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# The characterisation of the specialised knowledge of a university lecturer in linear algebra

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*This paper analyses the classroom practice of a university lecturer in Linear Algebra under the model Mathematics Teacher's Specialised Knowledge (MTSK). The knowledge revealed, related to the topic of Matrices and Determinants, is found to pertain chiefly to the sub-domain Knowledge of Topics (KoT). The categories of classification which comprise this sub-domain are of particular utility in the identification of this knowledge. Evidence of knowledge associated with other sub-domains comprising Subject Matter Knowledge and Pedagogical Content Knowledge is also found.*

**Keywords:** Specialised knowledge, linear algebra, university lecturer.

Studies into university lecturers' knowledge remain quite rare in the context of Mathematics Education, and thus it is that our study sets out to explore the specialised knowledge in evidence in the course of a lecturer's classroom practice while teaching Linear Algebra in the first year of a degree course. We use the model denominated *Mathematics Teacher's Specialised Knowledge* (MTSK) (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013) to analyse the lecturer's teaching sessions, with the aim of achieving a better understanding of the knowledge brought into play by topic *Matrices and Determinants*.

## THEORETICAL BACKGROUND

In the field of research into teachers' content knowledge of a particular subject, Shulman's (1986) study has proved particularly influential for its manner of differentiating components. Various models have drawn on Shulman's groundwork to describe the kind of knowledge deployed by mathematics teachers (e.g.,

Fennema & Franke, 1992; Bromme, 1994; Rowland, Huckstep, & Thwaites, 2005; Ball, Thames, & Phelps, 2008), most notable amongst which has been the model developed by Ball and associates (*Mathematical Knowledge for Teaching* – MKT), which applies the basic distinctions made by Shulman (*Subject Matter Knowledge* – SMK – and *Pedagogical Content Knowledge* – PCK) to the specific demands of teaching this subject. Based on our experiences of using the MKT model, and specifically the difficulties encountered in specifying and demarking certain sub-domains when applying it to teacher research (Flores, Escudero, & Carrillo, 2013), the Research Group for Mathematics Education at the University of Huelva is currently developing an alternative model denominated *Mathematics Teacher's Specialised Knowledge* (MTSK) (Carrillo et al., 2013), the aim of which is to capture the specialised nature of the teacher's knowledge as a whole. This specialisation is intrinsic to the teaching of mathematics, and as such permeates both SMK and PCK.

The MTSK model consists of two principle domains. The first of these is *Mathematical Knowledge* (MK), which covers “the whole universe of mathematics, comprising concepts and procedures, structuring ideas, connections between concepts, the reason for, or origin of, procedures, means of testing and any form of proceeding in mathematics, along with mathematical language and its precision” (Carrillo et al., 2013, p. 2990). MK is divided into the following subdomains: *knowledge of topics* (KoT); *knowledge of the structure of mathematics* (KSM); and *knowledge of practices in mathematics* (KPM). The second, *Pedagogical Content Knowledge* (PCK) concerns the teacher's knowledge of teaching and learning mathematics. PCK includes the subdomains *knowledge of*

*mathematics teaching* (KMT); *knowledge of features of learning mathematics* (KFLM); and *knowledge of mathematics learning standards* (KMLS). The MTSK model also contemplates teachers' conceptions about mathematics, its teaching and its learning (in line with Bromme, 1994).

KoT supposes a thoroughgoing, well founded knowledge of mathematical content. In order to facilitate the identification of this knowledge, five categories have been demarked. The first of these, *Procedures*, responds to the questions *How is something done?*, *Under what conditions can something be done?*, *Why is something done in this particular way?*, and *What are the key features of the result?* The second, *Phenomenology and applications*, concerns awareness of phenomena and contexts with which a particular topic is associated (Freudenthal, 1983) and relevant applications. As the name suggests, *Definitions*, the third category, refers to knowledge for describing and defining concepts, including appropriate examples and images. Fourthly, *Representations* comprise the knowledge of the different ways that a topic can be represented (Duval, 1995) including the associated notation and mathematical terminology. Finally, *Properties and fundamentals* encompasses knowledge of the properties inherent to a mathematical item or necessary for carrying out a particular procedure. This sub-domain receives a more detailed attention in comparison with the other sub-domains comprising MTSK for the reason that, as seen below, it has greatest relevance to the analysis of the work of the teacher participating in the study.

KSM consists of the connections between different contents. It implies being able to view content from different perspectives, that is, basic mathematics from a more advanced vantage point (*connections of increasing complexity*) and advanced mathematics from a more basic viewpoint (*connections of increasing simplicity*); it also includes connections between distinct concepts (Fernández, Figueiras, Deulofeu, & Martínez, 2010).

KPM corresponds to knowledge of the ways of proceeding in mathematics, such as the role of definitions, how to establish relations, correspondences and equivalences, the means of selecting representations, and forms of argumentation, generalisation and the examination of concepts (Carrillo et al., 2013).

KMT enables the teacher to select material appropriate to teaching a specific concept or procedure, and includes the categories of *theories of teaching*; *material and virtual resources*; *activities, tasks and examples for teaching*.

KFLM concerns the ways by which mathematical knowledge is acquired, with a clear focus on mathematical content as the object of learning. Categories in this sub-domain include *learning styles*, *areas of strengths and weaknesses associated with learning*, *students' forms of interacting with the content*, and *students' motivation with regard to mathematics*.

KMLS largely consists of knowledge of the curriculum and its varying demands, and the objectives and performance measures established by external agencies such as examination boards, professional associations and research groups.

A useful reference in the case of teachers of Linear Algebra is the study by McCrory, Floden, Ferrini-Mundy, Reckase, and Senk (2012), which places secondary teachers expertise on two planes, the first mapping their *Knowledge of Algebra for Teaching*, and the second describing their performance in terms of *Mathematical Uses of Knowledge in Teaching*. Some of the categories comprising the first of these dimensions overlap with corresponding categories within MTSK, for example *Knowledge of School Algebra*, detailing knowledge of the basics of the area, *Knowledge of Advanced Mathematics*, which establishes connections between contents, and *Mathematics for Teaching Knowledge*, in which elements of PCK and other teaching-specific content can be identified.

## METHODOLOGY

The study followed a qualitative and interpretative methodology, with a case-study design (Yin, 1989), focussing on the specialised knowledge of a university lecturer teaching Linear Algebra to first year students. The research question was, 'What specialised knowledge does a Linear Algebra lecturer reveal in the course of teaching matrices and determinants?'

The lecturer was invited to participate for his disposition and willingness to collaborate. Also relevant was that he worked at the same university as the lead author of this paper, whose professional interests lie in exploring the knowledge of mathematics teachers

working in university contexts, with a view to possible follow-up action.

The topic of *Matrices and Determinants* was chosen as it was the first topic in the programme for Linear Algebra and essential for subsequent topics. The lecturer had originally studied Educational Sciences (specialising in mathematics) at degree level, and could draw on 18 years' experience at secondary level and 10 years at the university. The main data collection instrument was non-participatory teaching observation. In total thirteen classes of around two hours each were video recorded. The observations were conducted in two blocks over two consecutive academic years (October 2011 to January 2012, and October 2012 to January 2013) to coincide with when the lecturer was teaching the topic of matrices and determinants. A semi-structured interview was also conducted to provide a degree of triangulation for the evidence garnered via the observation.

Data analysis followed the methodology of content analysis (Bardin, 1977), in which meant sifting the teacher's actions and utterances for evidence of the MTSK sub-domains. The theoretical basis for the analysis consisted of the categories outlined above in the background section. These categories, which we consider still somewhat in development, derive from the intersection between the theoretical readings and joint reflections of the research group and the practical application of the model to teachers' knowledge in various research projects in progress. Finally, it should be noted that although MTSK also contemplates teachers' *conceptions* about teaching and learning mathematics, this aspect of the model is not considered here.

### **ANALYSIS OF THE KNOWLEDGE OF A TEACHER OF LINEAR ALGEBRA**

In terms of the definitions of the MTSK sub-domains and constituent categories, the analysis of the teaching sessions revealed a predominance of KoT, although evidence of KFLM, KMT and KPM were also found. By contrast, there was no evidence of knowledge which could be attributed to the KSM and KMLS sub-domains. Emphasis is thus given below to the consideration of KoT and its corresponding categories. The units of information which form the basis of the discussion are drawn directly from the analysis of the teacher's performance over various

teaching sessions; in the case of KMT, these have been complemented by units of information taken from the interview so as to provide a degree of triangulation.

#### **Knowledge of Topics (KoT)**

Of the categories comprising this sub-domain, there is evidence of the teacher's knowledge in all five, albeit somewhat scant with regard to *Phenomenology and applications*. In terms of *Definitions*, those provided by the teacher generally tend towards the conventional. Examples include definitions of unitary matrices, equal matrices and orthogonal matrices. At the same time, he also shows knowledge of *Properties*, such as the fact that the determinant of a triangular matrix equals the product of the diagonal entries. The excerpts below illustrate these points:

Teacher: The unitary matrix is the identity matrix. The matrices which are the same are those with the same dimension and the elements placed in the same location and with the same value. When the inverse coincides with the transpose, we get the orthogonal matrix.

Teacher: This is a triangular matrix. What is the determinant of this matrix A? Well, we can compute the determinant using the properties; if we have a determinant with all its elements, we can make it triangular with the elementary transactions between rows and then we can calculate the determinant by multiplying the principle diagonal.

With regard to *Representations*, there is evidence of knowledge of the notation of mathematical objects, such as the notation for matrices and determinants.

Teacher: An important point. The 2x2 matrix is like this [...] When we calculate the determinant we no longer write it this way. I'm going to calculate the determinant of the matrix A, so here instead of brackets we have to put vertical bars [...] The determinant is denoted like this  $\langle \det(A) \rangle$  or like this  $\langle |A| \rangle$ .

The teacher uses *algebraic* and *arithmetic* registers (D'Amore, 2004) for representing basic operations between rows and for matrix algebra. In addition, and connected to the algebraic register, he reveals knowl-

edge of the matrix register (Ramírez, Romero, & Oktaç, 2013) for different types of matrices, linear equation systems and determinants; and finally the verbal register for setting exercises where the students apply their understanding. His knowledge allows him to encourage different types of representation, although he is not observed to go into the significance of these. However, he does make his awareness of the conventions of notation explicit, an element of knowledge pertaining to the subdomain *Knowledge of Practices in Mathematics* (see the section on KPM below).

There is little evidence for the teacher's knowledge of *Phenomenology and applications* of matrices. He makes mention, albeit briefly, of applications outside mathematics, such as real situations or situations connected to the degree course he teaches. In terms of examples of his knowledge of applications within mathematics itself, we can mention the determinant for finding the inverse matrix or for solving systems of linear equations using Cramer's rule, basic operations between rows applied in calculating the determinant, obtaining stepped and canonical matrices, inverse matrices by the Gauss Jordan method, and the solution to systems of linear equations by the Gauss and Gauss Jordan methods.

With respect to knowledge of *Procedures*, he shows that he knows the procedures appropriate to the topic, and is able to give a full description of them (*How is something done?*). Likewise, he knows where these procedures have their limits, and what is required to be able use them (*Under what conditions can something be done?*), which can be exemplified with his knowledge of the necessary conditions for multiplying matrices:

Teacher: We have matrix A. What are the dimensions of this matrix? [...] The dimension of this matrix is  $2 \times 3$ . In order to be able to multiply two matrices we need the number of columns of the first matrix to be the same as the number of rows of the second matrix. If A is like that, B needs to have three rows, it doesn't matter how many columns.

In some instances, he shows that he knows why certain procedures are used (especially in relation to ways of simplifying a procedure) (*Why is something done in this particular way?*). An example of this can

be seen in his explanation for calculating the determinant of a matrix:

Teacher: Likewise, you take the column or the row that you want, but here's an interesting thing. Here, there's a zero; this zero helps us to reduce the degree of difficulty because you are always going to multiply the determinant of the submatrix you've got by the value of the element you're focusing on, so it is always a good idea to take a row or a column which has a zero or lots of ones.

Regarding deeper knowledge relating to the mathematical foundations underpinning the procedures, no evidence came to light. It should be noted that the teacher emphasised the *features of the result* obtained by following certain procedures, as in the example below, summarising the characteristics of the product of matrices:

Teacher: The product of matrices has the same number of rows as the first matrix and the same number of columns as the second.

In his teaching, a tendency to rely on exercises of a mechanical nature was observed. This was closely connected to the learning of algorithms and syntactic conventions for the representation of the content, before conceptual aspects (in the terms of Hiebert & Lefevre, 1986). That said, the teacher does call the students' attention to connections between properties, different registers for representation, certain features and reasons for the procedures and their results. As can be seen, analysis of actual instances of the teacher's practice reveals knowledge pertaining chiefly to categories pertaining to KoT. Nevertheless, evidence of other sub-domains was found and this shall be discussed in the following sections.

### **Knowledge of Features of Learning Mathematics (KFLM)**

We also found evidence of the teacher's KFLM, in the category of *areas of strengths and weaknesses associated with learning* concerning unforeseen pitfalls associated with the topic. This knowledge prompts the teacher to warn his students about potential problems in putting certain procedures into effect. The unit of information below illustrates this knowledge with

a warning about changing rows in a matrix and in a determinant:

Teacher: So you need to be careful here. When we're working with matrices we can say that the matrices are equivalent if we change one row for another. But if we're working with determinants and we change one row for another, the determinant changes sign [...] Matrices and determinants are not the same thing. [...] I repeat, with determinants it's not the same as with matrices, with determinants a change in rows means a change in its sign.

There is abundant evidence of this category in the teacher's reiterated warnings about aspects of the topic which can easily lead to error.

### **Knowledge of Mathematics Teaching (KMT)**

In this sub-domain, the teacher's *Knowledge of Mathematics Teaching* (KMT) is reflected in the *examples* he selects in his teaching. In this instance, we can take his introduction to row echelon forms of matrices, in which his approach aims to ensure that students receive a broad coverage of the features of this type of matrix (as expressed in interview):

Teacher: Now, I'd like you to look this way at the blackboard. What do you notice about the first matrix? First, they are not square. [...] What do you notice about the zeros? [...] The zeros increase as you go down the rows. The important thing to note is that in this row there are various zeros before a non-zero element. [...] What do you notice in this second matrix? [...] What about the third matrix? Take good note that it is of no interest to us if there is a zero here, what is important is that in this row there are zeros before a non-zero element. This type of matrix is called the row echelon form. As you go down the rows, the number of zeroes increases until you get to the non-zero element or the whole row is zeroes.

Teacher: My intention was to give them three different matrices, each with a different row echelon form. I gave them those

so that the students would realise that there are different kinds of row echelon forms and realise what it is that fundamental to the row echelon form (interview excerpt).

The choice of these three examples constitutes evidence of the teacher's knowledge for teaching regarding the relevance and variability of examples, specifically three matrices of the same order, although with different row echelon forms.

### **Knowledge of Practices in Mathematics (KPM)**

Finally, as mentioned above, we have the example of *Knowledge of Practices in Mathematics* (KPM), regarding mathematical notation, to which the teacher attaches great importance and mentions several times over the course of the observed sessions. Here is just one of the examples:

Teacher: We need to be careful when we write on the blackboard, and not put signs and strokes in any old place, because there are signs in mathematics which mean something. Write the symbols properly. If you are going to calculate the determinant, don't put it in square brackets, you have to put it between vertical bars. This way, like that, is a matrix. But if you are going to calculate the determinant you need to put vertical bars instead of square brackets.

With respect to the two broad domains comprising MTSK, we have been able to note evidence of knowledge corresponding to both in the teacher's professional practice. In terms of MK, there is KoT and KPM (but not KSM, for which we failed to find evidence); and in terms of PCK, there is KFLM and KMT (but again, no KMLS).

### **CONCLUSIONS**

The purpose of this study has been to get a picture of the specialised knowledge in evidence during the observation of a university lecturer's teaching of Linear Algebra. The purpose was not to make any evaluation of the teaching that took place in terms of what might be considered desirable practice. The final picture that emerged is essentially one predominated by KoT, with indications of KFLM, KMT and KPM. We

failed to find evidence of knowledge related to the sub-domains KSM and KMLS, and it should be noted that the teacher's conceptions, whilst forming part of the model, were not made subject of the study. We can say that the teacher's practice is characterised by procedural knowledge, not only in terms of how to proceed, but also with regard to the conditions for proceeding, why to proceed and what characteristics can be expected as a result. In addition to this, he demonstrates knowledge of standard definitions, different representations of content, and applications to other areas within mathematics. His choice of examples, and his warnings to students of unforeseen pitfalls provide examples of his knowledge of the difficulties associated with learning certain content, and questions of exemplification.

The knowledge above was revealed through analysis of classroom observation (in conjunction with complementary data drawn from interview, and commenting on the former). To this extent, we cannot claim that the resultant snapshot corresponds to the teacher's knowledge of the topic, merely what was deployed in the course of his classroom practice. Evidence of knowledge pertaining to other sub-domains and categories of specialised knowledge might well be revealed through the use of other data collection tools. By the same token, the knowledge on display might be closely linked to teaching style and the teacher's beliefs and conceptions with regard to teaching and learning. To this end, the study aims to continue its investigation into better understanding the teacher's practice by bringing his conceptions into the analytical procedure.

Finally, we would highlight the utility of the MTSK model and its corresponding analytical categories, above all those relating to KoT, in enabling us to detail the knowledge in evidence during this teacher's classroom performance.

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