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Exploring pictorial representations in rational numbers: Struggles of a prospective teacher

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We aim to identify the knowledge that a prospective teacher, Maria, uses in practice, focusing on her struggles and what she learned from her practical experience teaching rational number multiplication. Data was collected from lesson plans, observations, written reflections and semi-structured interviews. Maria developed her knowledge for teaching when anticipating solutions and errors and selecting representations. Reflecting on her practice, she realised that she was able to solve some tasks with symbolic procedures but could not represent them pictorially.

Keywords: Knowledge, practice, rational numbers, representations.

INTRODUCTION

The knowledge for teaching mathematics that prospective teachers need to develop and the way they develop it are controversial issues (Ball, Thames, & Phelps, 2008; Ponte & Chapman, 2015; Shulman, 1986). The practicum is a particularly important site to study such knowledge since prospective teachers are faced with circumstances that allow noticing important weaknesses and strengths.

To foster students' understanding of mathematics concepts and procedures, teachers are called to engage them in making connections among representations (NCTM, 2007). They need to support students' fluent use of symbols, grounded in informal representations (Ball et al., 2008; Ma, 1999; Ponte & Chapman, 2015). Rational numbers raise many difficulties for students and challenge teachers to promote conceptual learning (Lamon, 2006; Ma, 1999). Research has brought attention to prospective teachers' knowledge of rational numbers in different ways. For example, Isiksal and Cakiroglu (2011) studied prospective teachers' pedagogical content knowledge of fraction multiplication.

Findings indicated that teachers have different perceptions of children's mistakes and employ different strategies including using multiple representations, using problem solving strategies, making clear explanations of questions, and focusing on meaning of concepts. However, no study has been found focusing on the use of informal and formal representations in teaching fraction multiplication and the struggles prospective teachers may experience in providing representations that enable students to develop their knowledge on this topic. The aim of this study is to identify the knowledge of a prospective teacher in the teaching and learning of rational number multiplication, with a focus on the use of informal and formal representations, analysing the knowledge mobilized in teaching practice, the struggles, and the knowledge built from the first practical experiences.

PROSPECTIVE TEACHERS' KNOWLEDGE

Teachers' knowledge includes mathematical and pedagogical content or didactical knowledge, both of which are of critical importance for teaching practice (Ball et al., 2008; Ponte & Chapman, 2015; Shulman, 1986). Mathematics knowledge involves conceptual and procedural aspects (Hiebert, 1988; Rittle-Johnson & Schneider, 2012). Conceptual knowledge is a network of concepts and procedural knowledge consists in rules or procedures for solving mathematical problems (Bartell, Webel, Bowen, & Dyson, 2012). Procedural knowledge may be part of conceptual knowledge. Procedures may be performed without understanding or may be performed knowing why and when, in which case we have mathematical conceptual knowledge. Conceptual mathematical knowledge of rational numbers involves knowing different representations and meanings and in order "to create one representation first we have to know what to represent" (Ma, 1999, p. 135). Conceptual knowledge

allows us to connect topics (e.g., seeing multiplication as repeated addition).

Didactical knowledge concerns with how teachers teach (Ponte & Chapman, 2015). Teachers must anticipate students' common mistakes and misconceptions (e.g., generalizing addition procedures in multiplication), to anticipate students' solutions in specific tasks, and also know what students will find challenging, interesting or confusing. Teachers also have to be able to sequence tasks, to recognize the value of using certain representations, to pose questions, and to explore students' strategies. In addition, they need to understand the main ideas of current curriculum documents, identifying principles of teaching (e.g., NCTM, 2007).

Ponte, Quaresma and Branco (2012) characterize teacher's practice into two main aspects: the tasks proposed to students and the communication established in the classroom. In respect to tasks, teachers may choose to offer just simple exercises or also propose challenging exploratory tasks, problems and investigations in which students need to design and implement solution strategies based on their previous knowledge (Ponte, 2005). Classroom communication may be univocal or dialogic, depending on the roles assumed by the teacher and the students and the types of teachers' questions, including inquiry, focusing or confirmation questions (Ponte et al., 2012). Representations are an important feature of tasks, and may be categorized as pictorial (images), iconic (points, lines, circles), and notational (number line, arrows, vertical columns, symbols) (Thomas, Mulligan, & Goldin, 2002).

RESEARCH METHODOLOGY

This study takes a qualitative and interpretative approach (Erickson, 1986), using a case study design. The participant is Maria, a prospective elementary school teacher. She always wanted to be a teacher but is in a higher education program at a late stage of her life. In school she had mathematics up to grade 9. Maria reflects with ease, addressing her difficulties in an explicit way. She said that she had to study hard to know the content that she was going to teach and to figure out how to implement the didactical ideas that she had learned at university. During teaching program, she experienced exploratory learning and she wishes to provide such approach to her students. Maria already

knew the grade 6 class with 28 students, in which her practicum took place. She interacted informally with her school mentor to discuss who would teach the different topics, deciding that they would give a total of six classes on rational numbers, three of each taught by each one of them, with Maria introducing the concepts and the school mentor providing practice.

Maria's lesson was observed and videotaped (Li). In addition, data was collected and analysed from initial (II) and final (FI) semi-structured interviews, and before (BIi) and after lesson (AIi) interviews. We also analysed the documents that she produced (lesson plans and reflections) and the field notes written by the first author during data collection. The interviews and videos were fully transcribed. The analysis is descriptive, seeking to characterize Maria's teaching. The transcribed conversations were first analysed according to four dimensions (conceptual/procedural mathematical knowledge and didactical knowledge about tasks and students). At a second moment the analysis was based on categories built from data. The intersection of the four dimensions enables us to highlight communication moves (Charmaz, 2006). We consider knowledge to be conceptual when there is evidence of understanding the reasons for using procedures and for knowing different representations and meanings of a situation. We consider knowledge as procedural when the teacher cannot relate informal and formal representations or when she cannot explain in the interviews why she did it. We also give attention to didactical knowledge in practice, focusing on knowledge about tasks, students and communication that takes place in the classroom. For example, teachers must design appropriate tasks, know what they will explore and relate the representations, anticipate student's solutions and plan how to orchestrate them. In addition, teachers should anticipate questions to help students understand the concepts in the context of productive classroom communication.

THE PRACTICE OF MARIA: LEARNINGS AND STRUGGLES

Despite her willingness to follow an exploratory approach and carry out considerable planning, Maria did not anticipate how to relate different representations. Comparing her agenda with her teaching practice, she realized that she did not clearly explore the concepts. That is, Maria is a case of a prospective teacher seeking to perform exploratory teaching

but with trouble in preparing and carrying it out as intended.

Mapping the topic and anticipating practice

Maria taught three lessons on rational numbers. She wanted to introduce fraction multiplication with an emphasis on understanding: “I’m more interested that students understand the why of the result and the meaning of the result. What it represents...” (BI3). She made decisions about the tasks to propose, considering their nature and value, and chose a pictorial rectangular representation and different symbolic representations of rational numbers. She intended to promote discussions with inquiry questions. Next Maria had to negotiate the tasks with the school mentor who wanted her to use the textbook. She used the tasks of the textbook, but felt that her choices were limited. She also read articles and documents about rational number multiplication. She reviewed procedures and solved several tasks: “I saw everything always supported what was in the textbook” (BI1). She tried to include ideas that represent “good practice”. In the end, she decided to explore first the multiplication of a natural number by a fraction and then the multiplication of fractions. However, she still had some unresolved questions and sought out the professors from her teacher education college: “It was only when the professor began making pictorial representation that it occurred to me! Only when I looked at this representation did I associate pictorial and symbolic representations” (BI3).

In her lesson plan, Maria anticipated students’ solutions, errors and explanations to prepare her to help students overcome these errors. She thought, “If they ask this, what will I say? If I ask this, what might they say?” (BI). And she added:

I consider students’ possible solutions (...) in the multiplication of fractions, whether they follow the rule denominator times denominator and numerator times numerator (...) multiply denominators and maintain the numerators, [or generalize ideas from] addition, finding common denominator ... (BI).

In her view, pictorial representations might serve as a useful support for solving the tasks because “sometimes... I think we do the mathematics, we give the results, but what are we talking about? Which unit? What part? Part of what?” (BI3). As she explored the

tasks, she encountered difficulties and said: “It’s very difficult to imagine students’ thinking, what will happen... Imagine them... It is a difficult exercise but very necessary for further practice” (BI2). In the lesson plan, she solved all tasks with fractions and pictorial representations except in the first task.

Maria used didactical knowledge when she considered the kind of tasks to propose, established a sequence of tasks, and anticipated students’ potential solutions and common mistakes. However, when anticipating the solutions of the tasks she did not fully realize how challenging and powerful they could be. As a result of a learning experience with her university professor, she recognized the value of using pictorial representations; thus, it seems that she had developed didactical knowledge. Yet when she anticipated different symbolic solutions for the tasks and some pictorial representations, her procedural knowledge became evident. However, her conceptual knowledge would only show up in the practical experience.

Instructional practice

Maria began the lesson reviewing the homework. Then, she told a story to engage students in solving a problem: “With the candy that came in a box, Luís separated 6 bags of $\frac{2}{5}$ kg each. Does the box weigh more or less than 3 kg?” This task combines discrete quantities (6 bags) and continuous quantities (weight of the bags). Dealing with these quantities requires suitable representations. Maria invited students to present their ideas, saying “How can we solve this problem? Who wants to help? Pedro!” The student proposed immediately “add $\frac{2}{5}$ six times”. She recorded Pedro’s idea and asked the result. After 5 seconds she wrote the sum with the students’ help “ $\frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{12}{5}$ ”. Immediately Pedro said “ $\frac{12}{5}$ isn’t more than 3 kg”. Maria again recorded this on the board and posed inquiry and focusing questions to help others students to understand it:

- Maria: Pedro said that $\frac{12}{5}$ is not more than 3 kg. Pedro, how did you think?
- Pedro: Because $\frac{5}{5} = 1$; $\frac{5}{5} + \frac{5}{5} = \frac{10}{5} = 2$; $\frac{15}{5} = 3$.
- Maria: Exactly! And what do I have (pointing to $\frac{12}{5}$)?
- Pedro: $\frac{12}{5}$.
- Maria: Then we know that we have at least 2 kg! Because $\frac{5}{5}$ more $\frac{5}{5}$ are $\frac{10}{5}$! And there is $\frac{2}{5}$ missing to one [more unit]. But how can we see that? Is there

another way of seeing this? And if we tried... (Turns to the board) How many bags do we have?

To illustrate Pedro's answer, she began the explanation with the support of a rectangular representation (Figure 1) and said:

Let's suppose we have here a rectangle and each part of the rectangle is a bag. How much does each bag weigh? (...) $2/5$ kg. And with the second bag? (...) $4/5$. So we have... (Together with the students) $4/5, 6/5, 8/5$ and the result is $12/5$. (L3)

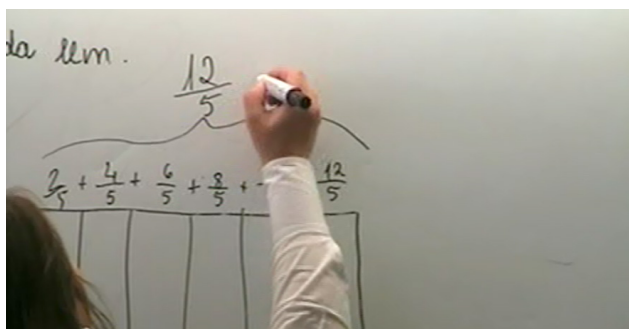


Figure 1: Representation of the first problem

This representation is rather confusing and students began complaining that they did not understand. Figure 1 illustrates that Maria was saying one thing and symbolically representing another. On a second trial, she divided each part in five parts unsuccessfully. Then she erased the figure and explained symbolically "we have to do $2/5+2/5+2/5+2/5+2/5+2/5=12/5$ ". The teacher did not relate the pictorial and symbolic representations. She went on posing focusing and inquiry questions:

Francisco: Ok! And what if we do... Six times two and the five stays the same?

Maria: Ok! How can we do it?

Francisco: Six times two is twelve and the five stays the same.

Maria: Why don't you multiply the 5 too? And what if I do like this $(6/1 \times 2/5)$? Now we have the numerator!

Pedro: Five times one is five...

Maria: And if I said that I multiply the (...) numerator and keep the denominator. Can I always do this? (L3)

The students rejected the rule and Maria, attempting to convince them, used another example with another

denominator, stressed the rule comparing the procedure with the result, and moved to the synthesis and final answer:

The problem said that Luís has six bags and each bag has $2/5$ of a kilo. Then we do $2/5+2/5=4/5$ and $4/5+2/5=6/5$ (...) $12/5$. Then Francisco said it would be faster to do 6 times $2/5$ which gives $12/5$. And, finally, we see that we can multiply numerator and numerator and keep the denominator. (...) Pedro said in the beginning that the answer is less than 3 kilos because $12/5$ isn't as much as 3 kilos... (L3)

In the sequence, Maria proposed a second task: "Rita cut a cake into 4 equal parts. But one quarter of the cake seemed to be a very big slice! After all, she only wants half a slice, or half of one quarter. Which part of the cake does Rita eat?"

Immediately a student explained that " $1/2$ represents half and one quarter is the final measure when we divide in 4 pieces. One half of a quarter is half of $1/4$ ". Again, Maria asked how to represent this situation and drew a rectangle divided in four parts (Figure 2). She went on asking confirmation and focusing questions:

Maria: Is this what Rita eats? (Pointing to half of a quarter of the cake)

Francisco: It is half of a quarter! (...) She only wants half!? Since all the pieces were the same... I have to divide everything in the middle...

Maria: Oh! Interesting, Francisco! (...) What Francisco is saying is very important! But maybe it is better to divide in columns! It's the same reasoning! (L3)

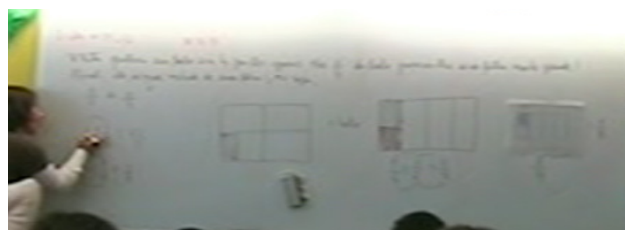


Figure 2: Different representations of the second problem

At this point, using the projector, Maria illustrated Francisco's idea. A student said:

She eats one eighth. Rita divided the unit and only ate one part... The cake was divided into four parts and then divided in eighth parts... Halves that gives eighths... She ate one out of eight parts! (L3)

In her teaching practice Maria mobilized mathematical and didactical knowledge and showed weaknesses in both. When she explored the first problem, she used fractions and she did not consider the pictorial representation of multiplying a whole number by a rational number. She showed a procedural nature of mathematics knowledge and revealed some weaknesses in her conceptual knowledge when she was not able to pictorially represent the operation. In the second task she anticipated the pictorial representations and showed conceptual knowledge. However, she did not explore the connections between the pictorial and symbolic representations. Her discourse had changed. In the beginning, she had a dialogue with three students (one at a time) and tried to focus the attention of all students on the emerging ideas. She posed different types of questions, including inquiry, focusing and confirmation questions. In the second moment, she posed fewer focusing and confirmation questions.

Evaluating, reflecting critically, reviewing, and restructuring knowledge

In her reflection, Maria considered that this lesson had presented unexpected difficulties with the pictorial representation of the task and regarding the exploration of mathematical ideas related to the representations and their connections. This aspect caused her great anxiety at the end of the lesson:

I shouldn't have used this kind of representation because they [the students] weren't used to it (...) Although I think that this representation helps explain why. The goal is for them to realize why in that case the result $12/5$ appeared. It didn't fall from the sky! (AI3)

Maria recognized the value of using certain representations but considered giving up on the use of

pictorial representation, which reveals some insecurity regarding her didactical ideas. She did not notice the mistakes that she had made on the board (shown in Figure 1) and merely felt that she had not handled the situation well and the students did not understand. After visualizing the lesson video she said: "I think this approach [using pictorial representations] is better for everyone, I am convinced. (...) The problem is not in the explanation of the problem! It's in me!" (AI3). She identified her mistakes in the first task: "What I wrote was not the same as what I was saying! And maybe that was the confusion! I caused the confusion because, in fact, what they are seeing is $2/5+4/5+...$ " (AI3).

During the interview, Maria tried to explain why she should not use this pictorial representation. She agreed that the reason was that she did not recognize the differences between quantities and that the result was an improper fraction. These difficulties raise questions about her conceptual mathematical knowledge.

Confronting the plan and the practice in the second task, Maria realized that she used different pictorial representations and different symbolic representations of the proposed problem (Figure 3). At the end of this lesson she evaluated her practice recognizing that: "I knew the procedure, I mastered the procedure! I memorized the procedure back and forth and not how to represent the concept [multiplication]!" (AI3).

Reflecting on her agenda and comparing it with her teaching practice, Maria concluded that the end result was positive but could have been better. She recognized that, although she had invested heavily in planning the lesson, she had not achieved her desired result. She noticed that in order to prepare these lessons she had mobilized her mathematics content knowledge that she learned in the teacher education program:

I see rational numbers in a different way (...) Before I had fractions and decimals arranged in

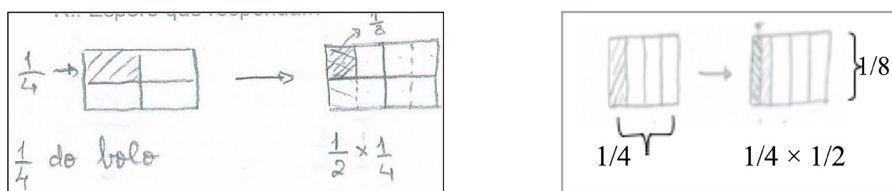


Figure 3: Representation of the second problem in the plan and in practice

different drawers (...) Now I see that they are different representations of the same number (FI).

When she studied the literature, she learned the relations between operations and said “when we multiply a whole number with a fraction we are simplifying repeated addition (like in $6 \times \frac{2}{5}$, when I work with students starting from $\frac{2}{5} + \frac{2}{5} + \dots$)” (FI). She did not explore the relationships between representations and she was able to see, during the interview, that the arithmetic expression that she wrote on the board did not represent the problem proposed.

Maria recalled general didactical ideas and these became meaningful for her. She realized the importance of planning and that her lesson plan was better after each practical experience. She realized that in order to be successful in her practice “the plans have to evolve (...). In the first plan [lesson before], I thought that it was perfect and then I went to practice and I realized that it could have been better...” (FI). She developed knowledge when anticipating solutions, errors and representations. Maria knew that a teacher must “lead students to understand the why of something” (FI) and that is why she knows that it is important to “unpack the pictorial representations” (FI). After the teaching practice, she continued to believe that good teaching leads to learning with understanding. She assumed that the teacher must support students in learning, by listening, asking good questions and helping them to build their knowledge.

CONCLUSION

Maria felt that she developed significant aspects of her knowledge about rational numbers during these lessons. Regarding her mathematics knowledge, she became more aware of the concepts related to rational number multiplication in fraction representation. Early in the lesson, her weakness in conceptual knowledge led her to take a procedural approach (Ma, 1999). However, she showed conceptual knowledge when she said that rational numbers may be represented by fractions and decimals and related addition and multiplication. However, she struggled to connect these with pictorial representations. As in other studies, we see issues related to connecting real-world situations and symbolic representations and connecting different representations of a concept (Ponte & Chapman, 2015). At the end of this teaching experience, Maria had learned about: how to repre-

sent improper fractions; the complexity of using both discrete and continuous quantities; connecting pictorial and symbolic representations; and the meanings of different expressions.

Concerning didactical knowledge, Maria was able to sequence tasks using ideas of several articles and recognized the value of using pictorial representations as tools to develop mathematical ideas. She mentioned that she had become more aware of anticipating students’ questions, common mistakes, and solutions (Bartell et al., 2012). She also became more aware of when she needed to explain. In her instructional practice she posed questions and explored students’ strategies, seeking to lead them to connect pictorial and symbolic representations. As in other studies, we see issues related to conceptual understanding, using multiple representations and curriculum materials and textbooks, planning, assessing students, and analysing mathematics teaching (Ponte & Chapman, 2015).

This study illustrates the struggles and learning that prospective teachers may experience when they strive to engage in an exploratory approach that requires strong mathematical and didactical knowledge (Ponte & Chapman, 2015). In order to propose challenging tasks and to use different representations, they need to develop a deep understanding of rational numbers. They must engage in analyzing students’ strategies, offering student-focused responses, anticipating practice by solving the tasks and discussing common mistakes (Son & Crespo, 2009). This study shows how anticipating practice in a careful way is essential for prospective teachers.

These results show us that some aspects may only become explicit in practice. Teachers have perceptions of children’s mistakes and different strategies that may be used including using multiple representations, using problem-solving strategies, providing clear explanations and focusing on the meaning of concepts (Isiksal & Cakiroglu, 2011). In this study we focused on the use of informal and formal representations in teaching fraction multiplication and on the struggles that a prospective teacher experienced in providing representations to help students to develop their knowledge about this topic. For example, Maria anticipated the solutions of the tasks in symbolic and pictorial representations but in class had trouble relating the two representations. In another words, practice was an appropriate context to see what she could do

and the nature of her knowledge. These issues are important to know how teacher educators may prepare and support prospective teachers who are learning to teach for meaningful conceptual learning.

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