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# GLOBAL PERSISTENCE OF GEOMETRICAL STRUCTURES FOR THE BOUSSINESQ EQUATION WITH NO DIFFUSION

RAPHAËL DANCHIN AND XIN ZHANG

ABSTRACT. Here we investigate the so-called *temperature patch problem* for the incompressible Boussinesq system with partial viscosity, in the whole space  $\mathbb{R}^N$  ( $N \geq 2$ ), where the initial temperature is the characteristic function of some simply connected domain with  $C^{1,\varepsilon}$  Hölder regularity. Although recent results in [1, 15] ensure that an initially  $C^1$  patch persists through the evolution, whether higher regularity is preserved has remained an open question. In the present paper, we give a positive answer to that issue globally in time, in the 2-D case for large initial data and in the higher dimension case for small initial data.

## 1. INTRODUCTION

This paper is devoted to the temperature patch problem for the following incompressible Boussinesq system with partial viscosity:

$$(B_{\nu,N}) \quad \begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi = \theta e_N, \\ \operatorname{div} u = 0, \\ (\theta, u)|_{t=0} = (\theta_0, u_0). \end{cases}$$

Above,  $e_N = (0, \dots, 0, 1)$  stands for the unit vertical vector in  $\mathbb{R}^N$  with  $N \geq 2$ . The unknowns are the scalar function  $\theta$  (the temperature), the velocity field  $u = (u^1, u^2, \dots, u^N)$  and the pressure  $\Pi$ , depending on the time variable  $t \geq 0$  and on the space variable  $x \in \mathbb{R}^N$ . We assume the viscosity  $\nu$  to be a positive constant.

The above Boussinesq system is a toy model for describing the convection phenomenon in viscous incompressible flows, and arises in simplified models for geophysics (see e.g. [23]). A number of works are dedicated to the global well-posedness issue of  $(B_{\nu,2})$  (see e.g. [1, 4, 15, 19, 21]). In particular, R. Danchin and M. Paicu proved in [15] (see also [18]) that  $(B_{\nu,2})$  has a unique global solution  $(\theta, u)$  such that  $\theta \in C(\mathbb{R}_+; L^2(\mathbb{R}^2))$  and

$$(1.1) \quad u \in \left( C(\mathbb{R}_+; L^2(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^2)) \cap \tilde{L}^1_{loc}(\mathbb{R}_+; H^2(\mathbb{R}^2)) \right)^2,$$

whenever the initial data  $(\theta_0, u_0)$  are in  $(L^2(\mathbb{R}^2))^3$  and satisfy  $\operatorname{div} u_0 = 0$ . Additionally, the following energy equality is fulfilled for all  $t \geq 0$ :

$$\|u(t)\|_{L^2(\mathbb{R}^2)}^2 + 2\nu \int_0^t \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 dt' = \|u_0\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \int_{\mathbb{R}^2} \theta u_2 dx dt'.$$

Global well-posedness results are also available in dimension  $N \geq 3$ , but exactly as for the standard Navier-Stokes equations, the initial data have to satisfy a suitable smallness condition, see for instance [14, 15].

To better explain the main motivation of our work, which is the *temperature patch problem*, let us assume that  $N = 2$  for a while. Based on the aforementioned well-posedness result, one may consider for  $\theta_0$  the characteristic function of some simply connected bounded domain  $\mathcal{D}_0$  of the plane. Given that  $\theta$  is just advected by the velocity field  $u$ , we expect to have  $\theta(t, x) = \mathbb{1}_{\mathcal{D}_t}(x)$  for all  $t \geq 0$ , where  $\mathcal{D}_t := \psi_u(t, \mathcal{D}_0)$  and  $\psi_u$  stands for the flow associated to  $u$ , that is to say the solution to the following (integrated) ordinary differential equation:

$$(1.2) \quad \psi_u(t, x) := x + \int_0^t u(t', \psi_u(t', x)) dt'.$$

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*Key words and phrases.* Boussinesq equations; Incompressible flows; Striated regularity, Para-vector field; Hölder spaces; Temperature patch problem.

<sup>1</sup>The notation  $\tilde{L}^1_{loc}(\mathbb{R}_+; H^2)$  designates a (close) superspace of  $L^1_{loc}(\mathbb{R}_+; H^2)$ , see (2.2).

If the regularity of  $u$  is given by (1.1), then it has been proved by J.-Y. Chemin and N. Lerner in [9] that (1.2) has a unique solution, which is in  $(C(\mathbb{R}_+; C^{0,1-\eta}))^2$  for any  $\eta > 0$ . Now, if we add a bit more regularity on the initial data, for instance<sup>2</sup>  $u_0 \in (B_{2,1}^0)^2$  and  $\theta_0 \in B_{2,1}^0$  then, according to [1], all the entries of  $\nabla u$  are in  $L_{loc}^1(\mathbb{R}_+; C_b)$ , and the flow  $\psi_u(t, \cdot)$  is thus  $C^1$ . Consequently, the  $C^1$  regularity of the temperature patch is preserved for all time.

Then a natural question arises: what if we start with a  $C^{1,\varepsilon}$  Hölder domain  $\mathcal{D}_0$  with  $\varepsilon \in ]0, 1[$ ? Our concern has some similarity with the celebrated vortex patch problem for the 2-D incompressible Euler equations. In that case, it has been proved (see e.g. [6, 7] and the references therein) that the  $C^{1,\varepsilon}$  regularity of the patch of the vorticity persists for all time. Proving that in our framework, too, the Hölder regularity of  $\mathcal{D}_t$  is conserved is the main purpose of the present paper. Just like for the vortex patch problem for Euler equations, our result will come up as a consequence of a much more general property of global-in-time persistence of *striated regularity*, a definition that originates from the work of J.-Y. Chemin in [5].

Before stating our main results, we need to introduce some notation, and to clarify what striated regularity is. Assume that<sup>3</sup>  $X = X^k(x)\partial_k$  is some vector field acting on functions in  $C^1(\mathbb{R}^N; \mathbb{R})$ . As usual, vector fields are identified with vector valued functions from  $\mathbb{R}^N$  to  $\mathbb{R}^N$ , and  $\partial_X f$  stands for the *directional derivative* of  $f \in C^1(\mathbb{R}^N; \mathbb{R})$  along the vector field  $X$ , namely

$$\partial_X f := X^k \partial_k f = X \cdot \nabla f.$$

The evolution  $X_t(x) := X(t, x)$  of any continuous initial vector field  $X_0$  along the flow of  $u$  is defined by:

$$X(t, x) := (\partial_{X_0} \psi_u)(\psi_u^{-1}(t, x)).$$

In the  $C^1$  case, combining the chain rule and the definition of the flow in (1.2) implies that  $X$  satisfies the transport equation<sup>4</sup>

$$(1.3) \quad \begin{cases} \partial_t X + u \cdot \nabla X = \partial_X u, \\ X|_{t=0} = X_0. \end{cases}$$

Applying operator  $\operatorname{div}$  to (1.3) and remembering that  $\operatorname{div} u = 0$ , we obtain in addition

$$(1.4) \quad \begin{cases} \partial_t \operatorname{div} X + u \cdot \nabla \operatorname{div} X = 0, \\ \operatorname{div} X|_{t=0} = \operatorname{div} X_0. \end{cases}$$

This implies that the divergence-free property is conserved through the evolution.

As we will see in Section 5, the temperature patch problem is closely related to the conservation of Hölder regularity  $C^{0,\varepsilon}$  for  $X$ . According to the classical theory of transport equations, if  $u$  is Lipschitz with respect to the space variable (a condition that will be ensured if the data of  $(B_{\nu,N})$  have critical Besov regularity), then conservation of  $C^{0,\varepsilon}$  regularity for  $X$  is equivalent to the fact that all the components of  $\partial_X u$  have the regularity  $C^{0,\varepsilon}$  with respect to the space variable.

In the 2-D case, it is natural to recast the regularity of  $u$  along the vector field  $X$  in terms of the vorticity  $\omega := \partial_1 u^2 - \partial_2 u^1$  as the simple transport-diffusion equation is fulfilled:

$$(1.5) \quad \begin{cases} \partial_t \omega + u \cdot \nabla \omega - \nu \Delta \omega = \partial_1 \theta, \\ \omega|_{t=0} = \omega_0, \end{cases}$$

and as it is known (see e.g. [2], Chap. 7) that<sup>5</sup>:

$$(1.6) \quad \|\partial_X u\|_{\mathcal{C}^\varepsilon} \lesssim \|\nabla u\|_{L^\infty} \|X\|_{\mathcal{C}^\varepsilon} + \|\operatorname{div}(X\omega)\|_{\mathcal{C}^{\varepsilon-1}},$$

where for any real number  $s$ , we denote<sup>6</sup>  $\mathcal{C}^s \equiv \mathcal{C}^s(\mathbb{R}^N) := B_{\infty,\infty}^s(\mathbb{R}^N)$ .

<sup>2</sup>See the definition of Besov spaces in the next section.

<sup>3</sup>We adopt Einstein summation convention in the whole text: summation is taken with respect to the repeated indices, whenever they occur both as a subscript and a superscript.

<sup>4</sup>Omitting the index  $t$  in  $X_t$  for notational simplicity.

<sup>5</sup>In all the paper, we agree that  $A \lesssim B$  means  $A \leq CB$  for some harmless constant  $C$ .

<sup>6</sup>Recall that  $\mathcal{C}^{k+\varepsilon}$  coincides with the standard Hölder space  $C^{k,\varepsilon}$  whenever  $k \in \mathbb{N}$  and  $\varepsilon \in ]0, 1[$ .

Now, applying operators  $\partial_X$  and  $\operatorname{div}(X \cdot)$  to the temperature and vorticity equations, respectively, we get the following system for  $(\partial_X \theta, \operatorname{div}(X\omega))$ :

$$(1.7) \quad \begin{cases} \partial_t \partial_X \theta + u \cdot \nabla \partial_X \theta = 0, \\ \partial_t \operatorname{div}(X\omega) + u \cdot \nabla \operatorname{div}(X\omega) - \nu \Delta \operatorname{div}(X\omega) = f, \end{cases}$$

with  $f := \nu \operatorname{div}(X \Delta \omega - \Delta(X\omega)) + \operatorname{div}(X \partial_1 \theta)$ .

Let us recap. Roughly speaking, to propagate the  $\mathcal{C}^\varepsilon$  regularity of  $X$ , it is sufficient to control  $\partial_X u$  in  $(L^1_{loc}(\mathbb{R}_+; \mathcal{C}^\varepsilon))^2$ , and this may be achieved, thanks to (1.6), if bounding the distribution  $\operatorname{div}(X\omega)$  in  $L^1_{loc}(\mathbb{R}_+; \mathcal{C}^{\varepsilon-1})$ . Then by taking advantage of smoothing properties of the heat flow, this latter information may be obtained through a bound of  $f$  in the very negative space  $L^1_{loc}(\mathbb{R}_+; \mathcal{C}^{\varepsilon-3})$ , if assuming that  $\operatorname{div}(X_0 \omega_0) \in \mathcal{C}^{\varepsilon-3}$ . Staring at the expression of  $f$ , we thus need to bound  $\partial_X \theta$  in  $L^\infty_{loc}(\mathbb{R}_+; \mathcal{C}^{\varepsilon-2})$ . As no gain of regularity may be expected from the transport equation satisfied by  $\partial_X \theta$ , we have to assume initially that  $\partial_{X_0} \theta_0 \in \mathcal{C}^{\varepsilon-2}$ . This motivates the following statement which is our main result of propagation of striated regularity in the 2-D case <sup>7</sup>.

**Theorem 1.1.** *Suppose that  $(\varepsilon, q) \in ]0, 1[ \times ]1, \frac{2}{2-\varepsilon}[$ . Let  $\theta_0$  be in  $B^{\frac{2}{q}-1}_{q,1}(\mathbb{R}^2)$  and  $u_0$  be a divergence-free vector field in  $(L^2(\mathbb{R}^2))^2$ , with vorticity  $\omega_0 := \partial_1 u_0^2 - \partial_2 u_0^1$  in  $B^{\frac{2}{q}-2}_{q,1}(\mathbb{R}^2)$ . Then there exists a unique global solution  $(\theta, u)$  of System  $(B_{\nu,2})$ , such that*

$$(1.8) \quad (\theta, u, \omega) \in C(\mathbb{R}_+; B^{\frac{2}{q}-1}_{q,1}) \times (C(\mathbb{R}_+; L^2))^2 \times (C(\mathbb{R}_+; B^{\frac{2}{q}-2}_{q,1}) \cap L^1_{loc}(\mathbb{R}_+; B^{\frac{2}{q}}_{q,1})).$$

Furthermore, if we consider some  $X_0$  in  $(\mathcal{C}^\varepsilon(\mathbb{R}^2))^2$  satisfying  $\partial_{X_0} \theta_0 \in \mathcal{C}^{\varepsilon-2}(\mathbb{R}^2)$  and  $\operatorname{div}(X_0 \omega_0) \in \mathcal{C}^{\varepsilon-3}(\mathbb{R}^2)$ , then there exists a unique global solution <sup>8</sup>  $X \in C_w(\mathbb{R}_+; \mathcal{C}^\varepsilon)$  to (1.3) and we have

$$(\partial_X \theta, \operatorname{div}(X\omega)) \in C_w(\mathbb{R}_+; \mathcal{C}^{\varepsilon-2}) \times (C_w(\mathbb{R}_+; \mathcal{C}^{\varepsilon-3}) \cap \tilde{L}^1_{loc}(\mathbb{R}_+; \mathcal{C}^{\varepsilon-1})).$$

Additionally, there is a constant  $C_{0,\nu}$  depending only on the initial data and viscosity constant such that for any  $t \geq 0$ ,

$$\|X\|_{L^\infty_t(\mathcal{C}^\varepsilon)} \leq C_{0,\nu} \exp\left(\exp(\exp(C_{0,\nu} t^4))\right).$$

A few comments are in order:

- The functional space  $B^{\frac{2}{q}-1}_{q,1}(\mathbb{R}^2)$  for  $\theta_0$  is large enough to contain the characteristic function of any bounded  $C^1$  domain. This will be needed to investigate the temperature patch problem later (see Corollary 1.2 and Section 5 below).
- It may be seen by means of elementary paradifferential calculus that if  $(\varepsilon, q) \in ]0, 1[ \times ]1, \frac{2}{2-\varepsilon}[$  then  $\operatorname{div}(X_0 \omega_0)$  and  $\partial_{X_0} \theta_0$  are distributions of  $\mathcal{C}^{-3}$  and  $\mathcal{C}^{-2}$ , respectively, and that the following (sharp) estimates are fulfilled:

$$\|\operatorname{div}(X_0 \omega_0)\|_{\mathcal{C}^{-3}} \lesssim \|X_0\|_{\mathcal{C}^\varepsilon} \|\omega_0\|_{B^{\frac{2}{q}-2}_{q,1}} \quad \text{and} \quad \|\partial_{X_0} \theta_0\|_{\mathcal{C}^{-2}} \lesssim \|X_0\|_{\mathcal{C}^\varepsilon} \|\theta_0\|_{B^{\frac{2}{q}-1}_{q,1}}.$$

Therefore, our striated regularity assumption on the initial data is indeed additional information.

- The required level of regularity is much lower than in the (inviscid) 2-D vortex patch problem where  $\operatorname{div}(X_0 \omega_0) \in \mathcal{C}^{\varepsilon-1}(\mathbb{R}^2)$  is needed. This is because the smoothing effect given by the heat flow enables us to gain two derivatives with respect to the initial data. Another difference is that to tackle the temperature patch problem, it is not necessary to consider a family of vector fields that does not degenerate on the whole  $\mathbb{R}^2$ : just one suitably chosen vector-field that does not vanish in the neighborhood of the boundary of the patch is enough, as we shall see just below.

<sup>7</sup>To fully benefit from the smoothing properties of the heat flow in the endpoint case, one has to work in (close) superspaces of  $L^1_{loc}(\mathbb{R}_+; \mathcal{C}^s)$  denoted by  $\tilde{L}^1_{loc}(\mathbb{R}_+; \mathcal{C}^s)$  and defined in (2.2).

<sup>8</sup>If  $E$  is a Banach space with predual  $E^*$  then  $C_w(\mathbb{R}_+; E)$  stands for the set of measurable functions  $h: \mathbb{R}_+ \rightarrow E$  such that for all  $\phi \in E^*$ , the function  $t \mapsto \langle h(t), \phi \rangle_{E \times E^*}$  is continuous on  $\mathbb{R}_+$ .

Let us now go to the temperature patch problem in the 2-D case. More precisely, consider a  $C^{1,\varepsilon}$  simply connected bounded domain  $\mathcal{D}_0$  of  $\mathbb{R}^2$  (in other words  $\partial\mathcal{D}_0$  is a  $C^{1,\varepsilon}$  Jordan curve on  $\mathbb{R}^2$ ). Let  $\mathcal{D}_0^*$  be any bounded domain of  $\mathbb{R}^2$  such that<sup>9</sup>

$$(1.9) \quad \overline{\mathcal{D}_0} \cap \overline{\mathcal{D}_0^*} = \emptyset.$$

Then the following result holds true.

**Corollary 1.2.** *Let  $(M_1, M_2)$  be in  $\mathbb{R}^2$ . Assume that  $\theta_0 = M_1 \mathbf{1}_{\mathcal{D}_0}$  and that the vorticity  $\omega_0$  of  $u_0$  may be decomposed into*

$$(1.10) \quad \omega_0 = M_2 \mathbf{1}_{\mathcal{D}_0} - \tilde{\omega}_0$$

for some  $\tilde{\omega}_0 \in L^r(\mathbb{R}^2)$  with  $r > 1$ , supported in  $\overline{\mathcal{D}_0^*}$  and such that

$$(1.11) \quad \int_{\mathbb{R}^2} \tilde{\omega}_0(x) dx = M_2 |\mathcal{D}_0|.$$

Then there exists a unique solution  $(\theta, u)$  to System  $(B_{\nu,2})$ , satisfying the properties listed in Theorem 1.1. Furthermore, we have  $\theta(t, \cdot) = M_1 \mathbf{1}_{\mathcal{D}_t}$  where  $\mathcal{D}(t) := \psi_u(t, \mathcal{D}_0)$  and  $\partial\mathcal{D}(t)$  remains a  $C^{1,\varepsilon}$  Jordan curve of  $\mathbb{R}^2$  for all  $t \geq 0$ .

Let us make some comments on that corollary.

- Hypothesis (1.11) ensures that the initial vorticity is mean free, which is necessary to have  $u_0 \in (L^2(\mathbb{R}^2))^2$ . As a matter of fact, the more natural assumption  $\omega_0 = M_2 \mathbf{1}_{\mathcal{D}_0}$  would require our extending Theorem 1.1 to infinite energy velocity fields, which introduces additional technicalities.
- Assumption (1.10) on the vorticity may seem somewhat artificial as it has no persistency whatsoever through the time evolution, even in the asymptotics  $\nu \rightarrow 0$  (in contrast with the slightly viscous vortex patch problem, see [10]). This is just to have a concrete example of initial velocity for which one can give a positive answer to the temperature patch problem.
- Corollary 1.2 may be generalized to the case where the initial velocity  $u_0 \in (L^2(\mathbb{R}^2))^2$  is such that  $\omega_0 \in B_{q,1}^{\frac{2}{q}-2}(\mathbb{R}^2)$  for some  $1 < q < \frac{2}{2-\varepsilon}$  and satisfies  $\operatorname{div}(X_0 \omega_0) \in \mathcal{C}^{\varepsilon-3}(\mathbb{R}^2)$  for some vector field  $X_0 \in \mathcal{C}^\varepsilon(\mathbb{R}^2)$  that does not vanish on  $\partial\mathcal{D}_0$  and is tangent to  $\partial\mathcal{D}_0$ . One just has to follow the proof that is proposed in Section 5 to get this more general result.

In space dimension  $N \geq 3$ , the vorticity equation has an additional stretching term, and it is thus less natural to measure the striated regularity by means of  $\operatorname{div}(X\Omega)$  with  $\Omega$  denoting the matrix of curl  $u$  (even though we suspect that our 2-D approach is adaptable to the high-dimensional case, like in [17]). We shall thus concentrate on the regularity of  $\partial_X \theta$  and  $\partial_X u$ . An additional (related) difficulty is that one cannot expect to prove global existence for general large initial data, since  $(B_{\nu,N})$  contains the standard incompressible Navier-Stokes equations as a particular case. Therefore we shall prescribe some smallness condition on the data (the same one as in [15]) to achieve a global statement. This leads to the following theorem:

**Theorem 1.3.** *Suppose that  $N \geq 3$  and that  $(\varepsilon, p) \in ]0, 1[ \times ]N, \frac{N}{1-\varepsilon}[$ . Assume that  $\theta_0$  is in  $B_{N,1}^0(\mathbb{R}^N) \cap L^{\frac{N}{3}}(\mathbb{R}^N)$  and that the components of the divergence-free vector field  $u_0$  are in  $B_{p,1}^{\frac{N}{p}-1}(\mathbb{R}^N)$  and in the weak Lebesgue space  $L^{N,\infty}(\mathbb{R}^N)$ . If there exists a (small) positive constant  $c$  independent of  $p$  such that*

$$\|u_0\|_{L^{N,\infty}} + \nu^{-1} \|\theta_0\|_{L^{\frac{N}{3}}} \leq c\nu,$$

then Boussinesq system  $(B_{\nu,N})$  has a unique global solution

$$(\theta, u, \nabla\Pi) \in C(\mathbb{R}_+; B_{N,1}^0) \times (C(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}-1}) \cap L_{loc}^1(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}+1}))^N \times (L_{loc}^1(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}-1}))^N.$$

Moreover, suppose that the vector field  $X_0$  is in the space  $(\tilde{\mathcal{C}}^\varepsilon(\mathbb{R}^N))^N$  defined by

$$(\tilde{\mathcal{C}}^\varepsilon(\mathbb{R}^N))^N := \{Y \in (\mathcal{C}^\varepsilon(\mathbb{R}^N))^N : \operatorname{div} Y \in \mathcal{C}^\varepsilon(\mathbb{R}^N)\},$$

<sup>9</sup>We denote by  $\overline{A}$  the closure of the subset  $A$  in  $\mathbb{R}^N$ ,  $N \geq 2$ .

and that the components of  $(\partial_{X_0}\theta_0, \partial_{X_0}u_0)$  are in  $\mathcal{C}^{\varepsilon-2}(\mathbb{R}^N)$ . Then the System (1.3) has a unique solution  $X \in C_w(\mathbb{R}_+; \tilde{\mathcal{C}}^\varepsilon)$ , that satisfies for all  $t \geq 0$ ,

$$\|X\|_{L_t^\infty(\tilde{\mathcal{C}}^\varepsilon)} \leq C_{0,\nu} \exp(\exp(C_{0,\nu}t))$$

with some constant  $C_{0,\nu}$  depending only on the initial data and on  $\nu$ .

Furthermore, the triplet  $(\partial_X\theta, \partial_Xu, \partial_X\nabla\Pi)$  belongs to

$$C_w(\mathbb{R}_+; \mathcal{C}^{\varepsilon-2}) \times (C_w(\mathbb{R}_+; \mathcal{C}^{\varepsilon-2}) \cap \tilde{L}_{loc}^1(\mathbb{R}_+; \mathcal{C}^\varepsilon))^N \times (\tilde{L}_{loc}^1(\mathbb{R}_+; \mathcal{C}^{\varepsilon-2}))^N.$$

As in the 2-D case, the above result will enable us to solve the temperature patch problem. Before giving the exact statement, let us recall what a  $C^{1,\varepsilon}$  domain is in dimension  $N \geq 2$ .

**Definition.** A simply connected bounded domain  $\mathcal{D} \subset \mathbb{R}^N$  is of class  $C^{1,\varepsilon}$  if its boundary  $\partial\mathcal{D}$  is some compact hypersurface of class  $C^{1,\varepsilon}$ .

Fix some domain  $\mathcal{D}_0$  of class  $C^{1,\varepsilon}$  and further consider another  $C^1$  simple bounded domain  $J_0$  such that  $\overline{\mathcal{D}_0} \subset J_0$ . Then we have the following statement<sup>10</sup>:

**Corollary 1.4.** Let  $N \geq 3$  and  $(m_1, m_2)$  be a pair of sufficiently small constants. Assume that  $\theta_0 = m_1 \mathbb{1}_{\mathcal{D}_0}$  and that the initial vorticity  $\Omega_0 := \text{curl } u_0$ , i.e.  $(\Omega_0)_j^i := \partial_j u_0^i - \partial_i u_0^j$  for any  $i, j = 1, \dots, N$ , satisfies  $\Omega_0 := m_2 \mathbb{1}_{J_0} \mathbb{A}_0$  where  $\mathbb{A}_0$  stands for the anti-symmetric matrix defined by  $(\mathbb{A}_0)_j^i = 1$  for  $i < j$ .

Then  $\theta(t, \cdot) = m_1 \mathbb{1}_{\mathcal{D}_t}$  where  $\mathcal{D}(t) := \psi_u(t, \mathcal{D}_0)$  and  $\mathcal{D}(t)$  remains a simply connected domain of class  $C^{1,\varepsilon}$ , for any  $t \geq 0$ .

The rest of the paper unfolds as follows. In the next section, we shortly introduce Besov spaces and present some linear or nonlinear estimates, which will be needed to achieve our results. Then the proofs of main theorems for the propagation of striated regularity will be revealed in Section 3 (for 2-D case) and Section 4 (for N-D case). Section 5 is devoted to the temperature patch problems. Some technical commutator bounds are proved in the Appendix.

## 2. BASIC NOTATIONS AND LINEAR ESTIMATES

We here introduce definitions and notations that are used throughout the text, and recall some properties of Besov spaces and transport or transport-diffusion equations.

Let us begin with the definition of the *nonhomogeneous Littlewood-Paley decomposition* (for more details see [2], Chap. 2). Set  $\mathcal{B} := \{\xi \in \mathbb{R}^N : |\xi| \leq 4/3\}$  and  $\mathcal{C} := \{\xi \in \mathbb{R}^N : 3/4 \leq |\xi| \leq 8/3\}$ . We fix two smooth radial functions  $\chi$  and  $\varphi$ , supported in  $\mathcal{B}$  and  $\mathcal{C}$ , respectively, and such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^N.$$

We then introduce the Fourier multipliers  $\Delta_{-1} := \chi(D)$  and  $\Delta_j := \varphi(2^{-j}D)$  with  $j \geq 0$  (the so-called *nonhomogeneous dyadic blocks*) and the low frequency cut-off operator

$$S_j := \sum_{j' \leq j-1} \Delta_{j'}.$$

With those notations, the nonhomogeneous Besov space  $B_{p,r}^s(\mathbb{R}^N)$  may be defined by

$$B_{p,r}^s(\mathbb{R}^N) := \{u \in \mathcal{S}'(\mathbb{R}^N) : \|u\|_{B_{p,r}^s} := \left\| 2^{js} \|\Delta_j u\|_{L^p} \right\|_{\ell^r(\mathbb{N} \cup \{-1\})} < \infty\},$$

for  $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$ .

Let us next introduce the following *paraproduct* and *remainder* operators:

$$T_u v := \sum_{j \geq -1} S_{j-N_0} u \Delta_j v \quad \text{and} \quad R(u, v) \equiv \sum_{j \geq -1} \Delta_j u \tilde{\Delta}_j v := \sum_{\substack{j \geq -1 \\ |j-k| \leq N_0}} \Delta_j u \Delta_k v,$$

where  $N_0$  stands for some large enough (fixed) integer.

<sup>10</sup>Like in the 2-D case, it goes without saying that much more general initial velocities may be considered

The following decomposition, first introduced by J.-M. Bony in [3]:

$$(2.1) \quad uv = T_u v + T_v u + R(u, v),$$

holds true whenever the product of the two tempered distribution  $u$  and  $v$  is defined. It will play a fundamental role in our study.

Bilinear operators  $R$  and  $T$  possess continuity properties in a number of functional spaces (see e.g. Chap. 2 in [2]). We shall recall a few of them throughout the text, when needed.

When investigating evolutionary equations in Besov spaces and, in particular, parabolic type equations, it is natural to use the following *tilde Besov spaces* first introduced by J.-Y. Chemin in [8]: for any  $t \in ]0, \infty[$  and  $(s, p, r, \rho) \in \mathbb{R} \times [1, \infty]^3$ , we set

$$\tilde{L}_t^\rho(B_{p,r}^s(\mathbb{R}^N)) := \left\{ u \in \mathcal{S}'(]0, t[ \times \mathbb{R}^N) : \|u\|_{\tilde{L}_t^\rho(B_{p,r}^s)} := \|2^{js} \|\Delta_j u\|_{L^\rho(]0, t[; L^p)}\|_{\ell^r} < \infty \right\}.$$

In the particular case where  $p = r = 2$  (resp.  $p = r = \infty$ ),  $B_{p,r}^s$  coincides with the Sobolev space  $H^s$  (resp. the generalized Hölder space  $\mathcal{C}^s$ ), and we shall alternately denote

$$(2.2) \quad \tilde{L}_t^\rho(H^s(\mathbb{R}^N)) := \tilde{L}_t^\rho(B_{2,2}^s(\mathbb{R}^N)) \quad \text{and} \quad \tilde{L}_t^\rho(\mathcal{C}^s(\mathbb{R}^N)) := \tilde{L}_t^\rho(B_{\infty,\infty}^s(\mathbb{R}^N)).$$

Let us next state some a priori estimates for the transport and transport-diffusion equations in (nonhomogeneous) Besov spaces.

**Proposition 2.1.** *Assume that  $v$  is a divergence free vector field. Let  $(p, p_1, r, \rho, \rho_1) \in [1, \infty]^5$  and  $s \in \mathbb{R}$  satisfy*

$$-1 - \min\left(\frac{N}{p_1}, \frac{N}{p'}\right) < s < 1 + \min\left(\frac{N}{p}, \frac{N}{p_1}\right) \quad \text{and} \quad \rho_1 \leq \rho.$$

Let  $f$  be a smooth solution of the following transport-diffusion equation with diffusion parameter  $\nu \geq 0$ :

$$(TD_\nu) \quad \begin{cases} \partial_t f + \operatorname{div}(fv) - \nu \Delta f = g, \\ f|_{t=0} = f_0. \end{cases}$$

Then there exists a constant  $C$  depending on  $N, p, p_1$  and  $s$  such that for all  $t \geq 0$ ,

$$(2.3) \quad \nu^{\frac{1}{p}} \|f\|_{\tilde{L}_t^\rho(B_{p,r}^{s+\frac{2}{p}})} \leq C e^{C(1+\nu t)^{\frac{1}{p}} V_{p_1}(t)} \left( (1+\nu t)^{\frac{1}{p}} \|f_0\|_{B_{p,r}^s} + (1+\nu t)^{1+\frac{1}{p}-\frac{1}{\rho_1}-\frac{1}{\nu\rho_1}-1} \|g\|_{\tilde{L}_t^{\rho_1}(B_{p,r}^{s-2+\frac{2}{\rho_1}})} \right),$$

where

$$V_{p_1}(t) := \int_0^t \|\nabla v\|_{B_{p_1, \infty}^{\frac{N}{p_1}} \cap L^\infty} dt'.$$

In the limit case  $s = -1 - \min(\frac{N}{p_1}, \frac{N}{p'})$ , one just need to refine  $\|\nabla v\|_{B_{p_1, \infty}^{\frac{N}{p_1}} \cap L^\infty}$  by  $\|\nabla v\|_{B_{p_1, 1}^{\frac{N}{p_1}}}$  in the definition of  $V_{p_1}$ .

*Proof.* The above statement has been proved in [12] in the case  $p \leq p_1$ . The generalization to the case  $p > p_1$  is straightforward: in fact that restriction came from the following commutator estimate<sup>11</sup>:

$$\|2^{js} \|[v \cdot \nabla, \Delta_j]f\|_{L^p}\|_{\ell^r} \lesssim \|\nabla v\|_{B_{p, 1}^{\frac{N}{p_1}}} \|f\|_{B_{p,r}^s},$$

that has been proved only in the case  $p \leq p_1$ .

To handle the case  $p > p_1$ , we proceed exactly as in [12], decomposing  $R_j := [v \cdot \nabla, \Delta_j]f$  into  $R_j = \sum_{i=1}^6 R_j^i$  with

$$\begin{aligned} R_j^1 &:= [T_{\tilde{v}^k}, \Delta_j] \partial_k f, & R_j^2 &:= T_{\partial_k \Delta_j f} \tilde{v}^k, \\ R_j^3 &:= -\Delta_j T_{\partial_k f} \tilde{v}^k, & R_j^4 &:= \partial_k R(\tilde{v}^k, \Delta_j f), \\ R_j^5 &:= -\partial_k \Delta_j R(\tilde{v}^k, f), & R_j^6 &:= [S_0 v^k, \Delta_j] \partial_k f, \end{aligned}$$

where  $\tilde{v} := v - S_0 v$ .

<sup>11</sup>We here adopt the usual notation  $[A, B]$  for the commutator  $AB - BA$ .

In [12], Condition  $p \leq p_1$  is used only when bounding  $R_j^3$ . Now, if  $p > p_1$  and  $s < 1 + \frac{N}{p}$  then combining standard continuity results for the paraproduct with the embedding  $B_{p,r}^{s-1}(\mathbb{R}^N) \hookrightarrow B_{\infty,r}^{s-1-\frac{N}{p}}(\mathbb{R}^N)$  implies that

$$\left\| 2^{j(s+\frac{N}{p_1}-\frac{N}{p})} \|R_j^3\|_{L^p} \right\|_{\ell^r} \lesssim \|\nabla v\|_{B_{p_1,\infty}^{\frac{N}{p_1}}} \|\nabla f\|_{B_{p,r}^{s-1}},$$

whence the desired inequality.  $\square$

**Remark 2.2.** *We shall often use the above proposition in the particular case  $\nu = 0$  and  $(p, r, \rho, \rho_1) = (\infty, \infty, \infty, 1)$ . Then Inequality (2.3) reduces to*

$$\|f\|_{L_t^\infty(\mathcal{C}^s)} \leq C e^{C\nu_{p_1}(t)} (\|f_0\|_{\mathcal{C}^s} + \|g\|_{\tilde{L}_t^1(\mathcal{C}^s)}) \quad \text{if} \quad -1 - \frac{N}{p_1} < s < 1.$$

We finally recall a refinement of Vishik's estimates for the transport equation [24] obtained by T. Hmidi and S. Keraani in [20], which is the key to the study of long-time behavior of the solution in critical spaces for Boussinesq system  $(B_{\nu,N})$  (see [1, 15]).

**Proposition 2.3.** *Assume that  $v$  is divergence-free and that  $f$  satisfies the transport equation  $(TD_0)$ . There exists a constant  $C$  such that for all  $(p, r) \in [1, \infty]^2$  and  $t > 0$ ,*

$$\|f\|_{\tilde{L}_t^\infty(B_{p,r}^0)} \leq C (\|f_0\|_{B_{p,r}^0} + \|g\|_{\tilde{L}_t^1(B_{p,r}^0)}) \left( 1 + \int_0^t \|\nabla v\|_{L^\infty} d\tau \right).$$

### 3. PROPAGATION OF STRIATED REGULARITY IN THE 2D-CASE

This section is devoted to the proof of Theorem 1.1. To simplify the computations, we shall first make the change of unknowns

$$(3.1) \quad \tilde{\theta}(t, x) = \nu^2 \theta(\nu t, x), \quad \tilde{u}(t, x) = \nu u(\nu t, x), \quad \tilde{P}(t, x) = \nu^2 P(\nu t, x)$$

so as to reduce the study to the case  $\nu = 1$ .

Throughout this section, we always assume that

$$(3.2) \quad 0 < \varepsilon < 1, \quad q > 1 \quad \text{and} \quad \frac{\varepsilon}{2} + \frac{1}{q} > 1,$$

in accordance with the hypotheses of Theorem 1.1.

**3.1. A priori estimates for the Lipschitz norm of the velocity field.** Those estimates will be based on the following global existence theorem (see [1]).

**Theorem 3.1.** *Let  $u_0$  be a divergence-free vector field belonging to the space  $(L^2(\mathbb{R}^2) \cap B_{\infty,1}^{-1}(\mathbb{R}^2))^2$  and let  $\theta_0$  be in  $B_{2,1}^0(\mathbb{R}^2)$ . Then there exists a unique global solution  $(u, \theta, \nabla \Pi)$  for System  $(B_{1,2})$  such that*

$$(3.3) \quad \begin{aligned} u &\in (C(\mathbb{R}_+; L^2 \cap B_{\infty,1}^{-1}) \cap L_{loc}^2(\mathbb{R}_+; H^1) \cap L_{loc}^1(\mathbb{R}_+; B_{\infty,1}^1))^2, \\ \theta &\in C_b(\mathbb{R}_+; B_{2,1}^0) \quad \text{and} \quad \nabla \Pi \in (L_{loc}^1(\mathbb{R}_+; B_{2,1}^0))^2. \end{aligned}$$

Moreover, for any  $t > 0$ , there exists a constant  $C_0$  depending only on the initial data such that

$$\|u\|_{L_t^1(B_{\infty,1}^1)} \leq C_0 e^{C_0 t^4}.$$

We claim that the data  $(\theta_0, u_0)$  of Theorem 1.1 fulfill the assumptions of the above theorem. Indeed, decompose  $u_0 = \Delta_{-1} u_0 + (\text{Id} - \Delta_{-1}) u_0$  and apply the following *Biot-Savart law*

$$\nabla u = -\nabla(-\Delta)^{-1} \nabla^\perp \omega \quad \text{with} \quad \nabla^\perp := (-\partial_2, \partial_1).$$

Then thanks to the obvious embedding  $B_{q,1}^{\frac{2}{q}-2}(\mathbb{R}^2) \hookrightarrow B_{2,1}^{-1}(\mathbb{R}^2)$ , we obtain that

$$(3.4) \quad \begin{aligned} \|u_0\|_{B_{\infty,1}^{-1}} &\lesssim \|u_0\|_{B_{2,1}^0} \lesssim \|\Delta_{-1} u_0\|_{L^2} + \|(\text{Id} - \Delta_{-1}) \nabla u_0\|_{B_{2,1}^{-1}} \\ &\lesssim \|u_0\|_{L^2} + \|\omega_0\|_{B_{q,1}^{\frac{2}{q}-2}}. \end{aligned}$$

Besides, the obvious embedding  $\theta_0 \in B_{q,1}^{\frac{2}{q}-1}(\mathbb{R}^2) \hookrightarrow B_{2,1}^0(\mathbb{R}^2)$  holds. Hence one may apply the above theorem to our data. The corresponding global solution  $(\theta, u)$  fulfills (3.3) and the following inequality for all  $t \geq 0$ ,

$$(3.5) \quad \|\nabla u\|_{L_t^1(L^\infty)} \lesssim \|u\|_{L_t^1(B_{\infty,1}^1)} \leq C_0 e^{C_0 t^4}.$$

**3.2. A priori estimates for  $\theta$  and  $\omega$ .** We now want to prove that  $(\theta, u)$  fulfills the additional property (1.8). To this end, the first observation is that  $\theta$  satisfies a free transport equation. Hence, from the standard theory of transport equations (apply Proposition 2.1 with  $\nu = 0$ ) and the bound (3.5), we deduce that

$$(3.6) \quad \|\theta\|_{\tilde{L}_t^\infty(B_{q,1}^{\frac{2}{q}-1})} \leq e^{C\|\nabla u\|_{L_t^1(L^\infty)}} \|\theta_0\|_{B_{q,1}^{\frac{2}{q}-1}} \leq \|\theta_0\|_{B_{q,1}^{\frac{2}{q}-1}} \exp(C_0 \exp(C_0 t^4)).$$

In order to bound  $\omega$ , one may apply Proposition 2.1 to the vorticity equation, which yields for all  $t \geq 0$ ,

$$\|\omega\|_{\tilde{L}_t^\infty(B_{q,1}^{\frac{2}{q}-2})} + \|\omega\|_{L_t^1(B_{q,1}^{\frac{2}{q}})} \leq C(1+t)e^{C(1+t)\|\nabla u\|_{L_t^1(L^\infty)}} (\|\omega_0\|_{B_{q,1}^{\frac{2}{q}-2}} + \|\nabla \theta\|_{L_t^1(B_{q,1}^{\frac{2}{q}-2})}).$$

Hence, taking advantage of (3.5) and (3.6), we get

$$(3.7) \quad \|\omega\|_{L_t^\infty(B_{q,1}^{\frac{2}{q}-2})} + \|\omega\|_{L_t^1(B_{q,1}^{\frac{2}{q}})} \leq C \exp(\exp(C_0 t^4)) (\|\omega_0\|_{B_{q,1}^{\frac{2}{q}-2}} + \|\theta_0\|_{B_{q,1}^{\frac{2}{q}-1}}).$$

Then Biot-Savart law allows to improve the regularity of the velocity  $u$  as follows:

$$(3.8) \quad U_q(t) := \|\nabla u\|_{L_t^1(B_{q,1}^{\frac{2}{q}})} \lesssim \|\omega\|_{L_t^1(B_{q,1}^{\frac{2}{q}})} \leq C_0 \exp(\exp(C_0 t^4)),$$

where  $C_0$  depends only on the Lebesgue and Besov norms of the data in Theorem 1.1.

**3.3. A priori estimates for the striated regularity.** We shall need the following lemma which is a straightforward generalization of Inequality (1.6).

**Lemma 3.2.** *For any  $\varepsilon \in ]0, 1[$ , there exists a constant  $C$  such that the following estimate holds true:*

$$(3.9) \quad \|\partial_X u\|_{\tilde{L}_t^1(\mathcal{C}^\varepsilon)} \lesssim \int_0^t \|\nabla u\|_{L^\infty} \|X\|_{\mathcal{C}^\varepsilon} dt' + \|\operatorname{div}(X\omega)\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-1})}.$$

As we explained in the introduction, the above lemma implies that it suffices to bound  $\operatorname{div}(X\omega)$  in  $\tilde{L}_t^1(\mathcal{C}^{\varepsilon-1})$  for all  $t \geq 0$  in order to propagate the Hölder regularity  $\mathcal{C}^\varepsilon$  of  $X$ . This will be based on Proposition 2.1, remembering that  $\operatorname{div}(X\omega)$  fulfills the transport-diffusion equation:

$$\partial_t \operatorname{div}(X\omega) + u \cdot \nabla \operatorname{div}(X\omega) - \Delta \operatorname{div}(X\omega) = f$$

with

$$f = \operatorname{div} F + \operatorname{div}(X\partial_1 \theta) \quad \text{and} \quad F := X\Delta\omega - \Delta(X\omega).$$

Thanks to Proposition 2.1, we have

$$(3.10) \quad \|\operatorname{div}(X\omega)\|_{L_t^\infty(\mathcal{C}^{\varepsilon-3})} + \|\operatorname{div}(X\omega)\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-1})} \lesssim (1+t) \left( \|\operatorname{div}(X_0\omega_0)\|_{\mathcal{C}^{\varepsilon-3}} + \int_0^t U'_q \|\operatorname{div}(X\omega)\|_{\mathcal{C}^{\varepsilon-3}} dt' + \|f\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-3})} \right),$$

where  $U_q$  has been defined in (3.8).

Let us now bound the source term  $f$  in  $\tilde{L}_t^1(\mathcal{C}^{\varepsilon-3})$  of (3.10). It is not hard to estimate  $F$  via the following decomposition:

$$F = [T_X, \Delta]\omega + T_{\Delta\omega}X + R(X, \Delta\omega) - \Delta T_\omega X - \Delta R(\omega, X).$$

Using the commutator estimates of Lemma A.1 and standard results of continuity for the paraproduct and remainder operators (see e.g. [2], Chap. 2), we get if Condition (3.2) is fulfilled,

$$(3.11) \quad \|F\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|\omega\|_{B_{q,1}^{\frac{2}{q}}} \|X\|_{\mathcal{C}^\varepsilon}.$$

The last part of  $f$  may be decomposed into

$$\operatorname{div}(X\partial_1 \theta) = \partial_1(\operatorname{div}(X\theta)) - \operatorname{div}(\theta\partial_1 X).$$

By Bony's decomposition (2.1),

$$\theta \partial_1 X = T_\theta \partial_1 X + T_{\partial_1 X} \theta + R(\theta, \partial_1 X).$$

As Condition (3.2) is fulfilled, we thus have

$$(3.12) \quad \|\theta \partial_1 X\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|\theta\|_{B_{q,1}^{\frac{2}{q}-1}} \|\partial_1 X\|_{\mathcal{C}^{\varepsilon-1}}.$$

Next, we see that

$$\operatorname{div}(X\theta) = \partial_X \theta + \theta \operatorname{div} X.$$

The last term  $\theta \operatorname{div} X$  may be bounded as in (3.12). Therefore, combining with Inequalities (3.11) and (3.12), integrating with respect to time, and using also the obvious embedding  $L_t^1(\mathcal{C}^{\varepsilon-3}) \hookrightarrow \tilde{L}_t^1(\mathcal{C}^{\varepsilon-3})$ , we end up with

$$(3.13) \quad \|f\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-3})} \lesssim \int_0^t (\|\omega\|_{B_{q,1}^{\frac{2}{q}}} + \|\theta\|_{B_{q,1}^{\frac{2}{q}-1}}) \|X\|_{\mathcal{C}^\varepsilon} dt' + \|\partial_X \theta\|_{L_t^1(\mathcal{C}^{\varepsilon-2})}.$$

Next, resuming to the transport equation (1.3) satisfied by  $X$ , combining with the last item of Proposition 2.1, and using also (3.9) and (3.10), we get

$$(3.14) \quad \begin{aligned} \|X\|_{L_t^\infty(\mathcal{C}^\varepsilon)} &\leq \|X_0\|_{\mathcal{C}^\varepsilon} + C \int_0^t \|\nabla u\|_{L^\infty} \|X\|_{\mathcal{C}^\varepsilon} dt' + C \|\partial_X u\|_{\tilde{L}_t^1(\mathcal{C}^\varepsilon)} \\ &\leq \|X_0\|_{\mathcal{C}^\varepsilon} + C \int_0^t \|\nabla u\|_{L^\infty} \|X\|_{\mathcal{C}^\varepsilon} dt' + C \|\operatorname{div}(X\omega)\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-1})} \\ &\leq \|X_0\|_{\mathcal{C}^\varepsilon} + C(1+t) \|\operatorname{div}(X_0\omega_0)\|_{\mathcal{C}^{\varepsilon-3}} \\ &\quad + C(1+t) \left( \int_0^t U'_q (\|X\|_{\mathcal{C}^\varepsilon} + \|\operatorname{div}(X\omega)\|_{\mathcal{C}^{\varepsilon-3}}) dt' + \|f\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-3})} \right). \end{aligned}$$

Set

$$Z(t) := \|X\|_{L_t^\infty(\mathcal{C}^\varepsilon)} + \|\operatorname{div}(X\omega)\|_{L_t^\infty(\mathcal{C}^{\varepsilon-3})}$$

and

$$W(t) := U_q(t) + \int_0^t (\|\omega\|_{B_{q,1}^{\frac{2}{q}}} + \|\theta\|_{B_{q,1}^{\frac{2}{q}-1}}) dt'.$$

Observe that the bounds (3.6), (3.7) and (3.8) imply that

$$(3.15) \quad W(t) \leq C_0 \exp(\exp(C_0 t^4)) (\|\omega_0\|_{B_{q,1}^{\frac{2}{q}-2}} + \|\theta_0\|_{B_{q,1}^{\frac{2}{q}-1}}).$$

Then putting together (3.10), (3.13) and (3.14) yields

$$Z(t) \lesssim (1+t) \left( Z(0) + \|\partial_X \theta\|_{L_t^1(\mathcal{C}^{\varepsilon-2})} + \int_0^t W'(t') Z(t') dt' \right).$$

Hence, by virtue of Gronwall lemma,

$$(3.16) \quad Z(t) \leq C(1+t) (Z(0) + \|\partial_X \theta\|_{L_t^1(\mathcal{C}^{\varepsilon-2})}) e^{C(1+t)W(t)}.$$

Now, Proposition 2.1 and the fact that  $\partial_X \theta$  satisfies a free transport equation imply that

$$(3.17) \quad \|\partial_X \theta\|_{L_t^\infty(\mathcal{C}^{\varepsilon-2})} \leq e^{CU_q(t)} \|\partial_{X_0} \theta_0\|_{\mathcal{C}^{\varepsilon-2}}.$$

Adding (3.15) and (3.17) to (3.16), we thus obtain

$$(3.18) \quad Z(t) \leq C_0 \exp \left( \exp(\exp(C_0 t^4)) \right).$$

Resuming to (3.10), one can eventually conclude that

$$(3.19) \quad \|\operatorname{div}(X\omega)\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-1})} + \|\operatorname{div}(X\omega)\|_{L_t^\infty(\mathcal{C}^{\varepsilon-3})} \leq C_0 \exp \left( \exp(\exp(C_0 t^4)) \right).$$

**3.4. Completing the proof of Theorem 1.1.** In order to make the previous computations rigorous, we have to work on smooth solutions. To this end, we solve System  $(B_{1,2})$  with smoothed out initial data

$$(\theta_0^n, u_0^n) := (S_n \theta_0, S_n u_0).$$

It is clear by embedding and (3.4) that the components of  $(\theta_0, u_0)$  belong to all Sobolev spaces  $H^s(\mathbb{R}^2)$ . Thanks to the result in [4], we thus get a unique global smooth solution  $(\theta^n, u^n, \nabla \Pi^n)$  having Sobolev regularity of any order. Furthermore, as  $\theta_0^n$  belongs to all spaces  $B_{q,1}^s$  and satisfies a linear transport equation with a smooth velocity field, we are guaranteed that  $\theta^n \in C(\mathbb{R}_+; B_{q,1}^s)$  for all  $s \in \mathbb{R}$ . Then as  $\omega^n$  is the solution of the transport-diffusion equation with  $\omega_0^n \in B_{q,1}^s(\mathbb{R}^2)$  and source term  $\partial_1 \theta^n$  in  $C(\mathbb{R}_+; B_{q,1}^s(\mathbb{R}^2))$  for all  $s \in \mathbb{R}$ , we conclude that  $(\theta^n, \omega^n)$  belongs to all sets  $E_q^s$  defined by

$$E_q^s := \left\{ (\vartheta, \sigma) : \vartheta \in L_{loc}^\infty(\mathbb{R}_+; B_{q,1}^{\frac{2}{q}-1+s}), \sigma \in \tilde{L}_{loc}^\infty(\mathbb{R}_+; B_{q,1}^{\frac{2}{q}-2+s}) \cap L_{loc}^1(\mathbb{R}_+; B_{q,1}^{\frac{2}{q}+s}) \right\}.$$

Finally, regularizing  $X_0$  into  $X_0^n := S_n X_0$  and setting  $X^n(t, x) := (\partial_{X_0^n} \psi_{u^n})(\psi_{u^n}^{-1}(t, x))$ , we see that  $X^n$  belongs to Hölder spaces of any order, and satisfies (1.3) with velocity field  $u^n$ . The estimates that we proved so far are thus valid for  $(\theta^n, u^n, X^n)$ . In particular, Inequalities (3.6), (3.15) and (3.18) are satisfied for all  $n \in \mathbb{N}$ , with their r.h.s. depending on  $n$  through the (regularized) initial data respectively.

Deducing from (3.6) and (3.15) that the sequence  $(\theta^n, \omega^n)_{n \in \mathbb{N}}$  is bounded in  $E_q^0$  is obvious as  $S_n$  maps  $L^p$  to itself for all  $n \in \mathbb{N}$ , with a norm independent of  $n$ . This guarantees that

$$(3.20) \quad \|\theta_0^n\|_{B_{q,1}^{\frac{2}{q}-1}} \lesssim \|\theta_0\|_{B_{q,1}^{\frac{2}{q}-1}}, \quad \|u_0^n\|_{L^2} \lesssim \|u_0\|_{L^2} \quad \text{and} \quad \|\omega_0^n\|_{B_{q,1}^{\frac{2}{q}-2}} \lesssim \|\omega_0\|_{B_{q,1}^{\frac{2}{q}-2}}.$$

To justify the uniform boundedness of  $(X^n, \partial_{X^n} \theta^n, \text{div}(X^n \omega^n))$  in

$$(L_{loc}^\infty(\mathbb{R}_+; \mathcal{C}^\varepsilon))^2 \times L_{loc}^\infty(\mathbb{R}_+; \mathcal{C}^{\varepsilon-2}) \times (L_{loc}^\infty(\mathbb{R}_+; \mathcal{C}^{\varepsilon-3}) \cap \tilde{L}_{loc}^1(\mathbb{R}_+; \mathcal{C}^{\varepsilon-1})),$$

it is only a matter of checking that the ‘constant’  $C_0^n$  that appears in the r.h.s. of (3.17) and (3.18) could be bounded independently of  $n$ . In fact, besides the norms appearing in (3.20),  $C_0^n$  depends only (continuously) on

$$\|X_0^n\|_{\mathcal{C}^\varepsilon}, \quad \|\partial_{X_0^n} \theta_0^n\|_{\mathcal{C}^{\varepsilon-2}} \quad \text{and} \quad \|\text{div}(X_0^n \omega_0^n)\|_{\mathcal{C}^{\varepsilon-3}}.$$

Arguing as in (3.20), we see that  $\|X_0^n\|_{\mathcal{C}^\varepsilon}$  can be uniformly controlled by  $\|X_0\|_{\mathcal{C}^\varepsilon}$ . Furthermore, combining Lemma A.3 and Lemma 2.97 in [2], we get

$$\|\partial_{X_0^n} \theta_0^n\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|\partial_{X_0} \theta_0\|_{\mathcal{C}^{\varepsilon-2}} + \|\theta_0\|_{B_{q,1}^{\frac{2}{q}-1}} \|X_0\|_{\mathcal{C}^\varepsilon}.$$

Finally, we claim that

$$\|\text{div}(X_0^n \omega_0^n)\|_{\mathcal{C}^{\varepsilon-3}} \lesssim \|\text{div}(X_0 \omega_0)\|_{\mathcal{C}^{\varepsilon-3}} + \|\omega_0\|_{B_{q,1}^{\frac{2}{q}-2}} \|X_0\|_{\mathcal{C}^\varepsilon}.$$

This is a consequence of the decomposition

$$\text{div}(Yf) - \mathcal{T}_Y f = \text{div}(T_f Y + R(f, Y)) + [\partial_k, T_{Y^k}] f,$$

applied to  $(Y, f) = (X_0^n, \omega_0^n)$  or  $(X_0, \omega_0)$ , and of the fact that

$$\mathcal{T}_{X_0^n} \omega_0^n = S_n \mathcal{T}_{X_0} \omega_0 + [T_{(X_0^n)^k}, S_n] \partial_k \omega_0 + \mathcal{T}_{(X_0^n - X_0)} \omega_0^n,$$

where the last term  $\mathcal{T}_{(X_0^n - X_0)} \omega_0^n$  vanishes if  $N_0$  in (2.1) is taken larger than 1.

Let us now establish that  $(\theta^n, u^n, \nabla \Pi^n)$  converges (strongly) to some solution  $(\theta, u, \nabla \Pi)$  of  $(B_{1,2})$  belonging to the space  $F_2^0(\mathbb{R}^2)$  where, for all  $s \in \mathbb{R}$ ,  $p \in [1, \infty]$  and  $N \geq 2$ , we set

$$F_p^s(\mathbb{R}^N) := \left\{ (\vartheta, v, \nabla P) : \vartheta \in \tilde{L}_{loc}^\infty(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}-1+s}(\mathbb{R}^N)), \nabla P \in \left( L_{loc}^1(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}-1+s}(\mathbb{R}^N)) \right)^N \right. \\ \left. \text{and } v \in \left( \tilde{L}_{loc}^\infty(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}-1+s}(\mathbb{R}^N)) \cap L_{loc}^1(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}+1+s}(\mathbb{R}^N)) \right)^N \right\}.$$

To this end, we shall first prove that  $(\theta^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$  is a Cauchy sequence in the space  $F_2^{-1}(\mathbb{R}^2)$ . Indeed, if setting

$$(\delta \theta_n^m, \delta u_n^m, \delta \Pi_n^m) := (\theta^m - \theta^n, u^m - u^n, \Pi^m - \Pi^n),$$

then we get from  $(B_{1,N})$  that

$$(B_{1,N}^{m,n}) \quad \begin{cases} \partial_t \delta \theta_n^m + \operatorname{div} (u^m \delta \theta_n^m) = -\operatorname{div} (\delta u_n^m \theta^n), \\ \partial_t \delta u_n^m + \operatorname{div} (u^m \otimes \delta u_n^m) - \Delta \delta u_n^m + \nabla \delta \Pi_n^m = \delta \theta_n^m e_N - \operatorname{div} (\delta u_n^m \otimes u^n), \\ \operatorname{div} \delta u_n^m = 0. \end{cases}$$

In dimension  $N = 2$ , we infer from Bony decomposition (2.1) and continuity results for the paraproduct and remainder, that

$$\|\operatorname{div} (\delta u_n^m \theta^n)\|_{B_{2,1}^{-1}} \lesssim \|\delta u_n^m\|_{B_{2,1}^1} \|\theta^n\|_{B_{2,1}^0} \quad \text{and} \quad \|\operatorname{div} (\delta u_n^m \otimes u^n)\|_{B_{2,1}^{-1}} \lesssim \|\delta u_n^m\|_{B_{2,1}^{-1}} \|u^n\|_{B_{2,1}^2}.$$

Using the above two inequalities and the estimates for the transport and transport-diffusion equations stated in Proposition 2.1, we end up with

$$\begin{aligned} \|\delta \theta_n^m\|_{\tilde{L}_t^\infty(B_{2,1}^{-1})} &\leq e^{C\|\nabla u^m\|_{L_t^1(B_{2,1}^1)}} \left( \|\delta \theta_n^m(0)\|_{B_{2,1}^{-1}} + \int_0^t \|\delta u_n^m\|_{B_{2,1}^1} \|\theta^n\|_{B_{2,1}^0} dt' \right), \\ \mathcal{E}_n^m(t) &\lesssim (1+t^2) e^{C(1+t)\|\nabla u^m\|_{L_t^1(B_{2,1}^1)}} \left( \|\delta u_n^m(0)\|_{B_{2,1}^{-1}} + \|\delta \theta_n^m(0)\|_{B_{2,1}^{-1}} \right. \\ &\quad \left. + \int_0^t (\|u^n\|_{B_{2,1}^1} + \|\theta^n\|_{B_{2,1}^0}) \mathcal{E}_n^m(t') dt' \right), \end{aligned}$$

where  $\mathcal{E}_n^m(t) := \|\delta u_n^m\|_{\tilde{L}_t^\infty(B_{2,1}^{-1})} + \|\delta \theta_n^m\|_{L_t^1(B_{2,1}^1)}$ .

Combining Gronwall lemma with the uniform bounds that we proved for  $(\theta^n, u^n)$ , we deduce that  $(\theta^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$  strongly converges in the norm of  $\|\cdot\|_{F_2^{-1}(\mathbb{R}^2)}$ . Then interpolating with the uniform bounds, we see that strong convergence holds true in  $F_2^{-\eta}(\mathbb{R}^2)$ , too, for all  $\eta > 0$ , which allows to justify that  $(\theta, u, \nabla \Pi)$  satisfies  $(B_{1,2})$ . Taking advantage of Fatou property of Besov spaces, we gather that the limit  $(\theta, u, \nabla \Pi)$  belongs to  $F_2^0(\mathbb{R}^2)$ . In addition, as the sequence  $(\theta^n, \omega^n)$  is bounded in  $E_q^0$ , Fatou property also implies that  $(\theta, \omega)$  belongs to  $E_q^0$ . Finally, using (complex) interpolation, we obtain that for any  $0 < \eta, \delta < 1$ , sequence  $(\theta^n, \omega^n)_{n \in \mathbb{N}}$  converges to  $(\theta, \omega)$  in the space  $E_{q_\delta}^{-\delta\eta}$  with  $q_\delta := \frac{\delta}{2} + \frac{1-\delta}{q}$ .

To complete the proof of Theorem 1.1, we have to check that the announced proprieties of striated regularity are fulfilled. In fact, taking advantage of the (uniform) Lipschitz-continuity of  $u^n$ , we may obtain that for all  $\eta > 0$  (see [7]),

$$(3.21) \quad X = \lim_{n \rightarrow \infty} X^n \text{ in } L_{loc}^\infty(\mathbb{R}_+; \mathcal{C}^{\varepsilon-\eta}),$$

which allows to justify that  $X$  satisfies (1.3), and also that for all  $\eta' > 0$ , Sequence  $(\partial_{X^n} \theta^n, \operatorname{div} (X^n \omega^n))_{n \in \mathbb{N}}$  converges in the space

$$L_{loc}^\infty(\mathbb{R}_+; \mathcal{C}^{\varepsilon-2-\eta'}) \times L_{loc}^1(\mathbb{R}_+; \mathcal{C}^{\varepsilon-1-\eta'}).$$

As we know in addition that (3.17) and (3.19) are fulfilled by  $\partial_{X^n} \theta^n$  and  $\operatorname{div} (X^n \omega^n)$  for all  $n \in \mathbb{N}$ , Fatou property enables us to conclude that we have

$$\partial_X \theta \in L_{loc}^\infty(\mathbb{R}_+; \mathcal{C}^{\varepsilon-2}) \quad \text{and} \quad \operatorname{div} (X\omega) \in L_{loc}^\infty(\mathbb{R}_+; \mathcal{C}^{\varepsilon-3}) \times \tilde{L}_{loc}^1(\mathbb{R}_+; \mathcal{C}^{\varepsilon-1}),$$

and that both (3.17) and (3.19) hold true.

#### 4. PROPAGATION OF STRIATED REGULARITY IN THE GENERAL CASE $N \geq 3$

This section is dedicated to the proof of Theorem 1.3. The first (easy) step is to extend the global existence result of R. Danchin and M. Paicu in [15], to nonhomogeneous Besov spaces. The second step is to propagate striated regularity. As pointed out in the introduction, it suffices to bound  $\partial_X u$  in the space  $\tilde{L}_t^1(\mathcal{C}^\varepsilon)$ , which requires our using the smoothing properties of the heat flow. By applying the directional derivative  $\partial_X$  to the velocity equation, we discover that  $\partial_X u$  may be seen as the solution to some evolutionary Stokes system with a *nonhomogeneous* divergence condition, namely (using the fact that  $\operatorname{div} u = 0$ ),

$$\operatorname{div} (\partial_X u) = \operatorname{div} (\partial_u X - u \operatorname{div} X).$$

In order to reduce our study to that of a solution to the standard Stokes system, it is thus natural to decompose  $\partial_X u$  into

$$\partial_X u = z + \partial_u X - u \operatorname{div} X.$$

Now,  $z$  is indeed divergence free. Unfortunately, the above decomposition introduces additional source terms in the Stokes system satisfied by  $z$ . One of those terms is  $\Delta \partial_u X$ , that we do not know how to handle without losing regularity. Moreover, this also leads to the failure of global estimates technically.

To overcome this difficulty, we here propose to replace the directional derivative  $\partial_X$  by the *para-vector field*  $\mathcal{T}_X$  that is defined by

$$\mathcal{T}_X u := T_{X^k} \partial_k u.$$

The reason why this change of viewpoint is appropriate is that, as we shall see below, the aforementioned loss of regularity does not occur anymore when estimating  $\mathcal{T}_X u$ , and that  $\mathcal{T}_X u$  may be seen of the leading order part of  $\partial_X u$ , as shown by the following two inequalities (that hold true whenever  $\frac{N}{p} + \varepsilon - 1 \geq 0$ , see Lemma A.3):

$$(4.1) \quad \|\partial_X u - \mathcal{T}_X u\|_{L_t^\infty(\mathcal{C}^{\varepsilon-2})} \lesssim \|X\|_{L_t^\infty(\tilde{\mathcal{C}}^\varepsilon)} \|u\|_{L_t^\infty(B_{p,1}^{\frac{N}{p}-1})},$$

$$(4.2) \quad \|\partial_X u - \mathcal{T}_X u\|_{L_t^1(\mathcal{C}^\varepsilon)} \lesssim \|X\|_{L_t^\infty(\mathcal{C}^\varepsilon)} \|u\|_{L_t^1(B_{p,1}^{\frac{N}{p}+1})}.$$

Thus the second step of the proof of Theorem 1.3 will be mainly devoted to proving a priori estimates for  $\mathcal{T}_X u$  in  $(L_t^\infty(\mathcal{C}^{\varepsilon-2}) \cap \tilde{L}_t^1(\mathcal{C}^\varepsilon))^N$ .

Finally, in the last step of the proof, we smooth out the data (as in the 2-D case) so as to justify that the a priori estimates of the first two steps are indeed satisfied.

**4.1. Global existence in nonhomogeneous Besov spaces.** The following result is an obvious modification of the work by the first author and M. Paicu in [15].

**Theorem 4.1.** *Let  $N \geq 3$  and  $p \in ]N, \infty[$ . Assume that  $\theta_0 \in B_{N,1}^0(\mathbb{R}^N) \cap L^{\frac{N}{3}}(\mathbb{R}^N)$  and that the initial divergence-free velocity field  $u_0$  is in  $(B_{p,1}^{\frac{N}{p}-1}(\mathbb{R}^N) \cap L^{N,\infty}(\mathbb{R}^N))^N$ . There exists a (small) positive constant  $c$  independent of  $p$  such that if*

$$(4.3) \quad \|u_0\|_{L^{N,\infty}} + \nu^{-1} \|\theta_0\|_{L^{\frac{N}{3}}} \leq c\nu,$$

then the Boussinesq system  $(B_{\nu,N})$  has a unique global solution

$$(\theta, u, \nabla \Pi) \in C(\mathbb{R}_+; B_{N,1}^0) \cap (C(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}-1}) \cap L_{loc}^1(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}+1}))^N \cap (L_{loc}^1(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}-1}))^N.$$

Moreover, there exists some positive constant  $C$  independent on  $N$  such that for any  $t \geq 0$ , we have

$$(4.4) \quad \|\theta(t)\|_{L^{\frac{N}{3}}} = \|\theta_0\|_{L^{\frac{N}{3}}}, \quad \|u(t)\|_{L^{N,\infty}} \leq C(\|u_0\|_{L^{N,\infty}} + \nu^{-1} \|\theta_0\|_{L^{\frac{N}{3}}}),$$

$$(4.5) \quad U_p(t) \leq A(t),$$

$$(4.6) \quad \|\theta\|_{\tilde{L}_t^\infty(B_{N,1}^0)} \leq C\|\theta_0\|_{B_{N,1}^0} (1 + \nu^{-1} A(t)),$$

$$(4.7) \quad \|(\partial_t u, \nabla \Pi)\|_{L_t^1(B_{p,1}^{\frac{N}{p}-1})} \leq C\nu^{-1} A^2(t) + Ct\|\theta_0\|_{B_{N,1}^0} (1 + A(t)),$$

where

$$U_p(t) := \|u\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}-1})} + \nu \|u\|_{L_t^1(B_{p,1}^{\frac{N}{p}+1})} \quad \text{and}$$

$$A(t) := C(\|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} + \nu) e^{C\nu^{-1}t\|\theta_0\|_{B_{N,1}^0}} + C\nu^2 \left( \frac{\|\theta_0\|_{B_{N,1}^0} + \nu^2}{\|\theta_0\|_{B_{N,1}^0}^2} \right) \left( e^{C\nu^{-1}t\|\theta_0\|_{B_{N,1}^0}} - 1 \right).$$

The above nonhomogeneous estimates are based on a very simple observation allowing to control the low frequency part of the velocity field : for  $r$  satisfying  $1 + \frac{1}{p} = \frac{1}{N} + \frac{1}{r}$ , we have as a consequence of refined Young's inequalities (see [2]):

$$(4.8) \quad \|\Delta_{-1}u(t)\|_{L^p} \lesssim \|\mathcal{F}^{-1}\chi\|_{L^r} \|u(t)\|_{L^{N,\infty}},$$

whose r.h.s. is bounded according to (4.4) thanks to Theorem 1.4 in [15].

For the reader convenience, let us outline the proof of (4.6) and (4.7) for a smooth solution of  $(B_{\nu,N})$ . Let  $\mathbb{P} := \text{Id} - \nabla\Delta^{-1}\text{div}$  be the Leray projector onto divergence-free vector fields. On one hand, applying  $\mathbb{P}\Delta_j$  to the momentum equation in  $(B_{\nu,N})$  and noting that Lemma 4.4 in [15] still holds for nonhomogeneous Besov space, we get some constant  $C$  and sequence  $(c_j)_{j \in \mathbb{N}}$  where  $\|(c_j)\|_{\ell^1} \leq 1$  such that for any  $j \geq 0$  and  $t \geq 0$ ,

$$(4.9) \quad \begin{aligned} \|\Delta_j u(t)\|_{L^p} &\leq e^{-C\nu t 2^{2j}} \|\Delta_j u_0\|_{L^p} + \int_0^t e^{-C\nu(t-t')2^{2j}} (\|\Delta_j \theta\|_{L^p} \\ &\quad + C2^{j(\frac{N}{p}-1)} c_j \|u\|_{\mathcal{C}^{-1}} \|u\|_{B_{p,1}^{\frac{N}{p}+1}}) dt'. \end{aligned}$$

On the other hand, the low frequency part of  $u$  could be controlled according to (4.8), and we get assuming the smallness condition (4.3) on initial data,

$$(4.10) \quad \begin{aligned} \|\Delta_{-1}u\|_{L_t^\infty(L^p)} + \nu \|\Delta_{-1}u\|_{L_t^1(L^p)} &\lesssim (1 + \nu t) \|u\|_{L_t^\infty(L^{N,\infty})} \\ &\lesssim (1 + \nu t) (\|u_0\|_{L^{N,\infty}} + \nu^{-1} \|\theta_0\|_{L^{\frac{N}{3}}}). \end{aligned}$$

Putting (4.9) and (4.10) together thus implies that

$$U_p(t) \lesssim \|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} + \|\theta\|_{L_t^1(B_{p,1}^{\frac{N}{p}-1})} + \nu^{-1} \|u\|_{L_t^\infty(\mathcal{C}^{-1})} U_p(t) + (1 + \nu t) (\|u_0\|_{L^{N,\infty}} + \nu^{-1} \|\theta_0\|_{L^{\frac{N}{3}}}).$$

As  $L^{N,\infty} \hookrightarrow \mathcal{C}^{-1}$  and (4.4) is fulfilled, the third term of the r.h.s. above may be absorbed by the l.h.s. if  $c$  is small enough in (4.3).

As regards  $\theta$ , thanks to Proposition 2.3 and to the embedding  $B_{N,1}^0 \hookrightarrow B_{p,1}^{\frac{N}{p}-1}$  for  $p \geq N$ , we have for all  $t \geq 0$ ,

$$(4.11) \quad \|\theta(t)\|_{B_{p,1}^{\frac{N}{p}-1}} \lesssim \|\theta\|_{\tilde{L}_t^\infty(B_{N,1}^0)} \lesssim \|\theta_0\|_{B_{N,1}^0} (1 + \|\nabla u\|_{L_t^1(L^\infty)}).$$

Adding up the above estimates to the bound of  $U_p(t)$  yields

$$U_p(t) \leq C(\|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} + c\nu) + C(\|\theta_0\|_{B_{N,1}^0} + c\nu^2)t + C\nu^{-1} \|\theta_0\|_{B_{N,1}^0} \int_0^t U_p dt'.$$

The fact that  $\int_0^t t' e^{B(t-t')} dt' \leq \frac{1}{B^2} (e^{Bt} - 1)$  for any positive constant  $B$ , allows to bound the term  $U_p(t)$  by  $A(t)$  according to Gronwall lemma.

The estimates for  $\nabla\Pi$  can be deduced from the equation, as

$$\nabla\Pi = -\mathbb{Q}(u \cdot \nabla u + \theta e_N) \quad \text{with} \quad \mathbb{Q} := \text{Id} - \mathbb{P}.$$

Finally, the bound for  $\partial_t u$  may be determined from the momentum equation.

**4.2. A priori estimates for striated regularity.** In this subsection, we assume that  $\nu = 1$  for simplicity (which is not restrictive, owing to (3.1)).

As explained at the beginning of this section, we shall concentrate on the proof of estimates for  $\mathcal{T}_X u$  in  $L_t^\infty(\mathcal{C}^{\varepsilon-2}) \cap \tilde{L}_t^1(\mathcal{C}^\varepsilon)$ , for all  $t \geq 0$ , and this will be (mainly) based on the smoothing properties of the heat flow. More precisely, applying the para-vector field  $\mathcal{T}_X$  to the velocity equation of  $(B_{1,N})$ , we discover that

$$\partial_t \mathcal{T}_X u + u \cdot \nabla \mathcal{T}_X u - \Delta \mathcal{T}_X u + \nabla \mathcal{T}_X \Pi = g,$$

with

$$(4.12) \quad g := -[\mathcal{T}_X, \partial_t + u \cdot \nabla]u + [\mathcal{T}_X, \Delta]u - [\mathcal{T}_X, \nabla]\Pi + (\mathcal{T}_X \theta) e_N.$$

In general, the divergence-free property is not satisfied by  $\mathcal{T}_X u$  as

$$\text{div} \mathcal{T}_X u = \text{div} (T_{\partial_k X} u^k - T_{\text{div} X} u).$$

In order to enter into the standard theory for the Stokes system, we set

$$v := \mathcal{T}_X u - w \quad \text{with} \quad w := T_{\partial_k X} u^k - T_{\text{div}_X} u.$$

Now,  $v$  satisfies:

$$(S) \quad \begin{cases} \partial_t v - \Delta v + \nabla \mathcal{T}_X \Pi = \tilde{g}, \\ \text{div } v = 0, \\ v|_{t=0} = v_0, \end{cases}$$

with  $\tilde{g} := g - u \cdot \nabla \mathcal{T}_X u - (\partial_t w - \Delta w)$  and  $g$  defined in (4.12).

Based on that observation, proving a priori estimates for striated regularity involves three steps. The first step is dedicated to bounding  $\tilde{g}$  (which mainly requires the commutator estimates of the appendix). In the second step, we take advantage of the smoothing effect of the heat flow so as to estimate  $v$ . In the third step, we resume to  $\mathcal{T}_X u$  and also bound  $X$  so as to complete the proof of our study of striated regularity.

*First step: bounds of  $\tilde{g}$ .* To start with, let us bound  $g$  in  $(\tilde{L}_t^1(\mathcal{C}^{\varepsilon-2}))^N$  for any positive  $t$ . According to Proposition A.5 and to the remark that follows, the first commutator of  $g$  satisfies

$$(4.13) \quad \begin{aligned} \|[\mathcal{T}_X, \partial_t + u \cdot \nabla] u\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-2})} &\lesssim \|u\|_{L_t^\infty(\mathcal{C}^{\varepsilon-1})} \|\mathcal{T}_X u\|_{\tilde{L}_t^1(\mathcal{C}^\varepsilon)} \\ &\quad + \int_0^t \|u\|_{B_{p,1}^{\frac{N}{p}+1}} \|\mathcal{T}_X u\|_{\mathcal{C}^{\varepsilon-2}} dt' + \int_0^t \|X\|_{\tilde{\mathcal{C}}^\varepsilon} \|u\|_{B_{p,1}^{\frac{N}{p}+1}} \|u\|_{B_{p,1}^{\frac{N}{p}-1}} dt'. \end{aligned}$$

Next, thanks to the commutator estimates (A.1) we have the following two bounds,

$$(4.14) \quad \|[\mathcal{T}_X, \Delta] u\|_{L_t^1(\mathcal{C}^{\varepsilon-2})} \lesssim \int_0^t \|\nabla X\|_{\mathcal{C}^{\varepsilon-1}} \|\nabla u\|_{\mathcal{C}^0} dt',$$

$$(4.15) \quad \|[\mathcal{T}_X, \nabla] \Pi\|_{L_t^1(\mathcal{C}^{\varepsilon-2})} \lesssim \int_0^t \|\nabla X\|_{\mathcal{C}^{\varepsilon-1}} \|\nabla \Pi\|_{\mathcal{C}^{-1}} dt'.$$

Taking Lemma A.3 into account and integrating from 0 to  $t$  gives

$$(4.16) \quad \|\mathcal{T}_X \theta\|_{L_t^1(\mathcal{C}^{\varepsilon-2})} \lesssim \|\partial_X \theta\|_{L_t^1(\mathcal{C}^{\varepsilon-2})} + \int_0^t \|X\|_{\tilde{\mathcal{C}}^\varepsilon} \|\theta\|_{B_{p,1}^{\frac{N}{p}-1}} dt'.$$

Putting together (4.13) to (4.16) thus yields

$$\begin{aligned} \|g\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-2})} &\lesssim \|u\|_{L_t^\infty(\mathcal{C}^{\varepsilon-1})} \|\mathcal{T}_X u\|_{\tilde{L}_t^1(\mathcal{C}^\varepsilon)} + \int_0^t \|u\|_{B_{p,1}^{\frac{N}{p}+1}} \|\mathcal{T}_X u\|_{\mathcal{C}^{\varepsilon-2}} dt' \\ &\quad + \int_0^t (\|u\|_{B_{p,1}^{\frac{N}{p}+1}} (\|u\|_{B_{p,1}^{\frac{N}{p}-1}} + 1) + \|\nabla \Pi\|_{\mathcal{C}^{-1}} + \|\theta\|_{B_{p,1}^{\frac{N}{p}-1}}) \|X\|_{\tilde{\mathcal{C}}^\varepsilon} dt' + \|\partial_X \theta\|_{L_t^1(\mathcal{C}^{\varepsilon-2})}. \end{aligned}$$

Bounding the second term of  $\tilde{g}$  is obvious : taking advantage of Bony's decomposition, and remembering that  $\frac{N}{p} + \varepsilon > 1$  and that  $\text{div } u = 0$ , we get

$$\|u \cdot \nabla \mathcal{T}_X u\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-2})} \lesssim \|u\|_{L_t^\infty(\mathcal{C}^{\varepsilon-1})} \|\mathcal{T}_X u\|_{\tilde{L}_t^1(\mathcal{C}^\varepsilon)} + \int_0^t \|u\|_{B_{p,1}^{\frac{N}{p}+1}} \|\mathcal{T}_X u\|_{\mathcal{C}^{\varepsilon-2}} dt'.$$

Finally, to bound the term  $\partial_t w - \Delta w$  in  $\tilde{L}_t^1(\mathcal{C}^{\varepsilon-2})$ , we use the decomposition

$$\partial_t w - \Delta w = \sum_{\alpha=1}^3 W_\alpha,$$

with

$$W_1 := T_{\partial_k X} \partial_t u^k - T_{\text{div}_X} \partial_t u, \quad W_2 := T_{\partial_k \partial_t X} u^k - T_{\text{div}_X \partial_t X} u, \quad W_3 := \Delta(T_{\text{div}_X} u - T_{\partial_k X} u^k).$$

By embedding, it suffices to bound the components of  $W_\alpha$  in  $L_t^1(B_{p,1}^{\frac{N}{p}+\varepsilon-2})$  with  $\alpha = 1, 2, 3$ .

Now, standard continuity results for the paraproduct ensure that

$$\begin{aligned}\|W_1\|_{L_t^1(B_{p,1}^{\frac{N}{p}+\varepsilon-2})} &\lesssim \int_0^t \|\nabla X\|_{\mathcal{C}^{\varepsilon-1}} \|\partial_t u\|_{B_{p,1}^{\frac{N}{p}-1}} dt', \\ \|W_2\|_{L_t^1(B_{p,1}^{\frac{N}{p}+\varepsilon-2})} &\lesssim \int_0^t \|\partial_t X\|_{\mathcal{C}^{\varepsilon-2}} \|u\|_{B_{p,1}^{\frac{N}{p}+1}} dt', \\ \|W_3\|_{L_t^1(B_{p,1}^{\frac{N}{p}+\varepsilon-2})} &\lesssim \int_0^t \|\nabla X\|_{\mathcal{C}^{\varepsilon-1}} \|u\|_{B_{p,1}^{\frac{N}{p}+1}} dt'.\end{aligned}$$

To estimate  $\partial_t X$  in the bound of  $W_2$ , we use the fact that

$$\partial_t X = -u \cdot \nabla X + \partial_X u.$$

The convection term  $u \cdot \nabla X = \operatorname{div}(u \otimes X)$  may be easily bounded (use Bony's decomposition and continuity results for the remainder and paraproduct operators) as follows:

$$\|u \cdot \nabla X\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|u\|_{B_{p,1}^{\frac{N}{p}-1}} \|X\|_{\mathcal{C}^{\varepsilon}}.$$

Bounding  $\partial_X u$  according to Lemma A.3 yields

$$\|W_2\|_{L_t^1(B_{p,1}^{\frac{N}{p}+\varepsilon-2})} \lesssim \int_0^t (\|X\|_{\tilde{\mathcal{C}}^{\varepsilon}} \|u\|_{B_{p,1}^{\frac{N}{p}-1}} + \|\mathcal{T}_X u\|_{\mathcal{C}^{\varepsilon-2}}) \|u\|_{B_{p,1}^{\frac{N}{p}+1}} dt'.$$

Putting together all the above estimates and using (slightly abusively) the notation  $U'_p(\tau) := \|u(\tau)\|_{B_{p,1}^{\frac{N}{p}+1}}$  for any  $\tau \geq 0$ , we eventually get the following estimate for  $\tilde{g}$ :

$$(4.17) \quad \begin{aligned}\|\tilde{g}\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-2})} &\lesssim \|u\|_{L_t^\infty(\mathcal{C}^{\varepsilon-1})} \|\mathcal{T}_X u\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon})} + \|\partial_X \theta\|_{L_t^1(\mathcal{C}^{\varepsilon-2})} + \int_0^t U'_p \|\mathcal{T}_X u\|_{\mathcal{C}^{\varepsilon-2}} dt' \\ &\quad + \int_0^t (U'_p (\|u\|_{B_{p,1}^{\frac{N}{p}-1}} + 1) + \|(\partial_t u, \nabla \Pi, \theta)\|_{B_{p,1}^{\frac{N}{p}-1}}) \|X\|_{\tilde{\mathcal{C}}^{\varepsilon}} dt'.\end{aligned}$$

*Second step: Bounds for  $v$ .* The second step is devoted to bounding  $v$  in  $(L_t^\infty(\mathcal{C}^{\varepsilon-2}) \cap \tilde{L}_t^1(\mathcal{C}^{\varepsilon}))^N$ . Granted with (4.17), this will essentially follow from the smoothing properties of the heat flow after spectral localization. More precisely, applying  $\mathbb{P}\Delta_j$  to the equation (S) yields for all  $j \geq -1$ ,

$$\begin{cases} \partial_t \Delta_j v - \Delta \Delta_j v = \mathbb{P}\Delta_j \tilde{g} \\ \Delta_j v|_{t=0} = \Delta_j v_0. \end{cases}$$

Now, Lemma 2.1 in [8] implies that if  $j \geq 0$ ,

$$\|\Delta_j v(t)\|_{L^\infty} \leq e^{-ct2^{2j}} \|\Delta_j v_0\|_{L^\infty} + \int_0^t e^{-c(t-t')2^{2j}} \|\Delta_j \tilde{g}(t')\|_{L^\infty} dt'.$$

Therefore, taking the supremum over  $j \geq 0$ , we find that the high frequency part of  $v$  satisfies

$$\sup_{j \geq 0} 2^{j(\varepsilon-2)} \|\Delta_j v\|_{L_t^\infty(L^\infty)} + \sup_{j \geq 0} 2^{j\varepsilon} \|\Delta_j v\|_{L_t^1(L^\infty)} \lesssim \|v_0\|_{\mathcal{C}^{\varepsilon-2}} + \|\tilde{g}\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-2})}.$$

Bounding the norm  $\|v_0\|_{\mathcal{C}^{\varepsilon-2}}$  stems from the definition of  $v_0$  and from Lemma A.3: we get

$$(4.18) \quad \|v_0\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|\mathcal{T}_{X_0} u_0\|_{\mathcal{C}^{\varepsilon-2}} + \|u_0\|_{\mathcal{C}^{\varepsilon-1}} \|X_0\|_{\mathcal{C}^{\varepsilon}} \lesssim \|\partial_{X_0} u_0\|_{\mathcal{C}^{\varepsilon-2}} + \|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} \|X_0\|_{\tilde{\mathcal{C}}^{\varepsilon}}.$$

To handle the low frequencies of  $v$ , we first observe from the definition of  $\|\cdot\|_{\mathcal{C}^s}$  that

$$\|\Delta_{-1} v\|_{L_t^\infty(L^\infty)} + \|\Delta_{-1} v\|_{L_t^1(L^\infty)} \lesssim \|v\|_{L_t^\infty(\mathcal{C}^{\varepsilon-2})} + \|v\|_{L_t^1(L^\infty)}.$$

Then we recall the definition  $v = \mathcal{T}_X u - T_{\partial_k X} u^k + T_{\operatorname{div} X} u$ , and hence continuity results for paraproduct yield

$$\|v\|_{L_t^\infty(\mathcal{C}^{\varepsilon-2})} + \|v\|_{L_t^1(L^\infty)} \lesssim \|X\|_{L_t^\infty(L^\infty)} (\|u\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}-1})} + \|u\|_{L_t^1(B_{p,1}^{\frac{N}{p}+1})}).$$

Moreover, the fact that  $X$  satisfies the transport equation (1.3) ensures

$$\|X\|_{L_t^\infty(L^\infty)} \leq \|X_0\|_{L^\infty} e^{\|\nabla u\|_{L_t^1(L^\infty)}}.$$

Hence, using the notation  $U_p$  of Theorem 4.1, we control the low frequency part of  $v$  via

$$\|\Delta_{-1}v\|_{L_t^\infty(L^\infty)} + \|\Delta_{-1}v\|_{L_t^1(L^\infty)} \lesssim \|X_0\|_{L^\infty} U_p(t) e^{CU_p(t)}.$$

Putting the estimates for low and high frequency parts of  $v$  together with (4.18), we end up with

$$(4.19) \quad \widetilde{\mathcal{H}}(t) \lesssim \|\partial_{X_0} u_0\|_{\mathcal{C}^{\varepsilon-2}} + \|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} \|X_0\|_{\widetilde{\mathcal{C}}^\varepsilon} + \|\widetilde{g}\|_{\widetilde{L}_t^1(\mathcal{C}^{\varepsilon-2})} + \|X_0\|_{L^\infty} U_p(t) e^{CU_p(t)},$$

where we denoted

$$\widetilde{\mathcal{H}}(t) := \|v\|_{L_t^\infty(\mathcal{C}^{\varepsilon-2})} + \|v\|_{\widetilde{L}_t^1(\mathcal{C}^\varepsilon)}.$$

*Third step: bounds for striated regularity.* Keeping (4.1) and (4.2) in mind, the core of the proof consists in deriving appropriate bounds for  $\mathcal{T}_X u$  and  $X$ . Now, remembering that

$$\mathcal{T}_X u = v + w \quad \text{with} \quad w = T_{\partial_k X} u^k - T_{\text{div} X} u,$$

it is easy to bound the following quantity:

$$\mathcal{H}(t) := \|\mathcal{T}_X u\|_{L_t^\infty(\mathcal{C}^{\varepsilon-2})} + \|\mathcal{T}_X u\|_{\widetilde{L}_t^1(\mathcal{C}^\varepsilon)}.$$

Indeed, continuity results for paraproduct operators guarantee that

$$(4.20) \quad \|w\|_{L_t^\infty(\mathcal{C}^{\varepsilon-2})} + \|w\|_{\widetilde{L}_t^1(\mathcal{C}^\varepsilon)} \lesssim \|u\|_{L_t^\infty(\mathcal{C}^{\varepsilon-1})} \|X\|_{L_t^\infty(\mathcal{C}^\varepsilon)} + \int_0^t \|u\|_{\mathcal{C}^1} \|\nabla X\|_{\mathcal{C}^{\varepsilon-1}} dt'.$$

Let us now bound  $X$  in  $L_t^\infty(\widetilde{\mathcal{C}}^\varepsilon)$ . As  $X$  and  $\text{div} X$  satisfy (1.3) and (1.4), respectively, Hölder estimates for transport equations and Lemma A.3 imply that

$$(4.21) \quad \begin{aligned} \|X\|_{L_t^\infty(\widetilde{\mathcal{C}}^\varepsilon)} &\leq \|X_0\|_{\widetilde{\mathcal{C}}^\varepsilon} + \int_0^t \|\nabla u\|_{L^\infty} \|X\|_{\widetilde{\mathcal{C}}^\varepsilon} dt' + \|\partial_X u\|_{\widetilde{L}_t^1(\mathcal{C}^\varepsilon)} \\ &\leq \|X_0\|_{\widetilde{\mathcal{C}}^\varepsilon} + \int_0^t U_p' \|X\|_{\widetilde{\mathcal{C}}^\varepsilon} dt' + \|\mathcal{T}_X u\|_{\widetilde{L}_t^1(\mathcal{C}^\varepsilon)}. \end{aligned}$$

Resuming to (4.20), one may refine the bound of  $w$  as follows:

$$(4.22) \quad \|w\|_{L_t^\infty(\mathcal{C}^{\varepsilon-2})} + \|w\|_{\widetilde{L}_t^1(\mathcal{C}^\varepsilon)} \lesssim \|u\|_{L_t^\infty(\mathcal{C}^{\varepsilon-1})} \|\mathcal{T}_X u\|_{\widetilde{L}_t^1(\mathcal{C}^\varepsilon)} + \|X_0\|_{\widetilde{\mathcal{C}}^\varepsilon} + \int_0^t U_p' \|X\|_{\widetilde{\mathcal{C}}^\varepsilon} dt'.$$

By the embedding  $L^{N,\infty}(\mathbb{R}^N) \hookrightarrow \mathcal{C}^{-1}(\mathbb{R}^N)$  and smallness condition (4.3), we know that the quantity  $\|u\|_{L_t^\infty(\mathcal{C}^{\varepsilon-1})}$  is small. Therefore, from (4.17), (4.19) and (4.22), we infer that

$$(4.23) \quad \begin{aligned} \mathcal{H}(t) &\lesssim \|\partial_{X_0} u_0\|_{\mathcal{C}^{\varepsilon-2}} + \|X_0\|_{\widetilde{\mathcal{C}}^\varepsilon} (\|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} + 1) + \|X_0\|_{L^\infty} U_p(t) e^{U_p(t)} \\ &\quad + \|\partial_X \theta\|_{L_t^1(\mathcal{C}^{\varepsilon-2})} + \int_0^t U_p' \mathcal{H} dt' + \int_0^t (U_p'(U_p + 1) + \|(\partial_t u, \nabla \Pi, \theta)\|_{B_{p,1}^{\frac{N}{p}-1}}) \|X\|_{\widetilde{\mathcal{C}}^\varepsilon} dt'. \end{aligned}$$

Denoting

$$\mathcal{K}(t) := \mathcal{H}(t) + \|X\|_{L_t^\infty(\widetilde{\mathcal{C}}^\varepsilon)}$$

and using (4.21) and (4.23), we obtain that

$$\begin{aligned} \mathcal{K}(t) &\lesssim (\|\partial_{X_0} u_0\|_{\mathcal{C}^{\varepsilon-2}} + \|X_0\|_{\widetilde{\mathcal{C}}^\varepsilon} (\|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} + 1) + \|X_0\|_{L^\infty} U_p(t) e^{CU_p(t)} + \|\partial_X \theta\|_{L_t^1(\mathcal{C}^{\varepsilon-2})}) \\ &\quad + \int_0^t (U_p'(U_p + 1) + \|(\partial_t u, \nabla \Pi, \theta)\|_{B_{p,1}^{\frac{N}{p}-1}}) \mathcal{K}(t') dt'. \end{aligned}$$

In order to bound  $\partial_X \theta$ , it suffices to remember that

$$\partial_t \partial_X \theta + u \cdot \nabla \partial_X \theta = 0.$$

As  $\frac{N}{p} + \varepsilon > 1$ , applying Proposition 2.1 gives

$$\|\partial_X \theta\|_{L_t^\infty(\mathcal{C}^{\varepsilon-2})} \lesssim \|\partial_{X_0} \theta_0\|_{\mathcal{C}^{\varepsilon-2}} e^{CU_p(t)}.$$

Furthermore, using embedding and Vishik type estimate (4.11) for  $\theta$  yields

$$\|\theta\|_{L_t^\infty(B_{p,1}^{\frac{N}{p}-1})} \lesssim \|\theta\|_{\tilde{L}_t^\infty(B_{N,1}^0)} \lesssim \|\theta_0\|_{B_{N,1}^0} (1 + U_p(t)).$$

Applying Gronwall lemma to  $\mathcal{K}(t)$  and mingling with the above bounds for  $(\partial_X \theta, \theta)$  and with the bounds of Theorem 4.1 for  $U_p$  and  $(\partial_t u, \nabla \Pi)$  yield

$$\begin{aligned} \mathcal{K}(t) &\leq C(\|\partial_{X_0} u_0\|_{\mathcal{C}^{\varepsilon-2}} + \|X_0\|_{\tilde{\mathcal{C}}^\varepsilon} (\|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} + 1) + \|X_0\|_{L^\infty} + \|\partial_{X_0} \theta_0\|_{\mathcal{C}^{\varepsilon-2}}) \\ &\quad \exp\left(C((A^2(t) + 1)(t + 1)(\|\theta_0\|_{B_{N,1}^0} + 1))\right) \leq C_0 \exp(\exp(C_0 t)), \end{aligned}$$

where  $A(t)$  has been defined in Theorem 4.1. Then we deduce from (4.1) and (4.2) that

$$\|\partial_X u\|_{L_t^\infty(\mathcal{C}^{\varepsilon-2})} + \|\partial_X u\|_{\tilde{L}_t^1(\mathcal{C}^\varepsilon)} \lesssim (1 + A(t))\mathcal{K}(t).$$

Finally, in order to bound  $\partial_X \nabla \Pi$ , we use the identity

$$\partial_X \nabla \Pi - \nabla \mathcal{T}_X \Pi = \operatorname{div} R(X, \nabla \Pi) - R(\operatorname{div} X, \nabla \Pi) + T_{\nabla^2 \Pi} \cdot X - \mathcal{T}_{\nabla X} \Pi,$$

from which we deduce that

$$\|\partial_X \nabla \Pi - \nabla \mathcal{T}_X \Pi\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-2})} \lesssim \|X\|_{L_t^\infty(\tilde{\mathcal{C}}^\varepsilon)} \|\nabla \Pi\|_{L_t^1(B_{p,1}^{\frac{N}{p}-1})}.$$

Now, the fact that  $\nabla \mathcal{T}_X \Pi$  may be estimated in  $(\tilde{L}_{loc}^1(\mathbb{R}_+; \mathcal{C}^{\varepsilon-2}))^N$  is a consequence of

$$\nabla \mathcal{T}_X \Pi = \mathbb{Q} \tilde{g}$$

and of (4.17) once noticed that the projector  $\mathbb{Q}$  on gradient-like vector fields is a 0-th order Fourier multiplier and that the Fourier transform of  $\nabla \mathcal{T}_X \Pi$  is supported away from 0. This completes the proof of a priori estimates for striated regularity in Theorem 1.3.

**4.3. End of the proof of Theorem 1.3.** Like in 2-D case, we start by smoothing out the initial data:

$$(\theta^n, u_0^n) := (S_n \theta_0, S_n u_0).$$

Since  $S_n$  is a (uniformly) bounded operator on all Besov spaces  $B_{q,r}^s$ , Lebesgue spaces  $L^q$  and weak Lebesgue spaces  $L^{q,\infty}$  with  $1 < q < \infty$ , we can apply Theorem 4.1 and obtain for all  $n \in \mathbb{N}$  a unique global solution  $(\theta^n, u^n, \nabla \Pi^n)$  which fulfills Inequalities (4.4) to (4.7) with right-hand sides independent of  $n$ . In particular, sequence  $(\theta^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$  is bounded in the space  $F_p^0(\mathbb{R}^N)$  of Subsection 3.4.

If we consider as in the 2-D case, System  $(B_{1,N}^{m,n})$  satisfied by

$$(\delta \theta_n^m, \delta u_n^m, \delta \Pi_n^m) := (\theta^m - \theta^n, u^m - u^n, \Pi^m - \Pi^n),$$

and combine estimates for transport and transport-diffusion equations, and product laws like e.g

$$\begin{aligned} \|\operatorname{div}(\delta u_n^m \theta^n)\|_{B_{p,1}^{\frac{N}{p}-2}} &\lesssim \|\delta u_n^m\|_{B_{p,1}^{\frac{N}{p}}} \|\theta^n\|_{B_{p,1}^{\frac{N}{p}-1}} \\ \text{and } \|\operatorname{div}(\delta u_n^m \otimes u^n)\|_{B_{p,1}^{\frac{N}{p}-2}} &\lesssim \|\delta u_n^m\|_{B_{p,1}^{\frac{N}{p}-2}} \|u^n\|_{B_{p,1}^{\frac{N}{p}+1}}, \end{aligned}$$

then we can easily prove that the sequence  $(\theta^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$  strongly converges in  $F_p^{-1}(\mathbb{R}^N)$  to some limit  $(\theta, u, \nabla \Pi)$ . By uniform estimates and interpolation, we see that in addition convergences holds true in  $F_p^{-\eta}(\mathbb{R}^N)$ , for any  $\eta > 0$ , and functional analysis arguments similar to those of Subsection 3.4 allow to show that  $(\theta, u, \nabla \Pi)$  is in indeed a solution to  $(B_{1,N})$  that belongs to  $F_p^0$ . Moreover,  $\theta \in \tilde{L}_{loc}^\infty(\mathbb{R}_+; B_{N,1}^0)$ , owing to (4.6).

Next, to establish conservation of striated regularity, we smooth out  $X_0$  into  $X_0^n := S_n X_0$  and define  $X^n(t, x) := (\partial_{X_0^n} \psi_{u^n})(\psi_{u^n}^{-1}(t, x))$ . By adapting the arguments of the 2-D case, it is easy to make all the bounds obtained in the previous step independent of  $n$ , and to conclude to the full statement of Theorem 1.3. The details are left to the reader.

## 5. THE TEMPERATURE PATCH PROBLEM WITH HÖLDER REGULARITY

This section is devoted to solving the temperature patch problem, through Corollaries 1.2 and 1.4. We start with general considerations that will be useful both in the 2-D and in the N-D cases.

Because  $\mathcal{D}_0$  is a simply connected bounded  $C^{1,\varepsilon}$  domain of  $\mathbb{R}^N$  (see Definition above Corollary 1.4), it is orientable (as  $\partial\mathcal{D}_0$  is compact), and one can adopt the so-called level-set characterization (see Chap. 2 of [16] and references therein). More precisely, for any (bounded) open neighborhood  $V_0$  of  $\overline{\mathcal{D}_0}$ , there exists some open set  $W_0$  with  $\overline{\mathcal{D}_0} \subset W_0 \subset V_0$  and a function  $f_0 \in C^{1,\varepsilon}(\mathbb{R}^N; \mathbb{R})$  such that  $\partial\mathcal{D}_0 = f_0^{-1}(\{0\}) \cap W_0$ ,  $\nabla f_0$  is supported in  $\overline{W_0}$  and does not vanish on  $\partial\mathcal{D}_0$ . The key to that global characterization is the *tubular neighborhood theorem*. In the case  $N = 2$ , we fix  $V_0$  so that in addition  $V_0 \cap \overline{\mathcal{D}_0^*} = \emptyset$  (see Corollary 1.2) while if  $N \geq 3$ , we assume that  $\overline{V_0} \cap (\mathbb{R}^N \setminus J_0) = \emptyset$ . Finally, we also fix some smooth cut-off function  $\chi_0$  with compact support in  $V_0$ , and value 1 on  $W_0$ .

As for the classical vortex patch problem for the incompressible Euler equations, the result will come up as a consequence of persistence of striated regularity. Indeed, at time  $t$  the boundary of  $\mathcal{D}_t = \psi_u(t, \mathcal{D}_0)$  is the level set of the function  $f(t, \cdot) := f_0(\psi_u^{-1}(t, \cdot))$  satisfying

$$(5.1) \quad \begin{cases} \partial_t f + u \cdot \nabla f = 0, \\ f|_{t=0} = f_0. \end{cases}$$

In order to control the regularity in  $C^{1,\varepsilon}$  of  $\partial\mathcal{D}_t$ , it is thus sufficient to show that  $\nabla f$  remains in  $(C^{0,\varepsilon})^N$  for all time. In the 2-D case, the proof is rather direct because it will be possible to apply directly Theorem 1.1 with the (divergence free) vector field  $X_0 := \nabla^\perp f_0$  provided  $u_0$  satisfies suitable striated regularity properties with respect to  $X_0$ . In the N-D case, more vector fields will be needed, and they will not be divergence free any longer.

Before going to the proof of our corollaries, we would like to point out two important properties. The first one is that the characteristic function of any bounded  $C^1$  bounded simply connected domain  $\mathcal{D}$  of  $\mathbb{R}^N$  for  $N \geq 2$  (see [13] and references therein) satisfies

$$(5.2) \quad \mathbf{1}_{\mathcal{D}} \in B_{q,\infty}^{\frac{1}{q}}(\mathbb{R}^N) \hookrightarrow B_{p,1}^{\frac{N}{p}-1}(\mathbb{R}^N), \quad \forall (N-1) < q \leq p \leq \infty.$$

The second observation concerns  $L^N$  boundedness of velocity field  $u_0$ .

In the 2-D case, having the vorticity compactly supported, in  $L^r$  for some  $r > 1$ , and *mean free* ensures that  $u_0 \in (L^2(\mathbb{R}^2))^2$  (see e.g. Chap.3 in [22]).

In dimension  $N \geq 3$ , the  $L^N$  type information is an easy consequence of Biot-Savart law and Hardy-Littlewood-Sobolev inequality. Indeed, we have

$$(5.3) \quad \|u_0\|_{L^N(\mathbb{R}^N)} \lesssim \| |\cdot|^{1-N} \star \Omega_0 \|_{L^N(\mathbb{R}^N)} \lesssim \|\Omega_0\|_{L^{\frac{N}{2}}(\mathbb{R}^N)}.$$

In what follows, whenever  $Y$  is a (tangential) vector field on  $\mathbb{R}^N$ , we shall denote by  $\tilde{Y}$  its restriction along the surface  $\partial\mathcal{D}_t$ .

**5.1. The 2-D case.** This short paragraph is devoted to proving Corollary 1.2. Of course, one can assume with no loss of generality that the exponent  $r$  is in  $]1, \frac{2}{2-\varepsilon}[$  so that we have  $\omega_0 \in B_{q,1}^{\frac{2}{q}-2}(\mathbb{R}^2)$  for some  $1 < q < \frac{2}{2-\varepsilon}$ . Thus, remembering that  $u_0$  is in  $(L^2(\mathbb{R}^2))^2$  (as explained above), and using (5.2), we see that both  $\theta_0$  and  $u_0$  fulfill the regularity assumptions of Theorem 1.1.

Next, we observe that the boundary  $\partial\mathcal{D}_0$  of the initial patch is just some Jordan curve of the plane. Setting  $X_0 := \nabla^\perp f_0$  provides us with a parametrization  $\gamma_0$  of the curve  $\partial\mathcal{D}_0$  through the following ODE:

$$\begin{cases} \partial_\sigma \gamma_0(\sigma) = \tilde{X}_0(\gamma_0(\sigma)), \quad \forall \sigma \in \mathbb{S}^1, \\ \gamma_0(\sigma_0) = x_0 \in \partial\mathcal{D}_0. \end{cases}$$

It is obvious that  $\partial_{X_0} \theta_0 = \operatorname{div}(X_0 \theta_0) \equiv 0$  in the sense of distributions. Similarly, because  $X_0 \equiv 0$  on some neighborhood of  $\overline{\mathcal{D}_0^*}$ , we have  $\operatorname{div}(X_0 \omega_0) = \partial_{X_0} \omega_0 \equiv 0$ . Hence Theorem 1.1 applies.

In order to conclude the proof of Corollary 1.2, we observe that a parametrization for  $\partial\mathcal{D}_t$  is given by  $\gamma_t(\sigma) := \psi_u(t, \gamma_0(\sigma))$  and that, obviously,

$$\begin{cases} \partial_\sigma \gamma_t(\sigma) = \widetilde{X}_t(\gamma_t(\sigma)), \quad \forall \sigma \in \mathbb{S}^1, \\ \gamma_t(\sigma_0) = \psi_u(t, x_0) \in \partial\mathcal{D}_t, \end{cases} \quad \text{with } X_t(x) := (\partial_{X_0} \psi_u)(\psi_u^{-1}(t, x)).$$

As pointed out before,  $X_t$  satisfies Equation (1.3), from which we can deduce that  $X_t$  is in  $(C^{0,\varepsilon}(\mathbb{R}^2))^2$  for all  $t \geq 0$ , and thus  $\gamma_t \in C^{1,\varepsilon}(\mathbb{S}^1; \mathbb{R}^2)$ .

**5.2. The N-D case.** If we take data  $\theta_0$  and  $u_0$  fulfilling the hypotheses of Corollary 1.4 then (5.2) and standard embedding ensure that

$$\theta_0 \in B_{N,1}^0(\mathbb{R}^N) \quad \text{and} \quad (\Omega_0)_j^i \in B_{p,\infty}^{\frac{1}{p}}(\mathbb{R}^N) \hookrightarrow B_{p,1}^{\frac{N}{p}-2}(\mathbb{R}^N) \quad \text{as long as } p > \frac{N-1}{2}.$$

Since in addition  $u_0^i \in L^N(\mathbb{R}^N)$  (see (5.3)), we have  $u_0^i \in B_{p,1}^{\frac{N}{p}-1}(\mathbb{R}^N)$  as may be seen through the following decomposition

$$u_0^i = \Delta_{-1} u_0^i - \sum_j (\text{Id} - \Delta_{-1})(-\Delta)^{-1} \partial_j (\Omega_0)_j^i,$$

and the fact that  $(\text{Id} - \Delta_{-1})(-\Delta)^{-1} \partial_j$  is a homogeneous multiplier of degree  $-1$  away from a neighborhood of the origin. Indeed, we have

$$(5.4) \quad \|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} \lesssim \|\Omega_0\|_{L^{\frac{N}{2}}} + \|\Omega_0\|_{B_{p,1}^{\frac{1}{p}}}, \quad \forall p > \frac{N-1}{2}.$$

Therefore, if  $m_1$  and  $m_2$  in Corollary 1.4 are sufficiently small, then we may apply the first part of Theorem 1.3 and get a unique global solution  $(\theta, u)$  for System  $(B_{\nu,N})$ , satisfying (4.4), (4.5), (4.6) and (4.7).

In order to propagate the regularity of the temperature patch, we shall borrow the ideas of [11, 17] pertaining to the standard vortex patch problem. Firstly, we claim that it is sufficient to verify that  $\partial_{X_0} \Omega_0$  has regularity of  $\mathcal{C}^{\varepsilon-3}$  in order to gain  $\partial_{X_0} u_0 \in (\mathcal{C}^{\varepsilon-2}(\mathbb{R}^N))^N$ , for any vector field  $X_0$  in  $(\widetilde{\mathcal{C}}^\varepsilon(\mathbb{R}^N))^N$ . Indeed, Lemma A.3 yields for  $\frac{N}{p} + \varepsilon > 1$

$$\|\partial_{X_0} u_0\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|\mathcal{T}_{X_0} u_0\|_{\mathcal{C}^{\varepsilon-2}} + \|X_0\|_{\widetilde{\mathcal{C}}^\varepsilon} \|u_0\|_{B_{p,1}^{\frac{N}{p}-1}}.$$

Then by Biot-Savart Law, we have the following identity

$$\begin{aligned} \mathcal{T}_{X_0} u_0^i &= \sum_j T_{X_0^k} \partial_k \Delta^{-1} \partial_j (\Omega_0)_j^i \\ &= \sum_j (\Delta^{-1} \partial_j \psi_{N_0}(D)) \mathcal{T}_{X_0} (\Omega_0)_j^i + [T_{X_0^k}, \Delta^{-1} \partial_j \psi_{N_0}(D)] \partial_k (\Omega_0)_j^i, \end{aligned}$$

where  $\psi_{N_0}$  is some suitable (smooth) cut-off function away from 0. From the above identity and Lemma A.1, we deduce that

$$\|\mathcal{T}_{X_0} u_0\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|\mathcal{T}_{X_0} \Omega_0\|_{\mathcal{C}^{\varepsilon-3}} + \|X_0\|_{\widetilde{\mathcal{C}}^\varepsilon} \|\Omega_0\|_{B_{q,\infty}^{\frac{1}{q}}} \quad \text{for } q \geq N-1,$$

whence, thanks to Lemma A.3 and (5.4),

$$(5.5) \quad \|\partial_{X_0} u_0\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|X_0\|_{\widetilde{\mathcal{C}}^\varepsilon} (\|\Omega_0\|_{L^{\frac{N}{2}}} + \|\Omega_0\|_{B_{q,\infty}^{\frac{1}{q}}}) + \|\partial_{X_0} \Omega_0\|_{\mathcal{C}^{\varepsilon-3}}.$$

Now it is easy to construct the first family of vector fields  $\{E_j\}_{j=1}^N$  according to (5.5) and Theorem 1.3 as follows. Let  $\{e_1, \dots, e_N\}$  be the canonical basis of  $\mathbb{R}^N$  and assume (with no loss of generality) that the cut-off function  $\chi_0$  defined at the beginning of this section is supported in the domain  $J_0$ . Define  $E_j$  be the solution to

$$\begin{cases} \partial_t E_j + u \cdot \nabla E_j = \partial_{E_j} u, \\ E_j|_{t=0} = E_{j,0} := \chi_0 e_j. \end{cases}$$

Because  $\nabla \theta_0 \in (B_{q,\infty}^{\frac{1}{q}-1}(\mathbb{R}^N))^N$  for all  $q \in [1, +\infty]$ , we have  $\nabla \theta_0 \in (\mathcal{C}^{\varepsilon-2}(\mathbb{R}^N))^N$  by embedding (just take  $q \geq \frac{N-1}{1-\varepsilon}$ ). Of course, as  $E_{j,0}$  is smooth and compactly supported, and as any nonhomogeneous Besov space is stable by multiplication by functions in  $\mathcal{C}_c^\infty$ , one can conclude that  $\partial_{E_j,0} \theta_0 \in \mathcal{C}^{\varepsilon-2}(\mathbb{R}^N)$ . Taking  $X_0 = E_{j,0}$

in (5.5), the last term  $\partial_{E_{j,0}}\Omega_0$  on the r.h.s. vanishes by the definition of  $J_0$ . Hence Theorem 1.3 applies and one can conclude that  $E_j \in (L_{loc}^\infty(\mathbb{R}_+; \tilde{\mathcal{C}}^\varepsilon))^N$  for all  $j = 1, \dots, N$ .

Next, let us introduce some notations before our construction of second family of vector fields  $\{Y_\lambda\}$ . For any family  $\mathcal{P} = (P_1, \dots, P_{N-1})$  of  $N - 1$  vectors of  $\mathbb{R}^N$ , we define  $\wedge \mathcal{P} \equiv P_1 \wedge \dots \wedge P_{N-1}$  to be the unique vector of  $\mathbb{R}^N$  satisfying

$$\wedge \mathcal{P} \cdot Q = \det(P_1, \dots, P_{N-1}, Q), \quad \forall Q \in \mathbb{R}^N.$$

Assume that  $\mathcal{X} = (X_1, \dots, X_{N-1})$  where the time dependent vector fields  $X_j$  satisfy System (1.3). Thanks to  $\operatorname{div} u = 0$ , the vector field  $\wedge \mathcal{X}$  satisfies the equation,

$$(5.6) \quad \partial_t \wedge \mathcal{X} + u \cdot \nabla \wedge \mathcal{X} = -(\nabla u)^{tr} \cdot (\wedge \mathcal{X}).$$

Set  $\Lambda = \{\lambda = (\lambda_1, \dots, \lambda_{N-1}) : 1 \leq \lambda_1 < \dots < \lambda_{N-1} \leq N\}$ . For any  $\lambda \in \Lambda$ , we define  $Y_\lambda$  to be the solution of

$$\begin{cases} \partial_t Y_\lambda + u \cdot \nabla Y_\lambda = \partial_{Y_\lambda} u, \\ Y_\lambda|_{t=0} = e_{\lambda_1} \wedge \dots \wedge e_{\lambda_{N-2}} \wedge \nabla f_0. \end{cases}$$

Extend  $\lambda$  to a permutation  $\bar{\lambda} := (\lambda_1, \dots, \lambda_{N-1}, \lambda_N)$  of  $\{1, \dots, N\}$ , with signature  $\tau(\bar{\lambda})$ . By definition of  $\bar{\lambda}$ , we have

$$Y_{\lambda,0} \equiv Y_\lambda|_{t=0} = \tau(\bar{\lambda})(\partial_{\lambda_{N-1}} f_0 e_N - \partial_{\lambda_N} f_0 e_{N-1}),$$

which is divergence free in  $(\mathcal{C}^\varepsilon(\mathbb{R}^N))^N$ . Because  $\tilde{Y}_{\lambda,0}$  is a (non-degenerate) tangential vector field along  $\partial \mathcal{D}_0$ , the function  $\partial_{Y_{\lambda,0}} \theta_0$  vanishes. In addition, from the fact

$$\operatorname{Supp} \nabla(\Omega_0)_j^i \cap \operatorname{Supp} \chi_0 = \emptyset \quad \text{and} \quad \nabla f_0 = \chi_0 \nabla f_0,$$

we gather that  $\partial_{Y_{\lambda,0}} \Omega_0$  vanishes for any  $\lambda$ . Applying (5.5) with  $X_0 = Y_{\lambda,0}$  and Theorem 1.3, we get  $Y_\lambda \in (L_{loc}^\infty(\mathbb{R}_+; \mathcal{C}^\varepsilon))^N$ . Note that in fact  $Y_\lambda \in (L_{loc}^\infty(\mathbb{R}_+; \tilde{\mathcal{C}}^\varepsilon))^N$  as all those vector fields are divergence free.

Finally, putting together what we proved hitherto, we deduce that the vector fields  $W_\lambda$  defined by

$$W_\lambda := E_{\lambda_1} \wedge \dots \wedge E_{\lambda_{N-2}} \wedge Y_\lambda,$$

are in  $(L_{loc}^\infty(\mathbb{R}_+; \mathcal{C}^\varepsilon))^N$ . Furthermore, according to the definition of  $E_j$  and  $Y_\lambda$ , they satisfy Equation (5.6). The expression of  $Y_{\lambda,0}$  and the fact that  $\nabla f_0 = \chi_0 \nabla f_0$  imply that

$$W_{\lambda,0} \equiv W_\lambda|_{t=0} = -(\partial_{\lambda_{N-1}} f_0 e_{N-1} + \partial_{\lambda_N} f_0 e_N),$$

and summing up over  $\lambda \in \Lambda$  thus yields

$$\sum_{\lambda \in \Lambda} W_{\lambda,0} = -(N-1) \nabla f_0.$$

Therefore

$$W := -\frac{1}{N-1} \sum_{\lambda \in \Lambda} W_\lambda \in (L_{loc}^\infty(\mathbb{R}_+; \mathcal{C}^\varepsilon))^N,$$

coincides with  $\nabla f$  thanks to (5.6) and to the uniqueness of solutions for the equation satisfied by  $\nabla f$ , namely,

$$\begin{cases} \partial_t \nabla f + u \cdot \nabla (\nabla f) = -(\nabla u)^{tr} \nabla f, \\ \nabla f|_{t=0} = \nabla f_0. \end{cases}$$

This completes the proof of the conservation of  $C^{1,\varepsilon}$  regularity for domain  $\mathcal{D}_t$  in dimension  $N \geq 3$ .

#### APPENDIX A. COMMUTATOR ESTIMATES

We here prove some commutator estimates that were needed in the previous sections. They strongly rely on continuity results in Besov spaces for the paraproduct and remainder operators, and on the following classical result (see e.g. [2], Chap. 2).

**Lemma A.1.** *Let  $A : \mathbb{R}^N \rightarrow \mathbb{R}$  be a smooth function, homogeneous of degree  $m$  away from a neighborhood of 0. Let  $(\varepsilon, s, p, r, p_1, p_2) \in ]0, 1[ \times \mathbb{R} \times [1, \infty]^4$  with  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ . Then there exists a constant  $C$ , depending only on  $s, \varepsilon, N$  and  $A$  such that,*

$$\| [T_g, A(D)]u \|_{B_{p_1, r}^{s-m+\varepsilon}} \leq C \| \nabla g \|_{B_{p_1, \infty}^{s-1}} \| u \|_{B_{p_2, r}^s}.$$

**Remark A.2.** A similar inequality holds for time-dependent distributions, as may be seen by following the proof of Lemma A.1, treating the time as a parameter, and applying (time) Hölder inequality when appropriate. For example, for any  $(\rho, \rho_1, \rho_2) \in [1, \infty]^3$  with  $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$ , and  $(\varepsilon, s, p, r, p_1, p_2)$  as above, we have

$$(A.1) \quad \|[T_g, A(D)]u\|_{\tilde{L}_T^\rho(B_{p,r}^{s-m+\varepsilon})} \leq C \|\nabla g\|_{\tilde{L}_T^{\rho_1}(B_{p_1,\infty}^{\varepsilon-1})} \|u\|_{\tilde{L}_T^{\rho_2}(B_{p_2,r}^s)}.$$

**Lemma A.3.** Let the vector field  $X$  be in  $(\mathcal{C}^\varepsilon(\mathbb{R}^N))^N$  for some  $\varepsilon \in ]0, 1[$ , and  $f$  be in  $B_{p,r}^s$  with  $(s, p, r) \in ]-\infty, 1 + \frac{N}{p}[\times[1, \infty]^2$ . Then we have:

(i). If in addition  $s + \varepsilon > 1$  or  $\{s + \varepsilon = 1, r = 1\}$ , then

$$\|\mathcal{T}_X f - \partial_X f\|_{\mathcal{C}^{s+\varepsilon-\frac{N}{p}-1}} \lesssim \|X\|_{\mathcal{C}^\varepsilon} \|\nabla f\|_{B_{p,r}^{s-1}}.$$

The previous estimate holds in the case  $s = 1 + \frac{N}{p}$ , if replacing  $\|\nabla f\|_{B_{p,r}^{\frac{N}{p}}}$  by  $\|\nabla f\|_{B_{p,\infty}^{\frac{N}{p}} \cap L^\infty}$ .

(ii). If in addition  $\operatorname{div} X \in \mathcal{C}^\varepsilon(\mathbb{R}^N)$  then for  $s + \varepsilon > 0$  or  $\{s + \varepsilon = 0, r = 1\}$ , we have

$$\|\mathcal{T}_X f - \partial_X f\|_{\mathcal{C}^{s+\varepsilon-\frac{N}{p}-1}} \lesssim \|X\|_{\tilde{\mathcal{C}}^\varepsilon} \|f\|_{B_{p,r}^s}.$$

*Proof.* One may start from the Bony decomposition as follows:

$$\begin{aligned} \partial_X f &= \mathcal{T}_X f + T_{\partial_k f} X^k + R(X^k, \partial_k f) \\ &= \mathcal{T}_X f + T_{\partial_k f} X^k + \partial_k R(X^k, f) - R(\operatorname{div} X, f). \end{aligned}$$

Then applying standard continuity results for paraproduct and remainder operators (see e.g. [2], Chap. 2) yields the desired inequalities.  $\square$

If the integer  $N_0$  in the definition of Bony's paraproduct and remainder is large enough (for instance  $N_0 = 4$  does), then the following fundamental lemma holds.

**Lemma A.4.** Let  $(\varepsilon, s, s_k, p, p_k, r, r_k) \in ]0, 1[\times\mathbb{R}^2 \times [1, \infty]^4$  for  $k = 1, 2$  satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.$$

(i). If  $s_2 < 0$  then we have

$$(A.2) \quad \|\mathcal{T}_X T_g f - T_g \mathcal{T}_X f - T_{\mathcal{T}_X g} f\|_{B_{p,r}^{s_1+s_2+\varepsilon-1}} \leq C \|X\|_{\mathcal{C}^\varepsilon} \|f\|_{B_{p_1,r_1}^{s_1}} \|g\|_{B_{\infty,r_2}^{s_2}}.$$

The above inequality still holds in the limit case  $(s_2, r_2) = (0, \infty)$ , if one replaces  $\|g\|_{B_{\infty,\infty}^{s_2}}$  by  $\|g\|_{L^\infty}$ .

(ii). If  $s_1 + s_2 + \varepsilon - 1 > 0$  then we have

$$\|\mathcal{T}_X R(f, g) - R(\mathcal{T}_X f, g) - R(f, \mathcal{T}_X g)\|_{B_{p,r}^{s_1+s_2+\varepsilon-1}} \leq C \|X\|_{\mathcal{C}^\varepsilon} \|f\|_{B_{p_1,r_1}^{s_1}} \|g\|_{B_{p_2,r_2}^{s_2}}.$$

The above inequality still holds in the limit case  $s_1 + s_2 + \varepsilon - 1 = 0$ ,  $r = \infty$  and  $\frac{1}{r_1} + \frac{1}{r_2} = 1$ .

*Proof.* The key to our proof is a generalized Leibniz formula for the para-vector field operators which was derived by J.-Y. Chemin in [5]. Define the following Fourier multipliers for  $k \in 1, \dots, N$

$$\Delta_{k,j} := \begin{cases} \varphi_k(2^{-j}D) & j \geq 0, & \text{with } \varphi_k(\xi) := i\xi_k \varphi(\xi), \\ \chi_k(D) & j = -1, & \text{with } \chi_k(\xi) := i\xi_k \chi(\xi), \\ 0 & j \leq -2. \end{cases}$$

Then we have

$$\begin{aligned} \mathcal{T}_X T_g f &= \sum_{j \geq -1} (S_{j-N_0} g \mathcal{T}_X \Delta_j f + \Delta_j f \mathcal{T}_X S_{j-N_0} g) + \sum_{j \geq -1} (T_{1,j} + T_{2,j}) \\ &= T_g \mathcal{T}_X f + T_{\mathcal{T}_X g} f + \sum_{\substack{j \geq -1 \\ \alpha=1,\dots,4}} T_{\alpha,j}, \end{aligned}$$

where

$$\begin{aligned} T_{1,j} &:= \sum_{\substack{j \leq j' \leq j+1 \\ j-N_0-1 \leq j'' \leq j'-N_0-1}} 2^{j'} \Delta_{j''} X^k (\Delta_{k,j'} (\Delta_j f S_{j-N_0} g) - \Delta_{k,j'} \Delta_j f S_{j-N_0} g), \\ T_{2,j} &:= \sum_{\substack{j' \leq j-2 \\ j'-N_0 \leq j'' \leq j-N_0-2}} 2^{j'} \Delta_{j''} X^k (\Delta_j f) \Delta_{k,j'} S_{j-N_0} g, \\ T_{3,j} &:= S_{j-N_0} g [T_{X^k}, \Delta_j] \partial_k f, \\ T_{4,j} &:= \Delta_j f [T_{X^k}, S_{j-N_0}] \partial_k g. \end{aligned}$$

Bounding  $T_{1,j}$  and  $T_{2,j}$  is straightforward : just use the definition of Besov norms. Lemma 2.100 of [2] allows to bound  $T_{3,j}$  and  $T_{4,j}$  provided  $\varepsilon < 1$ . We end up with the desired inequality.

In order to prove the second item, let us set  $A_{j,j'} := [j - N_0 - 1, j' - N_0 - 1] \cup [j' - N_0, j - N_0 - 2] \cap \mathbb{Z}$ . We thus have

$$\begin{aligned} \mathcal{T}_X R(f, g) &= \sum_{j \geq -1} (\tilde{\Delta}_j g \mathcal{T}_X \Delta_j f + \Delta_j f \mathcal{T}_X \tilde{\Delta}_j g) + \sum_{j \geq -1} (R_{1,j} + R_{2,j}) \\ &= R(\mathcal{T}_X f, g) + R(f, \mathcal{T}_X g) + \sum_{\substack{j \geq -1 \\ \alpha=1, \dots, 4}} R_{\alpha,j}, \end{aligned}$$

where, denoting  $\tilde{\Delta}_j := \Delta_{j-N_0} + \dots + \Delta_{j+N_0}$ ,

$$\begin{aligned} R_{1,j} &:= \sum_{\substack{|j'-j| \leq N_0+1 \\ j'' \in A_{j,j'}}} \text{sgn}(j' - j + 1) 2^{j'} \Delta_{j''} X^k (\Delta_{k,j'} (\Delta_j f \tilde{\Delta}_j g) - \Delta_j f \Delta_{k,j'} \tilde{\Delta}_j g) \\ &\quad + \sum_{\substack{j-1 \leq j' \leq j \\ j'-N_0 \leq j'' \leq j-N_0}} 2^{j'} \Delta_{j''} X^k (\Delta_{k,j'} \Delta_j f) \tilde{\Delta}_j g, \\ R_{2,j} &:= \sum_{\substack{j' \leq j-N_0-2 \\ j'-N_0 \leq j'' \leq j-N_0-2}} 2^{j'} \Delta_{j''} X^k \Delta_{k,j'} (\Delta_j f \tilde{\Delta}_j g), \\ R_{3,j} &:= \tilde{\Delta}_j g [T_{X^k}, \Delta_j] \partial_k f, \\ R_{4,j} &:= \Delta_j f [T_{X^k}, \tilde{\Delta}_j] \partial_k g. \end{aligned}$$

Here again, bounding  $R_{1,j}$  and  $R_{2,j}$  follows from the definition of Besov norms, while Lemma 2.100 of [2] allows to bound  $R_{3,j}$  and  $R_{4,j}$ .  $\square$

We are in position to prove our key commutator estimate that involves the convective derivative and some para-vector field evolving according to (1.3). A particular case of that estimate has been proved in [10], as it was needed to study the viscous vortex patch problem.

**Proposition A.5.** *Suppose that  $(\varepsilon, p) \in (0, 1) \times [1, \infty]$  with  $\frac{N}{p} + \varepsilon \geq 1$ . Consider a couple of vector fields  $(X, v)$  such that  $\text{div } v = 0$  and*

$$(X, v) \in (L_{loc}^\infty(\mathbb{R}_+; \tilde{\mathcal{C}}^\varepsilon))^N \times (L_{loc}^\infty(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}-1}) \cap L_{loc}^1(\mathbb{R}_+; B_{p,1}^{\frac{N}{p}+1}))^N,$$

satisfying the following equation

$$(A.3) \quad \begin{cases} (\partial_t + v \cdot \nabla) X = \partial_X v, \\ X|_{t=0} = X_0. \end{cases}$$

Then there is a constant  $C$  such that:

$$(A.4) \quad \begin{aligned} \|\mathcal{T}_X \partial_t + v \cdot \nabla v\|_{\mathcal{C}^{\varepsilon-2}} &\leq C (\|X\|_{\tilde{\mathcal{C}}^\varepsilon} \|v\|_{B_{p,1}^{\frac{N}{p}+1}} \|v\|_{B_{p,1}^{\frac{N}{p}-1}} \\ &\quad + \|v\|_{\mathcal{C}^{-1}} \|\mathcal{T}_X v\|_{\mathcal{C}^\varepsilon} + \|v\|_{B_{p,1}^{\frac{N}{p}+1}} \|\mathcal{T}_X v\|_{\mathcal{C}^{\varepsilon-2}}). \end{aligned}$$

*Proof.* Because  $\operatorname{div} v = 0$ , we may write

$$\begin{aligned} [\mathcal{T}_X, \partial_t + v^\ell \partial_\ell]v &= -v^\ell \partial_\ell \mathcal{T}_{X^k} \partial_k v - T_{\partial_t X^k} \partial_k v + T_{X^k} \partial_k (v^\ell \partial_\ell v) \\ &= -T_{\partial_t X^k} \partial_k v + \partial_\ell \mathcal{T}_X (v^\ell v) - \mathcal{T}_{\partial_\ell X} (v^\ell v) - v^\ell \partial_\ell \mathcal{T}_X v. \end{aligned}$$

Hence, decomposing  $v^\ell v$  according to Bony's decomposition, we discover that

$$[\mathcal{T}_X, \partial_t + v^\ell \partial_\ell]v = \sum_{\alpha=1}^{\alpha=5} R_\alpha$$

with

$$\begin{aligned} R_1 &:= -T_{\partial_t X^k} \partial_k v, & R_2 &:= \partial_\ell (\mathcal{T}_X T_{v^\ell} v + \mathcal{T}_X T_v v^\ell), \\ R_3 &:= \partial_\ell \mathcal{T}_X R(v^\ell, v), & R_4 &:= -\mathcal{T}_{\partial_\ell X} (v^\ell v), \\ R_5 &:= -v^\ell \partial_\ell \mathcal{T}_X v. \end{aligned}$$

To complete the proof, it suffices to check that all the terms  $R_j$  may be bounded by the r.h.s. of (A.4).

- Bound of  $R_1$ : From the equation (A.3), we have

$$R_1 = T_{v \cdot \nabla X^k} \partial_k v - T_{\partial_X v^k} \partial_k v.$$

Hence using standard continuity results for the paraproduct, we deduce that

$$\|R_1\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|\nabla v\|_{\mathcal{C}^0} (\|v \cdot \nabla X\|_{\mathcal{C}^{\varepsilon-2}} + \|\partial_X v\|_{\mathcal{C}^{\varepsilon-2}}).$$

The last term may be bounded according to item (ii) of Lemma A.3. As for the first term, we use the following decomposition

$$v \cdot \nabla X = \mathcal{T}_X v + T_{\partial_k v} X^k + \partial_k R(v, X^k) - R(v, \operatorname{div} X),$$

which allows to get, since  $\varepsilon + \frac{N}{p} \geq 1$ ,

$$\|R_1\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|\nabla v\|_{L^\infty} (\|v\|_{B_{p,1}^{\frac{N}{p}-1}} \|X\|_{\tilde{\mathcal{C}}^\varepsilon} + \|\mathcal{T}_X v\|_{\mathcal{C}^{\varepsilon-2}}).$$

- Bound of  $R_2$ : Due to Lemma A.4 (i) and continuity of paraproduct operator, we have

$$\|R_2\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|X\|_{\mathcal{C}^\varepsilon} \|v\|_{\mathcal{C}^1} \|v\|_{\mathcal{C}^{-1}} + \|v\|_{\mathcal{C}^{-1}} \|\mathcal{T}_X v\|_{\mathcal{C}^\varepsilon} + \|v\|_{\mathcal{C}^1} \|\mathcal{T}_X v\|_{\mathcal{C}^{\varepsilon-2}}.$$

- Bound of  $R_3$ : Applying Lemma A.4 (ii) and continuity of remainder operator under the assumption  $\frac{N}{p} + \varepsilon \geq 1$  yields

$$\|R_3\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|X\|_{\mathcal{C}^\varepsilon} \|v\|_{B_{p,1}^{\frac{N}{p}+1}} \|v\|_{\mathcal{C}^{-1}} + \|v\|_{B_{p,1}^{\frac{N}{p}+1}} \|\mathcal{T}_X v\|_{\mathcal{C}^{\varepsilon-2}}.$$

- Bound of  $R_4$ : From Bony decomposition, it is easy to get

$$\|v^\ell v\|_{B_{p,1}^{\frac{N}{p}}} \lesssim \|v\|_{\mathcal{C}^{-1}} \|v\|_{B_{p,1}^{\frac{N}{p}+1}}.$$

Hence

$$\|R_4\|_{B_{p,1}^{\frac{N}{p}+\varepsilon-2}} \lesssim \|\nabla X\|_{\mathcal{C}^{\varepsilon-1}} \|v\|_{\mathcal{C}^{-1}} \|v\|_{B_{p,1}^{\frac{N}{p}+1}}.$$

- Bound of  $R_5$ : Applying Bony decomposition and using that  $\operatorname{div} v = 0$  and  $\frac{N}{p} + \varepsilon \geq 1$  give

$$\|R_5\|_{\mathcal{C}^{\varepsilon-2}} \lesssim \|v\|_{\mathcal{C}^{-1}} \|\mathcal{T}_X v\|_{\mathcal{C}^\varepsilon} + \|v\|_{B_{p,1}^{\frac{N}{p}+1}} \|\mathcal{T}_X v\|_{\mathcal{C}^{\varepsilon-2}}.$$

Combining the above estimates for all  $R_\alpha$ , with  $\alpha = 1, \dots, 5$  yields (A.4).  $\square$

**Remark A.6.** *Slight modifications of the above proof allow to handle time dependent functions with tilde type Besov norms. In particular, one may prove the following estimate for all  $t > 0$ :*

$$\begin{aligned} \|[\mathcal{T}_X, \partial_t + v \cdot \nabla]v\|_{\tilde{L}_t^1(\mathcal{C}^{\varepsilon-2})} &\lesssim \|v\|_{L_t^\infty(\mathcal{C}^{-1})} \|\mathcal{T}_X v\|_{\tilde{L}_t^1(\mathcal{C}^\varepsilon)} \\ &\quad + \int_0^t \|v\|_{B_{p,1}^{\frac{N}{p}+1}} \|\mathcal{T}_X v\|_{\mathcal{C}^{\varepsilon-2}} dt' + \int_0^t \|X\|_{\tilde{\mathcal{C}}^\varepsilon} \|v\|_{B_{p,1}^{\frac{N}{p}+1}} \|v\|_{B_{p,1}^{\frac{N}{p}-1}} dt'. \end{aligned}$$

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