathbf{P}(\sigma \cap [x/2, x) = \emptyset) \leqslant \sum_{k=1}^{x/2} \mathbf{P}(k \in \sigma)\mathbf{P}(\sigma_1 > x/2) \leqslant c_5 \frac{x}{\mu_x} \varphi(x) x^{-\alpha} \to 0 \text{ as } x \to \infty.
$$

The convergence to 0 is straightforward when $\alpha > 1$, and uses that $\varphi(x)/\mu_x \to 0$ as $x \to \infty$ when $\alpha = 1$ (see for example Theorem 1 in [9, Ch. VIII, Sec. 9]). It follows that the sup in (4.28) approaches 0 as $n \to \infty$. The second probability on the right side of (4.27) is handled similarly, and this proves (4.23).

We now turn to (4.22). We show that for any $\eta > 0$, we can take n large enough so that for any $j \geqslant 1$,

$$
\mathbf{P}(J_{\widehat\tau} \geqslant j+1) \leqslant \eta \mathbf{P}\left(J_{\widehat\tau} \geqslant j\right). \tag{4.31}
$$

This easily gives that $\mathbf{E}\left[J_{\widehat\tau}\right] = \sum_{j\geqslant 1} \mathbf{P}(J_{\widehat\tau} \geqslant j) \leqslant \frac{1}{1-\eta}\mathbf{P}(J_{\widehat\tau} \geqslant 1)$, which is (4.22). To prove (4.31), we denote $T_{\widehat\tau}^{(j)}$ the endpoint of the jth large gap in $\widehat\tau$. Then, decomposing over the first $\widehat\sigma$ renewal in the interval $[T_{\widehat\tau}^{(j)} - n, T_{\widehat\tau}^{(j)})$, we get, similarly to (4.28)

$$
\begin{aligned}
\mathbf{P}(J_{\widehat\tau} \geqslant j+1) &\leqslant \mathbf{P}\left(J_{\widehat\tau} \geqslant j\right) \times \sup_{x\in(0,n]} \mathbf{P}_{0,-x}\left(J_{\widehat\tau} \geqslant 1\right) \\
&\leqslant \mathbf{P}\left(J_{\widehat\tau} \geqslant j\right) \sup_{x\in(0,n]} \mathbf{P}_{0,-x}\left(\rho_1 \geqslant n+1\right) \leqslant \eta\, \mathbf{P}\left(J_{\widehat\tau} \geqslant j\right),
\end{aligned}
$$

where the last inequality is valid provided that n is large enough, thanks to Lemma 4.3. This completes the proof of (4.22), and thus also of Theorem 1.7(iii).

5. Proof of Lemma 1.4: Stretching of gaps

By assumption ρ is recurrent, and we need to show that when n is large $\mathbf{P}(\rho_1 = n) \approx \mathbf{P}(\rho_1 = n+k)$ for all $k \in (0, \varepsilon n)$, with $\varepsilon \ll 1$. The idea is to take the set of trajectories of τ and σ such that $\rho_1 = n$, and to *stretch* them slightly so that $\rho_1 = n+k$, see Figure 1. In Section 5.1, we prove that for some $\delta > 0$, conditioned on $\rho_1 = n$, the largest gap of τ and σ in $[0,n]$ is larger than δn with high probability; see Lemma 5.1. Assume that it is a τ-gap, and that it has length m. Then, in Section 5.2, we show that for $\varepsilon \ll \delta$ we can stretch this τ-gap by $k \leqslant \varepsilon n \ll m$, and stretch σ inside this τ-gap by the same k, without altering the probability significantly.

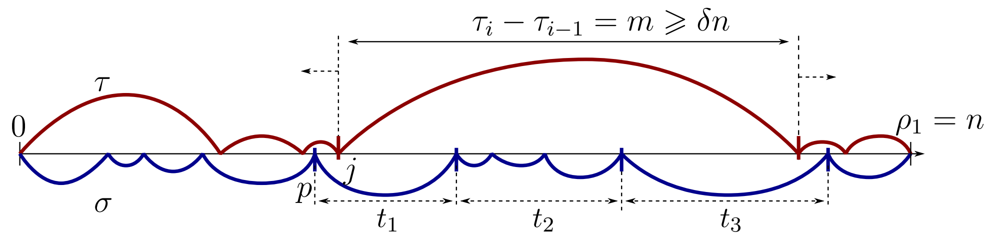

Figure 1. How to "stretch" trajectories, to go from $\rho_1 = n$ to $\rho_1 = n+k$: we identify the largest gap in τ (which is larger than δn with great probability, see Lemma 5.1) and we stretch it by k, while at the same time stretching one of the three associated σ-intervals (the largest of t_1, t_2, t_3). See the proof of Lemma 5.2 for more detailed explanations.

5.1. **Probability of having a large gap.**

Denote by $\mathcal{A}_\delta$ the event that there is a gap (either in σ or τ) longer than δn:

$$\mathcal{A}_\delta := \Big\{ \exists\, i :\, \tau_i - \tau_{i-1} > \delta n\,,\, \tau_i \leqslant n \ \text{ or } \ \sigma_i - \sigma_{i-1} > \delta n\,,\, \sigma_i \leqslant n \Big\}\,. \tag{5.1}$$

We will show that $\mathcal{A}_\delta^c$ contributes only a small part of $\{\rho_1 = n\}$. Recall that

$$v_n = \mathbf{P}(\rho_1 > n)^2 \mathbf{P}(n \in \rho)\,.$$

Lemma 5.1. *Assume* (1.1). *There exist $c_6 > 0$ and δ_0 such that if $\delta \in (0, \delta_0)$, then for n sufficiently large,*

$$\mathbf{P}\Big(\rho_1 = n\,;\, \mathcal{A}_\delta^c\Big) \leqslant e^{-c_6/\delta} v_n\,.$$

Proof On the event $\{\rho_1 = n\} \cap \mathcal{A}_\delta^c$, all τ and σ gaps are smaller than δn, and therefore all blocks of length at least δn are visited by both τ and σ. We control probabilities in each third of $[0,n]$ separately. To that end, define

$$\ell_\tau = \max \tau \cap (0, n/3), \quad \ell_\sigma = \max \sigma \cap (0, n/3),$$

and define events

$$G_1 : \tau \cap \sigma \cap (0, n/8) = \emptyset, \qquad G_2 : \tau \cap \sigma \cap [n/3, 2n/3] = \emptyset, \tag{5.2}$$
$$G_3 : \tau \cap \sigma \cap (7n/8, n) = \emptyset,$$
$$\mathcal{D}_{\delta\tau} : \tau_i - \tau_{i-1} \leqslant \delta n \text{ for all } i \text{ with } [\tau_{i-1}, \tau_i] \cap [n/3, 2n/3] \neq \emptyset,$$
$$\mathcal{D}_{\delta\sigma} : \sigma_i - \sigma_{i-1} \leqslant \delta n \text{ for all } i \text{ with } [\sigma_{i-1}, \sigma_i] \cap [n/3, 2n/3] \neq \emptyset,$$

$$\mathcal{L}_1 : \ell_\tau, \ell_\sigma \in (n/4, n/3).$$

Assuming $\delta < 1/12$, we have $\mathcal{A}_\delta^c \subseteq \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \subseteq \mathcal{L}_1$.

End thirds. By Lemma A.1, there exists C_0 such that

$$\max_{i,j\in(n/4,n/3)} \mathbf{P}\big(G_1 \mid \ell_\tau = i, \ell_\sigma = j\big) = \max_{i,j\in(n/4,n/3)} \mathbf{P}\big(G_1 \mid i \in \tau, j \in \sigma\big) \leqslant C_0 \mathbf{P}(G_1). \tag{5.3}$$

It follows that

$$\begin{aligned}\mathbf{P}(\rho_1 =&n, \mathcal{A}_\delta^c \mid n \in \rho) \leqslant \mathbf{P}\left(G_1 \cap G_2 \cap G_3 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \;\middle|\; n \in \rho\right) \\ &= \mathbf{P}\left(G_1 \mid G_2 \cap G_3 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \cap \{n \in \rho\}\right) \mathbf{P}\left(G_2 \cap G_3 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \mid n \in \rho\right) \\ &= \mathbf{E}\left(\mathbf{P}(G_1 \mid \ell_\tau, \ell_\sigma) \;\middle|\; G_2 \cap G_3 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \cap \{n \in \rho\}\right) \\ &\qquad\qquad\qquad\qquad \times \mathbf{P}\left(G_2 \cap G_3 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \mid n \in \rho\right) \\ &\leqslant C_0 \mathbf{P}(G_1) \mathbf{P}\left(G_2 \cap G_3 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \mid n \in \rho\right). \end{aligned} \tag{5.4}$$

Symmetrically we obtain

$$\mathbf{P}\left(G_2 \cap G_3 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \;\middle|\; n \in \rho\right) \leqslant C_0 \mathbf{P}(G_3) \mathbf{P}\left(G_2 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \;\middle|\; n \in \rho\right) \tag{5.5}$$

so, using Theorem 1.3,

$$\begin{aligned}\mathbf{P}(\rho_1 = n, \mathcal{A}_\delta^c \mid n \in \rho) &\leqslant C_0^2 \mathbf{P}(\rho_1 \geqslant n/8)^2 \mathbf{P}\left(G_2 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \;\middle|\; n \in \rho\right) \\ &\leqslant c_7 \mathbf{P}(\rho_1 > n)^2 \mathbf{P}\left(G_2 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \;\middle|\; n \in \rho\right). \end{aligned} \tag{5.6}$$

Middle third. We need to bound the last probability in (5.6). We divide the interval $[n/3, 2n/3]$ into blocks $B_i = [a_{i-1}, a_i]$ of length $A\delta n$ where A is a (large) constant to be specified. We denote by $d_\tau^{(i)}$ and $f_\tau^{(i)}$ the first and last renewals, respectively, of τ in B_i, and similarly for $d_\sigma^{(i)}, f_\sigma^{(i)}$. Let $B_{i,\ell} := [a_{i-1}, a_{i-1} + \delta n]$ and $B_{i,r} := [a_i - \delta n, a_i]$. On the event $\mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma}$, we have $d_\tau^{(i)}, d_\sigma^{(i)} \in B_{i,\ell}$ and $f_\tau^{(i)}, f_\sigma^{(i)} \in B_{i,r}$. Let $B_i^{(1)} := [a_{i-1}, a_{i-1} + A\delta n/3]$ denote the first third of B_i. Define events

$$\mathcal{D}_{\delta\tau}^{(i)} : \tau_j - \tau_{j-1} \leqslant \delta n \text{ for all } j \text{ with } \tau_{j-1} \in B_i^{(1)},$$

$$\mathcal{D}_{\delta\sigma}^{(i)} : \sigma_j - \sigma_{j-1} \leqslant \delta n \text{ for all } j \text{ with } \sigma_{j-1} \in B_i^{(1)}.$$

Using again Lemma A.1, we obtain

$$\begin{aligned}&\mathbf{P}\left(G_2 \cap \mathcal{D}_{\delta\tau} \cap \mathcal{D}_{\delta\sigma} \;\middle|\; n \in \rho\right) \\ &\quad\leqslant \prod_{i \,\leqslant\, 1/3A\delta} \max_{h,k\in B_{i,\ell},\, j,m\in B_{i,r}} \mathbf{P}\Bigg(\tau \cap \sigma \cap B_i^{(1)} = \emptyset, \mathcal{D}_{\delta\tau}^{(i)}, \mathcal{D}_{\delta\sigma}^{(i)} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \Bigg| \, d_\tau^{(i)} = h, f_\tau^{(i)} = j, d_\sigma^{(i)} = k, f_\sigma^{(i)} = m\Bigg) \\ &\quad\leqslant \prod_{i \,\leqslant\, 1/3A\delta} \max_{h,k\in B_{i,\ell}} C_0\, \mathbf{P}\big(\tau \cap \sigma \cap B_i^{(1)} = \emptyset, \mathcal{D}_{\delta\tau}^{(i)}, \mathcal{D}_{\delta\sigma}^{(i)} \,\big|\, d_\tau^{(i)} = h, d_\sigma^{(i)} = k\big). \end{aligned} \tag{5.7}$$

We claim that for any $\eta > 0$, there exists $A > 0$ such that, for δ small, for n large enough, for all $h, k \in [0, \delta n)$,

$$\mathbf{P}_{h,k}\big(\tau \cap \sigma \cap (0, \tfrac{1}{3}A\delta n] = \emptyset\,,\, \mathcal{D}^{(1)}_{\delta\tau}, \mathcal{D}^{(1)}_{\delta\sigma}\big) \leqslant \eta\,. \tag{5.8}$$

This bounds all the probabilities on the right side of (5.7) by η, which with (5.6) and (5.7) shows that, provided η is small,

$$\mathbf{P}(\rho_1 = n, \mathcal{A}_\delta^c \mid n \in \rho) \leqslant c_7 \mathbf{P}(\rho_1 > n)^2 (C_0\eta)^{1/3A\delta} \leqslant e^{-c_6/\delta}\mathbf{P}(\rho_1 > n)^2,$$

which completes the proof of the lemma.

It remains to prove (5.8). In the case of $\alpha \geqslant 1, \widetilde{\alpha} \geqslant 1$, we can drop the events $\mathcal{D}^{(1)}_{\delta\tau}, \mathcal{D}^{(1)}_{\delta\sigma}$ and (5.8) follows from Lemma 4.3. So suppose $\alpha < 1$; we will show that $\mathbf{P}_{0,0}(\mathcal{D}^{(1)}_{\delta\tau}) \leqslant \eta$. (This is sufficient, since $\mathbf{P}_{h,k}(\mathcal{D}^{(1)}_{\delta\tau}) \leqslant \mathbf{P}_{\delta n,0}(\mathcal{D}^{(1)}_{\delta\tau})$ for all $h, k \in [0, \delta n)$ and the last probability is unchanged if we replace δn with 0 and $\frac{1}{3}A$ with $\frac{1}{3}A - 1$.) We therefore drop the subscript $0,0$ in the notation.

Let $J := \min\{j \geqslant 1 : \tau_j - \tau_{j-1} > \delta n\}$, let $\bar{\tau}_1$ have the distribution of τ_1 given $\tau_1 \leqslant \delta n$, and let $\bar{\tau}$ be a renewal process with gaps distributed as $\bar{\tau}_1$. We have for $k \geqslant 1$:

$$\begin{aligned}
\mathbf{P}(\mathcal{D}^{(1)}_{\delta\tau}) &\leqslant \sum_{j=0}^{k-1} \mathbf{P}(J = j+1, \tau_j > A\delta n/3) + \mathbf{P}(J > k) \\
&\leqslant \sum_{j=0}^{k-1} \mathbf{P}(J = j+1)\mathbf{P}\left(\bar{\tau}_j > \tfrac{1}{3}A\delta n\right) + \mathbf{P}\left(\max_{i \leqslant k}(\tau_i - \tau_{i-1}) \leqslant \delta n\right), \\
&\leqslant \mathbf{P}\left(\bar{\tau}_k > \tfrac{1}{3}A\delta n\right) + e^{-k\mathbf{P}(\tau_1 > \delta n)}\,.
\end{aligned} \tag{5.9}$$

Then we use that for any $\alpha \in [0,1)$ there exist some $c_8, c_9 > 0$ such that for large n, $\mathbf{E}[\bar{\tau}_1] \leqslant c_8 \varphi(n)(\delta n)^{1-\alpha}$, and $\mathbf{P}(\tau_1 > \delta n) \geqslant c_9 \varphi(n)(\delta n)^{-\alpha}$ (in fact $\mathbf{P}(\tau_1 > \delta n) \gg \varphi(n)$ for $\alpha = 0$.) We obtain that

$$\mathbf{P}(\mathcal{D}^{(1)}_{\delta\tau}) \leqslant \frac{3c_8}{A} k\varphi(n)(\delta n)^{-\alpha} + e^{-c_9 k\varphi(n)(\delta n)^{-\alpha}}. \tag{5.10}$$

Choosing $k = A^{1/2}\varphi(n)^{-1}(\delta n)^{\alpha}$ with A large enough, we get that $\mathbf{P}(\mathcal{D}^{(1)}_{\delta\tau}) \leqslant \eta$. This completes the proof of (5.8). □

5.2. **Stretching argument.** We next show that, on the event $\mathcal{A}_\delta$, we can formalize the stretching previously described, and the cost of the stretching is small.

Lemma 5.2. *Assume* (1.1). *Given* $\delta > 0$, *if* n *is sufficiently large, then for any* $k \in [0, 2\delta^3 n]$ *we have*

$$\mathbf{P}(\rho_1 = n\,;\, \mathcal{A}_\delta(n)) \leqslant (1+\delta)\mathbf{P}(\rho_1 = n+k)\,.$$

Proof Fix n and denote

$$M_\tau := \max\{\tau_i - \tau_{i-1} : \tau_i \leqslant n\} \quad \text{and} \quad M_\sigma := \max\{\sigma_i - \sigma_{i-1} :\, \tau_i \leqslant n\}\,,$$

$$\mathcal{A}^\tau_\delta(n) := \{\rho_1 = n\} \cap \mathcal{A}_\delta \cap \{M_\tau \geqslant M_\sigma\}, \quad \mathcal{A}^\sigma_\delta(n) := \{\rho_1 = n\} \cap \mathcal{A}_\delta \cap \{M_\sigma > M_\tau\}.$$

We will show that provided that δ is small enough, for n large enough and $k \in [0, 2\delta^3 n]$

$$\mathbf{P}(\mathcal{A}_\delta^\tau(n)) \leqslant (1+\delta)\mathbf{P}(\rho_1 = n+k\,,\, M_\tau \geqslant M_\sigma)\,. \tag{5.11}$$

The analogous statement also holds with $\mathcal{A}_\delta^\sigma(n)$ instead of $\mathcal{A}_\delta^\tau(n)$; combining the two completes the proof.

To prove (5.11), define random indices

$$i_0 := \min\{i \geqslant 1 : \tau_i - \tau_{i-1} = M_\tau\}, \quad \ell_0 := \min\{\ell \geqslant 1 : \sigma_\ell > \tau_{i_0-1}\},$$
$$\ell_1 := \min\{\ell \geqslant 1 : \sigma_\ell \geqslant \tau_{i_0}\}.$$

We call $[\tau_{i_0-1}, \tau_{i_0}]$ the *maximal gap*, and the three intervals $[\sigma_{\ell_i-1}, \sigma_{\ell_i}], i = 0,1$ and $[\sigma_{\ell_0}, \sigma_{\ell_1-1}]$ are called *associated σ-intervals.* We decompose the probability according to the locations of this gap and the intervals: define the events

$$\begin{aligned}\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3) := \mathcal{A}_\delta^\tau(n) \cap \big\{&\tau_{i_0} = j, \tau_{i_0} - \tau_{i_0-1} = m, \sigma_{\ell_0-1} = p,\\ &\sigma_{\ell_0} - \sigma_{\ell_0-1} = t_1, \sigma_{\ell_1-1} - \sigma_{\ell_0} = t_2, \sigma_{\ell_1} - \sigma_{\ell_1-1} = t_3\big\}\,.\end{aligned}$$

This means the maximal gap (in τ) is from j to $j+m$, and σ has gaps from p to $p+t_1$ and from $p+t_1+t_2$ to $p+t_1+t_2+t_3$, each containing an endpoint of the maximal τ gap, see Figure 1. For the event to be nonempty, we must have $m \geqslant \delta n$ and

$$0 \leqslant p < j < p+t_1 \leqslant p+t_1+t_2 < j+m \leqslant p+t_1+t_2+t_3 \leqslant n. \tag{5.12}$$

Given such indices let us define $I \leqslant 3$ by $t_I = \max\{t_1,t_2,t_3\}$, with ties broken arbitrarily. Consider now the map Φ_k which assigns to each nonempty event $\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3)$ the event

$$\Phi_k(\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3)) := \begin{cases} \mathcal{A}_\delta^\tau(n+k,j,m+k,p,t_1+k,t_2,t_3) & \text{if } I=1,\\ \mathcal{A}_\delta^\tau(n+k,j,m+k,p,t_1,t_2+k,t_3) & \text{if } I=2,\\ \mathcal{A}_\delta^\tau(n+k,j,m+k,p,t_1,t_2,t_3+k) & \text{if } I=3.\end{cases}$$

Applying Φ_k corresponds to stretching the maximal gap and the longest of the associated σ-intervals by the amount k. It is easy to see that for distinct tuples (j,m,p,t_1,t_2,t_3), the corresponding events $\Phi_k(\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3))$ are disjoint subsets of $\mathcal{A}_\delta^\tau(n+k)$; this just means that the relevant interval and gap lengths in the original configuration are identifiable from the stretched configuration. We claim that provided δ is small enough, for n large enough and $k \in [0, 2\delta^3 n]$,

$$\mathbf{P}\left(\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3)\right) \leqslant (1+\delta)\mathbf{P}\left(\Phi_k\big(\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3)\big)\right) \tag{5.13}$$

whenever $\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3) \neq \emptyset$. Due to the aforementioned disjointness, summing this over (j,m,p,t_1,t_2,t_3) immediately yields (5.11). To prove (5.13), note that if $I = 1$ then $t_1 \geqslant m/3$, so $k/t_1 \leqslant 6\delta^2$, while $k/m < 2\delta^2$, so provided δ is small,

$$\frac{\mathbf{P}\left(\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3)\right)}{\mathbf{P}\left(\Phi_k(\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3))\right)} = \frac{\mathbf{P}(\tau_1 = m)}{\mathbf{P}(\tau_1 = m+k)}\frac{\mathbf{P}(\sigma_1 = t_1)}{\mathbf{P}(\sigma_1 = t_1+k)} < 1+\delta.$$

The same bound holds if $I = 3$. If $I = 2$ we have $t_2 \geqslant m/3$, so $k/t_2 \leqslant 6\delta^2$, and provided that δ is small

$$\frac{\mathbf{P}\left(\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3)\right)}{\mathbf{P}\left(\Phi_k(\mathcal{A}_\delta^\tau(n,j,m,p,t_1,t_2,t_3))\right)} = \frac{\mathbf{P}(\tau_1 = m)}{\mathbf{P}(\tau_1 = m+k)} \frac{\mathbf{P}(t_2 \in \sigma)}{\mathbf{P}(t_2 + k \in \sigma)} < 1 + \delta.$$

The claim (5.13), and hence the lemma, now follow. □

We proceed with the proof of Lemma 1.4. Indeed, the second inequality in (1.8) is immediate from Lemmas 5.1 and 5.2. Also, since v_n is regularly varying, Lemma 5.1 gives that for δ small, for any $j \in (0, \delta^3 n]$,

$$\mathbf{P}(\rho_1 = n - j\,;\, \mathcal{A}_\delta^c(n-j)) \leqslant 2e^{-c_6/\delta} v_n\,.$$

This and Lemma 5.2 yield that for any $k \in (0, \delta^3 n] \subseteq (0, 2\delta^3(n-k)]$,

$$\mathbf{P}(\rho_1 = n-k) \leqslant (1+\delta)\mathbf{P}(\rho_1 = n) + 2e^{-c_6/\delta} v_n\,. \tag{5.14}$$

and the first inequality in (1.8) follows.

6. Proof of Theorem 1.5

Let

$$A_n^+(\varepsilon) := \frac{\mathbf{P}(\rho_1 > n) - \mathbf{P}(\rho_1 > (1+\varepsilon)n)}{\varepsilon n}\,,$$

$$A_n^-(\varepsilon) := \frac{\mathbf{P}(\rho_1 > (1-\varepsilon)n) - \mathbf{P}(\rho_1 > n)}{\varepsilon n}\,.$$

We claim that, if ρ is recurrent, there is a constant $c_{10} > 0$ such that for sufficiently small $\varepsilon > 0$, when n is large,

$$v_n \leqslant c_{10} A_n^{\pm}(\varepsilon)\,. \tag{6.1}$$

It is sufficient to prove this for $A_n^+(\varepsilon)$, since v_n is regularly varying. Consider first $\alpha^* = 0$. It follows readily from (3.1) and Theorem 2.2 that for small ε, when n is large we have

$$A_n^+(\varepsilon) \geqslant \frac{1}{2}(U_n^*)^{-2}\mathbf{P}(n \in \rho) \geqslant \frac{1}{4} v_n\,. \tag{6.2}$$

Next consider $\alpha^* \in (0,1)$. Here $\alpha^* = 1 - \theta^*$, so by Theorem 1.3, for some c_{11}, for small ε we have for large n

$$A_n^+(\varepsilon) \geqslant c_{11} n^{-(1+\alpha^*)}\psi^*(n)^{-1} = c_{11} n^{-\theta^*}\psi^*(n) n^{-2\alpha^*}\psi^*(n)^{-2} \geqslant c_{12} v_n\,.$$

Finally consider $\alpha^* \geqslant 1$; here $1 \leqslant \alpha \leqslant \widetilde{\alpha}$. Since $\mathbf{P}(\tau_1 = n)$ and $\mathbf{P}(\sigma_1 = n)$ are regularly varying, it follows from Theorem 1.3 that for ε small and large n,

$$A_n^+(\varepsilon) \geqslant \frac{1}{2}\left(\widetilde{\mu}_n \mathbf{P}(\tau_1 = n) + \mu_n \mathbf{P}(\sigma_1 = n)\right) = \widetilde{\mu}_n \frac{\varphi(n)}{2n^{1+\alpha}} + \mu_n \frac{\widetilde{\varphi}(n)}{2n^{1+\widetilde{\alpha}}}\,.$$

Using $(a+b)^2 \leqslant 2a^2 + 2b^2$, we obtain

$$v_n \leqslant 2\left(\widetilde{\mu}_n \frac{\varphi(n)}{n^\alpha} + \mu_n \frac{\widetilde{\varphi}(n)}{n^{\widetilde{\alpha}}}\right)^2 \frac{1}{\mu_n \widetilde{\mu}_n} \leqslant 4\left(\widetilde{\mu}_n \frac{\varphi(n)}{n^{2\alpha}}\frac{\varphi(n)}{\mu_n} + \mu_n \frac{\widetilde{\varphi}(n)}{n^{2\widetilde{\alpha}}}\frac{\widetilde{\varphi}(n)}{\widetilde{\mu}_n}\right) \leqslant A_n^+(\varepsilon), \tag{6.3}$$

where for last inequality we used that $\frac{\varphi(n)}{n^{\alpha-1}\mu_n} \to 0$ as $n \to \infty$ (since $\varphi(n)/\mu_n \to 0$ when $\alpha = 1$), and similarly $\frac{\widetilde{\varphi}(n)}{n^{\widetilde{\alpha}-1}\widetilde{\mu}_n} \to 0$. The claim (6.1) is now proved.

For δ sufficiently small, applying Lemma 1.4 and (6.1) we get that for n large and $c_{13} = c_{10} + 1$,

$$\mathbf{P}(\rho_1 = n) \geqslant (1 - c_{13}\delta) A_n^-(\delta^3)\,. \tag{6.4}$$

Similarly, we get

$$\mathbf{P}(\rho_1 = n) \leqslant (1 + c_{13}\delta) A_n^+(\delta^3)\,. \tag{6.5}$$

If $\alpha^* = 0$, as with (6.2) it follows easily from Theorem 2.2 that for large n we have

$$A_n^-(\delta^3) \geqslant (1-\delta)(U_n^*)^{-2}\mathbf{P}(n \in \rho) \quad \text{and} \quad A_n^+(\delta^3) \leqslant (1+\delta)(U_n^*)^{-2}\mathbf{P}(n \in \rho),$$

and then part (ii) of the theorem follows from (6.4) and (6.5).

If $\alpha^* \in (0,1)$, then by Theorem 1.3(i), when δ is small we have for large n

$$A_n^-(\delta^3) \geqslant (1-\delta)\frac{\alpha^* \sin(\pi\alpha^*)}{\pi}\,\psi^*(n)^{-1} n^{-(1+\alpha^*)},$$
$$A_n^+(\delta^3) \leqslant (1+\delta)\frac{\alpha^* \sin(\pi\alpha^*)}{\pi}\,\psi^*(n)^{-1} n^{-(1+\alpha^*)},$$

and again part (i) of the theorem follows from (6.4) and (6.5).

If $\alpha^* \geqslant 1$, then by Theorem 1.3(iii), when δ is small we have for large n

$$A_n^-(\delta^3) \geqslant (1-\delta)\left(\widetilde{\mu}_n \frac{\varphi(n)}{n^{1+\alpha}} + \mu_n \frac{\widetilde{\varphi}(n)}{n^{1+\widetilde{\alpha}}}\right),$$
$$A_n^+(\delta^3) \leqslant (1+\delta)\left(\widetilde{\mu}_n \frac{\varphi(n)}{n^{1+\alpha}} + \mu_n \frac{\widetilde{\varphi}(n)}{n^{1+\widetilde{\alpha}}}\right),$$

and part (iii) of the theorem follows once more from (6.4) and (6.5). □

Appendix A. Extension of Lemma A.2 in [12]

We generalize here Lemma A.2 of [12], which covers $\alpha > 0$, to include $\alpha = 0$. The idea is essentially unchanged, but the computations are different.

Lemma A.1. *Assume that $\mathbf{P}(\tau_1 = k) = \varphi(k)k^{-(1+\alpha)}$ for some $\alpha \geqslant 0$ and some slowly varying function $\varphi(\cdot)$. Then, there exists a constant $C_0 > 0$ such that, for all sufficiently large n, for any non-negative function $f_n(\tau)$ depending only on $\tau \cap \{0,\dots,n\}$, we have*

$$\mathbf{E}[f_n(\tau) \mid 2n \in \tau] \leqslant C_0 \mathbf{E}[f_n(\tau)]\,.$$

Proof We define X_n to be the last τ-renewal up to n. It is sufficient to show that there exists $c_{14} > 0$ such that for large n, for any $0 \leqslant m \leqslant n$

$$\mathbf{P}(2n \in \tau \mid X_n = m) \leqslant c_{14}\mathbf{P}(2n \in \tau)\,. \tag{A.1}$$

To prove this, we write

$$\mathbf{P}(2n \in \tau \mid X_n = m) = \sum_{j=1}^{n} \mathbf{P}(\tau_1 = j + n - m | \tau_1 \geqslant n - m)\mathbf{P}(n - j \in \tau). \tag{A.2}$$

We split this sum into $j \leqslant n/2$ and $j > n/2$.

For $j \leqslant n/2$, we use that $\mathbf{P}(k \in \tau)$ is regularly varying and $n-j \geqslant n/2$, to bound the corresponding part of the sum in (A.2) by

$$\sup_{k \geqslant n/2} \mathbf{P}(k \in \tau) \times \sum_{j=1}^{n} \mathbf{P}(\tau_1 = j+n-m | \tau_1 \geqslant n-m) \leqslant c_{15} \mathbf{P}(2n \in \tau)\,.$$

For $j > n/2$, we use that for $n \geqslant j > n/2$ and $n \geqslant m \geqslant 0$,

$$\mathbf{P}(\tau_1 = j+n-m \mid \tau_1 \geqslant n-m) \leqslant c_{16} \mathbf{P}(\tau_1 = n \mid \tau_1 \geqslant n-m) \leqslant c_{16} \mathbf{P}(\tau_1 = n \mid \tau_1 \geqslant n)$$

to bound the corresponding part of the sum in (A.2) by

$$c_{16} \frac{\mathbf{P}(\tau_1 = n)}{\mathbf{P}(\tau_1 \geqslant n)} U_n \leqslant \begin{cases} c_{17} \mathbf{P}(2n \in \tau) & \text{if } \alpha = 0; \\ c_{18} n^{-1} U_n \leqslant c_{19} \mathbf{P}(2n \in \tau) & \text{if } \alpha > 0. \end{cases}$$

Here for $\alpha = 0$ we used (2.5), and for $\alpha > 0$ we used the regular variation of $\mathbf{P}(n \in \tau)$. This completes the proof of (A.1). □

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Department of Mathematics, KAP 108, University of Southern California, Los Angeles, CA 90089-2532 USA

LPMA, Université Pierre et Marie Curie, Campus Jussieu, case 188, 4 place Jussieu, 75252 Paris Cedex 5, France

E-mail address: quentin.berger@upmc.fr