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What does it mean as a teacher to “know infinity”? The case of convergence series

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This paper aims to explore the knowledge of infinity that three teachers deploy in making methodological suggestions for dealing with the convergence of a geometric series in a secondary level lesson. We will also illustrate the potential of the model of professional knowledge known as Mathematics Teacher’s Specialised Knowledge (MTSK) as a tool for analysing the knowledge of infinity underpinning the three proposals. The mathematical knowledge revealed is considered, alongside suggestions for further exploration of the pedagogical knowledge of the three teachers regarding infinity.

Keywords: Infinity, professional knowledge, MTSK, series, secondary teacher knowledge.

INTRODUCTION

It is widely acknowledged that the nature of a mathematics teacher’s knowledge of the subject is different to that of other people involved in mathematics. To this end, Ball, Thames and Phelps (2008) specifically identify the sub-domain *Specialised Content Knowledge* within their analytical model, MKT, in contrast to that of *Common Knowledge*, with the aim of underlining the kind of knowledge about certain topics required of a teacher, but not necessarily required of others who habitually draw on mathematical knowledge. An alternative model was recently presented, *Mathematics Teacher’s Specialised Knowledge*, henceforth MTSK (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013), the premise of which is that the specialisation derives not from the unique nature of the teacher’s knowledge in itself, but from the use to which this knowledge is put in teaching. It is from this perspective of specialisation, following Montes, Carrillo and Ribeiro (2014), that we consider the concept of infinity and its significance to the teacher’s store of professional knowledge.

The concept of infinity is an interesting case because it touches many areas of the secondary curriculum, and is hence especially relevant to analysing teachers’ mathematical knowledge. From our perspective, a teacher’s knowledge of infinity is specialised by virtue of the fact that, as practitioners, they need to know how to deploy their knowledge in contexts of teaching and learning. This paper applies this perspective to the question of how this knowledge influences the methodological considerations involved in planning a lesson on the convergence of a series generated by successive powers of $\frac{1}{2}$. In doing so, it hopes to illustrate the effectiveness of the MTSK model in revealing the specialised knowledge underlying teachers’ decisions.

THEORETICAL FRAMEWORK

The MTSK model (Figure 1), which provides the theoretical background to this paper, consists of six sub-domains divided between Shulman’s (1986) original dichotomy of *Subject Matter Knowledge* and *Pedagogical Content Knowledge*.

In terms of purely mathematical knowledge, it proposes three sub-domains. The first of these, *Knowledge of Topics*, brings together all topics involved in the teaching-learning binomial, including many aspects which are seldom made explicit to students, but can nevertheless be understood as fundamental to the teacher’s knowledge. The second, *Knowledge of the Structure of Mathematics*, concerns the connections between areas, such as concepts which range across the curriculum (as in the case of the concept of infinity). Finally, *Knowledge of Practices in Mathematics* encompasses ways of doing and proceeding with mathematics. There are likewise three sub-domains within *Pedagogical Content Knowledge*, which aim to enhance the specifications proposed by Ball and colleagues (2008). The first of these, *Knowledge of*

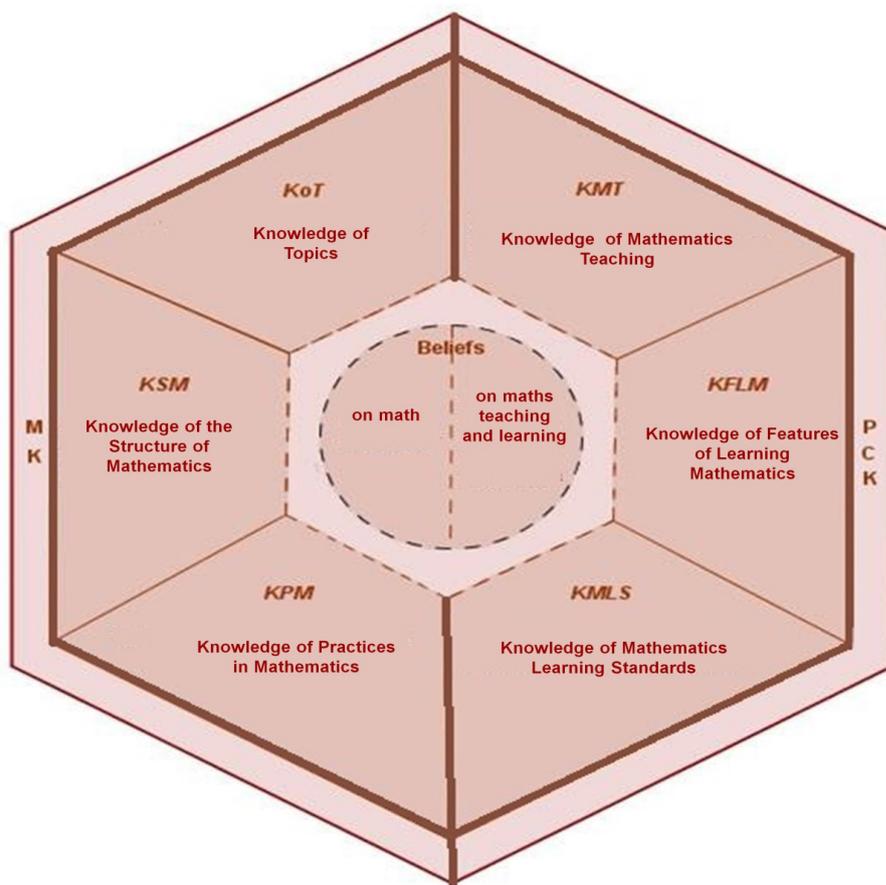


Figure 1: Mathematics Teacher’s Specialised Knowledge

Mathematics Teaching, refers to knowledge about ways of presenting mathematical content in the classroom. The second, *Knowledge of the Features of Learning Mathematics*, concerns knowledge about how pupils learn, understand and interact with mathematical content. The last, *Knowledge of Mathematics Learning Standards*, includes knowledge relating to the curriculum and learner expectations in respect of year group, age and educational stage.

A professional understanding of infinity

The concept of infinity, as an item of human knowledge, has been widely studied over the centuries and at the present time is understood as a property of various sets (Moreno & Waldegg, 1991). With respect to research into infinity and learning, studies have considered the *intuition of infinity* (Fishbein, Tirosh, & Hess, 1979), the relation between conceptions of infinity and understanding the notion of limit (e.g., Sierpinski, 1987), how infinity is dealt with in class (e.g., Hannula, Pehkonen, Maijala, & Soro, 2006; Roa-Fuentes, 2013), and the development of students’ understanding of situations closely associated with the concept (Arnon, Cottrill, Dubinsky, Oktaç, Roa-Fuentes, Trigueros, & Weller, 2014). Nevertheless, the concept of infinity

has received scant attention as a component of professional knowledge, that is as something useful for education as opposed to a mathematical item to be studied in depth.

Efforts to research teachers’ knowledge of infinity have thus tended towards regarding their subjects as “possessors of advanced knowledge” on account of their training background, including mathematics training, and their familiarity with the various concepts pertaining to school mathematics. But interest in teachers’ knowledge also derives from their capacity to satisfy their students’ curiosity and provide valid responses to their questions. As Hannula and colleagues (2006) state, “Most primary children are very interested in infinity, and they enjoy discussing the concept, if the teacher is only ready for it” (p. 1). Any such responses must be founded on mathematics; the teacher needs to be aware of what to say and why this is valid. In this respect, there have been numerous studies into the misconceptions about infinity of different groups of teachers, focusing on the epistemological basis for their thinking (e.g., Kattou, Michael, Kontoyianni, Christou, & Philippou, 2009), the nature of the misconceptions themselves, including taxonom-

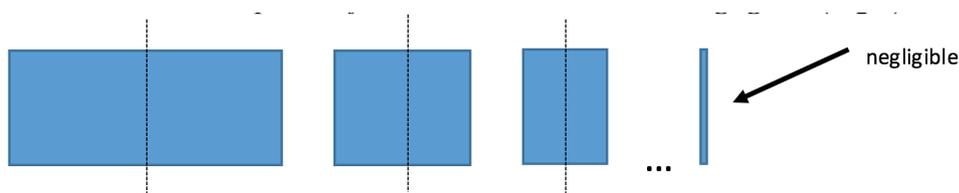


Figure 1: Graphical reconstruction of the explanation of Sandro

ical analysis (e.g., Yopp, Burroughs, & Lindaman, 2011), and the degree of cognitive development in relation to concepts associated with infinity (e.g., Weller, Arnon, & Dubinsky, 2009). Finally, some studies give special emphasis to how such misconceptions can be transmitted to students (Tall & Schwarzenberger, 1978). However, we would argue that the mere fact of being in possession of a valid response does not account for the full range of knowledge that a teacher mobilises when they construct an example, as shall be seen in the following section.

The potential of various models has recently been illustrated in relation to teachers’ knowledge of infinity (Montes, Carrillo, & Ribeiro, 2014). In this paper, we consider the use of one such model, MTSK, in revealing the impact of the teachers’ understanding of infinity on their planning for a lesson on the addition of geometric progressions.

METHOD

The data used in this paper are drawn from two separate but related case studies (Montes, 2011; Montes, in press), studying the knowledge of infinity of three teachers by means of an interview structured around the question, “How would you approach the addition of successive powers of 1/2 with a class of 16-year-olds?”

Here, we consider the different approaches adopted by the three teachers in response to this question. The first two, Sandro and Aaron, are mathematics graduates with 5 and 8 years’ teaching experience behind them respectively, whilst the third, Manuel, also a mathematics graduate, was following a teacher training course at the time of the study. As noted above, all three were taking part in studies into their knowledge of infinity, and this meant that all of them took for

granted that the interview about successive powers was concerned with the notion of convergence of an infinite series. Their suggestions for dealing with the concept in class are given below:

Sandro: *The usual thing is you say that they have a cake [draws a rectangular cake], and you eat half, and then half of the remaining bit, and then half again, and carry on like that a few times, and you see [...] You can also do it with a piece of paper, which you cut again and again, and each time you are left with a smaller piece, and they see that there comes a moment when the piece left over is so small as to be negligible (Figure 1).*

Aaron: *I’d take a piece of paper a metre wide as reference, and then one of half a metre, which I’d place on top. After that, another of a quarter of a metre, which I’d place on the right, and then one of an eighth, which I’d also lay on top. At some point, I’d get them to see that if you zoom in on the process, you get back to what was essentially the starting point, making a comparison between the piece left over and the original half-metre piece. I’d continue a few more times in the same way and get them to see that you can never get beyond that, however close you get (Figure 2).*

Manuel: *I’d get them to cut a sheet of paper in half, and then one of the halves in half again, and carry on like that five or six times until they see that you could continue with some ‘super-sharp scissors’ as many times as you like. Then I’d get them to write on each piece the fraction it represents, put all the pieces together*

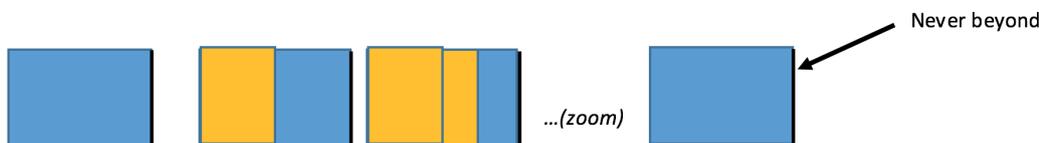


Figure 2: Graphical reconstruction of the explanation of Aaron

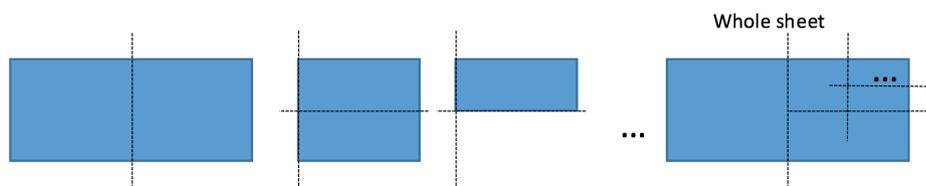


Figure 3: Graphical reconstruction of the explanation of Manuel

and imagine them all together, including the ones we could have cut but didn't, so that they became aware that all the pieces together comprise the whole sheet, which has an area of 1 (Figure 3).

ANALYSIS

This section takes the examples of three teachers talking about the same exercise and suggesting ways of dealing with it in class, and illustrates the potential these situations offer for exploring the knowledge displayed by the teachers beyond the strictly mathematical.

The three teachers take the same methodological approach in that they all recommend cutting paper in half, albeit that two – Sandro and Manuel – suggest the pupils themselves do the cutting, whilst in Aaron's version it is the teacher who does this. In terms of the mathematical knowledge deployed, all three responses show understanding of the proposed scenario, predicated on convergence when adding a geometric series. However, the three scenarios reflect different understanding of the concept of infinity.

In the case of Sandro, his argument is based on the fact that after repeatedly cutting the sheet in half (in order to represent division by two) and putting the resultant pieces together, there comes a moment when the remaining piece “is so small as to be negligible.” This alludes to a significant property of the remaining item in a series, underlying which is the idea of the infinitesimal. The knowledge revealed by the answer is a clear example of *Knowledge of Topics*, in which the topic in question is ‘convergence of series’. Nevertheless, the concept underlying the solution and making it coherent, although not actually explicitly articulated, is that of the tendency to zero of the remainder in a series. This suggests a potential understanding of infinity (as the process is unfinished) which, given that the concept ranges across topics, is most appropriately located in the sub-domain *Knowledge of the Structure of Mathematics*.

Turning now to Aaron's answer, we can see that the mathematical foundation of his answer concerns the limiting effect of adding successive halves. Aaron diverts attention away from the total value of the addition towards the impossibility of this total ever reaching the total dimensions of the sheet. In mathematical terms, the element underlying this procedure is the concept of convergence of a sequence (of partial additions). Aaron makes the process a recurrent one so as to illustrate the fact that, however many times one cuts the paper in halves, there will always be one half lacking to complete the whole. The completion point constitutes a limit, which can be approached as closely as one wishes, but which can never actually be reached. This process view reflects a process-oriented and iterative perspective of infinity, again also associated with a potential understanding (Lakoff & Nuñez, 2000). Once again we are in the domain of *Knowledge of Topics*, in this case the topic being convergence of series. Aaron's familiarity with the definition of convergence means he puts the emphasis on the possibility of choosing any term and recognising that it never goes beyond the established limit, although it might be *very close*. This familiarity, rooted in his understanding of infinity, is likewise associated with *Knowledge of the Structure of Mathematics*.

Finally, Manuel's answer is suggestive of elements which are different from the previous answers, less in terms of the knowledge mobilised as the nature of this knowledge itself. It follows the same dynamic as the previous two in that it generates the $(\frac{1}{2})^n$ series via the expedient of dividing a sheet of paper in two. During the procedure, Manuel introduces the idea of ‘super-sharp scissors’ to achieve the required degree of abstraction suggested by being able to cut the paper indefinitely, beyond the physical limitations. To the extent that this approach illustrates awareness of potential difficulties in the students' capacity for abstraction, the extract pertains to the sub-domain *Knowledge of Features in Learning Mathematics*. Afterwards, Manuel introduces an element which diverges significantly from the previous suggestions. Instead of centring attention on the process of repeatedly di-

viding in two, he pieces the sheet together again. This helps the students to make sense of the final result, graphically illustrating that the process of addition has a finite result equivalent to the surface area of the original sheet. The knowledge involved in this instance is that of *Knowledge of Topics*, which enables him to construct an example based on the meaning of the convergence of a series, and to take into consideration the implicit role of infinity in the expression ‘all the pieces together’, which provides an objective correlative for the notion of convergence of a series.

In each case, the proposed activity is underpinned by an understanding of the role of the concept of completion in demonstrating convergence, which is most pertinent to the sub-domain of *Knowledge of Practices in Mathematics*. Sandro makes use of the characteristic of series that if the remainder tends to zero, then the series is convergent. Aaron, for his part, draws on the fact that the annotation of partial additions implies convergence. On the other hand, Manuel bases his approach on the result of the addition, and as such it does not constitute a demonstration in the strictest sense of the word, although it is an interesting strategy to use with students.

Opportunities for further research

The analysis above is concerned chiefly with the purely mathematical considerations brought into play by the three teachers. Nevertheless, this is not the only kind of knowledge we believe they mobilise. The fact that the three suggestions are significantly different, in terms of both the mathematics involved and the manner of understanding infinity alerts us to the presence of *opportunities for further research* (Flores *et al* 2013), specifically into the responses which these three teachers might give to different questions about the appropriateness of their suggestions for dealing with this topic. With a view to developing teacher training materials such as vignettes for discussion, we think that further research into other aspects of these three teachers’ knowledge is of interest as it will enhance the consistency of analysis. Some of the areas we feel would reward study are indicated below.

We think it would be interesting to ask the three teachers about the effectiveness of their approaches in terms of the students’ potential understanding of the scenario. In this respect, although the first two approaches demand an understanding of the concept of limit if they are to be successful, they imply a potential

understanding of infinity, something usually expected amongst students. By contrast, the third scenario implies considering the process as a whole, given that it demands an actual understanding of the infinite process, which is a more difficult concept to acquire. Exploring with teachers the use of the paper-cutting technique as a means of approaching the task also has the potential to reveal useful information about mathematics teachers’ knowledge, going beyond the purely manipulative dimension of the technique towards understanding how it might help pupils to understand the concept in question. Finally, it would be interesting to go into greater depth regarding the suitability of the lesson plans in relation to the pupils’ age, previous learning experiences of the topic, and syllabus expectations regarding the concept of infinity.

CONCLUSIONS

Infinity matters for teaching certain concepts, many associated with the area of mathematical analysis. For certain items on the syllabus, its importance to mathematical procedures at a conceptual level, such as the calculation of limits, is explicit. In the examples above, each teacher’s consideration of infinity leads them to tackle the same question from different mathematical perspectives, each of which provides the epistemological underpinnings for the chosen approach, specifically the infinitesimal of the remainder, the iterative and recurrent elements of the limiting process, and the conceptualization of a sequence as a whole. We are aware that the examples covered in this paper offer nothing more than an initial consideration of the knowledge of infinity brought into play by teachers when they plan a lesson around the concept at an epistemological level. Nevertheless, we believe that they demonstrate the involvement of infinity in the teaching process, and explore a little further the big idea of ‘*dealing with infinity*’ (Kuntze, Lerman *et al.*, 2011). The examples also enable us to map the contours of the knowledge of infinity which the teachers bring to bear in their classes, which is extremely closely related to how they conceive of infinity, and complements studies of a purely cognitive nature (e.g., Weller *et al.*, 2009).

Nevertheless, beyond purely mathematical considerations, we would argue that it is necessary to further explore those aspects of infinity which come under the auspices of *Pedagogical Content Knowledge*, as it is

these that make the mathematical content relevant to the classroom. There are, too, the questions of the potential of different methodological approaches for different concepts, and the features of the pupils’ learning of concepts involving infinity, as well as those of a curricular nature or associated with the performance expected at a particular age or level, all of which are also important for the teacher to be aware of.

In this paper, we have outlined some of the areas of pedagogical content knowledge associated with infinity which we believe merit further attention and which are of special interest to our work. The understanding and awareness of infinity which a teacher is required to acquire is necessarily one which takes a specialised perspective of mathematics teaching. The role of PCK is relevant from this perspective in terms of opportunities for further research beyond the purely mathematical aspects of the concept, such as understanding which features of the concept are involved in the process of learning. It is in this respect that the MTSK model is especially useful.

The different aspects arising from analysing teachers’ professional practice with regard to infinity or related items should also be taken into account in the training of both primary and secondary teachers. For example, paradoxes could be used to nudge teachers to develop their understanding of key mathematical concepts, especially those associated with infinity, or could be a “vehicle for raising their pedagogical awareness of the development of mathematical knowledge” (Moshovitz-Hadar & Hadass, 1990).

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