Teachers' resources in analysing mathematical content and classroom situations: The case of using multiple tepresentations

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Using multiple representations for mathematical objects in the classroom is a key for fostering students' understanding. As teachers have to analyse both mathematical content and classroom situations in order to effectively accompany learning processes, they need corresponding professional knowledge and resources. As specific empirical research is scarce, this study aims at collecting first evidence about whether and how different samples of pre-service and in-service teachers analyse content against the idea of using multiple representations and whether these content-related resources interdepend with their awareness of the role of dealing with representations in classroom situations. The findings indicate expected differences in resources and interdependencies between CK and PCK.

Keywords: Teachers' analysing, PCK, CK, multiple representations.

INTRODUCTION

Teacher expertise is related to personal resources in various domains of content knowledge (CK) and pedagogical content knowledge (PCK, cf. Shulman, 1986). A reservoir of such CK and PCK is needed in particular when teachers have to analyse learning opportunities and subject matter on the one hand or specific interaction situations with learners in the mathematics classroom on the other hand. Making connections with professional knowledge in these two contexts belongs to the core of what is required from mathematics teachers in their profession. This holds in particular for being able to analyse content matter and classroom situations against criteria related to the use of multiple representations, as abstract mathematical objects can hardly be accessed without any representations (cf. Duval, 2006).

Consequently, there is a high interest in investigating whether secondary teachers from different school types are able to analyse (A) how mathematical content affords using multiple representations and (B) how problems for learners may emerge from a non-optimal use of multiple representations in classroom situations. To our knowledge this is the first study with a quantitative empirical scope on the potential interrelatedness of these domains of teachers' profession-related analysing.

In order to clarify our focus on teachers' analysing, we will in the following introduce into the theoretical background of this study, derive the research questions from these thoughts, inform about sample and methods, present results and discuss their implications for the theoretical and practical contexts.

THEORETICAL BACKGROUND

As mathematical objects are only accessible through representations (Duval, 2006), representations play an important role in the process of mathematical learning and its support in the classroom. Different representations can stand for a mathematical object (Goldin & Shteingold, 2001), so using multiple representations may support learners in learning more about the mathematical object behind these different representations (Lesh, Post, & Behr, 1987) and in distinguishing the mathematical object from its representations. As different representations mostly emphasise different aspects of the corresponding mathematical object, the use of multiple representations enables learners to build up a rich concept image (Tall, 1988; Ainsworth, 2006). Connecting different representations as well as changing between representations provide thus crucial learning opportunities (e.g., Ainsworth, 2006; Duval, 2006; Lesh, Post, & Behr, 1987; Ainsworth, 2006).

Using multiple representations for mathematical objects in the classroom is hence a key for fostering students' conceptual learning (see also Dreher & Kuntze, 2013). Accordingly, teachers need professional knowledge so as to foster their students' learning – and they have to be able to *use* this knowledge in analysis processes. Analysing mathematical content against the background of multiple representations (A) is probably as important as analysing related to ways of dealing with representations in the learning process (B). In the following, we will concentrate on the teachers' analysing and the underlying professional knowledge they refer to.

As a framework, this study uses a multi-layer model (Kuntze, 2012), which combines Shulman's (1986) domains with the spectrum between knowledge and prescriptive convictions/views (Pajares, 1992), which are considered as individual professional knowledge as well. As a third dimension, the degree to which professional knowledge is bound to content or classroom situations is used to distinguish different levels of globality (cf. Törner, 2002), resp. situatedness (vertically ordered layers in Figure 1). The cells of the model in Figure 1 should not be seen as completely separable – indeed, the extent to which components of professional knowledge are consistent across different cells may even be interpreted as an indicator of teacher expertise (cf. Doerr & Lerman, 2009). A more detailed description of this theoretical background is given in Kuntze (2012) and Dreher and Kuntze (2015a).

Professional knowledge related to using multiple representations is a resource (Schoenfeld, 2011) for teachers, which is the base on which they can draw when they analyse content matter related to learning opportunities or the interaction with students in classroom situations. By *analysing* we understand *an awareness-driven, knowledge-based process which connects the subject of analysis with relevant criterion knowledge and is marked by criteria-based explanation and argumentation*. The subject of analysis can be for instance an area of content knowledge (A) or a classroom situation (B). Both of these are in the focus of this study.

Analysing mathematical content with respect of criterial knowledge related to using multiple representations (A) means in particular that teachers can identify and connect different representations of mathematical objects, but also that they can think of examples of content in which the use of multiple representations plays an important role for example, for gaining in-depth conceptual insight or for simplifying problems (cf. Kuntze et al., 2011). Such analysis steps draw above all on content knowledge (CK). An example of a question which requires analysing content in such a way is given in Figure 2 (in the methods section): For answering this task, teachers have to be able to review examples of content matter according to criterial knowledge related to using multiple representations and hence to make connections between examples of content matter according to this criterial knowledge.

Analysing classroom situations against the background of dealing with representations (B) is more based on aspects of pedagogical content knowledge (PCK). In particular, the following processes play a role which are also connected to aspects of teacher noticing (cf. Friesen et al., in this proceedings book): 1) Identifying relevant situations concerning the use of representations, which marks the "starting point" of an analysis, 2) evaluating such situations in a critical way which is based on connecting relevant situations and arguments to corresponding elements of theory regarding the use of representations, and 3) presenting/articulating the results of the analysis. These processes should not be considered as ordered or completely

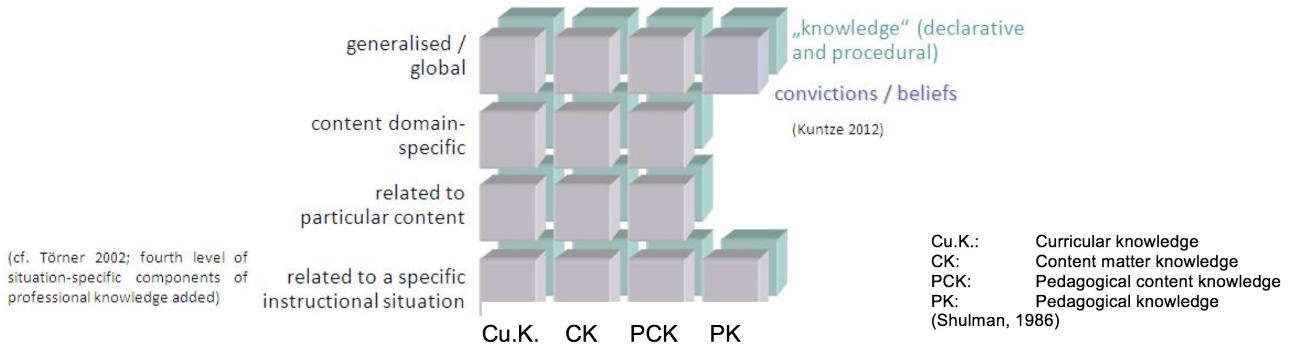


Figure 1: Multi-layer model for components of professional knowledge

separable, as there may be jumps or simultaneous processes.

In both cases, the analysis needs an initial awareness of criteria linked with using multiple representations. Moreover, making connections between this criterial knowledge on the one hand and specific content or characteristics of a specific classroom situation on the other hand is needed. Drawing on criterial knowledge affords explaining the subject of analysis and identifying arguments for the conclusions which are necessary for an appropriate answer. For this reason, potential interdependencies between teachers' personal resources in analysing merit attention.

Despite a clear relevance of the requirement of analysing the use of multiple representations in content matter and classroom situations for teaching mathematics, research focusing on teachers' analysing related to these aspects is scarce (cf. overviews in Dreher & Kuntze, 2015a, 2015b). More specifically, to our knowledge this is the first quantitative empirical study which focuses on interdependencies of teachers' analysing related to these domains of professional knowledge. For exploratory insight and against the background of differences (cf. Dreher & Kuntze, 2015a, 2015b), it is also necessary to explore whether there are differences between teachers from different school types and between in-service and pre-service teachers.

RESEARCH QUESTIONS

Against the background outlined in the previous section, there is a need of research exploring teachers' analyses of content matter (A) and of classroom situations (B) related to using multiple representations. Using multiple representations is thus an overarching concept (cf. Kuntze et al., 2011) which affords analysing and making connections between professional knowledge and the subjects of analysis, namely examples of content and classroom situations.

This study hence aims at answering the following research questions:

- To what extent do in-service teachers have professional knowledge necessary for analysing whether content matter can be connected with the overarching concept of using multiple representations?
- Are there differences between in-service teachers from different school types?
- How big are these differences in comparison with the knowledge and analyses of pre-service teachers?
- Does the quality of analysing content matter regarding the use of multiple representations interdepend with the in-service teachers' analysis of classroom situations based rather on PCK?

SAMPLE AND METHODS

The in-service teachers participating in this study belonged to two sub-samples: There were in total 58 German in-service teachers which consisted of:

- 34 teachers from academic-track secondary schools (20 female, 14 male; mean teaching experience 14.8 years, SD=13.4 years)
- 24 teachers from secondary schools for lower-attaining students (9 female, 15 male; mean teaching experience 10.7 years, SD=9.5 years)

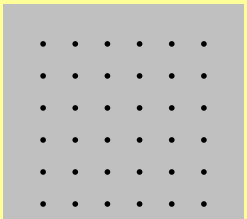
Moreover for answering the third research question, 117 German pre-service teachers were included (78 female, 35 male, 4 without data; mean age of 22.33 years, SD = 3.56 years) who had been studying on average for 2.19 semesters (SD = 1.12). Within this sub-sample of pre-service teachers, 61 were preparing for being teachers in primary schools, 32 in secondary schools for lower-attaining students (HSWRS), and 15 for working in schools for students with special needs (9 without data). An additional sample of 42 Austrian pre-service teachers preparing for teaching in academic-track schools were included for exploratory comparisons.

In order to tap the teachers' analyses of contents, a test was conceived with a focus on analysing contents and perceiving links across contents according to overarching concepts. The instrument concentrated on CK-related analyses connected with the overarching concepts *using multiple representations*, *dealing with infinity*, and *giving arguments/proving*. (cf. Kuntze et al., 2011). This study only focuses on the teachers' answers to two tasks related to using multiple representations, a sample task is presented in Figure 2. The answers were collected in an open format, the

On the right, there is a graphical representation of the definition of "square number". This representation affords an additional access compared to the symbolic definition ("if $q=n^2$ for a positive integer n, the q is called a square number").

Can you think of other mathematical concepts, for which a symbolic definition can complement with a non-symbolic representation in a similar way?

Please give as many examples as possible.



„A number that can be visualised like shown here in the form of a square is called a square number."

Figure 2: Sample task for analysing and connecting content according to the overarching concept of using multiple representations (Kuntze et al., 2011, p. 2722)

teachers were given as much time as they required to complete the test.

The answers were coded according to a top-down coding method (see Kuntze et al., 2011, for details and sample answers). Among others, the coding categories *existence of a codable answer*, *quality and transfer level of the examples*, and *embedding of examples provided* were used. An indicator score was generated in the following way: If the teachers gave at least one appropriate example, their answer was coded with 1 point, if at least one of the examples was from a different content area than the example given in the task, the full score of two points was assigned to the corresponding answer, as in this case we can assume that the teacher had shown awareness of criterial knowledge related to using multiple representations and analysed successfully at least one example of content matter.

In classroom situations, the quality of the use of multiple representations often depends on how carefully teachers link different representations and whether they accompany the students' understanding with this respect. For tapping the teachers analyses in this area, we used an instrument which is described in detail in (Dreher & Kuntze, 2015a). In this instrument, the teachers are given fictitious transcripts of classroom situations (related to the content area of fractions) in which the teacher changes the representation register unnecessarily and without explicitly providing help or explanation for linking the corresponding representations. In the situation in Figure 3, this is the case for the teachers' change to the pizza representation. The participating teachers were asked to evaluate how the given teacher's reaction supports the student's understanding in this situation and to give reasons for their evaluation. According to the design of the classroom situations, the participating teachers' analyses of these situations based on their PCK related to using multiple representations should hence lead to a critical evaluation of the way how multiple representations are used (Dreher & Kuntze, 2015a). A

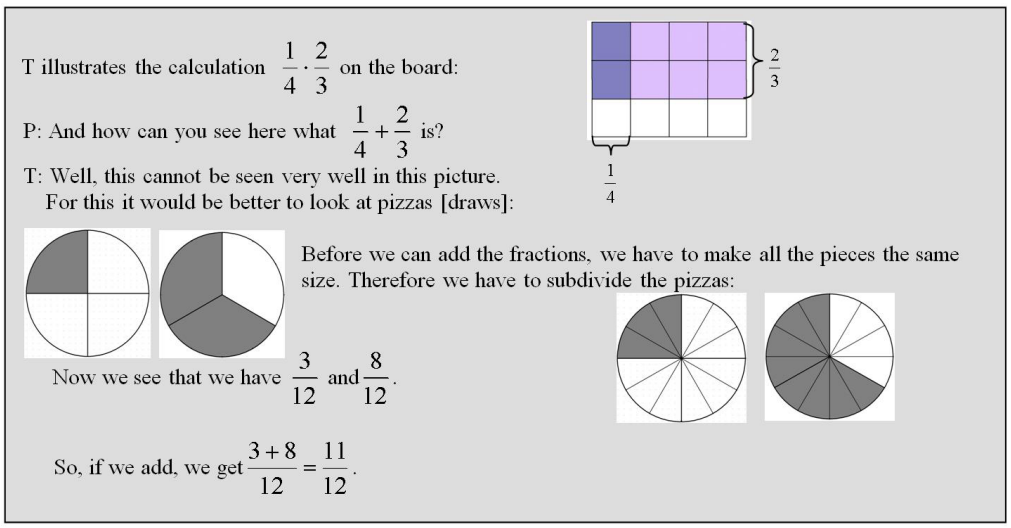


Figure 3: Classroom situation to be analysed according to the use of multiple representations (Dreher & Kuntze, 2015a)

corresponding top-down coding was consequently done with the teachers answers which were collected in an open format. Based on these codes, a score was calculated by counting successful answers. As there were four classroom situations, these scores ranged from 0 to 4 points (Dreher & Kuntze, 2015a).

For answering the fourth research question which focuses on interdependencies between the two analysis scores, correlations (Pearson) were calculated. As there was the possibility of controlling for specific CK related to changing between representations of fractions (for details about this additional instrument see Dreher & Kuntze, 2015a), this variable could be included in a regression model which affords describing the interdependencies in the focus of the fourth research question.

RESULTS

The first, second, and third research questions focus on the extent to which different samples of in-service and pre-service teachers analyse content matter against the background of the overarching concept of dealing with multiple representations. Figure 4 shows an overview of these results. The in-service teachers working at academic track secondary schools reached on average about two thirds of the full test score related to analysing and connecting content matter through the concept of using multiple representations. In contrast, the in-service teachers from HSWRS secondary schools were significantly less successful in these analyses (T-test; T=3.40; df=56; p=.001, d=0.90). However, each sub-sample of in-service teachers scored significantly better than the corresponding sub-sample of pre-service teachers (academic-track teachers: T=3.32; df=74; p=.001, d=0.76); HSWRS teachers: T=4.75; df=54; p<.001, d=1.23). The values of Cohen's d indicate that these are strong effects.

The average score of the academic-track pre-service teachers was very similar to the score of the HSWRS in-service teachers.

As the data in Figure 4 is rather connected to the teachers' CK it may be of additional interest that we found in a corresponding study (Dreher & Kuntze, 2015a, 2015b) that there was a relatively similar pattern for teachers' analyses of classroom situations against the background of dealing with multiple representations, which is rather based on teachers' PCK. This points to the fourth research question, which concentrates on connections between CK-related and PCK-related variables.

CK, especially CK related to the particular content, is probably important for analysing classroom situations. For this reason specific CK (in this case related to the use of multiple representations with fractions) was also included in the study as an additional variable. Table 1 shows correlations, which suggest that for academic-track teachers the more overarching CK-related variable has played a more important role than the more specific CK variable (fractions) which is more close to the classroom situations the teachers had to analyse.

Deepening this insight, we included the variables in regression analyses (cf. Table 2). The results show that analysing contents according to the overarching concept of using multiple representations (B) interdepended more strongly with the quality of the analyses of classroom situations (A, i.e., the corresponding score as independent variable), than specific CK related to using multiple representations of fractions – and that in particular it plays a significant role even if the data is controlled for the fraction-related "local" CK component, as far as the in-service academic-track teachers are concerned.

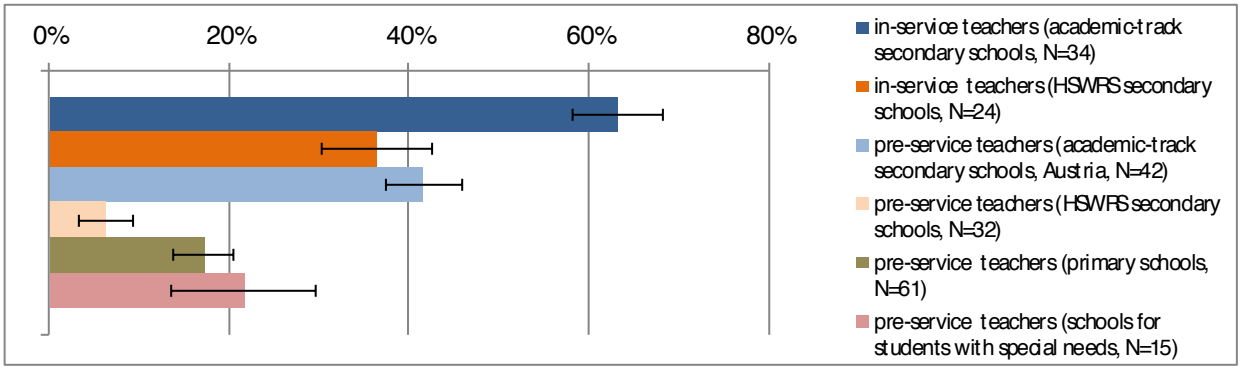


Figure 4: Analysing contents against the overarching concept of using multiple representations: Means and their standard errors

Correlations with score related to analysing classroom situations with respect to using multiple representations (B) (Dreher & Kuntze, 2015a)	Subsample of all in-service teachers (N=58)	academic-track in-service teachers (N=34)	HSWRS in-service teachers (N=24)
CK-related score „using representations in the content area of fractions" (Dreher & Kuntze, 2015a)	,128 ,333	-,031 ,863	,217 ,309
Score "analysing content matter with respect of using multiple representations" (A) (Kuntze et al., 2011)	,319* ,014	,365* ,034	-,021 ,922

Table 1: Correlations (Pearson) and significance values (2-sided)

		B	St. error	Beta	t	p
HSWRS in-service teachers	(Intercept)	,235	,077		3,046	,006
	CK-related score „using representations (fractions)"	-,206	,197	-,238	-1,050	,306
	Score "analysing content matter with respect of using multiple representations"	,028	,105	,061	,269	,791
academic-track in-service teachers	(Intercept)	,201	,132		1,523	,138
	CK-related score „using representations (fractions)"	-,146	,195	-,129	-,752	,458
	Score "analysing content matter with respect of using multiple representations"	,300	,130	,396	2,318	,027

Table 2: Regression analyses for subsamples of in-service teachers

DISCUSSION AND CONCLUSIONS

Before discussing the results in detail, it is important to recall that the study has clear limitations, as the findings hardly allow for generalisation. For instance, the samples are not representative for German teachers, and the research instruments should be extended in follow-up research. However, the findings afford answering the research questions and call for further investigation in related empirical studies.

The following key findings can be summarised:

— The results related to the first research question indicate that many of the academic-track secondary teachers were not able to connect the overarching concept *using multiple representations* to at least one example of mathematical content. Analysing content against this criterial knowledge appears hence to be very difficult even for in-service teachers from academic-track secondary schools.

— Teachers from HSWRS schools showed, however, even less elaborated analyses and a significantly lower professional knowledge related to connections with the overarching concept of using multiple representations. In line with findings from Dreher and Kuntze (2015b), the findings related to the second research question thus suggest that the culture associated with different school types or selection effects of teachers might play an important role and that there are school-type-specific professional development needs.

— As appropriate answers of the pre-service teachers were – as expected (cf. Kuntze et al., 2011) – even less frequent, the findings associated with the third research question point to a substantial need for teacher education.

— As far as the fourth research question is concerned, the results related to interdependencies may be seen as consistent with corresponding findings from the COACTIV study (e.g., Kunter et al., 2013), suggesting that there might be general interdependencies between CK and PCK. This study can add preliminary, but more specific insight by focusing on the teachers' analysing.

Follow-up questions arising from the findings open up several directions for further research: First of

all, there is a clear relevance for the development of theory: What status does analysing related to overarching concepts (cf. Kuntze et al., 2014) have as a component of mathematics teacher expertise and to what extent do different contexts of profession-related analysing interdepend? But there are also more practice-related follow-up questions, such as: How can CK and PCK related to overarching concepts be supported best in professional development programs? When and how can growth in the corresponding domains of professional knowledge take place? Which role may be played by classroom experience or curricular knowledge? Finally, also aspects of the further development of the test instrument are in the focus (e.g., extensions to more overarching concepts, other test formats) and may contribute to a deepened understanding of the findings and teachers' profession-related analysing.

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