Supporting students' development of mathematical explanation: A case of explaining a definition of fraction

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University of Michigan, Ann Arbor, USA, mskwon@umich.edu

This paper aims to conceptualize the work of supporting students' development of mathematical explanation. To provide an empirical basis, I analyse instructional interactions managed by the same teacher for teaching the brown rectangle problem for different cohorts of students across five years. The four core tasks of teaching are (1) attending to the organic structure of the mathematical task; (2) mapping the scope of answers onto the targeted mathematical ideas; (3) hearing the mathematical needs embedded in students' explanations; and (4) distributing and building a mathematical talk collectively.

Keywords: Decomposition, the work of teaching, mathematical explanation, fraction.

INTRODUCTION

Teaching is often described as a complex activity because it involves managing multiple relationships simultaneously with students and with content over time (Cohen, Raudenbush, & Ball, 2003; Lampert, 2001). To make this complex work doable and learnable for teachers, especially for beginning teachers, several scholars (Boerst et al., 2011; Grossman et al., 2009; Sleep, 2012; Thames, 2009) have addressed the need to decompose the work of teaching into its constituent components. As a result, a core practice of teaching has begun to be decomposed into nested practices with varying grain sizes. Despite these initiatives, the call for "a specific technical language for describing the implicit grammar and for naming the parts" (Grossman et al., 2009, p. 2069) has not met an agreed-upon robust framework yet. For example, in decomposing the work of steering instruction toward the mathematical point, Sleep (2009) identifies seven core tasks of teaching and further decomposes each core task into strategies and problematic issues. The core tasks she identifies are not mutually exclusive but rather might be enacted simultaneously. Furthermore, in decomposing each core task into strategies, she does not associate it with particular teaching moves. On the other hand, in decomposing the work of leading a mathematical discussion, Boerst and colleagues (2011) start with the larger grain size of domains (e.g., leading a discussion) and then specify it into a smaller grain size of techniques (e.g., revoicing), while articulating intermediate practices (e.g., clarifying student thinking) that connect between domains and techniques. A brief review of literature on decomposition, despite focusing on a different domain of teaching practices, gives a particular prominence to the structure of decomposition, the level of decomposition, and the link to teaching moves or discourse moves.

Given the lack of agreed-on grammar for decomposing the work of teaching, this study aims to decompose one of the key teaching practices that are crucial for accomplishing the ambitious goal of developing mathematical power and mathematical proficiency for all students: the work of supporting students' development of mathematical explanation. The practice of giving, hearing, and evaluating explanation has been considered an important goal for learning because it resolves cognitive dissonance and facilitates cognitive development in the process of knowledge construction. More specifically, giving explanations can serve as opportunities for students to reflect on their own thinking and to reconstruct their existing knowledge, while hearing others' explanations provides opportunities for students to appropriate language that a teacher or more advanced students use, to recognize any cognitive dissonance that contradicts

their own understanding, and to use others' explanation as a resource to extend their own knowledge.

Despite its crucial role for learning, there is a general consensus that most students do not have sufficient opportunities to develop their own explanations in U.S. mathematics classrooms (e.g., Stigler & Hiebert, 1999). One reason might be that many teachers believe that giving an explanation to students is more efficient and less complicated than eliciting an explanation from students. Even if this belief is being challenged as greater emphasis is being placed on eliciting an explanation from students, it is pedagogically demanding work for teachers. This is well captured by Cohen's (2011) metaphor in describing challenges in extending students' knowledge as he writes:

> Teachers and learners face the same gulfs of ignorance, but from different sides. Learners must somehow build bridges across the gulf, but these bridges are often fragile because the learners work from relative ignorance. The teacher's assignment is to help learners build those bridges, but they work from greater knowledge. ... Rather than helping learners construct and reconstruct bridges of their own, teachers present the finished results of their learning. That reduces the likelihood that teachers can cultivate a practice of teaching, for it can limit learners' understanding. (Cohen, 2011, p. 106)

This metaphor also applies to challenges in supporting students' development of mathematical explanation. On the one side, students do not have sufficient language to explain their mathematical ideas (Forman & Larreamendy-Joerns, 1998) and their explanations are distant from disciplinary explanation (Leinhardt, 2001). On the other side, teachers often present the compressed, polished, and finished form of mathematical explanations to students rather than helping students construct their own explanation. Considering this demanding but crucial work, this study examines what is entailed in supporting students' development of mathematical explanation, and particularly, the ways of using instructional resources to that end.

METHOD

The methods for studying teaching have adopted terms, concepts, and techniques from other disciplines (e.g., grounded-theory; ethnography), but have not further articulated how the selected method addresses issues that particularly matter for teaching. To make an explicit connection between the phenomenon being studied and the method being chosen, I briefly articulate the study design, which is situated in the instructional triangle (Cohen et al., 2003).

Teaching is often examined as a single case in which a teacher teaches a particular topic for a single group of students, but multiple cases of teaching are also examined. In such an examination, a variety of methods are employed. One way of examining multiple cases is to maximize the variation of components within the instructional triangle—a teacher, students, and contents. For example, to identify elements of expertise of teaching, Leinhardt (1985) contrasts performance of expert teachers with that of novice teachers. In another example, in order to identify the common model of instructional explanation and to specify the features of instructional explanation in each subject, Leinhardt (2001) analyses instructional explanations in history and in mathematics.

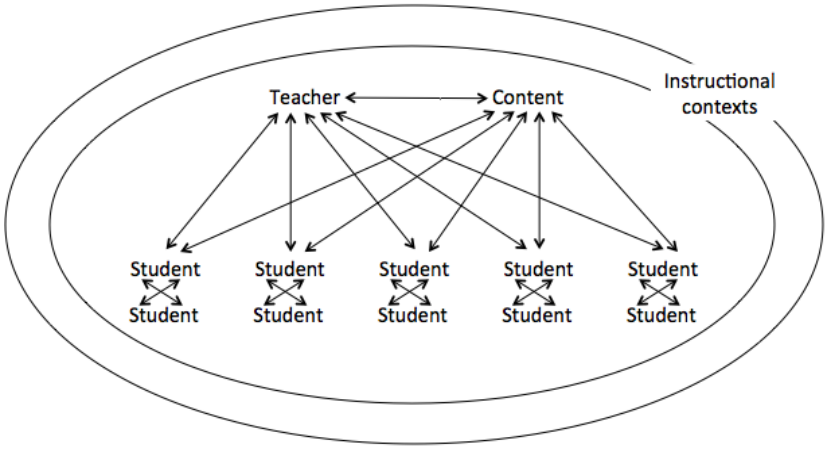

Figure 1: The study design

This paper approaches the problem in another way, minimizing much of the variation among components of the instructional triangle in a more controlled context in which only students vary (see Figure 1). As one of the greatest predicaments of teaching is its dependence on students (Cohen, 2011), analysing instructional interactions managed by the same teacher teaching the same mathematical task to different cohorts of students without substantial differences in students' mathematical abilities is crucial to identify the core tasks of teaching across the particulars of students and unfolding instructions. This method untangles the ways in which the same teacher adjusts the work of supporting students' development of mathematical explanation wherein each cohort of students brings different mathematical ideas, stances, issues, language, ambiguity, and difficulties in explaining the same mathematical task.

To provide an empirical basis, I analyse a longitudinal data set from the Elementary Mathematics Laboratory (EML), a two-week summer mathematics program for entering fifth graders taught by Professor Deborah Ball at the University of Michigan's School of Education, across five years (EML2007, EML2008, EML2009, EML2010, and EML2013). There are no prerequisites to participate in the EML, but it mainly focuses on students who are struggling with learning mathematics rather than students who are outperformed in mathematics. Considering the process of recruitment, there were no substantial differences in students' abilities in mathematics across five years. Each year, approximately 25–30 students, who are ethnically, racially, linguistically, and socioeconomically diverse, participate in a whole-group mathematics class every morning during the two-week program.

As part of a large-scale study which analyses instructional interactions managed by the same teacher for teaching four different mathematical tasks across multiple years, this paper mainly focuses on the brown rectangle problem (see Figure 2). The brown rectangle problem has been used with slight variations in the layout of the rectangle (i.e., where the shaded part is located; the rotation of the drawing), the colour of shaded parts, the inclusion of written problem statement on the poster, the presentation of two sub-problems (posting together vs. posting separately), and the wording of the problem statement (the big rectangle vs. the rectangle; shaded in vs. shaded in brown), but the mathematical demand remains the same across years.

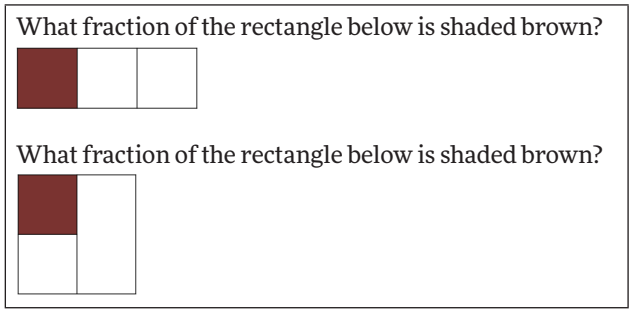

Figure 2: The brown rectangle problem

The analysis of each individual year provides detailed images of explanations that individual students produce, the process of constructing a mathematical explanation collectively by each cohort, and instructional supports that the teacher provides to develop a mathematical explanation for the brown rectangle problem. The cross-year analysis illustrates that instruction for teaching the same mathematical task unfolds somewhat differently, even by the same teacher. The similarities across multiple cases become strong candidates to be scaled up into the coherent structure of supporting students' development of mathematical explanation, whereas the differences across multiple cases offer analytical opportunities to examine whether or not the particular instructional feature plays a role in supporting students' development of mathematical explanation. In doing so, I do not treat such differences as discrepant or disconfirming evidence, but use the differences as the data to reveal the underlying structure of the work of teaching to support students' development of mathematical explanation. In addition, the differences observed across years do not necessarily represent the characteristics of the expert teacher's teaching practice. Looking at the multiple uses of the brown rectangle problem by the same teacher to different groups of students allows to elicit the demands entailed in the work of supporting students' development of mathematical explanation and framing the underlying structure that serves to meet such demands.

PROBLEMS OF STUDENTS' INITIAL EXPLANATIONS FOR THE BROWN RECTANGLE PROBLEM

In order to understand what is entailed in supporting students' development of mathematical explanation,

it is critical to first diagnose problems that students have in explaining the brown rectangle problem. Some of these problems are more generic and apply to all mathematical tasks, but others are more unique to a particular mathematical task. If each mathematical task would require a specialized form of reasoning to develop a mathematical explanation, the problems that students struggle with would be different and the supports that a teacher needs to provide would be different accordingly. Identifying such problems, both generic and unique to the type of mathematical task, contributes to revealing how a mathematical task plays a role in the work of teaching and the use of instructional resources. The following list below is a more general characterization and a more comprehensive collection of problems that individual students have in offering an initial explanation for the brown rectangle problem across five years.

- Having difficulties providing, hearing, and constructing an explanation
- Not establishing the mathematical grammar to describe the objects to be explained
- Using inaccurate language in which its intended meaning is different from the accepted mathematical definition
- Using pre-defined mathematical terms
- Skipping the logical structure of naming a fraction or paying attention to the partial components of naming a fraction
- Losing the purpose and focus of what is being explained
- Not building correspondences between an answer, an explanation, and representations
- Heavily using demonstrative pronouns
- Grounding explanation on non-mathematical reasons or procedural knowledge
- Missing the key definitional ideas of naming a fraction

Despite sharing these similar problems in explaining the brown rectangle problem, the collective process of constructing mathematical explanation does not always remain the same across five years.

THE CONSTRUCTION OF COLLECTIVE RESOURCES: DIFFERENCES ACROSS COHORTS AND IMPLICAITONS FOR TEACHING

The process of recruiting and selecting EML students is quite similar from one year to another, so it is presumed that there are no substantial differences in students' mathematical abilities across five years. Despite the homogeneous features of the EML cohorts across five years, the cohort's mathematical ideas, stances, dispositions, and issues do not always remain the same. The observed differences are (1) the answers that the students collectively discuss in a public space; (2) the proportion of the students who produce correct answer to the students who produce incorrect answers; (3) the intensity of counterarguments made against a competing proposal and the process of being convinced by a competing proposal; (4) when the key idea of "equal" emerges; and (5) mathematical issues that matter the most for each cohort.

First, the answers that each cohort discussed collectively in a public space are not the same. For the first part of the brown rectangle problem, only one correct answer (1/3) was proposed in the EML 2007, the EML 2008, and the EML 2010, but three answers (1/3, 2/3, and 2/6) were proposed in the EML 2009 and two answers (1/3, 1/2) were proposed in the EML 2013. For the second part of the brown rectangle problem, four answers (1/4, 1/3, 1 and 1/3, and 1/2) were proposed in the EML 2007, two answers (1/4 and 1/3) were proposed in the EML 2008, one answer (1/4) was proposed in the EML 2009, five answers (1/4, 1/3, 1/6, 2/8, and 4/16) were proposed in the EML 2010, and three answers (not a fraction, 1/4, and 1 and 1/2) were proposed in the EML 2013. Even though the teacher made similar attempts to elicit multiple answers, different groups of students brought a different set of answers in a public space. Beyond attending to the number of answers elicited in a public space, an important task of teaching includes (1) not dismissing any proposals made in a public space but unpacking the reasoning behind the proposals; (2) introducing the key incorrect answers if they are not brought by students; (3) mapping the proposed answers to the targeted mathematical ideas; (4) deciding what needs an immediate agreement or disagreement and what needs to be preserved; and (5)

customizing questions, probes, and prompts based on the dynamics of proposed answers.

Second, the proportion of the students who produced correct answer to the students who produced incorrect answer is not the same across cohorts. For the first part of the brown rectangle problem, nearly all of the students came up with the correct answer. On the other hand, for the second part of the brown rectangle problem, the incorrect answers were more prevalent than the correct answer in the EML 2007, the EML 2008, the EML 2010, and the EML 2013, but most of the students recorded the correct answer in the EML 2009. If the proportion of correct answer to incorrect answers might be related to the mathematical stance that students bring to the instruction, an important task of teaching includes (1) surveying the composition of students' mathematical ideas; (2) ensuring that a mathematical stance is not influenced by the idea held either by the majority of students or by advanced students; and (3) customizing questions, probes, and prompts based on the proportion of the students who produce correct answer to the students who produce incorrect answers.

Third, the intensity of counterarguments made against a competing proposal and the process of being convinced by a competing proposal was not the same. Except in the EML 2009, students proposed the key incorrect answer of 1/3 for the second part of the brown rectangle problem, but the degree of defending the incorrect answer and what made them being convinced by the correct answer was not the same. Some cohorts were more easily convinced by the idea that adding a line makes equal parts, but others were more resistant and hesitant to accept the idea because it contradicts their non-mathematical perception that adding a line changes the problem. The process of reconciling the competing proposals was not the same across five cohorts, but all of the cohorts ultimately arrived on the agreement that making equal parts is an important idea for naming a fraction and drawing a line provides an easy access to seeing the equal parts. As the intensity of counterargument and the resistance of accepting the competing proposal increased, the cohort constructed richer collective resources to convince others who had a competing proposal. It is not an easy task for a teacher to support students to have a strong stance on their mathematical ideas and to have them sustain their perseverance, but detecting such a moment, confronting competing ideas, and providing sufficient opportunities to defend one's proposal is an important task for supporting students' development of a mathematical explanation.

Fourth, the key idea of "equal" emerged at different stages of developing a mathematical explanation. It was early proffered by a student who proposed the answer of 1/4 in the EML 2007 as well as by a student who proposed the answer of "not a fraction" in the EML 2013, but emerged in the process of comparing between the equally partitioned rectangle and the unequally partitioned rectangle in the EML 2008, 2009, and 2010. Eliciting the targeted mathematical idea and developing the accurate mathematical language is key for developing a mathematical explanation, but an important task is not just accepting the targeted mathematical idea offered by a single individual student, but providing supports for students to use those collective resources.

Lastly, the mathematical issues that matter the most for each cohort are not always the same. For the second part of the brown rectangle problems, the EML 2007 cohort spent a significant amount of time to make sense of 1/2, the EML 2008 cohort discussed whether or not the line changes the problem, the EML 2010 cohort engaged in removing the existing line or adding an additional line to make unequal parts, and the EML 2013 cohort spent time making sense of 1 and 1/2. An important task of teaching is to adjust the instructional time according to the mathematical issues that each cohort struggles with the most.

In basic ways, the students and their mathematical proficiency were similar across years, but each cohort brought different mathematical ideas, stances, dispositions, and issues to explain the brown rectangle problem. Thus each cohort developed different collective resources that became available for use either by the teacher or by students. In comparing the mathematical ideas, stances, dispositions, and issue brought by different groups of students, I offer the following observations. First, the mathematical scope and terrain of collective resources that each cohort establishes varies to a certain degree, but all of the five cohorts develop the key ideas for naming a fraction. Second, there are variations in what collective resources are available for use to develop a mathematical explanation across cohorts, but the practice of constructing collective resources is quite the same. Third, some collective resources are for immediate or

necessary use, but others remain in reservoir or are optional for use either by a teacher of by students. Fourth, the same mathematical issue is treated differently based on the established knowledge that each cohort constructs. Lastly, eliciting multiple answers has been considered an important pedagogical practice for fostering students' mathematical abilities and enriching mathematical discussion, but how the proposed answers could be used as resources for maximizing the development of mathematical explanation needs to be further examined.

THE CORE TASKS OF TEACHING

The four core tasks in supporting students' development of mathematical explanation for the brown rectangle problem are: (1) attending to the organic structure of the mathematical task; (2) mapping the scope of answers onto the targeted mathematical idea; (3) hearing the mathematical needs embedded in students' explanation; and (4) distributing and building a mathematical talk collectively.

The first core task is attending to the organic structure of the mathematical task. This core task includes (1) focusing on mathematical or non-mathematical attributes which impact the construction of an explanation (e.g., "big rectangle"; the affordance of sticky line; drawing the rectangle on the grids); (2) not attending to mathematical or non-mathematical attributes which substantially distract from the construction of an explanation; (3) recognizing how the design of the mathematical task creates or eliminates confusions and how the design of the mathematical task makes the key ideas implicit or explicit.

The second core task is mapping the scope of answers onto the targeted mathematical ideas. This includes (1) being aware of the scope of answers that students propose; (2) deciding which of the proposed answers needs an immediate acceptance or denial and which needs to be preserved; (3) not delving into the ideas that students do not have a shared access to; (4) not diverging into the ideas that seriously deviate from the targeted mathematical ideas; and (5) spending sufficient instructional time on scaling up the proposed answers to the targeted mathematical ideas.

The third core task is hearing the mathematical needs embedded in students' explanation. This core task includes (1) recognizing inaccurate or inconsistent language use that impedes building a mathematically acceptable form of common knowledge; (2) deciphering the vague, unclear, or implicit idea conveyed by students' explanations; (3) providing supports to build mathematical connections or correspondences instead of repeatedly asking general questions; and (4) recognizing the skip of or the deviation from the logical structure of building an explanation.

The last core task is distributing and building a mathematical task collectively. This core task includes (1) not exclusively relying on one students' contribution; (2) being attentive to the trajectory of constructing a mathematical explanation; (3) appropriately or sufficiently using a private space and a public space; and (4) making each other's contribution accessible in a public space.

DISCUSSION

Given that one of the greatest predicaments of teaching is its dependence of students, it is important to figure out how instruction might unfold with different groups of students. On the one hand, one might speculate that instruction would unfold in the same way by the same teacher teaching the same mathematical task because a teacher might make the same decisions based on his or her knowledge, skills, disposition, and instructional goals. On the other hand, one might suggest that instruction would unfold in a dramatically different way even by the same teacher teaching the same mathematical task because teaching entails being responsive to students. The question of how instruction unfolds with different groups of students might be answered based on one's personal sensibilities or perceptions built through years of their own teaching experiences, but it is not yet rigorously examined in the field how instruction managed by a teacher teaching the same mathematical task is likely to unfold differently with different groups of students; how collective resources are likely to be constructed differently with different groups of students; and what is the underlying structure of using collective resources with different groups of students. By analysing instructional interactions managed by the same teacher teaching the same mathematical task for different cohorts of students, this study contributes to identifying core tasks of teaching across the particulars of students and unfolding instructional dynamics.

The four core tasks of teaching are not just a mere collection of temporal stages, general pedagogical strategies, instructional routines, or discourse moves, but devised to structurally and attentively capture the essential elements of instructional interactions. Approaching through pedagogical strategies or discourse moves might be one way of examining what is entailed in supporting students' development of mathematical explanation, but it entails the risk of losing some key elements of instructional interactions. Instead, this study conceptualizes core tasks of teaching by taking into serious account the three-pronged arrows that a teacher has relationships with in the instructional triangle (students, content, and students-content) and by anchoring the core tasks of teaching into these relationships. These four core tasks are neither sequential nor mutually exclusive. Even though there exist differences in what bring to foreground and what leaves as background, all four core tasks of teaching attend to the coordination between students and mathematics.

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REFERENCES

Boerst, T. A., Sleep, L., Ball, D. L., & Bass, H. (2011). Preparing teachers to lead mathematics discussions. *Teachers College Record, 113*(12), 2844–2877.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119–142.

Cohen, D. K. (2011). *Teaching and its predicaments*. Cambridge, MA: Harvard University Press.

Forman, E, A., & Larreamendy-Joerns, J. (1998). Making explicit the implicit: Classroom explanation and conversational implicatures. *Mind, culture, and activity, 5*(2), 105–113.

Grossman, P., Hammersness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. Teachers and Teaching. *Theory and Practice, 15*(2), 273–289.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Leinhardt, G., & Greeno, J. G. (1986). The cognitive skill of teaching. *Journal of Educational Psychology, 78*(2), 75–95.

Leinhardt, G. (2001). Instructional explanations: A commonplace for teaching and location for contrast. In V. Richardson (Ed.), H*andbook for research on teaching (4th Ed.).* Washington, DC: American Educational Research Associates.

Sleep, L. (2012). The work of steering instruction toward the mathematical point: A decomposition of teaching practice. *American Educational Research Journal, 49*(5), 935–970.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York, NY: Free Press.

Thames, M. H. (2009). *Coordinating mathematical and pedagogical perspectives in practice-based and discipline-grounded approaches to studying mathematical knowledge for teaching (K-8)*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, MI.