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# A contingent opportunity taken investigating in-between fractions

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*In this paper, we discuss a contingent opportunity from a 7<sup>th</sup> grade mathematics lesson about decimal numbers, percentages and fractions in Norway. The question if adding numerators and denominators of two fractions was a way to find an in between fraction, occurred. The teacher's response to this question, which we saw as contingent opportunity, is our focus here. Although the teacher did not have the substantive mathematics knowledge needed to give an answer to the problem, many mathematics activities took place in the lesson.*

**Keywords:** Contingent opportunity, teachers' mathematical knowledge, mathematising activities, argumentation and proofs.

## INTRODUCTION AND RESEARCH QUESTIONS

When teaching mathematics, teachers are faced with many opportunities – both expected and unexpected. Awareness of opportunities and teachers' decision-making when they occur, is crucial with regard to students' learning outcome (Bishop, 1976) and thus important issues in mathematics teacher education. In the mathematics research literature, several possible responses to contingent moments are discussed (Kleve, 2010; Rowland, Huckstep, & Thwaites, 2005; Rowland, Turner, Thwaites, & Huckstep, 2009; Rowland & Zazkis, 2013). In responding to contingent opportunities, teachers may run a risk.

In a mathematics lesson about fractions, decimal numbers and percentages in a 7<sup>th</sup> grade in Norway, which we observed, the problem finding an in between fraction arose, and data from this classroom event create the background for this paper.

Our research question is:

- How can a teacher's decision making to an unplanned opportunity open up for mathematics discussions, even if the teacher does not know the answer?

In order to answer this question, we focus in particular on the mathematizing activities that took place and how the teacher took advantage of the contingent opportunity when it arose. We also investigate how the teacher's knowledge appeared to influence the course of the lesson and thereby supported the students' learning opportunities.

## A CONTINGENT MOMENT FROM THE RESEARCH LITERATURE

Earlier research have suggested ways in which teachers may respond when a student suggests how to find a fraction between two given fractions (Bishop, 1976; Rowland & Zazkis, 2013; Star & Stylianides, 2013; Stylianides & Stylianides, 2010). However, authentic data, which display teachers' responses to students' contributions to the task or from classroom discussions, have not been presented. In these studies, how students *may* respond to the task to find a fraction between fractions and how teachers *may* respond to a student's response, have created the background for discussions among prospective teachers.

Referring to the fractions example<sup>1</sup>, Bishop (1976) asked the question: "How would you deal with that response?" (p. 41) and he emphasized teachers' decision making, to be at the heart of the teaching process.

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1 Teacher: Give me a fraction which lies between  $\frac{1}{2}$  and  $\frac{3}{4}$   
Pupil:  $\frac{2}{3}$

Teacher: How do you know that  $\frac{2}{3}$  lies between  $\frac{1}{2}$  and  $\frac{3}{4}$ ?  
Pupil: Because 2 is between the 1 and the 3 and 3 is between the 2 and the 4

Rowland and Zazkis (2013) presented possible (tacit) conjectures made by the pupil who responded to Bishop's fraction example<sup>2</sup>. However, they "did not know how the lesson proceeded mathematically nor whether the teacher took the advantage of the opportunity" (p. 150).

Stylianides and Stylianides (2010) referred a similar scenario in order to exemplify Mathematics Knowledge for Teaching. In this scenario, a student, Mark, suggested adding the numerators and denominators of two fractions to find one in between, and he illustrated that on a number line. They focused on implementation of special tasks in teacher education and discussed how the task (a student's response to finding an in between fraction) could support the development of mathematical knowledge for teaching.

Star and Stylianides (2013) used the fractions task for prospective teachers in mathematics teacher education in order to discuss teachers' mathematical knowledge asking the question: "What can be considered as procedural/conceptual knowledge?" (p. 172).

## TEACHERS' MATHEMATICAL KNOWLEDGE

In order to investigate teachers' mathematical knowledge, several frameworks have been developed (Ball, Thames, & Phelps, 2008; Fennema & Franke, 1992; Rowland et al., 2005). Both Ball et al. and Rowland et al. based their work on Shulman's categories of knowledge (Shulman, 1986, 1987). Stylianides and Stylianides (2010) related their work to Ball and Bass (2000; 2003). Using the abbreviation, MKfT, they emphasised mathematical knowledge *for* teaching:

This specialised kind of mathematical knowledge is important for solving the barrage of mathematical problems of teaching that teachers face as they teach mathematics: offering mathematically accurate mathematical explanations that are understandable to students of different ages, evaluating the correctness of students' methods, identifying mathematical correspondences be-

tween different student solutions of a problem etc. (Stylianides & Stylianides, 2010, p. 161)

They suggested that the teacher's mathematical knowledge would influence the course of the lesson and students' learning opportunities. Linked to argumentation and proof they discuss both teacher's mathematical knowledge, which sometimes may deviate from conventional mathematical knowledge (misconception), and knowledge, which is consistent with conventional mathematical knowledge. Harel and Sowder (2007) carried out a study, finding that students' use of numerical examples, as a way of proving, was prominent. In everyday discourse "no rule without exceptions" and "the exception proves the rule" are accepted ways of "proving" something. Studies on teachers' mathematical knowledge and their conceptions about proofs in mathematics, show similar conceptions as revealed by Harel and Sowder (Martin & Harel, 1989; Stylianides & Stylianides, 2010).

*Contingency* is one of four dimensions in the Knowledge Quartet, KQ, which is a framework for Mathematical Knowledge *in* Teaching (Rowland, 2008; Rowland et al., 2005; Rowland et al., 2009). The KQ provides us with tools for analysing how teachers draw on different kinds of mathematical knowledge in order to support learners in the classroom situations. Contingency is informed by the three other dimensions in the KQ (Foundation, Transformation and Connection) and is about situations in mathematics classrooms, which are not planned for. Identifying "contingent moments" or contingent opportunities in order to analyse aspects of a teacher's mathematical knowledge has proven to be helpful (Kleve, 2010; Solem & Hovik, 2012). The teacher's choice whether to deviate from what s/he had planned and the teacher's readiness to respond to pupils' ideas, are important classroom events within this dimension. The Contingency dimension has also been expanded in a later research (Rowland, 2008; [www.knowledgequartet.org](http://www.knowledgequartet.org); Rowland et al., 2009; Weston, Kleve, & Rowland, 2012). One such expansion is "Teacher's Insight" which is demonstrated when a teacher is realizing that children are constructing the mathematical ideas and something that sounds 'half baked' which means that they are in what Vygotsky (1978) termed Zone of Proximal Development (ZPD) where teacher can help with a scaffolding question or two ([www.knowledgequartet.org](http://www.knowledgequartet.org)). In their article, Rowland and Zazkis (2013) discussed several contingent opportu-

2 (C1): Whenever the numerators of three fractions are consecutive integers, and the denominators likewise, the second fraction will be between the two

(C2): Whenever the numerators of three fractions are in arithmetic progression, and the denominators likewise, the second fraction will be between the two

nities, which may occur in a mathematics classroom and whether they are taken or missed. One contingent opportunity that they discussed was similar to the one that actually happened in the lesson we analyse here.

## METHODOLOGY

Our data are taken from an ongoing action research project which is about classroom conversations in mathematics focusing on what questions teachers ask (Solem & Ulleberg, 2013). Seven experienced teachers participated in the project. In this project, how questions could be used reflectively in planning a lesson, implementing the lesson and in order to analyse the lesson, were investigated (Ulleberg & Solem, in press). All teachers were observed in teaching a mathematics lesson in own class. We have chosen an episode from a lesson with a teacher, here called Kim. The reason for choosing this episode was not that it was representative for the research project, but that we found it especially interesting with regard to the contingent possibility, which occurred, and the teacher's response to students' input. Our data are field notes (including pictures), transcribed audio recordings from the lesson, and the teacher's prepared notes for the lesson. The lesson is from grade 7 (12–13 years old), and a whole class discussion had taken place. How the teacher drew on his mathematical and pedagogical knowledge in orchestrating the whole class discussion in this lesson is discussed in Kleve and Solem (2014). The teacher's plan for the lesson was to place fraction, percentages and decimal numbers on a number line.

## ANALYSIS

As a challenge, the teacher wanted the students to find a fraction that was bigger than  $\frac{3}{5}$  but smaller than  $\frac{4}{5}$ . According to the teacher's notes for the lesson, finding the fraction in between was supposed to be done by finding equivalent fractions with denominator 10. A student, Lea, suggested  $\frac{7}{10}$ , because "I only doubled both and found what was in between". Then the following contributions from other students in class occurred:

- Stud 1: Now it is the same again. 4 plus 3 makes 7.  
 Stud 2: It was the same last time we did it.  
 Stud 3: It happened then too, but you said it was a coincident.  
 Kim: But, may be, there is a pattern here?  
 Students: yes, yes..

- Kim: Wait a minute....  
 Stud 4: But last time it wasn't ten, but now it is ten, isn't it?  
 Stud 5: We can add them both, getting ten

Kim then went back to how to get from  $\frac{3}{5}$  to  $\frac{6}{10}$  ensuring that all students were following the discussion. He asked Ada to explain to the others. Another student supported Ada saying:

- Stud: You just multiply by 2 both in the upper and in the lower.  
 Kim: We double. Ok. Then back to the exciting thing, we talked about. Some of you are starting to see a pattern, which I am not sure I can see. Hanna?  
 Hanna: Yesterday we had the same, or we multiplied both with two. And then we saw that- that those numbers, I mean, the numerators, if we added them, it became the same as, yes, as the answer. And it happened now again.

Kim did not miss this contingent opportunity (Rowland & Zazkis, 2013), but responded by letting the students come up with suggestions, and after stud 3, he asked if there might be a pattern. Thus, he deviated from his agenda, which was to find a fraction between two fractions through expanding the two. He eagerly joined the students in searching for a pattern.

In their article, Stylianides and Stylianides (2010) discussed two possibilities from a similar scenario. One was that the teacher might consider the students' method to be correct because it works in different examples. The other possibility was that unless the method was shown to work for all possible cases, it could not be accepted as correct. Let us see what happened in our lesson:

- Kim: Ok. But if it has happened twice, are we then sure it will happen next time?  
 Tiril: No, but there is a rule which is that if it happens once it is a coincident, twice, it is especially, three times, it is Ok.  
 Kim: Three times, then it must be like that? Does such rule exist?  
 Students: it is, ... no, no  
 Students: Can just be luck  
 Kim: OK, let us try

This suggests that Kim had knowledge about the distinction between empirical argumentation and proofs in mathematics. Although he expressed doubt about Tiril's rule (a method is correct if it works in three different examples), he invited his students to try a third time (the first was from yesterday, the second was  $3/5$  and  $4/5$ ), suggesting  $1/4$  and  $2/4$  and asked for a fraction in between. He challenged Januscha to explain:

Januscha: One plus two, I added the numerators, which made three, and the denominators, made eight. And to be sure, I doubled it too..

Kim: And then you got 3

Januscha: of 8

Kim: Did you all find the same? (Students confirmed)

Kim: OK, So we can see a pattern here now. But are we sure, does this apply to absolutely all fractions?

Students: That we don't know.

Ronja: I think, I don't know if it applies to fractions with different denominators

Kim: Ok?

Student: I think it applies to all fractions, because if  $1/4$  - if you multiply by two you get  $2/8$  and then you get  $4/8$  and that is in between (explaining that  $3/8$  is between  $1/4$  and  $2/4$ ).

Kim now suggested that they were seeing a pattern, but he emphasized uncertainty that it applied to all fractions. Thus, he exposed his students to hypothesize and to come up with conjectures. Ronja's question, if it applied to fractions with different denominators, was one conjecture, which opened up the possibility to find a counter example.

Kim did not follow up the students' hypotheses at this point, but he insisted further investigations with fractions with same denominators, and drew attention to the distance between the numerators. Proposing  $3/5$  and  $5/5$ , he asked: "what if the distance between the numerators are bigger than one"? The students did not find this challenging at all:

Student: But if it is bigger distance between them, there is no challenge. What is it that is between  $3/5$  and  $5/5$ ? It's got to be  $4/5$ !

Kim: Good. I agree that the challenge may disappear a little, but we have to check this out now. We are curious to know, aren't we?

Here, we suggest that the task had changed. Kim's point was no longer to find a fraction between two given fractions but to investigate if the "rule" -adding numerators and denominators to find an in between fraction- worked in different situations, also when the solution was obvious. Kim's proposal to have more than one between the numerators resulted in an "obvious example" ( $3/5$  and  $5/5$ ) suggested by a student, who expressed that it was not challenging at all. Instead of being engaged in Kim's obvious example, the students contributed with further conjectures (which are partly contracting):

- I think it always goes when there is one even and one odd numerator
- When there are two even numbers or two odd numbers there is always a number between
- When there is one number of each, then it goes
- When there are two even or two odd, it will not work, but if there are one even and one odd, it works.

This demonstrates that the way Kim was teaching opened up for students' engagement and participation in mathematizing activities. Making conjectures and hypotheses are important ingredients in the learning of mathematics. The conjectures the students made, were general (about odd, even, and different numbers) and not linked to concrete numbers. Kim's response to these hypotheses were:

Kim: Ok- good. We are getting short of time, but this we *have* to check out. How can we check it all out?

Students: Do it many times

Kim: Shall we use "Tiril's method" and try: if it works the third time, it is ok, or what, how shall we do this?

Students: Why don't we try?

Kim: Ok, Then, firstly, I would like to try once having more than one in between. Let us do that first. Afterwards we can do it like you suggest with different denomi-

nators. So if we have.... We can take 3 of 8 and 5 of 8, then we have, the distance here is 2.

The students now agreed to go for Kim's suggestion first, for which he gave the following reason:

Just so you do not believe.... that we can do it this way with the rest – that we just can add them together.

Now an eager discussion now took place. They had expanded the fractions to  $\frac{6}{16}$  and  $\frac{10}{16}$ . We can hear students saying, “yes it goes”, “it is four in between”. Kim asked if they could choose between 7, 8 and 9, when a student said:

Student: The middle is  $\frac{8}{16}$  and if we take  $3+5$  you get 8.

Kim: Yes, good, it works.

Student: It would have been the same if we just took different fractions. 3 of 8, and 4 of 8, or 4 of whatever. Just making different fractions.

This discussion ended up with the following statement from Kim:

You know, now we are within an area in which I haven't checked either. Neither do I know for sure where this entire end up. I am not in the position to see all the patterns here either. Therefore, I find it very exciting. [] At least. Here we have lots of material we can work on in the future. But, you know what? We haven't done what we were supposed to do in this lesson (to place the fractions, percentages and decimal numbers on a number line)

This statement shows that Kim neither knew if it was possible to find an in-between fraction generally by adding the numerators and denominators, and consequently nor to prove it. Although he expressed that he did not know where it all would end up, he expressed excitement and was challenged to work further with the issue.

## DISCUSSION

In this lesson a contingent opportunity occurred. Several students suggested that it was possible to

find a fraction between two fractions by adding the numerators and denominators (Fareys mediant). The analysis above shows that the teacher deviated from the agenda and followed up the suggestions from the students. In grasping this contingent opportunity Kim took a risk. Although he neither knew if the conjectures were correct, nor had the advanced mathematical knowledge needed for proving the conjectures made, he deviated from the agenda and incorporated the students' suggestions in the further course of the lesson. Hence we see, that the mathematical knowledge he had, and also the mathematical knowledge missed, influenced the course of the lesson and thus the students' learning possibilities. Together with the students, he investigated their suggestions and claims. In asking “But are we sure, does it apply to absolutely all fractions?” Kim demonstrated mathematical knowledge about mathematical proving and signaled that it is not sufficient with numerical examples to generalize in mathematics. On the contrary, trying out different examples could have opened up for a counter example, which we suggest that he was looking for. This indicates that his substantive content knowledge incorporated what is a mathematical proof.

Since Kim did not have the mathematical knowledge needed to answer if adding the numerators and the denominators would make an in between fraction, he was not in a position to say “yes, it is correct”, nor “it is not correct”. The only tool Kim had was to try, and this he did in collaboration with the students. He encouraged his students to come up with several conjectures. The mathematical activities which took place incorporated hypothesizing if the rule, adding the numerators and the denominators to find an in between fraction, would work in different situations: fractions with different denominators, fractions where both numerators are odd, both numerators are even, one numerator is even and one is odd, different distances between the numerators.

The conjectures made:

- Adding numerators and denominators made an in between fraction (common denominators).
- Does it apply to all fractions?
- Does it apply to fractions with different denominators?

- No, but there is a rule which is that if it happens once it is a coincident, twice, it is especially, three times, it is Ok.

Formulating hypothesis, making conjectures and searching for patterns are important activities in learning mathematical thinking. The way Kim took advantage of students' input and questions, offered possibilities for the students to participate in these activities. Kim was sensitive to the students' mathematical knowledge (knowledge of content and students) and posed questions accordingly (knowledge of content and teaching). Posing appropriate questions as Kim did, is extremely demanding (Kleve and Solem, in press) and this episode displayed how he drew on his mathematical knowledge for teaching.

Although Kim did not have the necessary command of substantive knowledge to prove the rule (Fareys mediant), he did not close the door for mathematizing activities, and a fruitful mathematical discussion to take place.

### IMPLICATIONS FOR TEACHER EDUCATION AND RESEARCH

Research referred in the beginning of this paper, have suggested ways in which teachers *may* respond to a similar contingent opportunity, which actually happened in the lesson we observed and have analyzed here. Authentic records of teaching, whether they are videos or transcripts of data, are extremely powerful sites for learning: "Teachers and researchers are finding that analyses grounded in actual practice allow a kind of awareness and learning that has not previously been possible" (Boaler & Humphreys, 2005, p. 4).

Our teacher did not know the answer. However, these data demonstrate for student teachers that taking the risk and grasping the contingent opportunity may lead to fruitful mathematizing activities. In mathematics teacher education, proving  $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$  algebraically is possible. Another possibility is to show with functions and graphs (Stylianides & Stylianides, 2010). However, these proofs are not accessible for children in 7<sup>th</sup> grade. Therefore discussions how to illustrate Fareys mediant for these children may be relevant. For example, if the juice analogy referred in Rowland and Zazkis (2013), is an accepted argument for "the adding numerators and adding denomina-

tors procedure" can lead to good discussions among student teachers.

In further analysis of data from this teacher's teaching, we are interested in finding out what socio-mathematical norms were established in his class and what role the norms played for the mathematical activities that took place, such as use of language and classroom discussions, and hence students' learning possibilities.

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