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The dynamical behaviour of nonlinear elastic spherical shells

CARMEN CALDERER

Heriot-Watt University, Department of Mathematics, Edinburgh EH12 4AS, Scotland, U.K.

University of Maryland, Department of Mathematics, College Park, MD 20742, U.S.A.

Introduction

We consider the problem of the radial motion of a thick spherical shell under a constant pressure difference between the inner and the outer surfaces, and under zero body forces. We assume that the material of the shell is homogeneous, isotropic, hyperelastic and incompressible.

The problem of inflation under pressure of an arbitrary (i.e. nonspherical) compressible hyperelastic body has been studied by Ball [14]. He applied the concavity method of Knops, Levine and Payne [15] to show that weak solutions to the equations of motion with appropriate initial data do not exist for all positive time. The method of proof suggests, but does not establish, that the mechanism for non-existence is blow-up of the displacement in finite time. However, it seems that this method does not apply if the material is almost incompressible. This is related to the fact that the concavity hypothesis made on the strain-energy function $W(F)$ in [14] cannot hold if the material is almost incompressible. Also the set of initial data corresponding to non-existence is not easy to characterize. In this paper we show that for the radial motion of an incompressible spherical or cylindrical shell the special geometry enables us to determine completely the behaviour of all solutions, and in particular to describe exactly for which initial data the solution exists for all time, and for which the solution becomes unbounded in finite time. This description is possible because the equation of motion reduces to a conservative autonomous ordinary differential equation in the plane.

The constitutive properties of the material enter the differential equation through a function $g(x^3, \delta)$ of the tangential strain x and a parameter δ giving the thickness of the shell in reference configuration. The strain-energy function $W(x^3)$ enters into the expression for $g(x^3, \delta)$ in a complicated way.

The phase-plane analysis is carried out initially under assumptions on the function $g(x^3, \delta)$, rather than $W(x^3)$. We make hypotheses on the growth of this function for large and

small values of x , on its sign, on the number of its critical points, and on the number of roots $x > 1$ of the equation $g(x^3, \delta) = P$ for a given $P > 0$.

The growth conditions on g are related to the growth properties of $W(F)$ for large F . The importance of such growth conditions in nonlinear elasticity was first emphasized by Antman [16]. In [14] Ball interprets the concavity inequality in terms of the slow growth of $W(F)$ as some norm of the deformation gradient F tends to infinity, this condition being appropriate for a “weak material”. This slow growth of $W(F)$ is responsible for the global non-existence of solutions. Following this idea, this paper shows that for “strong materials”, those for which $g(x^3, \delta)$ grows faster than certain powers of x for small and large x , all solutions exist and remain bounded for all time, while “weak” materials have unbounded solutions that blow up in finite time, for some boundary pressures and some initial data. (As an example of these growth conditions, let us consider the isotropic strain energy function $\sigma(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha$, where $\lambda_1, \lambda_2, \lambda_3$ describe the principal stretches, and $\alpha > 0$. For the radial deformation under study, $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1/2} = \lambda_3$ and $x = \lambda^{-1/2}$, where λ is the radial stretch. Then the material is weak for x large, if $\alpha < 3$, and strong if $\alpha > 3$, and it is strong for x small for all $\alpha > 0$).

Since the dynamical description is given in terms of $g(x^3, \delta)$, in Section 3 we consider the problem of determining for $\delta > 0$ a strain energy function corresponding to a given $g(x^3, \delta)$. Using finite difference equation methods, we find a unique solution given by an infinite series.

The hypothesis on the sign of $g(x^3, \delta)$ are interpreted in terms of constitutive restrictions on the strain-energy function.

Critical situations arise for some particular values of the external pressure P_{crit} at which the number of equilibrium solutions decreases by two as P passes through P_{crit} . Under some hypotheses on $g(x^3, \delta)$, we, in Section 4, use invariant manifolds to determine when $P = P_{\text{crit}}$ and the behaviour of the solutions in the neighbourhood of the relevant equilibrium point.

In Section 5 we consider the dynamical equations of a membrane, for which it is possible to carry out the same analysis. We show that the solutions of the equation of motion are the uniform limit on finite time intervals of the solutions for the thick shell as δ tends to zero. In this case we construct some examples to show that some of the phase-plane results obtained in Section 2 can occur under constitutive hypotheses on the strain-energy function such as convexity of $W(F)$. (For incompressible materials this is not an unreasonable assumption).

In Section 6 we summarise the corresponding results for a cylindrical shell.

Section 1

We consider a Cartesian system and a set of spherical coordinates (R, Θ, Φ) in \mathbb{R}^3 .

Let the reference configuration of the shell be the domain $\Omega \subset \mathbb{R}^3$,

$$\Omega = \{(R, \Theta, \Phi), R_1 < R < R_2\}, \quad (1.1)$$

where R_1, R_2 denote the radii of the inner and the outer surfaces of the shell, respectively.

Let the position $(r(t), \theta(t), \phi(t))$ at time t of the particle labelled by (R, Θ, Φ) in the reference configuration be given by

$$\begin{aligned} r^3(t) &= R^3 + r_1^3(t) - R_1^3, \\ \theta(t) &= \Theta, \\ \phi(t) &= \Phi. \end{aligned} \tag{1.2}$$

(We refer to [2] for the following elements in continuum mechanics). The physical components of the deformation gradient matrix are

$$\begin{aligned} F &= \text{diag} \left[\frac{R^2}{r^2}, \frac{r}{R}, \frac{r}{R} \right], \\ J &= \det F = 1. \end{aligned} \tag{1.3}$$

In spherical components the right and left Cauchy-Green tensor C, B are given by

$$C = B = \text{diag} \left[\frac{R^4}{r^4}, \frac{r^2}{R^2}, \frac{r^2}{R^2} \right]. \tag{1.4}$$

The principal stretches $\lambda_1, \lambda_2, \lambda_3$ of the deformation (1.2) are

$$\lambda_1 = \lambda = \frac{R^2}{r^2}, \quad \lambda_2 = \lambda^{-1/2} = \lambda_3 = \frac{r}{R}. \tag{1.5}$$

The principal invariants of (1.4) are

$$I_1 = \lambda^2 + \frac{2}{\lambda}; \quad I_2 = \frac{1}{\lambda^2} + 2\lambda, \quad I_3 = 1. \tag{1.6}$$

We assume that the solid is hyperelastic and isotropic, i.e. the strain energy function depends on F through the invariants of B , or equivalently, it can be expressed as a symmetric function of the principal stretches λ_i ,

$$W(F) = W(I_1, I_2) = \sigma(\lambda_1, \lambda_2, \lambda_3). \tag{1.7}$$

Because the solid is incompressible, the Cauchy stress tensor has the form

$$T(\mathbf{x}, t) = -pI + \psi_1 B + \psi_{-1} B^{-1}, \tag{1.8}$$

where

$$\psi_1 = 2 \frac{\partial W}{\partial I_1}(I_1, I_2), \quad \psi_{-1} = -2 \frac{\partial W}{\partial I_2}(I_1, I_2) \tag{1.9}$$

and

$$p = p(\mathbf{x}, t)$$

is an arbitrary hydrostatic pressure.

In particular, for B given by (1.4), (1.8) becomes

$$T = \text{diag} \left(-p + \psi_1 \frac{R^4}{r^4} + \psi_{-1} \frac{r^4}{R^4}, -p + \psi_1 \frac{r^2}{R^2} + \psi_{-1} \frac{R^2}{r^2}, -p + \psi_1 \frac{r^2}{R^2} + \psi_{-1} \frac{R^2}{r^2} \right). \quad (1.10)$$

The spherical components of the acceleration \mathbf{a} for the motion (1.2) are

$$a_r = \ddot{r}, \quad a_\theta = 0 = \alpha_\phi \quad (1.11)$$

The balance laws of linear and angular momentum, with zero body force, are given in point-wise form by

$$\begin{aligned} \text{div } \mathbf{T}(\mathbf{x}, t) &= \rho \mathbf{a}(\mathbf{x}, t), \\ T &= T^T. \end{aligned} \quad (1.12)$$

The stress vector \mathbf{t} measured per unit area of the deformed configuration is

$$\mathbf{t} = T\hat{n}, \quad (1.13)$$

where \hat{n} denotes the unit outward normal to the boundary of Ω . We follow the work of Knowles and Jacub [23] to deduce the differential equation of the motion of the spherical shell. We write the field equations (1.12) in spherical coordinates [2], [3], for \mathbf{T} , a given by (1.10), (1.11), respectively,

$$\begin{aligned} \frac{\partial}{\partial r} T_{rr} + \frac{2}{r} [T_{rr} - T_{\theta\theta}] \\ = \frac{\partial}{\partial r} \left(-p + \psi_1 \frac{R^4}{r^4} + \psi_{-1} \frac{r^4}{R^4} \right) + \frac{2}{r} \left[\psi_1 \left(\frac{R^4}{r^4} - \frac{r^2}{R^2} \right) \right. \\ \left. + \psi_{-1} \left(\frac{r^4}{R^4} - \frac{R^2}{r^2} \right) \right] = \rho \ddot{r}, \\ \frac{\partial p}{\partial \theta} = 0 = \frac{\partial p}{\partial \phi}. \end{aligned} \quad (1.14)$$

From the two last equations we can assert that p is independent of θ and ϕ , i.e.,

$$p = p(r, t). \quad (1.15)$$

Differentiating with respect to t in (1.2), we deduce that

$$\ddot{\mathbf{r}} = -\frac{\partial \S}{\partial r}, \quad (1.16)$$

where

$$\S = r^{-1}(2r_1 \dot{r}_1^2 + r_1^2 \ddot{r}_1) - \frac{1}{2} r^{-1} r_1^4 \dot{r}_1^2. \quad (1.17)$$

is the acceleration potential.

We assume that the outer surface is free of traction and the inner one is subjected to a constant pressure $p > 0$ measured per unit area in the present configuration,

$$\begin{aligned} T(\mathbf{x}, t) \hat{n}(\mathbf{x}, t) &= -p \hat{n}(\mathbf{x}, t), & \mathbf{x}(r, \theta, \phi) &= \mathbf{x}(r_1, \theta, \phi) \\ &= 0, & \mathbf{x}(r, \theta, \phi) &= \mathbf{x}(r_2, \theta, \phi), \end{aligned} \quad (1.18)$$

where

$$r_2 = r_2(R_2, t).$$

Hence,

$$T_{rr}(r_2, t) - T_{rr}(r_1, t) = p. \quad (1.19)$$

We integrate (1.14) with respect to r using (1.16) and (1.17). Then, we can write (1.19) as

$$-\rho(\xi_1 - \xi_2) + 2 \int_{r_2}^{r_1} \left(\frac{r^2}{R^2} - \frac{R^4}{r^4} \right) \left(\psi_1 - \frac{r^2}{R^2} \psi_{-1} \right) \frac{dr}{r} = -p, \quad (1.20)$$

where

$$\begin{aligned} \xi_1 &= r_1 \ddot{r}_1 + \frac{3}{2} \dot{r}_1^2, \\ \xi_2 &= \frac{1}{r_2} (2r_1 \dot{r}_1^2 + r_1^2 \ddot{r}_1) - \frac{1}{2} r_2^{-4} r_1^4 \dot{r}_1^2. \end{aligned} \quad (1.21)$$

Next we set

$$x(t) = \frac{r_1(t)}{R_1} > 0, \quad \xi(R, t) = \frac{r^3}{R^3}, \quad \delta = \frac{R_2^3}{R_1^3} - 1 > 0. \quad (1.22)$$

Then

$$\begin{aligned} \frac{r_2^3}{R_2^3} &= \frac{\delta + x^3}{1 + \delta}, \quad \frac{r_1^3}{R_1^3} = 1 + \frac{\delta}{x^3}, \\ \xi(R, t) &= 1 + \frac{R_1^3}{R^3} (x^3 - 1), \end{aligned} \quad (1.23)$$

and

$$\lambda = \xi^{-2/3}.$$

Substituting (1.21), (1.22), (1.23) in (1.20), we finally get

$$\begin{aligned} 2x\ddot{x} \left[1 - \left(1 + \frac{\delta}{x^3} \right)^{-1/3} \right] + \dot{x}^2 \left[3 - 3 \left(1 + \frac{\delta}{x^3} \right)^{-1/3} \right. \\ \left. - \frac{\delta}{x^3} \left(1 + \frac{\delta}{x^3} \right)^{-4/3} \right] + g(x^3, \delta) = P, \quad x > 0, \end{aligned} \quad (1.24)$$

$$P = \frac{2}{\rho R_1^2} p,$$

and

$$g(x^3, \delta) = \frac{4}{3\rho R_1^2} \int_{\frac{x^3+\delta}{1+\delta}}^{x^3} \xi^{-7/3} (1 + \xi) (\psi_1 - \xi^{2/3} \psi_{-1}) d\xi. \quad (1.25)$$

We consider the density ρ as a constant.

It is important to notice that since P is constant (1.24) is autonomous.

We introduce the one-dimensional strain energy function $W(\xi)$,

$$W(\xi) = W(I_1(\xi), I_2(\xi)),$$

or (1.26)

$$\sigma(\lambda) = W(\xi(\lambda)).$$

Differentiating (1.26) and using (1.9), we can write (1.25) as

$$g(x^3, \delta) = \frac{2}{\rho R_1^2} \int_{\frac{\delta+x^3}{1+\delta}}^{x^3} \frac{W'(\xi)}{\xi-1} d\xi. \quad (1.27)$$

We seek the solutions of equation (1.24) satisfying the initial conditions

$$\begin{aligned} x(0) &= x_0, \\ \dot{x}(0) &= \dot{x}_0, \end{aligned} \quad (1.28)$$

for $(x_0, \dot{x}_0) \in \mathcal{D} = (0, \infty) \times \mathbb{R}$

We consider equation (1.24) for a given $g(x^3, \delta)$, $\delta > 0$ fixed. We regard $g(x^3, \delta)$ as a function of x only, δ being a parameter. We make the following hypotheses on $g(x^3, \delta)$ (1.29):

(A) $g \in C^2(0, \infty)$.

(B) $g(x^3, \delta) > 0 \quad x > 1; \quad g(x^3, \delta) < 0 \quad x < 1$.

(C) There exist $D \geq C > 0$ and $\alpha > 0$, for x large, such that $Cx^{\alpha-3} \leq g(x^3, \delta) \leq Dx^{\alpha-3}$.

We consider the three cases

(1) $\alpha > 3$,

(2) $\alpha < 3$,

(3) $\alpha = 3$.

(D) There exist $M \geq N > 0$, and $\beta > 0$, for x small, such that $-Mx^{-\beta} \leq g(x^3, \delta) \leq -Nx^{-\beta}$.

We consider the two cases

(1) $\beta \geq 3$,

(2) $\beta < 3$.

Remark. In some examples the exponents α and β may be related. See the examples in Section 5.

Moreover, we will make further assumptions later on

(a) The number of nondegenerate critical points with respect to ξ of $g(\xi, \delta)$.

(b) The number of roots $x > 1$ of the equation $g(x^3, \delta) = P$, for $P > 0$ a given constant.

Since we want to analyse equation (1.24) in the phase-plane, it is useful to consider the equivalent first order system of two equations

$$\dot{\mathbf{x}}(t) = \mathbf{h}(x_1, x_2), \quad (1.30)$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$\mathbf{h}: \mathcal{D} \times \mathbb{R} \rightarrow \mathbb{R}^2,$$

$$\mathbf{h}(x_1, x_2) = \begin{pmatrix} x_2 \\ \frac{1}{2x_1 \left[1 - \left(1 + \frac{\delta}{x_1^3} \right)^{-1/3} \right]} \left\{ P - g(x_1^3, \delta) - x_2^2 \left[3 - 3 \left(1 + \frac{\delta}{x_1^3} \right)^{-1/3} - \frac{\delta}{x_1^3} \left(1 + \frac{\delta}{x_1^3} \right)^{-4/3} \right] \right\} \right\} \end{pmatrix} \quad (1.31)$$

$$(1.32)$$

and (1.28),

$$\mathbf{x}(0) = \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix}. \quad (1.33)$$

It is easy to check that hypothesis (A) on $g(x^3, \delta)$ and the smoothness of the functions involved in (1.32) imply that $h(\cdot, \cdot) \in (C^2(\mathcal{D}))^2$, and, therefore h is Lipschitz in \mathcal{D} [4], [5].

Proposition:

Let $(x_0, \dot{x}_0) \in \mathcal{D}$.

The initial boundary value problem (1.30)–(1.33) has a unique maximally defined C^2 solution $\mathbf{x}(t)$ on $[0, t_{\max})$, $0 < t_{\max} \leq \infty$, satisfying $x_1(0) = x_0, x_2(0) = \dot{x}_0$.

Moreover if $t_{\max} < \infty$, then

$$|x(t)| + |\dot{x}(t)| \rightarrow \infty \quad \text{as} \quad t \rightarrow t_{\max}.$$

Proof: Local existence and uniqueness of the solution through an initial data point (x_0, \dot{x}_0) , follow from standard results in ordinary differential equations.

The last statement will be proved in Section 2, using the energy equation. In particular, we will show that if

$$\lim_{t \rightarrow t_{\max}} x(t) = 0, \quad \text{then} \quad \lim_{t \rightarrow t_{\max}} \dot{x}(t) = -\infty.$$

Therefore, the solution cannot reach a point $(0, b)$ of the boundary of the domain in a finite time.

Energy Equation

Next we want to find the energy equation. We multiply (1.24) by $x^2 \dot{x}$ and integrate respect to t obtaining

$$\left[1 - \left(1 + \frac{\delta}{x^3}\right)^{-1/3}\right] x^3 \dot{x}^2 + G(x^3, \delta) - \frac{1}{3} P x^3 = E, \quad (1.34)$$

where

$$G(x^3, \delta) = \frac{1}{3} \int_1^{x^3} g(\lambda, \delta) d\lambda. \quad (1.35)$$

E is the total energy, which depends on (x_0, \dot{x}_0) . We define the potential energy

$$V(x, \delta) = G(x^3, \delta) - \frac{1}{3} P x^3. \quad (1.36)$$

Next we want to characterise the equilibrium solutions of the initial boundary value problem (1.30)–(1.33). We consider a boundary pressure $P > 0$. We state the following.

Proposition (1.1)

Let $P > 0, R_1 > 0, \rho > 0, \delta > 0$, be given constants. We write

$$g'(\xi, \delta) = \frac{d}{d\xi} g(\xi, \delta).$$

Then:

- (i) The equilibrium solutions $x_1(t) = \bar{x}, x_2(t) = 0, t \in [0, \infty)$ of (1.30) are the roots of

$$g(x^3, \delta) = P, \quad (1.37)$$
 and they satisfy $\bar{x} > 1$.
- (ii) If $g'(\bar{x}^3, \delta) > 0$ then $(\bar{x}, 0)$ is a centre and $V(\bar{x}, \delta)$ is a relative minimum.
- (iii) If $g'(\bar{x}^3, \delta) < 0$ then $(\bar{x}, 0)$ is a saddle point and $V(\bar{x}, \delta)$ is a relative maximum.

Proof.

- (i) This follows from hypothesis (B).
- (ii), (iii) We consider the eigenvalues of the linearised equation about $(\bar{x}, 0)$ of (1.31),

$$\lambda = -\mu = \left[\frac{-3\bar{x}g'(\bar{x}^3, \delta)}{2 \left[1 - \left(1 + \frac{\delta}{\bar{x}^3}\right)^{-1/3} \right]} \right]^{1/2}. \quad (1.38)$$

Standard results on ODE [4], [5] imply that the equilibrium solution $(\bar{x}, 0)$ is a centre of the eigenvalues of the linearised equation are pure imaginary, equal, with opposite sign. This result applies in case (ii).

If $g' < 0$ then the eigenvalues are real with the same absolute value and opposite sign. By the same results $(\bar{x}, 0)$ is a saddle point. This applies in case (iii).

The results on $V(x, \delta)$ follow using (1.36) and its derivatives. □

Remark: If $g'(\bar{x}^3, \delta) = 0$, then $\lambda = 0 = \mu$ and the linearisation method fails. In Section 4 we consider this case under further hypotheses on $g(\bar{x}^3, \delta)$.

In order to study continuation of the solution of (1.24) for given initial data, we will use the energy equation (1.34). The behaviour of the potential energy $V(x, \delta)$ for large and small values of x is a basic ingredient in obtaining such results. Therefore, we are going to study the asymptotic growth of $V(x, \delta)$ for large and small values of x , for all the cases included in (C) and (D).

Proposition (1.2)

The asymptotic behaviour of $V(x, \delta)$, for x large, is as follows:

(1) If (C1) holds, then, for x large,

$$\frac{C}{2\alpha} x^\alpha \leq V(x, \delta) \leq \frac{(D+1)}{\alpha} x^\alpha. \tag{1.39}$$

(2) If (C3) holds, then, for x large, and a given $P > 0$,

$$\frac{1}{2}(C-P)x^3 \leq V(x, \delta) < (D-P)x^3. \tag{1.40}$$

(3) If (C2) holds, then

$$\lim_{x \rightarrow \infty} \frac{V(x, \delta)}{\frac{P}{3}x^3} = -1. \tag{1.41}$$

Proof:

(1) For a sufficiently large fixed $a > 0$ and for sufficiently large x , we can write

$$\begin{aligned} \int_1^{x^3} g(\xi, \delta) d\xi &= \int_1^{a^3} g(\xi, \delta) d\xi + \int_{a^3}^{x^3} g(\xi, \delta) d\xi \\ &\leq \int_1^{a^3} g(\xi, \delta) d\xi + \frac{3D}{\alpha} (x^\alpha - a^\alpha). \end{aligned}$$

Using (1.35), (1.36), we have

$$V(x, \delta) \leq \frac{(D+1)}{\alpha} x^\alpha - \frac{P}{3} x^3.$$

Therefore, for $\alpha > 3$,

$$\frac{C}{2\alpha} x^\alpha \leq V(x, \delta) \leq \frac{(D+1)}{\alpha} x^\alpha.$$

(2) and (3) follow similarly.

Proposition (1.3)

The asymptotic behaviour of $V(x, \delta)$, for $x > 0$ small, is as follows:

(1) If (D1) holds, then,

$$\text{for } \beta > 3, \frac{M+1}{\beta-3} \frac{1}{x^{\beta-3}} \geq V(x, \delta) \geq \frac{N}{2(\beta-3)} \frac{1}{x^{\beta-3}} > 0, \quad (1.42)$$

$$\text{for } \beta = 3, \lim_{x \rightarrow 0^+} \frac{V(x, \delta)}{-\log x} = 1.$$

(2) If (D2) holds, then

$$V_0 = \lim_{x \rightarrow 0^+} V(x, \delta) \quad (1.43)$$

exists, is finite and positive.

Proof.

We consider a sufficiently small fixed $a > 0$. For a sufficiently small $x > 0$, we can apply (D) to the following inequality

$$\begin{aligned} \int_1^{x^3} g(\xi, \delta) d\xi &= \int_1^{a^3} g(\xi, \delta) d\xi + \int_{a^3}^{x^3} g(\xi, \delta) d\xi \geq \int_1^{a^3} g(\xi, \delta) d\xi \\ &+ \frac{3N}{3-\beta} (a^{-\beta+3} - x^{-\beta+3}). \end{aligned} \quad (1.44)$$

Using (1.35), (1.36), we see that (1.42) follows for $\beta \neq 3$ and x small. It follows from (1.44) and (B) that

$$\infty > \int_0^1 -g(\xi, \delta) d\xi > 0.$$

Then (2) follows immediately. (1), for $\beta = 3$, follows similarly.

Section 2

Now we analyse the initial-boundary-value problem in the phase-plane under the hypotheses on $g(x^3, \delta)$ previously given. Moreover, we assume

- (1) The number of critical points of $g(x^3, \delta)$ is finite.
- (2) If $g(x^3, \delta) = c$, for $x \in (a, b)$, where $b > a > 1$, $c > 0$, are some given constants, then the boundary pressure $P \neq c$. (The case $P = c$, can be treated similarly.)

We need the following notation.

$$\text{Let } (x_0, \dot{x}_0) \in \mathcal{D}, \quad \text{and } E = E(x_0, \dot{x}_0).$$

We let

$$\Omega = \{y \in \mathbb{R} : V(y, \delta) = E\}.$$

It is obvious from Proposition (1.1) under the previous assumptions on $g(x^3, \delta)$, that Ω is a finite set. We let

$$\Omega = \{y_1, \dots, y_n\}, \quad \text{with, } y_{j-1} \leq y_j, \quad \text{for } j \leq n$$

We state the following proposition.

Proposition 2.1

Let $(x_0, \dot{x}_0) \in \mathcal{D}$, $E = E(x_0, \dot{x}_0)$.

(1) If (C1) and (D1) hold, then

- (a) $\Omega = \{y_1, \dots, y_{2r}\}, r \geq 1,$
- (b) $x_0 \in [y_{2s-1}, y_{2s}],$ for some $s \leq r,$
- (c) $V(x, \delta) \leq E$ for all $x \in [y_{2s-1}, y_{2s}].$

(2) If (C2) and (D1) hold, then

- (a) $\Omega = \{y_4, \dots, y_{2r-1}\}, r \geq 1,$
- (b) $x_0 \geq y_1.$

The intergers r, s are determined by $V(x, \delta), (x_0, \dot{x}_0)$ and $E(x_0, \dot{x}_0)$.

Proof.

1(a) and 2(a) follow from Propositions (1.2) and (1.3).

1(b) We prove first that $x_0 \in [y_1, y_{2r}].$

Let us assume that $x_0 \notin [y_1, y_{2r}],$ then $V(x_0, \delta) > E,$ which is impossible by the energy equation. In particular $x_0 \in [y_{2s-1}, y_{2s}].$ for a integer $s \leq r,$ which is determined by $V(\cdot, \delta), (x_0, \dot{x}_0)$ and $E.$

1(c) follows by similar arguments. □

Remark (2.1)

(A) If (C3) and (D1) or (D3) hold, then if $P \geq D$ the results (2) hold, while if $P \leq C$ then the results (1) hold.

(B) If (C2) and (D2) hold, there exist some values of E such that $\Omega = \phi.$

Related to the the above proposition we introduce the following notation:

$$(i) \quad V_s = \min_{x \in [y_{s-1}, y_s]} V(x, \delta). \tag{2.1}$$

$$(ii) \quad v_s^2 = \frac{1}{y_s^3 - 1 \left[1 - \left(1 + \frac{\delta}{y_s} \right)^{-1/3} \right]} [E - V_s]. \tag{2.2}$$

Definition.

(i) The positive orbit of the solution $x(\cdot)$ is the set $C^+(x(\cdot))$ given by

$$C^+ = Ux(t).$$

$$t \in [0, t_{\max})$$

Now we are going to state the results of the phase-plane analysis of the initial-boundary-value problem. We fix $\delta > 0$ and $P > 0$. Several examples are shown at the end to illustrate some of the different cases that may occur.

Proposition 2.2

We suppose that

- (i) (C1) and (D1) hold,
- (ii) $g(x^3, \delta)$ has $2n$ critical points,
- (iii) the roots of $g(x^3, \delta) = P$ are

$$\{\bar{x}_i \in \mathbb{R}^+, \quad 1 \leq i \leq 2k + 1, \quad k \leq n\}.$$

Then, any initial data $(x_0, \dot{x}_0) \in \mathcal{D}$ satisfies one of the following possibilities.

- (a) If (x_0, \dot{x}_0) is such that $E \neq V(\bar{x}_{2s})$, $j = 1, \dots, k$, the solution is periodic, with period τ given by

$$\tau = 2 \int_{y_{2s-1}}^{y_{2s}} \left\{ \frac{x^3 \left[1 - \left(1 + \frac{\delta}{x^3} \right)^{-1/3} \right]}{E - V(x, \delta)} \right\}^{1/2} dx, \quad \text{for some } s \geq 1. \quad (2.3)$$

- (b) For (x_0, \dot{x}_0) such that $E = V(\bar{x}_{2s})$ for some $s \in \{1, 2, \dots, k\}$, \bar{x}_{2s} is a saddle point, and,

- (1) If $x_0 \leq \bar{x}_{2s}$ then

$$x(t) \in [y_{2s-1}, \bar{x}_{2s}) \quad \text{and} \quad |\dot{x}(t)| \leq v_s \quad \text{for } t \in [0, \infty).$$

Moreover,

$$\lim_{t \rightarrow \infty} x(t) = \bar{x}_{2s} \quad \text{and} \quad \lim_{t \rightarrow \infty} \dot{x}(t) = 0.$$

- (2) If $y_{2(s+1)} > x_0 > \bar{x}_{2s}$, then

$$x(t) \in (\bar{x}_{2s}, y_{2(s+1)}] \quad \text{and} \quad |\dot{x}(t)| \leq v_s, \quad t \in [0, \infty).$$

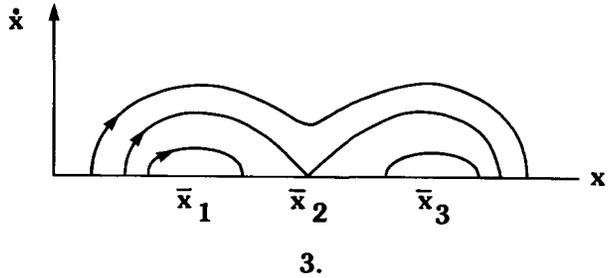
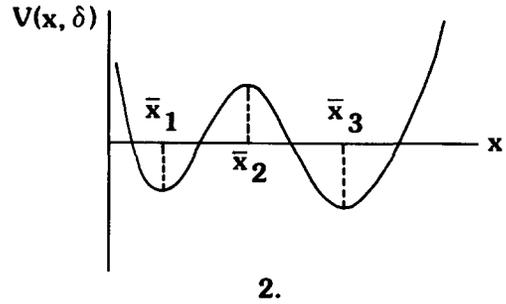
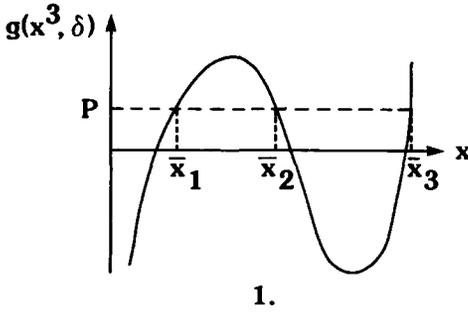
Also

$$\lim_{t \rightarrow \infty} x(t) = \bar{x}_{2s} \quad \text{and} \quad \lim_{t \rightarrow \infty} \dot{x}(t) = 0.$$

- (3) If x_0 does not belong to any of the above intervals then the solution is periodic.

Proof.

- (a) We consider y_{2s-1}, y_{2s} such that $x_0 \in [y_{2s-1}, y_{2s}]$. Then, the solution $(x(t), \dot{x}(t))$ starting at (x_0, \dot{x}_0) at $t = 0$ is such that



1. $g(x^3, \delta)$ satisfying hypotheses (C1), (D1). 2. $V(x, \delta)$ for a given $P > 0$. 3. Phase-plane diagram.

$x(t) \in [y_{2s-1}, y_{2s}]$ for all $t \in [0, t_{\max})$,

for otherwise the energy equation would be contradicted. Also

$0 \leq \dot{x}^2(t) < v_1^2, t \in [0, t_{\max})$.

Hence, the orbit C^+ through (x_0, \dot{x}_0) is bounded. Moreover, $(y_{2s-1}, 0), (y_{2s}, 0)$ are not equilibrium points by the choice of E . Therefore, the orbit is periodic.

(b) This follows by similar arguments and the fact that $(\bar{x}_{2j}, 0)$ is a saddle point.

Proposition (2.3)

We suppose that

- (i) (C1) and (D2) hold,
- (ii) and (iii) are as in Proposition (2.2).

Then, any initial data $(x_0, \dot{x}_0) \in \mathcal{D}$ satisfies one of the following possibilities:

- (a) If $E < V_0$ all solutions are bounded.
- (b) If $E \geq V_0$, then solutions with initial data $x_0 > y_1$ remain bounded, and solutions with $x_0 < y_1$ blow up in finite time. In this case

$$\lim_{t \rightarrow t_{\max}} (x(t), \dot{x}(t)) = (0, -\infty).$$

Moreover, if x_0 is sufficiently small, we have the following estimate for t_{\max}

$$t_{\max} = \frac{2}{3}(E - V_0)^{-1/2} x_0^{5/2} + \epsilon(x_0), \quad (2.4)$$

where

$$|\epsilon(x_0)| \leq \frac{1}{2(3 - \beta)} \left(M + 1 - \frac{N}{2} \right) (E - V_0)^{-3/2} x_0^{11/2 - \beta} \quad (2.5)$$

and M, N, β are given in hypotheses (D2).

Proof.

(a) By the choice of E this case is included in Proposition (2.2). The same occurs in (b) for $x_0 > y_1$.

(b) If $x_0 < y_1$, then $x(t) \leq y_1, t \in [0, t_{\max})$, by definition of y_1 . This implies that

$$E - V(x(t), \delta) > 0 \quad \text{for } t \in [0, t_{\max}).$$

Therefore, if we suppose that $\dot{x}_0 < 0$, then $\dot{x}(t) < 0, t \in [0, t_{\max})$, i.e., $x(t) \leq x_0$ is monotonically decreasing.

Since

$$\lim_{x \rightarrow 0^+} x^3 \left[1 - \left(1 + \frac{\delta}{x^3} \right)^{-1/3} \right] = \lim_{x \rightarrow 0^+} x^3 = 0, \quad (2.6)$$

then

$$\lim_{x \rightarrow 0^+} \dot{x}^2 = \lim_{x \rightarrow 0^+} \frac{[E - V(x, \delta)]}{x^3 \left[1 - \left(1 + \frac{\delta}{x^3} \right)^{-1/3} \right]} = \lim_{x \rightarrow 0^+} \frac{E - V_0}{x^3} = \infty. \quad (2.7)$$

By energy equation (1.34), it can not occur that

$$\lim_{t \rightarrow t_{\max}} (x(t), \dot{x}(t)) = (0, C), \quad \text{for some constant } C.$$

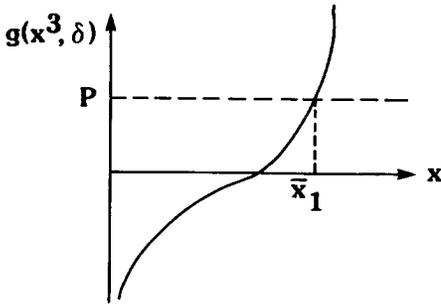
Hence,

$$\lim_{t \rightarrow t_{\max}} x(t) = 0^+ \quad \text{and} \quad \lim_{t \rightarrow t_{\max}} \dot{x}(t) = -\infty.$$

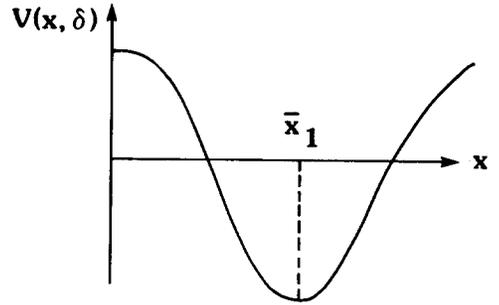
Now we are going to prove that $t_{\max} < \infty$. Suppose $t_{\max} = \infty$. From (1.34) we can write

$$\int_x^t \frac{ds}{\left\{ \frac{1}{s^3 \left(1 - \left(1 + \frac{\delta}{s^3} \right)^{-1/3} \right)} [E - V(s, \delta)] \right\}^{1/2}} = - \int_0^x dt. \quad (2.8)$$

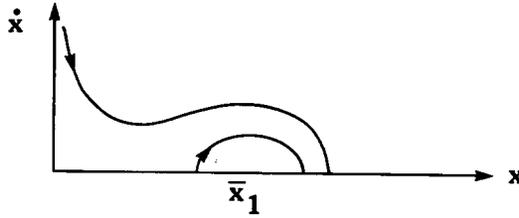
(The minus sign is because $\dot{x}(t) < 0$).



1.



2.



3.

1. $g(x^3, \delta)$ satisfying hypotheses (C1), (D2). 2. $V(x, \delta)$ for a given $P > 0$. 3. Phase-plane diagram.

By the previous statements, it follows that the left integral remains bounded and the right one is unbounded, as $t \rightarrow \infty$. This contradiction shows that $t_{\max} < \infty$.

To obtain estimates for t_{\max} , we consider x_0 sufficiently small and apply (1.44) to the left integral (2.8). We thus get upper and lower bounds for this integral. Retaining the higher order terms in x_0 , using (2.6) and integrating the remaining expression, we see that (2.4) and (2.5) follow immediately.

We have assumed that $\dot{x}_0 < 0$. The same conclusions hold in the case $\dot{x}_0 > 0$, since this case reduces to the previous one once \dot{x} becomes negative for $x = y_1$. \square

Proposition (2.4)

We suppose that

- (i) (C2) and (D1) hold,
- (ii) $g(x^3, \delta)$ has $2n + 1$ critical points, $n \geq 0$,
- (iii) the roots of $g(x^3, \delta) = P$ are

$$\{\bar{x}_i > 1, \quad 1 \leq i \leq 2k + 2, \quad k \leq n\}.$$

Then, any initial data $(x_0, \dot{x}_0) \in \mathcal{D}$ satisfies one of the following possibilities:

- (a) For (x_0, \dot{x}_0) such that $E > \bar{V} = \text{Max} \{V(\bar{x}_i), i = 1, \dots, 2(k + 1)\}$, the solution blows up in finite time.

(b) For (x_0, \dot{x}_0) such that $E < V(\bar{x}_{2k+2})$ the solution blows up in finite time if $x_0 > \bar{x}_{2k+2}$, and remains bounded if $x_0 < \bar{x}_{2k+2}$.

(c) For (x_0, \dot{x}_0) such that $\bar{V} > E > V(\bar{x}_{2k+2})$, then if $x_0 > \bar{x}$ the solution blows up in finite time and if $x_0 < \bar{x}$ it remains bounded for all time, where

$$\bar{x} = \max \{x \in \mathbb{R}^+, V(x, \delta) = E\}.$$

If $\bar{V} = V(\bar{x}_{2k+2})$, case (c) is included in (b).

(d) For $E \in \{V(\bar{x}_{2j}), j = 1, \dots, k+1\}$, if $x \geq \bar{x}_{2j}$ then $V(\bar{x}_{2j}) > V(x, \delta)$, and

(1) If $x_0 > \bar{x}_{2j}$ and $\dot{x}_0 < 0$, or $x_0 < x_{2j}$, then

$$\lim_{t \rightarrow \infty} x(t) = \bar{x}_{2j} \quad \text{and} \quad \lim_{t \rightarrow \infty} \dot{x}(t) = 0,$$

(2) If $x_0 > x_{2j}$, $\dot{x}_0 > 0$ the solution blows up in finite time. Whenever the solution blows up in finite time, we have the following estimate for t_{\max} , provided x_0 is sufficiently large:

$$t_{\max} = \sqrt{\frac{\delta}{P x_0}} + \mu(x_0) \quad (2.14)$$

where

$$\frac{3C}{2P} \sqrt{\frac{\delta}{P}} \frac{1}{7-2\alpha} \frac{1}{x_0^{\frac{7}{2}-\alpha}} \leq \mu(x_0) \leq \frac{3(D+1)}{P} \sqrt{\frac{\delta}{P}} \frac{1}{7-2\alpha} \frac{1}{x_0^{\frac{7}{2}-\alpha}}. \quad (2.15)$$

Proof.

(a) We consider $\dot{x}_0 > 0$. Also $\Omega = \{y_1\}$, and by Proposition (2.1) $y_1 \leq x_0$. Then $E - V(x, \delta) > 0$ and $\dot{x}(t) > 0$, $t \in [0, t_{\max})$.

Hence, $x(t)$ increases monotonically. Also

$$\lim_{x \rightarrow \infty} x^3 \left[1 - \left(1 + \frac{\delta}{x^3} \right)^{-1/3} \right] = \frac{\delta}{3}. \quad (2.16)$$

We consider (2.8) with positive sign on the right integral. By the previous statements, it follows that the left integral remains bounded and the right one is unbounded, as $t \rightarrow \infty$. This contradiction shows that $t_{\max} < \infty$.

To estimate t_{\max} we choose x_0 sufficiently large in the previous integral to apply Proposition (1.2). Using (2.16), taking the higher order terms in x_0 in the remaining expression and integrating, we obtain (2.14) and (2.15).

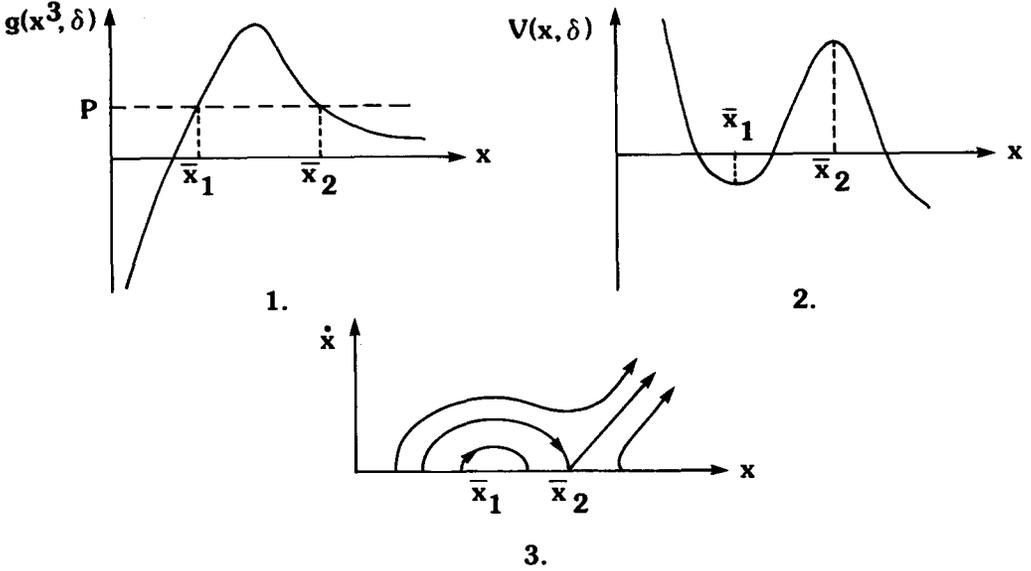
For any $x_0 > x_1$, we can proceed similarly.

If $\dot{x}_0 < 0$ the same arguments can be applied after $\dot{x}(t)$ changes sign in $x = y_1$.

The first case in (b) and (c) and (d.2) correspond to Proposition (2.1). The second case in (b) and (c) and (d.2) correspond to case (a) in the present Proposition. \square

Remark (2.3)

We suppose that (C2) and (D2) hold, and (ii) and (iii) are as in (2.4). Then, any $(x_0, \dot{x}_0) \in \mathcal{D}$ corresponds to one of the following situations:



1. $g(x^3, \delta)$, satisfying hypotheses (C2), (D1). 2. $V(x, \delta)$ for a given $P > 0$. 3. Phase-plane diagram.

For $V_0 < \bar{V}$,

- (a) If $E < V_0$, the conclusions of Proposition (2.4) hold.
- (b) If $\bar{V} > E > V_0$, either
 - (1) $x_0 < y_1$, then, the conclusions of Proposition (2.3) hold, or
 - (2) $x_0 > y_1$, then, the conclusions of Proposition (2.4) hold.
- (c) $E > \bar{V}$, the solutions are unbounded and blow up in finite time.

If $V_0 > \bar{V}$, then the conclusions of Proposition 2.4 hold for $E < V_0$ and (c) for $E > V_0$.

Remark (2.4)

We suppose that (C3) and (D1) hold. Then

- (i) If $P \leq C$, the conclusions of Proposition (2.2) hold,
- (ii) If $P \geq D$, the conclusions of Proposition (2.4) hold, where C and D are given in (1.29).

Section 3

The constitutive properties of the material enter the differential equation through the function $g(x^3, \delta)$.

Motivated by this fact we want to find which strain-energy function $W(x^3)$ corresponds to a given $g(x^3, \delta)$ for $\delta > 0$ and $W(1)$ fixed, i.e., we look for a solution of equation (1.27) under our previous hypotheses.

It is interesting to notice, however, that in the derivation of the differential equation

(1.24) and its energy integral (1.34), it is not essential to require the material to be hyper-elastic [6], [7].

We construct a solution to (1.27) via a finite difference equation [13] as follows.

We introduce new variables,

$$x^3 = 1 + e^y, \quad y \in (-\infty, \infty), \quad (3.1)$$

$$w(y) = W'(1 + e^y).$$

$$\mathcal{L} = \log(1 + \delta) > 0, \quad (3.2)$$

$$\gamma = \frac{\rho R_1^2}{2}.$$

We differentiate equation (1.27) with respect to x^3 . Using (3.1), (3.2), we obtain the finite difference equation

$$\begin{aligned} w(y) - w(y - \mathcal{L}) &= \gamma e^y g'(1 + e^y), \\ w(-\infty) &= \mu. \end{aligned} \quad (3.3)$$

(In this section we suppress the dependence of g on δ .) A solution of (3.3) is obtained as follows:

(1) We find a solution $w_N = w_N(y)$, $y \in \mathbb{R}$, for $N > 1$ with the following boundary conditions

$$w_N(y) = \mu, \quad y \in (-N, -N + \mathcal{L}). \quad (3.4)$$

We can write from equation (3.3) that

$$w_N(y + r\mathcal{L}) = w_N(y) + \gamma \sum_{j=1}^r e^{y+j\mathcal{L}} g'(1 + e^{y+j\mathcal{L}}) \quad (3.5)$$

and

$$w_N(y - r\mathcal{L}) = w_N(y) - \gamma \sum_{j=0}^{r-1} e^{y-j\mathcal{L}} g'(1 + e^{y-j\mathcal{L}}), \quad (3.6)$$

where r is a positive integer.

Now we consider (3.5). We let $z = y + r\mathcal{L}$. Then

$$w_N(z) = w_N(z - r) + \gamma \sum_{j=1}^r e^{z+(j-r)\mathcal{L}} g'(1 + e^{z+(j-r)\mathcal{L}}). \quad (3.7)$$

If we take

$$z \geq -N + \mathcal{L}$$

and

$$r = \frac{1}{\mathcal{L}} [z + N], \quad (\text{the brackets mean the integer part})$$

then $-N \leq z - r\mathcal{L} \leq -N + \mathcal{L}$.

Therefore with (3.4), (3.7) becomes

$$w_N(z) = \mu + \gamma \sum_{j=1}^r e^{z+(j-r)\mathcal{L}} g'(1 + e^{z+(j-r)\mathcal{L}}),$$

$$z \geq -N + \mathcal{L}.$$

Similarly, for (3.6) using $z = y - r\mathcal{L}$, we obtain

$$w_N(z) = \mu - \gamma \sum_{j=0}^{r-1} e^{z+\mathcal{L}(r-j)} g'(1 + e^{z+\mathcal{L}(r-j)}), \quad z \leq -N \quad (3.9)$$

and

$$r = \left\lceil -\frac{1}{\mathcal{L}}(N+z) + 1 \right\rceil.$$

(3.8) and (3.9) give the solution (3.3) satisfying (3.4) for $z \in \mathbb{R}$.

(2) We take limits in (3.4) as $N \rightarrow \infty$:

$$\mu = w(-\infty); \quad (3.10)$$

$$w(z) = \mu + \gamma \sum_{K=0}^{\infty} e^{z-K\mathcal{L}} g'(1 + e^{z-K\mathcal{L}}), \quad z \in (-\infty, \infty). \quad (3.11)$$

Letting $N \rightarrow \infty$ in (3.9), we again get (3.10).

It is easy to check that w given by (3.11) is the unique solution of (3.3). With the original variables, (3.11) becomes

$$W'(x^3) = \mu + \gamma(x^3 - 1) \sum_{j=0}^{\infty} (1 + \delta)^{-j} g' \left(1 + \frac{x^3 - 1}{(1 + \delta)^j} \right). \quad (3.12)$$

Now we want to prove uniform convergence of this series in order to integrate it term by term, following classical results in analysis.

Let us consider $N \geq 1$ and $M \in \mathbb{R}$ large enough, and a fixed $x \leq M$. Using the mean value theorem we find that, for a fixed j

$$g' \left(1 + \frac{x^3 - 1}{(1 + \delta)^j} \right) - g'(1) = g''(\xi) \frac{x^3 - 1}{(1 + \delta)^j}, \quad (3.13)$$

where

$$\xi \in \left[1, \frac{x^3 - 1}{(1 + \delta)^j} \right].$$

We let

$$K = K(x^3, N) = \text{Sup} |g''(\xi)| \quad (3.14)$$

$$\xi \in \left[1, \frac{x^3 - 1}{(1 + \delta)^N} \right]$$

and

$$\bar{K} = \sup_{x \leq M} K(x^3, N) = \bar{K}(N).$$

For $j \geq N$,

$$|g''(\xi)| \leq K$$

and

$$\left| g' \left(1 + \frac{x^3 - 1}{(1 + \delta)^j} \right) \right| \leq g'(1) + K \frac{x^3 - 1}{(1 + \delta)^j}. \quad (3.16)$$

We let

$$W'_n(x^3) = \mu + \gamma(x^3 - 1) \sum_{j=0}^n (1 + \delta)^{-j} g' \left(1 + \frac{x^3 - 1}{(1 + \delta)^j} \right).$$

Now, for $n \geq N$,

$$\begin{aligned} |W'(x^3) - W'_n(x^3)| &\leq \frac{\gamma(x^3 - 1)}{\delta} \left[\frac{g'(1)}{(1 + \delta)^{N-1}} + \frac{(x^3 - 1)K}{(\delta + 2)(1 + \delta)^{2N-2}} \right] \\ &\leq \frac{\gamma(M^3 - 1)}{\delta} \left[\frac{g'(1)}{(1 + \delta)^{N-1}} + \frac{\bar{K}(M^3 - 1)}{(\delta + 2)(1 + \delta)^{2N-2}} \right] \\ &= \epsilon(N), \end{aligned} \quad (3.17)$$

i.e., given $\epsilon > 0$ there exists an N such that the above difference is less than ϵ for $n \geq N$. \square

Hence integrating (3.12) for $x \leq M$, we get

$$W(x^3) = \mu x^3 + \gamma \sum_{j=0}^{\infty} \int_1^{x^3} \left[g \left(1 + \frac{x^3 - 1}{(1 + \delta)^j} \right) - g \left(1 + \frac{s - 1}{(1 + \delta)^j} \right) \right] ds \quad (3.18)$$

We have thus proved the following:

Proposition (3.1)

For a given $g(x^3, \delta)$ satisfying hypotheses (A), (B) and for fixed constants $\delta > 0, \mu \in \mathbb{R}$, the integral equation

$$g(x^3, \delta) = \frac{2}{\rho R_1^2} \int_{\frac{x^3 + \delta}{1 + \delta}}^{x^3} \frac{W'(\xi)}{\xi - 1} d\xi, \quad W(1) = \mu \quad (3.19)$$

has a unique C^2 solution $W(\xi)$, given by (3.18).

Next we want to find what asymptotic growth of $W(x^3)$ is implied by hypotheses (C) and (D) on $g(x^3, \delta)$.

Proposition (3.2)

The asymptotic growth of $W(x^3)$ for x large, is as follows:

(1) If hypotheses (C1) holds, then

$$x^\alpha \frac{(1+\delta)^{(\alpha-3)/3}}{(1+\delta)^{(\alpha-3)/3} - 1} \left(D - \frac{3C}{\alpha} \right) \geq \frac{1}{\gamma} W(x^3) \geq x^\alpha \frac{(1+\delta)^{(\alpha-3)/3}}{(1+\delta)^{(\alpha-3)/3} - 1} \left(C - \frac{3D}{\alpha} \right). \quad (3.20)$$

(2) If (C2) holds, then

$$\begin{aligned} \frac{1}{\gamma} W(x^3) &\geq x^\alpha \left(C - \frac{3D}{\alpha} \right) \frac{(1+\delta)^{(3-\alpha)(N+1)/3} - 1}{(1+\delta)^{(3-\alpha)/3} - 1} + \frac{\mu}{\gamma} x^3 \\ &\leq x^\alpha \left(D - \frac{3C}{\alpha} \right) \frac{(1+\delta)^{(3-\alpha)(N+1)/3} - 1}{(1-\delta)^{(3-\alpha)/3} - 1} + \frac{\mu}{\gamma} x^3 + \epsilon, \end{aligned} \quad (3.21)$$

where

$$\epsilon = \epsilon(x^3, N) = k_1(x^3, N) \frac{(x^3 - 1)^2}{\delta(1+\delta)^{N-1}}$$

and

$$\begin{aligned} k_1(x^3, N) &= \sup |g'(\xi)| \\ \xi &\in \left[1, 1 + \frac{x^3 - 1}{(1+\delta)^N} \right]. \end{aligned}$$

(3) If (C3) holds, then

$$\frac{\mu}{\gamma} x^3 + (D - C)x^3 \geq \frac{1}{\gamma} W(x^3) \geq (C - D)x^3 + \frac{\mu}{\gamma} x^3 \quad (3.22)$$

Proof.

(1), (3) follow from (3.18) using (C1), (C3), respectively. (2) We approximate the terms with $j \geq N$, in (3.18) for a N sufficiently large by $\epsilon(x^3, N)$, and we use (C2). It is important to notice that k_1 tends to $g'(1)$ as N tends to ∞ . \square

Similarly, we can state, for x small, the following.

Proposition (3.3)

We suppose that hypotheses (D) hold. Then the asymptotic growth of $W(x^3)$, for x small is

$$\frac{N}{2} x^{-\beta} \leq \frac{1}{\gamma} W(x^3) \leq (1 + M)x^{-\beta}, \quad (3.23)$$

where M, N, β are given in hypotheses (D).

Now we want to discuss the meaning of hypotheses (B) in terms of constitutive restrictions on the strain-energy function. Such restrictions have been widely studied [6], [7], [8], [9], [10], [17].

Strong ellipticity or Hadamard's condition [9] for the stability of equilibrium solutions is in some cases physically reasonable. Necessary and sufficient conditions on the isotropic strain-energy function for strong ellipticity to hold are known [12] in two dimensions but not in three, and this makes difficult the application of such a condition in specific problems.

It is well known [6], [8], [17], that a convexity of $W(F)$ is not physically reasonable in compressible elasticity because it implies the uniqueness of equilibrium solutions to mixed boundary-value problems. It is, however, a good condition in incompressible elasticity; the strain energy function for a Neo-Neocean material, $W(F) = \text{tr}(FF^T)$, is convex.

It is easy to check that strong ellipticity applied to our problem implies that $\sigma'(\lambda) > 0$ for $\lambda > 1$ and $\sigma'(\lambda) < 0$ for $\lambda < 1$. From this, (B) follows.

We also can state the following.

Proposition (3.4)

We suppose that $W(F)$ is convex for $F \in M^{3 \times 3}$ such that $\det F = 1$.

If $\sigma(\lambda_1, \lambda_2, \lambda_3)$ is the corresponding isotropic strain-energy function, then

$$\sigma(\lambda) = \sigma(\lambda, \lambda^{-1/2}, \lambda^{-1/2}) \text{ is convex on } \mathbb{R}^+.$$

Proof.

It follows from [8] that $W(F)$ convex on $M^{3 \times 3}$ implies that the isotropic strain-energy function of the principal stretches $\sigma(\lambda_1, \lambda_2, \lambda_3) = W(F)$ is convex and non-decreasing in each variable. Therefore,

$$\begin{aligned} \sigma(\alpha\lambda + (1-\alpha)\mu) &= \sigma(\alpha\lambda + (1-\alpha)\mu, (\alpha\lambda + (1-\alpha)\mu)^{-1/2}, (\alpha\mu + (1-\alpha)\mu)^{-1/2}) \\ &\leq \alpha\sigma(\lambda, \lambda^{-1/2}, \lambda^{-1/2}) + (1-\alpha)\sigma(\mu, \mu^{-1/2}, \mu^{-1/2}) \\ &= \alpha\sigma(\lambda) + (1-\alpha)\sigma(\mu). \end{aligned}$$

So, $\sigma(\lambda)$ is convex. □

Later, we are going to produce specific examples to show that some of the phase-plane results obtained in Section 2 can occur under convexity of $\sigma(\lambda)$.

Section 4

The purpose of this section is to study the stability properties of the equilibrium solutions in critical cases. We consider a $\bar{x} > 1$ and $P > 0$ such that

$$g(\bar{x}^3, \delta) = P \quad \text{and} \quad g'(\bar{x}^3, \delta) = 0.$$

The eigenvalues of the linearised equation are $\lambda = 0 = -\mu$, and therefore, the arguments used in Proposition (1.1) do not hold. Following [18], [19] we will compute local invariant manifolds through the equilibrium point.

These values P_{crit} of P are called “critical pressures” or “bifurcating pressures”. Under certain conditions the number of equilibrium solutions decreases by two as P passes through P_{crit} . This problem has been previously studied mainly from a static point of view, cf. [11].

We want to show the existence of an invariant manifold through $(\bar{x}, 0)$ and also that this manifold can be computed to any degree of accuracy given hypotheses of smoothness on $g(x^3, \delta)$.

Definition: A set $S \subset \mathbb{R}^2$ is an invariant manifold for (1.31) if for any $(x_0, \dot{x}_0) \in S$ the solution $(x(t), \dot{x}(t))$ of (1.31) starting at (x_0, \dot{x}_0) is such that $(x(t), \dot{x}(t)) \in S, t \in [0, T]$ for some $T > 0$ sufficiently small.

Proposition (4.1)

For a given $P > 0$, let $\bar{x} > 1$ be a solution of $g(\bar{x}^3, \delta) = P$ such that $g'(\bar{x}^3, \delta) = 0$ and $g''(\bar{x}^3, \delta) \neq 0, g$ smooth. Then, there exists an invariant manifold, $x_2 = h(x_1)$ through $(\bar{x}, 0)$ defined for $x_1 \in [\bar{x}, \bar{x} + a)$, if $\mathcal{L}(x) \geq 0$ for $x \in [\bar{x}, \bar{x} + a)$, for a given $a > 0$, and for $x_1 \in (\bar{x} - b, \bar{x}]$, if $\mathcal{L}(x) \leq 0$ for $x \in (\bar{x} - b, \bar{x}]$, for a given $b > 0$.

In this case,

$$x_2 = h^\pm(x_1) = \pm \left\{ e^{\int_{\bar{x}}^{x_1} r(x) dx} \int_{\bar{x}}^{x_1} e^{\int_{\bar{x}}^s r(u) du} \mathcal{L}(s) ds \right\}^{1/2}, \tag{4.1}$$

where

$$r(x) = \frac{1}{x \left[1 - \left(1 + \frac{\delta}{x^3} \right)^{-1/3} \right]} \left[\frac{\delta}{x^3} \left(1 + \frac{\delta}{x^3} \right)^{-4/3} + 3 \left(1 + \frac{\delta}{x^3} \right)^{-1/3} - 3 \right], \tag{4.2}$$

$$\mathcal{L}(x) = \frac{1}{x \left[1 - \left(1 + \frac{\delta}{x^3} \right)^{-1/3} \right]} [P - g(x^3, \delta)].$$

Proof.

Let $x_2 = h(x_1)$ be an invariant manifold for equation (1.30). Then

$$\dot{x}_2 = h'(x_1)\dot{x}_1 = h(x_1)h'(x_1) = \frac{1}{2} \frac{d}{dx_1} (h^2(x_1)), \tag{4.3}$$

$$\frac{d}{dx_1} [h^2(x_1)] = \mathcal{L}(x_1) + r(x_1)h^2(x_1). \tag{4.4}$$

We let $y(x_1) = h^2(x_1)$.

Then, we can write the previous equation as

$$\frac{d}{dx_1} y(x_1) = r(x_1)y(x_1) = \mathcal{L}(x_1)$$

and (4.5)

$$Y(\bar{x}) = 0.$$

Using the variation of constants formulae, we can write the solution of the above equation as

$$Y(x_1) = e^{\int_{\bar{x}}^{x_1} r(x) dx} \int_{\bar{x}}^{x_1} e^{-\int_{\bar{x}}^s r(u) du} \mathcal{L}(s) ds. \quad (4.6)$$

The conclusion follows by considering the sign of $y(x_1)$.

Proposition (4.2)

Let the above hypotheses hold. Let $\epsilon > 0$ be a given constant. The invariant manifold through $(\bar{x}, 0)$ is given locally as follows:

- (1) If $g''(\bar{x}^3, \delta) < 0$, then $h^\pm(x_1)$ is defined for x_1 such that $0 \leq x_1 - \bar{x} < \epsilon$.
- (2) If $g''(\bar{x}^3, \delta) > 0$ then $h^\pm(x_1)$ is defined for x_1 such that $0 \leq \bar{x} - x_1 < \epsilon$.

In both cases

$$h^\pm(x_1) = C(\bar{x})|x_1 - \bar{x}|^{3/2} + O(\epsilon^{3/2}), \quad (4.7)$$

where

$$C(\bar{x}) = \sqrt{\frac{3}{2}} \bar{x}^{3/2} \frac{|g''(\bar{x}^3, \delta)|^{1/2}}{\left[1 - \left(1 + \frac{\delta}{\bar{x}^3}\right)^{-1/3}\right]^{1/2}}. \quad (4.8)$$

Proof.

We consider (4.6) for x_1 such that $|x_1 - x_0| < \epsilon$. We compute all the integrals involved in (4.6) developing $\mathcal{L}(s)$ in a power series about $x_1 = \bar{x}$. The result follows by considering both possible signs for $g''(\bar{x}^3, \delta)$ and using Proposition (4.1). □

Proposition (4.3)

Let the previous hypotheses hold. Then $(\bar{x}, 0)$ is unstable.

Proof.

This follows by energy arguments, considering a solution corresponding to $E = V(\bar{x}, \delta)$ starting near $(\bar{x}, 0)$, taking $\dot{x}_0 > 0$ and $\dot{x}_0 < 0$ successively, and considering both possible signs of $g''(\bar{x}^3, \delta)$. □

To get a more specific idea of the asymptotic behaviour of the solutions in the invariant manifold, we consider the following initial value problems:

$$\begin{aligned} \text{(a)} \quad \dot{x}_1 &= h^+(x_1), \quad \dot{x}_1(0) = h^+(u_0), \\ \text{(b)} \quad \dot{x}_1 &= h^-(x_1), \quad \dot{x}_1(0) = h^-(v_0) \end{aligned} \tag{4.9}$$

for given u_0, v_0 such that $|u_0 - \bar{x}| < \epsilon$, and $|v_0 - \bar{x}| < \epsilon$.

The solution of (4.9) for $|x_1 - \bar{x}| < \epsilon$ is as follows.

(1) If $g''(\bar{x}^3, \delta) < 0$, then:

In case (a),

$$+t = \frac{2}{C(\bar{x})} \left[\frac{1}{(u_0 - \bar{x})^{1/2}} - \frac{1}{(x_1 - \bar{x})^{1/2}} \right] + O(\epsilon^{1/2})$$

$$\text{and } \lim_{t \rightarrow -\infty} x_1 = \bar{x}_+.$$

In case (b),

$$-t = \frac{2}{C(\bar{x})} \left[\frac{1}{(v_0 - \bar{x})^{1/2}} - \frac{1}{(x_1 - \bar{x})^{1/2}} \right] + O(\epsilon^{1/2})$$

$$\text{and } \lim_{t \rightarrow \infty} x_1 = \bar{x}_+.$$

(2) If $g''(\bar{x}^3, \delta) > 0$, then:

In case (a),

$$t = \frac{2}{C(\bar{x})} \left[-\frac{1}{(\bar{x} - u_0)^{1/2}} + \frac{1}{(\bar{x} - x_1)^{1/2}} \right] + O(\epsilon^{1/2})$$

$$\text{and } \lim_{t \rightarrow +\infty} x_1 = \bar{x}_-.$$

In case (b),

$$-t = \frac{2}{C(\bar{x})} \left[-\frac{1}{(\bar{x} - v_0)^{1/2}} + \frac{1}{(\bar{x} - x_1)^{1/2}} \right] + O(\epsilon^{1/2})$$

$$\text{and } \lim_{t \rightarrow -\infty} x_1 = \bar{x}_-.$$

Section 5

The membrane approximation.

The differential equation describing the radial motion of a membrane is given by [3], [6], [7], [21], [22].

$$\ddot{x} + \frac{3}{\rho R_1^2} x^2 W'(x^3) = \frac{3}{2} p x^2, \tag{5.1}$$

This equation is obtained by approximating (1.24) up to the first order in δ [(1.22)]. For the thin shell, δ is proportional to the thickness e of the shell: $\delta = 3(e/R_1)$.

The same equation is obtained using a membrane approximation up to the first order in δ [11]. In this case x is the average of the tangential stretch divided by the initial thickness of the shell. The energy equation is

$$\dot{x}^2 + V(x) = E, \quad (5.2)$$

where

$$V(x) = \frac{2}{\rho R_1^2} W(x^3) - px^3 \quad (5.3)$$

is the potential energy.

The equilibrium solutions of (5.1) are the roots of the equation

$$\frac{2}{\rho R_1^2} W'(x^3) = p. \quad (5.4)$$

Results similar to Proposition (2.2), (2.3), (2.4) can be obtained by exactly the same method under suitable assumptions on $W'(x^3)$. In (5.4) p is measured per unit of thickness.

Next we show that in the neighbourhood of an equilibrium solution for the membrane we can find equilibrium solutions for sufficiently thin shells, and also that dynamic solutions for the membrane are the uniform limits of solutions for the thick shell as δ tends to zero.

Proposition (5.1)

We suppose that:

- (i) $x_0 > 1$ is a solution of (5.4) for a given $p > 0$,
- (ii) $W''(x_0^3) \neq 0$,
- (iii) Hypotheses (A) and (B) hold for $W'(x^3)$.

Then, there exists an interval $(0, \delta_1)$ and a function $x = x(\delta) \in C^2(0, \delta_1)$ the values of which give equilibrium solutions for a shell with thickness parameter δ corresponding to an external pressure $P = \delta p$.

Proof.

We are going to apply the implicit function theorem to

$$h(y, \delta) = \frac{g(y, \delta)}{\delta}.$$

Let $y = x^3$. From

$$h(y, 0) = \lim_{\delta \rightarrow 0} \frac{g(y, \delta)}{\delta} = \frac{2}{\rho R_1^2} W'(y), \quad (5.5)$$

and with (i), it follows that

$$h(x_0^3, 0) - p = 0. \quad (5.6)$$

It is easy to check that

$$\lim_{\delta \rightarrow 0} \frac{d}{dy} h(y, \delta) = \frac{2}{\rho R_1^2} y W''(y).$$

In particular,

$$\frac{d}{dx^3} h(x_0^3, 0) = \lim_{\delta \rightarrow 0} \frac{d}{dy} h(y, \delta)|_{y=x_0^3} = \frac{2}{\rho R_1^2} x_0^3 W''(x_0^3) \neq 0. \quad (5.7)$$

By the implicit function theorem, there exists a $\delta_1 > 0$ and a $x \in C^2 [0, \delta_1)$ such that

$$x(0) = x_0, \quad x(\delta) > 1$$

and

$$h(x^3(\delta), \delta) = p, \text{ i.e.}$$

$$g(x^3(\delta), \delta) = \frac{2}{\rho R_1^2} \int_{\frac{x^3(\delta)+\delta}{1+\delta}}^{x^3(\delta)} \frac{W'(\xi)}{\xi-1} d\xi = p\delta.$$

Hence, $x = x(\delta)$ is an equilibrium solution for a shell of thickness parameter δ and $P = p\delta$.

Remark. The interval $(0, \delta_1)$ can be extended up to a δ_{\max} .

$$\frac{d}{dx^3} \left(\frac{g(x^3, \delta_{\max})}{\delta_{\max}} \right) = 0.$$

If $W''(x_0^3) = 0$, the previous result for this x_0 cannot be guaranteed.

Proposition (5.2)

Let $E, x_0 > 0, \delta > 0, p > 0, T > 0$ be fixed constants, and let $x = x(t, \delta)$ be the solution of (1.34) for $0 \leq t \leq T$ with energy $\delta E/3$, and $x(0, \delta) = x_0$. Then

$$\lim_{\delta \rightarrow 0} x(t, \delta) = x(t), \quad t < T,$$

where $x(t)$ is the solution of (5.2) with $x(0) = x_0$ and energy E .

Proof.

Let $\dot{x}_0 > 0$. We consider $t_1 < T$ such that $E - V(x(t_1)) = 0$. Then $E - V(x) > 0$ for $t < t_1$. The result for $x(t), t \in [0, t_1], x(t, \delta), t \in [0, t_1 - \epsilon], \epsilon > 0$ follows by considering (5.2), (5.3), (1.34), (1.36) and also that

$$\lim_{\delta \rightarrow 0} \frac{3}{\delta} V(x, \delta) = V(x).$$

For $t > t_1$, we consider the minus sign in the square root of (1.34) and (5.2). If there exists

t_2 such that $E - V(x(t_2)) = 0$, then the result follows by the same arguments for $x(t)$, $t \in [t_1, t_2]$ and $x(t, \delta)$, $t \in [t_1 + \epsilon, t_2 - \epsilon]$. We proceed similarly up to $t = T$. This also holds for $\dot{x}_0 < 0$. \square

Remark. It follows from (5.4) that a necessary condition for the existence of more than one equilibrium solution for appropriate values of p is that

$$W''(x^3) = 0 \quad (5.9)$$

for at least one $x > 1$.

In terms of $\sigma(\lambda)$ the above condition says that

$$\sigma''(\lambda) = -\frac{10}{4\lambda} \sigma'(\lambda) \quad (5.10)$$

must have a least one solution for $\lambda < 1$. It is interesting to note that the assumption of convexity of $\sigma(\lambda)$ does not contradict (5.10).

We are going to give examples for a class of materials studied by Ogden [20] satisfying $\sigma'' > 0$. Moreover, by choosing the appropriate growth conditions we can obtain two of the cases studied in Section 2.

The materials studied by Ogden have the following isotropic stored energy function

$$\begin{aligned} \sigma(\lambda_1, \lambda_2, \lambda_3) = & \sum_{i=1}^N \alpha_i (\lambda_1^{\mu_i} + \lambda_2^{\mu_i} + \lambda_3^{\mu_i} - 3) \\ & + \sum_{j=1}^M \beta_j ((\lambda_1 \lambda_2)^{\nu_j} + (\lambda_1 \lambda_3)^{\nu_j} + (\lambda_2 \lambda_3)^{\nu_j} - 3). \end{aligned}$$

For the radial deformation under study,

$$\begin{aligned} \sigma(\lambda) \equiv \sigma(\lambda, \lambda^{-1/2}, \lambda^{-1/2}) = & \sum_{j=1}^M \alpha_j (\lambda^{\mu_j} + 2\lambda^{-\mu_j/2} - 3) \\ & + \sum_{i=1}^N \beta_i (2\lambda^{\nu_i/2} + \lambda^{-\nu_i} - 3). \end{aligned}$$

Hypotheses (C1) and (D1) for $W'(\xi)$:

$$(1) \quad N = 1, \quad M = 0$$

$$\alpha = 1, \quad \mu = 3/2$$

A root of (5.10) in $(0, 1)$ is $\lambda = 1/256^{1/9}$.

$$(2) \quad N = 3, \quad M = 0$$

$$\alpha_1 = 0.153, \quad \alpha_2 = -0.092, \quad \alpha_3 = 0.5$$

$$\mu_1 = 2.8, \quad \mu_2 = 2.5, \quad \mu_3 = 2.0$$

Two of the roots of (5.10) for $\lambda < 1$ are $\lambda_1 = 0.5$, $\lambda_2 = 0.45$.

Hypotheses (D1) and (C2) for $W'(\xi)$:

$$(1) \quad N = 1, \quad M = 0$$

$$\alpha = 1, \quad \mu = \frac{3}{2}$$

A root of (5.10) in $(0, 1)$ is $\lambda = 1/256^{1/9}$.

$$(2) \quad N = 1, \quad M = 1$$

$$\alpha = 1, \quad \mu = \frac{5}{2}$$

$$\beta = 1, \quad \nu = 1$$

There is one root of (5.10) in $(0, 1)$.

Remark: For this class of materials hypothesis (D2) cannot be satisfied.

Section 6

The differential equation describing the radial motion of the cylindrical shell is

$$x \log \left(1 + \frac{\delta}{x^2} \right) \ddot{x} + \left[\log \left(1 + \frac{\delta}{x^2} \right) - \frac{\delta}{\delta + x^2} \right] \dot{x}^2 + f(x^2, \delta) = P,$$

where

$$f(x^2, \delta) = \frac{1}{\rho R_1^2} \int_{\frac{x^2+\delta}{1+\delta}}^{x^2} \frac{W'(\xi)}{\xi-1} d\xi.$$

and

$$x = \frac{r_1(t)}{R_1}, \quad \xi = \frac{r^2}{R^2}, \quad \delta = \frac{R_2^2}{R_1^2} - 1 > 0.$$

The energy equation is

$$\frac{1}{2} x^2 \log \left(1 + \frac{\delta}{x^2} \right) \dot{x}^2 + F(x^2, \delta) - \frac{1}{2} P x^2 = E$$

where

$$F(x^2, \delta) = \frac{1}{2} \int_1^{x^2} f(\lambda, \delta) d\lambda.$$

The hypotheses for $f(x^2, \delta)$ are the same as (1.29) except that x^2 replaces x^3 .

The phase-plane analysis yields similar results to the spherical shell with the exception that under conditions (C2) and (D1) for $f(x^2, \delta)$ and the other usual hypotheses, the solutions exist for all time and are of the following type:

(1) Periodic solutions under the same kind of initial data as in the spherical shell.

- (2) Unbounded solutions, with exponential growth for large values of t , under the analogous class of initial data that give blow-up for the spherical shell.

The remaining results are similar with the number 2 in this case playing the same role as 3 for the spherical shell.

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List of symbols

α	Ψ_1	D	\mathcal{L}
β	Ψ_{-1}	P, p	$g(\cdot)$
γ	0	\hat{n}	G
δ	ξ	x	t_{\max}
μ	σ	R, r	V, v
ν	ϵ	\mathbb{R}	$[\cdot]$
λ	\rightarrow	F, f	(\cdot)
θ, Θ	∞	J, j	x_0
ρ	∂	a	\dot{x}_0
ϕ, Φ	B	h	\S
Ω	C, c	\mathcal{L}	T, t, \mathbf{t}