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► **To cite this version:**

| Aernout van Enter, Arnaud Le Ny. Decimation of the Dyson-Ising Ferromagnet. 2016. hal-01290716

HAL Id: hal-01290716

<https://hal.science/hal-01290716>

Preprint submitted on 18 Mar 2016

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Decimation of the Dyson-Ising Ferromagnet

Aernout van Enter* and Arnaud Le Ny†

March 17, 2016

Abstract:

We study the decimation to a sublattice of half the sites of the one-dimensional Dyson-Ising ferromagnet with slowly decaying long-range pair potentials of the form $\frac{1}{|i-j|^\alpha}$, in the phase transition region ($1 < \alpha \leq 2$ and low temperature). We prove non-Gibbsianness of the decimated measures at low enough temperatures by exhibiting a point of essential discontinuity for the (finite-volume) conditional probabilities of decimated Gibbs measures. This result complements previous work proving conservation of Gibbsianness for fastly decaying potentials ($\alpha > 2$) and provides an example of a "standard" non-Gibbsian result in one dimension, in the vein of similar results in higher dimensions for short-range models. We also discuss how these measures could fit within a generalized (almost vs. weak) Gibbsian framework.

AMS 2000 subject classification: Primary- 60K35 ; secondary- 82B20

Keywords and phrases: Long-range Ising models, hidden phase transitions, generalized Gibbs measures.

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1 Introduction

In this paper, we focus on properties of transformed equilibrium measures of one-dimensional Ising models with long-range, polynomially decaying, pair interactions called *Dyson-Ising models* or sometimes shortly *Dyson Models*. These models display a phase transition at low temperature, for appropriate values of the decay parameter. Varying this decay parameter plays a similar role as varying the dimension in short-range models. This can be done in a continuous manner, so one has analogues of well-defined models in continuously varying non-integer dimensions, which is a major reason why these models have attracted a lot of attention in the study of phase transitions and critical behaviour (see e.g. [5] and references therein). Here we show that under a decimation transformation the low-temperature measures of the Dyson models are mapped to non-Gibbsian measures, similarly to what happens for short-range interactions in higher dimensions. We also discuss possible extensions within the generalized Gibbs framework and some related issues.

The paper is organized as follows. In Section 2, we describe the standard DLR approach to Gibbs measures in mathematical statistical mechanics, introduce our long-range Dyson-Ising models and introduce "global specifications". In Section 3, we introduce the decimation transformation – an elementary renormalization transformation that keeps odd or even spins only – and prove non-Gibbsianness at low temperature for the decimated Dyson-Ising models whose interactions are so slowly decaying that a phase transition occurs. Eventually, in Section 4, we extend previous results to show that this decimated measure is included in the class of Almost Gibbsian measures, and comment on some related issues.

2 Gibbs Measures, Background and Notation

2.1 Specifications and Measures

We thus deal with long-range ferromagnetic Ising models with pair interactions in one dimension. They are part of the more general class of lattice (spin) models with Gibbs measures, as discussed for example in [14, 23, 24, 11]. The finite-spin state space is the usual Ising space (E, \mathcal{E}, ρ_0) with $E = \{-1, +1\}$, $\mathcal{E} = \mathcal{P}(\{-1, +1\})$ and the a priori counting measure $\rho_0 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$. We denote by \mathcal{S} the set of the finite subsets of \mathbb{Z} and, for any $\Lambda \in \mathcal{S}$, write $(\Omega_\Lambda, \mathcal{F}_\Lambda, \rho_\Lambda)$ for the finite-volume configuration space $(E^\Lambda, \mathcal{E}^{\otimes \Lambda}, \rho_0^{\otimes \Lambda})$. At infinite volume, configurations are denoted by σ, ω , etc., lying in an infinite-volume *configuration space*, the infinite-product probability space $(\Omega, \mathcal{F}, \rho) = (E^{\mathbb{Z}}, \mathcal{E}^{\otimes \mathbb{Z}}, \rho_0^{\otimes \mathbb{Z}})$, equipped with the product topology of the discrete topology on E . For this topology, continuous functions coincide with *quasilocal* functions, that is, uniform limits of local functions, the latter being \mathcal{F}_Λ -measurable functions for some $\Lambda \in \mathcal{S}$. A function is said to be *right-continuous* (resp. *left-continuous*) when for every $\omega \in \Omega$, $\lim_{\Lambda \uparrow \mathcal{S}} f(\omega_\Lambda + \Lambda^c) = f(\omega)$ (resp. $\lim_{\Lambda \uparrow \mathcal{S}} f(\omega_\Lambda - \Lambda^c) = f(\omega)$), where one writes ω_Λ for its projection on Ω_Λ , and $+$ (resp. $-$) for the configurations whose value are respectively $+1$ (resp. -1) everywhere. We also generically consider infinite subsets $S \subset \mathbb{Z}$, for which all the preceding notations defined for finite Λ extend naturally ($\Omega_S, \mathcal{F}_S, \rho_S, \sigma_S$, etc.). Important events to be considered are the *asymptotic events*, which are the elements of the *tail σ -algebra* $\mathcal{F}_\infty = \bigcap_{\Lambda \in \mathcal{S}} \mathcal{F}_{\Lambda^c}$. These events typically do not depend on local behaviors and are mostly obtained by some limiting procedure.

Within the product topology, configurations are close when they coincide on large finite regions Λ , and the larger the region, the closer they are. For a given configuration $\omega \in \Omega$, a

neighborhood base is thus provided by the family $(\mathcal{N}_\Lambda(\omega))_{\Lambda \in \mathcal{S}}$ with, for any $\Lambda \in \mathcal{S}$,

$$\mathcal{N}_\Lambda(\omega) = \left\{ \sigma \in \Omega : \sigma_\Lambda = \omega_\Lambda, \sigma_{\Lambda^c} \text{ arbitrary} \right\}.$$

We also consider particular open subsets of neighborhoods $\mathcal{N}_\Lambda(\omega)$ on which the configuration is + (resp. -) on an annulus $\Delta \setminus \Lambda$ for $\Delta \supset \Lambda$, defined for all $\Lambda \in \mathcal{S}$, $\omega \in \Omega$ as

$$\begin{aligned} \mathcal{N}_{\Lambda, \Delta}^+(\omega) &= \left\{ \sigma \in \mathcal{N}_\Lambda(\omega) : \sigma_{\Delta \setminus \Lambda} = +_{\Delta \setminus \Lambda}, \sigma_{\Delta^c} \text{ arbitrary} \right\} \\ \mathcal{N}_{\Lambda, \Delta}^-(\omega) &= \left\{ \sigma \in \mathcal{N}_\Lambda(\omega) : \sigma_{\Delta \setminus \Lambda} = -_{\Delta \setminus \Lambda}, \sigma_{\Delta^c} \text{ arbitrary} \right\}. \end{aligned}$$

We denote by $C(\Omega)$ the set of continuous functions on Ω . In our finite state-space set-up, continuity is equivalent to uniform continuity and to *quasilocality*¹, so that one has

$$f \in C(\Omega) \iff \lim_{\Lambda \uparrow \mathcal{S}} \sup_{\sigma, \omega: \sigma_\Lambda = \omega_\Lambda} |f(\omega) - f(\sigma)| = 0. \quad (2.1)$$

States are represented by the set \mathcal{M}_1^+ of probability measures on the configuration space $(\Omega, \mathcal{F}, \rho)$. To describe such measures on the infinite product space Ω in a way that would *not* necessarily lead to uniqueness, and thereby allow to mathematically describe phase transitions, Dobrushin [7] and Lanford/Ruelle [33] introduced in the late 60's an approach where a measure is required to have prescribed conditional probabilities w.r.t. the outside of *finite* sets. Such a system of conditional probabilities extended to be defined *everywhere*, rather than almost everywhere, is called a specification.

Definition 1 (Specification) A specification $\gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ on (Ω, \mathcal{F}) is a family of probability kernels $\gamma_\Lambda : \Omega \times \mathcal{F} \rightarrow [0, 1]$; $(\omega, A) \mapsto \gamma_\Lambda(A | \omega)$ s.t. for all $\Lambda \in \mathcal{S}$:

1. For all $\omega \in \Omega$, $\gamma_\Lambda(\cdot | \omega)$ is a probability measure on (Ω, \mathcal{F}) .
2. For all $A \in \mathcal{F}$, $\gamma_\Lambda(A | \cdot)$ is \mathcal{F}_{Λ^c} -measurable.
3. For all $\omega \in \Omega$, $\gamma_\Lambda(B | \omega) = \mathbf{1}_B(\omega)$ when $B \in \mathcal{F}_{\Lambda^c}$.
4. For all $\Lambda \subset \Lambda' \in \mathcal{S}$, $\gamma_{\Lambda'} \gamma_\Lambda = \gamma_{\Lambda'}$ where

$$\forall A \in \mathcal{F}, \forall \omega \in \Omega, (\gamma_{\Lambda'} \gamma_\Lambda)(A | \omega) = \int_{\Omega} \gamma_\Lambda(A | \sigma) \gamma_{\Lambda'}(d\sigma | \omega). \quad (2.2)$$

These kernels also act on functions and on measures: for all $f \in C(\Omega)$ or $\mu \in \mathcal{M}_1^+$,

$$\gamma_\Lambda f(\omega) := \int_{\Omega} f(\sigma) \gamma_\Lambda(d\sigma | \omega) = \gamma_\Lambda[f | \omega] \text{ and } \mu \gamma_\Lambda[f] := \int_{\Omega} (\gamma_\Lambda f)(\omega) d\mu(\omega) = \int_{\Omega} \gamma_\Lambda[f | \omega] \mu(d\omega).$$

These objects are designed to represent consistent systems of conditional probabilities, with the important additional property that they are defined everywhere and not only almost-surely as ordinarily conditional probabilities would have been required to be. However, as we do not have a measure to begin with, the notion of "almost surely" a priori does not make sense. For a given specification, different measures can then have their conditional

¹The continuous functions are the uniform limits of local functions, explaining the terminology *quasilocal* [14, 24].

probabilities represented by the same specification (and satisfy the *DLR equations* (2.3)) but live on different full-measure sets. This leaves the door open to a mathematical description of phase transitions, which is well known e.g. for the ferromagnetic (n.n.) Ising model on the square lattice \mathbb{Z}^2 [27], but also for our long-range Ising models on the one-dimensional lattice \mathbb{Z} , see next section.

Definition 2 (DLR measures) *A probability measure μ on (Ω, \mathcal{F}) is said to be consistent with a specification γ (or specified by γ) when for all $A \in \mathcal{F}$ and $\Lambda \in \mathcal{S}$*

$$\mu[A|\mathcal{F}_{\Lambda^c}](\omega) = \gamma_{\Lambda}(A|\omega), \mu\text{-a.e. } \omega. \quad (2.3)$$

We denote by $\mathcal{G}(\gamma)$ the set of measures consistent with γ .

A specification is said to be quasilocal when the set of quasilocal functions is conserved by its kernels. More formally, for any local function, its image by the kernels constituting γ is a continuous function of the boundary condition :

$$\gamma \text{ quasilocal} \iff \gamma_{\Lambda}f \in C(\Omega) \text{ for any } f \text{ local (or any } f \text{ in } C(\Omega)). \quad (2.4)$$

A measure is said to be quasilocal when it is specified by a quasilocal specification. In fact, such quasilocal measures are very close to Gibbs measures, introduced next.

2.2 Gibbs and Quasilocal Measures

Gibbs measures were originally designed to represent equilibrium states, typically also satisfying a variational principle for a (formal) Hamiltonian H . The latter is defined *via* a potential Φ , i.e. a family $(\Phi_A)_{A \in \mathcal{S}}$ of local functions $\Phi_A \in \mathcal{F}_A$ that provide the contributions of spins in finite sets A to the total energy through the *finite-volume Hamiltonians* – or *Hamiltonians with free boundary conditions* – defined for all $\Lambda \in \mathcal{S}$ by

$$H_{\Lambda}(\omega) = \sum_{A \subset \Lambda} \Phi_A(\omega), \forall \omega \in \Omega. \quad (2.5)$$

To define Gibbs measures, we require Φ to be *Uniformly Absolutely Convergent* (UAC):

$$\forall i \in \mathbb{Z}, \sum_{A \ni i} \sup_{\omega} |\Phi_A(\omega)| < \infty. \quad (2.6)$$

For such a potential, one can give sense to the *Hamiltonian at volume $\Lambda \in \mathcal{S}$ with boundary condition ω* defined for all $\sigma \in \Omega$ as

$$H_{\Lambda}^{\Phi}(\sigma|\omega) := \sum_{A \cap \Lambda \neq \emptyset} \Phi_A(\sigma_{\Lambda} \omega_{\Lambda^c}) (< \infty). \quad (2.7)$$

The *Gibbs specification at inverse temperature $\beta > 0$* is then defined by

$$\gamma_{\Lambda}^{\beta\Phi}(\sigma|\omega) = \frac{1}{Z_{\Lambda}^{\beta\Phi}(\omega)} e^{-\beta H_{\Lambda}^{\Phi}(\sigma|\omega)} (\rho_{\Lambda} \otimes \delta_{\omega_{\Lambda^c}})(d\sigma) \quad (2.8)$$

where the normalization $Z_{\Lambda}^{\beta\Phi}(\omega)$ –the partition function– is a normalizing constant related to a free energy. Such a specification is *non-null²* and has the property that it is *quasilocal*, thanks

²In the sense that $\forall \Lambda \in \mathcal{S}, \forall A \in \mathcal{F}_{\Lambda}, \rho(A) > 0$ implies that $\gamma_{\Lambda}(A|\omega) > 0$ for any $\omega \in \Omega$.

to the convergence properties (2.6) of the defining potential (see e.g. [24, 37]). Gibbs measures are measures consistent with a Gibbs specification defined in terms of a UAC potential, but Kozlov [31] and Sullivan [49] established that **being Gibbs is in fact also equivalent to being non-null and quasilocal**. We take then the following

Definition 3 (Gibbs measures) $\mu \in \mathcal{M}_1^+$ is a Gibbs measure iff $\mu \in \mathcal{G}(\gamma)$, where γ is a non-null and quasilocal specification.

While non-nullness prevents hard-core exclusions and only allows a proper exponential factor to alter the product structure of the measure – to get correlated random fields –, quasilocality allows us to interpret Gibbs measures as natural extensions of the class of Markov fields³.

Indeed, when $\mu \in \mathcal{G}(\gamma)$ is quasilocal, then for any f local and $\Lambda \in \mathcal{S}$, the conditional expectations of f w.r.t. the outside of Λ are μ -a.s. given by $\gamma_\Lambda f$, by (2.2), and this is itself a continuous function of the boundary condition by (2.1). Thus, one gets for any ω

$$\lim_{\Delta \uparrow \mathbb{Z}} \sup_{\omega^1, \omega^2 \in \Omega} \left| \mu[f|\mathcal{F}_{\Lambda^c}](\omega_\Delta \omega_{\Delta^c}^1) - \mu[f|\mathcal{F}_{\Lambda^c}](\omega_\Delta \omega_{\Delta^c}^2) \right| = 0 \quad (2.9)$$

which yields an (almost-sure) asymptotically weak dependence on the conditioning, which can be seen as an extended Markov property. In particular, for Gibbs measures the conditional probabilities always have continuous versions, or equivalently there is no point of essential discontinuity, in the following sense:

Definition 4 (Essential discontinuity) A configuration ω is said to be a point of essential discontinuity for a conditional probability of $\mu \in \mathcal{M}_1^+$ if there exists $\Lambda_0 \in \mathcal{S}$, f local, $\delta > 0$, such that for all Λ with $\Lambda_0 \subset \Lambda$ there exist $\mathcal{N}_\Lambda^1(\omega)$ and $\mathcal{N}_\Lambda^2(\omega)$, two open (or at least positive-measure) neighborhoods of ω , such that

$$\forall \omega^1 \in \mathcal{N}_\Lambda^1(\omega), \forall \omega^2 \in \mathcal{N}_\Lambda^2(\omega), \left| \mu[f|\mathcal{F}_{\Lambda^c}](\omega^1) - \mu[f|\mathcal{F}_{\Lambda^c}](\omega^2) \right| > \delta$$

or equivalently

$$\lim_{\Delta \uparrow \mathbb{Z}} \sup_{\omega^1, \omega^2 \in \Omega} \left| \mu[f|\mathcal{F}_{\Lambda^c}](\omega_\Delta \omega_{\Delta^c}^1) - \mu[f|\mathcal{F}_{\Lambda^c}](\omega_\Delta \omega_{\Delta^c}^2) \right| > \delta. \quad (2.10)$$

In the generalized Gibbsian framework, one also says that such a configuration is a *bad configuration* for the considered measure, see e.g. [37]. The existence of such bad configurations implies non-Gibbsianness of the associated measures.

2.3 Dyson-Ising models: Ferromagnets in One Dimension

In our framework⁴, for a given probability measure μ , it is always possible to construct a specification γ such that $\mu \in \mathcal{G}(\gamma)$ (see e.g. Goldstein [25], Preston [43] or Sokal [48]). Nevertheless, even in such a framework, there exist specifications γ for which $\mathcal{G}(\gamma) = \emptyset$ (see e.g. [24, 37]), others where $\mathcal{G}(\gamma) = \{\mu\}$ but also – and this is more interesting for us – some for which this set contains more than one element. In the latter, we say in mathematical statistical mechanics that there is a *phase transition*. The set of DLR measures is then known to be a convex set whose extremal elements are trivial on the tail algebra \mathcal{F}_∞ . Any other element of $\mathcal{G}(\gamma)$ admits a unique⁵ convex combination of the extremal elements and is characterized by

³In fact Sullivan used the term of *Almost Markovian* instead of quasilocal in [49].

⁴Or more generally when the configuration space is *standard Borel*, see [24].

⁵It is a *Choquet simplex*, see [8, 24].

its action on the tail σ -field \mathcal{F}_∞ [11, 24]. We focus here on such a case in dimension one:

Definition 5 (Dyson-Ising model) *Let $\beta > 0$ be the inverse temperature, consider $1 < \alpha \leq 2$. We call Dyson-Ising specification with decay parameter α the Gibbs specification (2.8) with (pair-)potential Φ^D defined for all $\omega \in \Omega$ by*

$$\Phi_A^D(\omega) = -\frac{1}{|i-j|^\alpha} \omega_i \omega_j \text{ when } A = \{i, j\} \subset \mathbb{Z}, \text{ and } \Phi_A^D \equiv 0 \text{ otherwise.} \quad (2.11)$$

Remark: We also will make at various points use of the existence of a partial order (FKG) \leq on Ω : $\sigma \leq \omega$ if and only if $\sigma_i \leq \omega_i$ for all $i \in \mathbb{Z}$. Its maximal and minimal elements are the configurations $+$ and $-$, and this order extends to functions: $f : \Omega \rightarrow \mathbb{R}$ is called *monotone increasing* when $\sigma \leq \omega$ implies $f(\sigma) \leq f(\omega)$. It induces then a stochastic domination on probability measures on Ω for which we write $\mu \leq \nu$ if and only if $\mu[f] \leq \nu[f]$ for all f monotone increasing, where we denote $\mu[f]$ for the expectation $\mathbb{E}_\mu[f]$.

The Dyson-Ising specification is *monotonicity-preserving* (or *attractive*) in the sense that for all bounded increasing functions f , and $\Lambda \in \mathcal{S}$, the function $\gamma_\Lambda^D f$ is increasing.⁶ Using as boundary conditions the extremal (maximal and minimal) elements of this order \leq already allows to define the extremal elements of $\mathcal{G}(\gamma^D)$. Indeed, one can learn in e.g. [16, 28, 34] that

Proposition 1 *The weak limits*

$$\mu^-(\cdot) := \lim_{\Lambda \uparrow \mathbb{Z}} \gamma_\Lambda^D(\cdot | -) \text{ and } \mu^+(\cdot) := \lim_{\Lambda \uparrow \mathbb{Z}} \gamma_\Lambda^D(\cdot | +) \quad (2.12)$$

are well-defined, translation-invariant and extremal elements of $\mathcal{G}(\gamma^D)$. For any f bounded increasing, any other measure $\mu \in \mathcal{G}(\gamma^D)$ satisfies

$$\mu^-[f] \leq \mu[f] \leq \mu^+[f]. \quad (2.13)$$

Moreover, μ^- and μ^+ are respectively left-continuous and right-continuous.

Putting together results of Dyson [10] and Fröhlich/Spencer [21], we get the main fundamental fact on Dyson-Ising models:

Proposition 2 *The Dyson-Ising model with potential (2.11), for $1 < \alpha \leq 2$, exhibits a phase transition at low temperature:*

$$\exists \beta_c^D > 0, \text{ such that } \beta > \beta_c^D \implies \mu^- \neq \mu^+ \text{ and } \mathcal{G}(\gamma^D) = [\mu^-, \mu^+]$$

where the extremal phases μ^+ and μ^- are translation-invariant. They have in particular opposite magnetisations $\mu^+[\sigma_0] = -\mu^-[\sigma_0] = M_0(\beta, \alpha) > 0$ at low temperature.

Furthermore it is known that all Gibbs measures for our Dyson-Ising models are translation-invariant ([24], Theorem 9.5).

The DLR-measures share the same expression for their conditional probabilities w.r.t. the outside of *finite* sets, thanks to the local specification (2.11), but the expressions are valid

⁶It is a consequence of the FKG property [19, 28]: spins have a tendency to align.

almost surely w.r.t to the DLR-measure itself. It is important to notice that this does not hold anymore when one wants to get the conditional probabilities w.r.t. the outside of *infinite* sets for which highly phase-dependent asymptotic effects could yield different expressions depending on the DLR-measure considered. For this reason, the extension of the DLR property to infinite sets is rather direct in case of uniqueness of the DLR-measure for a given local specification [16, 18, 26], but it can be more problematic otherwise. Another positive solution to this extension question, beyond the uniqueness case, was found in the case of phase transitions for attractive models by Fernández and Pfister [16] and, as we will make essential use of it, we describe it now in our particular case. The terminology used is that of *global specifications*, and it is in fact a central tool in studying various Gibbs vs. non-Gibbs questions.

2.4 Global specification

Definition 6 (Global specification) A global specification Γ on \mathbb{Z} is a family of probability kernels $\Gamma = (\Gamma_S)_{S \subset \mathbb{Z}}$ on (Ω, \mathcal{F}) such that for any S subset of \mathbb{Z} :

1. $\Gamma_S(\cdot|\omega)$ is a probability measure on (Ω, \mathcal{F}) for all $\omega \in \Omega$.
2. $\Gamma_S(A|\cdot)$ is \mathcal{F}_{S^c} -measurable for all $A \in \mathcal{F}$.
3. $\Gamma_S(B|\omega) = \mathbf{1}_B(\omega)$ when $B \in \mathcal{F}_{S^c}$.
4. For all $S_1 \subset S_2 \subset \mathbb{Z}$, $\Gamma_{S_2}\Gamma_{S_1} = \Gamma_{S_2}$ where the product of kernels is made as in (2.2).

Definition 7 Let Γ be a global specification. We write $\mu \in \mathcal{G}(\Gamma)$, or say that $\mu \in \mathcal{M}_1^+$ is Γ -compatible, if for all $A \in \mathcal{F}$ and any $S \subset \mathbb{Z}$,

$$\mu[A|\mathcal{F}_{S^c}](\omega) = \Gamma_S(A|\omega), \mu\text{-a.e. } \omega. \quad (2.14)$$

Note, by considering $S = \mathbb{Z}$, that $\mathcal{G}(\Gamma)$ contains at most one element. Following the construction of [16] in the general monotonicity-preserving case, we get:

Theorem 1 Consider any Dyson-Ising model on \mathbb{Z} at inverse temperature $\beta > 0$, i.e. the specification γ^D with potential (2.11) and its extremal Gibbs measures μ^+ and μ^- defined by (2.12). Define $\Gamma^+ = (\Gamma_S^+)_{S \subset \mathbb{Z}}$ to be the family of probability kernels on (Ω, \mathcal{F}) as follows:

- For $S = \Lambda$ finite, for all $\omega \in \Omega$, $\Gamma_\Lambda^+(d\sigma|\omega) := \gamma_\Lambda^D(d\sigma|\omega)$.
- For S infinite, for all $\omega \in \Omega$,

$$\Gamma_S^+(d\sigma|\omega) := \mu_S^{+, \omega} \otimes \delta_{\omega_{S^c}}(d\sigma) \quad (2.15)$$

where $\mu_S^{+, \omega}$ is the constrained measure on $(\Omega_S, \mathcal{F}_S)$ (well-)defined as the weak limit

$$\mu_S^{+, \omega}(d\sigma_S) := \lim_{I \in \mathcal{S}, I \uparrow S} \gamma_I^D(d\sigma | +_S \omega_{S^c}). \quad (2.16)$$

Then Γ^+ is a global specification such that $\mu^+ \in \mathcal{G}(\Gamma^+)$. It is moreover monotonicity-preserving and right-continuous. Similarly, one defines a monotonicity-preserving and left-continuous global specification Γ^- such that $\mu^- \in \mathcal{G}(\Gamma^-)$.

Remark that when the set S is infinite, one proceeds in two steps, the order of which is crucial: Freeze first the configuration into ω on S^c and perform afterwards the weak limit with +-boundary condition *in* S , to get the constrained measure $\mu_S^{+,\omega}$ on $(\Omega_S, \mathcal{F}_S)$. Note also that **the global specification obtained need not to be quasilocal in general**, even when the original specification is itself quasilocal. This failure of quasilocality, caused by long-range ordering due to hidden phase-transitions, is in fact crucial, as we see now.

3 Decimation of the Dyson Ising Model

3.1 Set-up : Decimation Transformation

We start at low temperature in the phase transition region of the Dyson-Ising model with any Gibbs measure μ , mainly considering the +-phase μ^+ got by weak limit with +-boundary conditions, and introduce the following *decimation transformation* defined as

$$T: (\Omega, \mathcal{F}) \longrightarrow (\Omega', \mathcal{F}') = (\Omega, \mathcal{F}); \omega \longmapsto \omega' = (\omega'_i)_{i \in \mathbb{Z}}, \text{ with } \omega'_i = \omega_{2i} \quad (3.17)$$

This transformation acts on measures in a canonical way and we denote $\nu^+ := T\mu^+$ the decimation of the +-phase. It is formally defined as an image measure via

$$\forall A' \in \mathcal{F}', \nu^+(A') = \mu^+(T^{-1}A') = \mu^+(A) \text{ where } A = T^{-1}A' = \{\omega : \omega' = T(\omega) \in A'\}.$$

When necessary, we distinguish between original and image sets using primed notation⁷.

This type of transformation was also the basic example in [11], where non-quasilocality is proved in dimension 2 at low enough temperature, as soon as a phase transition is possible for an Ising model on the decorated lattice, which consists of a version of \mathbb{Z}^2 where the "even" sites have been removed. In our one-dimensional set-up, the role of this decorated lattice will be played by the set of odd sites, $2\mathbb{Z} + 1$, which again can be identified with \mathbb{Z} itself, and we observe that when a phase transition holds for the Dyson specification – at low enough temperature for $1 < \alpha \leq 2$ – the same is true for the constrained specification (2.16) *with alternating constraint*, leading to non-Gibbsianness of ν^+ . Once the +-phase is shown to be non-Gibbsian after being subjected to a decimation transformation, the same holds true for all other Gibbs measures of the model.

3.2 Non-Gibbsianness at Low Temperature

Theorem 2 *For any $1 < \alpha \leq 2$, at low enough temperature the decimation ν of any Gibbs measure μ of the Dyson-Ising model, $\nu = T\mu$ is non-quasilocal, hence non-Gibbs.*

Proof of Theorem 2:

We know from Section 2.2 – and basically from [11] – that to get non-Gibbsianness via a failure of quasilocality, one exhibits an essential discontinuity, i.e. a local function f , a finite subset Λ' and a configuration ω' so that the conditional expectation of f when the outside of Λ' is fixed under ω' cannot be made continuous by changes on zero-measure sets, i.e. by taking other versions. Such a point of essential discontinuity is also called a *bad configuration*. Here,

⁷Notice that by rescaling the configuration spaces Ω (original) and Ω' (image) are identical.

the bad configuration for the image measure ν^+ will be, just as in [11] in the two-dimensional case, the so called *alternating configuration* ω'_{alt} defined for any $i \in \mathbb{Z}$ as $(\omega'_{\text{alt}})_i = (-1)^i$. To get the essential discontinuity, the choice of $f(\sigma') = \sigma'_0$ and conditioning outside $\{0\}$ will be enough.

Intuitively, because any non-fixed site at all odd distances has a positive and a negative spin whose influences cancel, conditioning by this alternating configuration yields a constrained model that is again a model of Dyson-type which has a low-temperature transition in our range of decays $1 < \alpha \leq 2$. The non-Gibbsianness proof essentially goes along the lines sketched in [11], with the role the “annulus” played by two large intervals $[-N, -L - 1]$ and $[L + 1, N]$ to the left and to the right of the central interval $[-L, +L]$. If we constrain the spins in these two intervals to be either plus or minus, within these two intervals the measures on the unfixed spins are close to those of the Dyson-type model in a positive, c.q. negative, magnetic field. As those measures are unique (due to e.g. a Yang-Lee argument [35], see also [30]) no influence from the boundary can be transmitted by via the “annulus”. Due to the long range of the Dyson interaction, there may be also a direct influence from the boundary to the central interval, however. But by choosing $N(L)$ large enough – e.g. $N = L^{\frac{1}{\alpha-1}}$ – we can make this direct influence as small as we want, so the strategy of [11], there worked out for finite-range models, does also work here.

Lemma 1 *The alternating configuration ω'_{alt} is a point of essential discontinuity for ν^+ .*

Proof of Lemma 1:

We want to study the continuity of various conditional expectations under decimated Dyson measures of the spin at the origin when the outside is fixed under ω'_{alt} . As the conditioning takes place on an infinite set with infinite complement, we need here global specifications for the decimated measures, and we first build them following [16] or [38]. To build these specifications, we first note that

$$\nu^+[\sigma'_0 | \mathcal{F}_{\{0\}^c}](\omega') = \mu^+[\sigma_0 | \mathcal{F}_{S^c}](\omega), \quad \nu^+ - \text{a.s.} \quad (3.18)$$

where $S^c = (2\mathbb{Z}) \cap \{0\}^c$, i.e. with $S = (2\mathbb{Z})^c \cup \{0\}$ is not finite: *the conditioning is **not** on the complement of a finite set.* To get an expression for it, we need thus to use the global specification Γ^+ such that $\mu^+ \in \mathcal{G}(\Gamma^+)$, built in Theorem 1, with $S = (2\mathbb{Z})^c \cup \{0\}$ consisting of the *odd integers plus the origin*. Hence (3.18) reduces for ν^+ -a.e. $\omega' \in \mathcal{N}_{\Lambda'}(\omega'_{\text{alt}})$ to

$$\nu^+[\sigma'_0 | \mathcal{F}_{\{0\}^c}](\omega') = \Gamma_S^+[\sigma_0 | \omega] \quad \mu^+ - \text{a.e.}(\omega) \quad (3.19)$$

with $S = (2\mathbb{Z})^c \cup \{0\}$ and $\omega \in T^{-1}\{\omega'\}$ is defined as the alternating configuration, so that $\omega_{2i} = (-1)^i$. Now, by (2.15) we have an expression of the latter in terms of the constrained measure $\mu_{(2\mathbb{Z})^c \cup \{0\}}^{+, \omega}$, with $\omega \in T^{-1}\{\omega'\}$ so that we get for any $\omega' \in \mathcal{N}_{\Lambda'}(\omega'_{\text{alt}})$,

$$\nu^+[\sigma'_0 | \mathcal{F}_{\{0\}^c}](\omega) = \mu_{(2\mathbb{Z})^c \cup \{0\}}^{+, \omega} \otimes \delta_{\omega_{2\mathbb{Z} \cap \{0\}^c}}[\sigma_0].$$

Thanks to monotonicity-preservation, the constrained measure is explicitly built as the weak limit (2.16) obtained by $+$ -boundary conditions fixed after a freezing of ω on the even sites, so that:

$$\forall \omega' \in \mathcal{N}_{\Lambda'}(\omega'_{\text{alt}}), \forall \omega \in T^{-1}\{\omega'\}, \mu_{(2\mathbb{Z})^c \cup \{0\}}^{+, \omega}(\cdot) = \lim_{I \in \mathcal{S}, I \uparrow (2\mathbb{Z})^c \cup \{0\}} \gamma_I^D(\cdot | +_{(2\mathbb{Z})^c \cup \{0\}} \omega_{2\mathbb{Z} \cap \{0\}^c}). \quad (3.20)$$

and it is enough to consider this limit on a sequence of intervals $I_n = [-n, +n] \cap \mathbb{Z}$ in the original space. Now, one obtains an essential discontinuity if we can get a difference in the expectation of the spin at the origin of this constrained measure conditioned on two different open subsets of arbitrary neighborhoods of ω'_{alt} . As we shall see, this is indeed the case as soon as the temperature is low enough in order to get a phase transition for the Dyson-Ising ferromagnet on the odd sites – the hidden phase transition –. The key observation is the following

Lemma 2 *Consider a Dyson-Ising model with long-range decay parameter $1 < \alpha \leq 2$, at sufficiently low temperature. Let $\Lambda' \subset \Delta' \in \mathcal{S}$ and consider two arbitrary configurations $\omega'^+ \in \mathcal{N}_{\Lambda', \Delta'}^+(\omega'_{\text{alt}})$ and $\omega'^- \in \mathcal{N}_{\Lambda', \Delta'}^-(\omega'_{\text{alt}})$. Then $\exists \delta > 0$, and $\exists \Lambda'_0$ big enough s.t. for some $\Delta' \supset \Lambda' \supset \Lambda'_0$ with $\Delta' \setminus \Lambda'$ chosen big enough compared to Λ' , for all $\omega^+ \in T^{-1}\{\omega'^+\}$ and all $\omega^- \in T^{-1}\{\omega'^-\}$*

$$\left| \mu_{(2\mathbb{Z})^c \cup \{0\}}^{+, \omega^+}[\sigma_0] - \mu_{(2\mathbb{Z})^c \cup \{0\}}^{+, \omega^-}[\sigma_0] \right| > \delta. \quad (3.21)$$

The main reason is that, conditioned on all primed spins being alternating, the conditioned model is a Dyson-like model in zero field, due to cancellations, so that a phase transition occurs at low temperature, making it possible to select the phase by boundary conditions arbitrarily far away. On the contrary, when conditioned on all primed spins to be plus (resp. minus), there is no phase transition, but the system of unprimed spins as a Dyson model in a homogeneous external field has a unique Gibbs measure⁸ with positive (resp. negative) magnetisation $M_0(\beta, \alpha) > 0$. What then has to be shown is that it is possible to prescribe plus or minus spins on a large enough annulus so that they select the above phases.

Proof of Lemma 2 : With the notation of the lemma, let us consider

$$M^+ = \mu_{(2\mathbb{Z})^c \cup \{0\}}^{+, \omega^+}[\sigma_0] \text{ and } M^- = \mu_{(2\mathbb{Z})^c \cup \{0\}}^{+, \omega^-}[\sigma_0].$$

They are both magnetisations for some +-phase, but for different constrained specifications. We first check that both these expectations are almost insensitive to asymptotic influences from beyond the annulus, for an annulus large enough. We will show that this is indeed the case when the size of the annulus increases properly. Afterwards we shall check that changes inside the annulus will on the contrary substantially change local expectations.

Write $\Lambda' = \Lambda'(L) = [-L, +L]$ and $\Delta' = \Delta'(N) = [-N, +N]$, with $N > L$ and denote formally by H the Hamiltonian of the constrained specifications for ω_1^+ and ω_2^+ as prescribed. One can bound uniformly in L the relative Hamiltonians with either ω_1^+ and ω_2^+ b.c. to get

$$\left| H_{\Lambda, \omega_1^+}(\sigma_\Lambda) - H_{\Lambda, \omega_2^+}(\sigma_\Lambda) \right| \leq C < \infty. \quad (3.22)$$

as soon as one takes $N = N(L) = O(L^{\frac{1}{\alpha-1}})$. Then one gets by [4] (see also [23]) that all of the limiting Gibbs states obtained by these boundary conditions have the same measure zero sets, an equivalent decomposition into extremal Gibbs states (presumably trivial here, as the Gibbs measure will be unique, as we shall see), and thus yield the same magnetisation : $M^+ = M^+(\omega, N, L) = M^+(\omega_1^+, N, L) = M^+(\omega_2^+, N, L)$ is indeed independent of ω as soon as

⁸Provable e.g. by the Lee-Yang theorem [35].

it belongs to the pre-image of the $+$ -neighborhood of the alternating configuration. To get (3.22), we use the long-range structure of the interaction to get a uniform bound

$$\left| H_{\Lambda, \omega_1^+}(\sigma_\Lambda) - H_{\Lambda, \omega_2^+}(\sigma_\Lambda) \right| \leq 2 \sum_{x=-L}^L \sum_{k>N} \frac{1}{k^\alpha} < 2L \frac{N^{1-\alpha}}{1-\alpha}$$

so that choosing $N = N(L)$ such that $2L \frac{N^{1-\alpha}}{\alpha-1} = 1$, or any bigger values of N , will do the job and one can choose

$$N(L) = L^{\frac{1}{\alpha-1}}. \quad (3.23)$$

For example, for $\alpha = \frac{3}{2}$, one has thus to take some annulus of the order of $N(L) = O(L^2)$.

Once we got rid of any possible direct asymptotic effects due to the long range by choosing a large enough annulus as above, the main point is now that freezing the primed spins to be minus can overcome the $+$ -boundary condition⁹ when the frozen annulus $\Delta' \setminus \Lambda'$ is in a $--$ state, for L (and $N(L)$) large enough. Thus the second expectation of (3.21) can be made as close as possible to the magnetisation of the Dyson-Ising model with an homogeneous external field $h_x = -$ everywhere, which at low enough temperature and for L large enough is close to (or even smaller than) the magnetisation of the Dyson-Ising model under the $--$ phase, i.e to $-M_0(\beta, \alpha) < 0$ (and this $--$ phase is also unique, see [30]). The magnetisation with the constraint ω^+ will thus be close to or bigger than $+M_0(\beta, \alpha)$ so that a non-zero difference is created at low enough temperature. One needs to again to adjust the sizes of L and N to be sure that boundaries effects are negligible. Let us be a bit more precise now.

We use the expression (3.20) with $\omega'^+ \in \mathcal{N}_\Lambda^+(\omega'_{\text{alt}})$ and to facilitate the proof we will make use of (3.20), and freely change between regular versions of conditional probabilities on arbitrarily small neighborhoods of configurations (all $+$, all $-$, all ω'_{alt} , all ω^+ , etc.) with conditioning by the considered configuration itself (to avoid the problem of conditioning on zero measure sets). Recall that ω'^+ is generic for a configuration coinciding with the alternating configuration around the origin, and with the $+$ one on sets depending of N and L . To be still able to neglect boundary effects, we take $N(L)$ big compared to L just as in the previous part of the proof. We have seen that conditioning of the primed sites to be all $+$ reduces (3.20) to the magnetisation obtained by taking a weak limit of a Dyson Ising specification with an everywhere¹⁰ homogeneous strictly positive external field h^+ , with well-defined strictly positive¹¹ value

$$h^+ = 2 \sum_{k=0}^{\infty} \frac{1}{(2k)^\alpha} = F(\alpha) < \infty.$$

For this choice of $\omega' = +$, the constrained magnetisation (the lhs of (3.18)) is thus $M_0(\beta, \alpha)$ of Proposition 2, which is strictly positive at low temperature in our range $1 < \alpha \leq 2$.

Now, consider the case of ω'^+ . It again reduces to a Dyson-Ising model with external field, but the latter $(h_x)_{x \in \mathbb{Z}}$ depends on $x \in \mathbb{Z}^d$ and is not homogeneous anymore. Nevertheless,

⁹From the initial measure, we decimate the $+$ -state and this is visible in the weak limit with $+$ -b.c. performed to get the global specification consistent with the decimated measure ν^+ .

¹⁰Modulo an adaptation to fix and unfix the spin at the origin, as in [11].

¹¹Its value could probably be expressed in terms of a Riemann ζ -function.

we see now that the difference to the homogeneous part is negligible on the large interval I of (3.20), and the field is always non-negative. Write, for all $x \in \mathbb{Z}$,

$$h_x = h^+ - \epsilon_x \asymp h^+$$

where the perturbation ϵ_x depends on the location of x :

$$\begin{aligned} x < -L \text{ or } x > 0 : \epsilon_x &= \sum_{k=-L}^L \frac{2}{(2k+1)^\alpha} = o(h^+). \\ -L \leq x \leq L : \epsilon_x &\leq \sum_{k=-L}^L \frac{2}{(2k+1)^\alpha} = o(h^+). \end{aligned}$$

It is maximal for (the worst case) $x = 0$ with a maximal perturbation dominated by the homogenous one so that $h_0 \ll h_+$.

A similar computation holds with the all $-$'s-constrained specification, with the same perturbation to the homogeneous external field $h^- = -h^+$. This perturbation is thus small compared to either h^+ or h^- so that for any arbitrary L , $\exists 0 < \delta(L) < 1$ with

$$M_0^+ > \delta(L)M_0(\beta, \alpha) > 0 \text{ and } M_0^- < -\delta(L)M_0(\beta, \alpha) < 0.$$

The magnetisations of the Dyson models in an external field are larger in absolute value than the plus and minus phases in zero field, so taking them as boundary conditions everywhere produces the plus and minus phases. Changing any spins, primed or not, outside Δ' makes a negligible change when $N(L)$ is chosen large enough, and the Lemma follows, as choosing plus spins in the annulus produces a magnetisation at the origin of at least $\frac{1}{2}\delta$ and choosing minus spins a magnetisation lower than $-\frac{1}{2}\delta$. Then we are reduced to compare the "complete" magnetisations of the extremal phases of the Dyson long-range Ising model, and the lemma follows with $\delta = \delta(L)M_0(\beta, \alpha)$ which is strictly positive at low enough temperature for any $1 < \alpha \leq 2$.

◇

Now standard arguments as in [11] provide the non-Gibbsianness.

4 Extensions, related issues and comments

We have shown that the alternating configuration is a point of essential discontinuity for expectations in the decimation from \mathbb{Z} to $2\mathbb{Z}$, implying that the associated decimated Gibbs measures are non-Gibbsian. In our choice of decimated lattice we made use of the fact that the constrained system, due to cancellations, again formed a zero-field Dyson-like model. In the case of decimations from \mathbb{Z} to a more diluted lattice $b\mathbb{Z}$ the constrained models could form ferromagnetic models in a periodically varying external field, with zero mean. Although the original proofs of Dyson and of Fröhlich and Spencer or the Reflection Positivity proof of [20] do no longer apply to such periodic-field cases, the contour-like arguments of [5] and [29] could presumably still be modified to include these cases. Compare also [30].

The analysis of [6] which proves existence of a phase transition for Dyson models in random magnetic fields for a certain interval of α -values should imply that in that case there are many

more, random, configurations with all are points of discontinuity. We note that choosing independent spins as a constraint provides a random field which is correlated. However, these correlations decay enough that this need actually not spoil the argument. Similarly, one should be able to prove that decimation of Dyson models in a weak external field will result in a non-Gibbsian measure.

Estimating the measure of the discontinuity points leads one to the question of "almost Gibbsian" [39], "intuitively weakly Gibbsian" [13] and "weakly Gibbsian" properties [39]. The analysis of [16] and [38] extends, due to monotonicity and right-continuity properties, to prove almost Gibbsianness of the transformed measures both with and without a field. This implies as usual (see e.g. [39]) weak Gibbsianness with an a.s. convergent potential as the telescoping one given in [44]. The latter possesses extra asymptotic properties such as a uniform polynomial decay that should be weaker here. An interesting question would be to perform the analysis of [40]) or [38] to get configuration-dependent a.s. correlation decay.

On the other hand, an example of almost surely non-quasilocal transformed measure is given by the joint measure of the random-field Dyson-Ising model studied in [6], who, similarly to the 3-dimensional nearest-neighbour random-field Ising model, lacks the property of being almost Gibbs and may therefore violate the variational principle [32].

We have thus extended results which were known before for nearest-neighbour Ising models to a class of long-range models of Dyson type. It turns out that the analogy between varying the dimension and varying the decay parameter of the Dyson models also holds regarding the non-Gibbsianness of various transformed measures, under decimation transformations. In particular, it turns out that at sufficiently low temperatures the Gibbs measures of the zero-field models, as well as the models in a weak magnetic field under decimation are mapped to non-Gibbsian measures. We expect that, as in the nearest-neighbour case, the nature of the transformation (decimation, average, majority rule, stochastic evolutions, factor maps...) should not play that much of a role either but we have not pursued our investigations further in this direction. The case of stochastic evolutions (in particular subjecting the Dyson measures to an infinite-temperature evolution) should be fairly immediate, for short times the results of [36] imply Gibbsianness for a wide class of evolutions, for large times non-Gibbsianness follows from an analysis more or less along the lines of [12], and the observations made above that Dyson models in weak periodic or random fields will have phase transitions at low temperatures should imply a Gibbs-non-Gibbs transition.

Another class of one-dimensional systems which has attracted a lot of attention over the last years is the class of g -measures. In the presence of phase transitions [2, 3, 14], it seems plausible that transforming them also will often map them to non-Gibbsian, cq "non- g " measures. In fact, although it is known that g -measures need not be Gibbs measures [15] it appears at this point not known if the Gibbs measures of the Dyson-Ising models can be represented as g -measures.

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