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This paper presents a didactic experience with problem posing carried out with in-service secondary teachers. We propose a strategy specifically designed to modify a given problem and enrich its mathematical and didactic potential. The starting point is a teacher's class episode, which includes a previously designed problem as well as the reactions of the teacher's students when solving it. We ask the participating teachers to pose 'pre-problems' and 'post-problems', working individually at first and then in groups. The experience shows that problem posing contributes to the development of teachers' didactic and mathematical competencies.

Keywords: Problem posing, problem solving, teacher competencies, teacher training.

INTRODUCTION

In 1989 the National Council of Teachers of Mathematics (NCTM) recommended providing opportunities for students to think mathematically and develop knowledge by creating problems: "Students in grade 9–12 should also have some experience recognizing and formulating their own problems, an activity that is at the heart of doing mathematics" (p. 138). In addition to this, the NCTM recommended offering opportunities to formulate problems from a given situation as well as opportunities to create new problems by modifying the conditions of a given problem (NCTM, 1991, p. 95). In this sense, teachers must obviously develop their problem posing skills in order to work in this way with their students; they should not be limited to using the problems found in books or online (Ellerton, 2013; Singer & Voica, 2013; Malaspina, 2013b; Bonotto, 2013). The cited authors also emphasize the importance of the relationship between problem solving and problem posing. In Bonotto's words (2013): "There is a certain degree of agreement in recommending problem-posing and problem-solving activities to promote creative thinking in the students and assess it." (p. 40).

Some research on problem posing and its relation to the problem solving process has led to new research on the benefits of incorporating problem posing in teacher training programs (Ellerton, 2013; Tichá & Hošpesová, 2013; Malaspina, Gaita, Flores, & Font, 2012; Malaspina, 2013b). We agree with Ellerton (2013) when she says: "For too long, successful problem solving has been lauded as the goal; the time has come for problem posing to be given a prominent but natural place in mathematics curricula and classrooms" (pp. 100–101) and our research shares this idea. We have designed activities with the purpose of motivating pre-service teachers and current teachers to create math problems and reflect on their didactic aspects. The problems were posed starting from a given problem or from a given situation. In this paper, we describe some cases of the first type.

FRAMEWORK

Mathematics teacher's competencies

At the international level we have observed a tendency for convergence among university curricula design. Some countries have opted for a curriculum model organized by professional competencies that differentiates general (or transversal) competencies from specific ones.

Many of the tasks proposed in order to develop and evaluate students' mathematical competencies are problem-based. A teacher must not only be good at solving problems, but also needs to know how to choose, modify and create them with a didactic purpose. A teacher also needs to be able to critically evaluate the quality of the mathematical activity required to solve the problem proposed and, if necessary, to be able to modify the problem in order to facilitate a richer mathematical activity.

Teachers should already have mathematical competence to solve problems, but if they want to select, modify or create them with a didactic purpose, they need to be competent in didactic analysis of the mathematic activity (Rubio, 2012). The first competence is common in many of the professions developed by mathematicians. The second one is, however, more specific to the mathematics teacher.

According to Giménez, Font and Vanegas (2013) and Rubio (2013), we understand the competence of didactic analysis as the ability to design, apply and evaluate learning sequences by means of didactic analysis techniques and quality criteria. In teacher training, this competence has to be developed by proposing tasks to the future and current teachers that require them to carry out didactic analysis. One of these tasks consists of creating problems and thinking about them didactically.

Problem posing

Stoyanova and Ellerton (1996) summarize the meaning of creating mathematical problems from different points of view:

> Problem posing has been viewed as the generation of a new problem or reformulation of a given problem (Duncer, 1945); as the formulation of a sequence of mathematical problems from a given situation (Shukkwan, 1993); or as a resultant activity when a problem is inviting the generation of other problems (Mamona Downs, 1993). Dillon (1982) conceptualized „problem finding as a process resulting in a problem to solve." Silver (1993, 1995) referred to problem posing as involving the creation of a new problem from a situation or experience, or the reformulation of given problems. (p. 518)

As Silver, we consider that posing mathematical problems is a process through which one produces a new problem from a given one (problem's variation) or a new problem from a situation (problem's elaboration), whether it is real or imagined.

In order to develop this perspective of problem posing, it is necessary to identify the four key elements of a problem: Information, Requirement, Context and Mathematical Environment (Malaspina, 2013c). The Information consists of the quantitative or relational data that are given in the problem. The Requirement is what is asked to be found, examined or concluded; it can be quantitative or qualitative, and it can include graphics and demonstrations. With respect to Context, a "contextualized problem" usually relates to any real situation, to everyday life; but we consider that the Context can also be strictly formal or mathematical. In this sense, we affirm that the Context can be intra mathematical or extra mathematical. In the first case, as its name implies, the problem is more linked to an abstract situation and, in the second case, the problem is more linked to a real situation. Finally, the Mathematical Environment refers to the mathematical concepts needed to solve the problem.

Therefore, we understand the problem's variation as a process that builds a new problem by modifying one or more of the four key problem elements.

OBJECTIVES AND METHODOLOGY

We have two objectives:

— To show that an appropriate strategy helps stimulate the ability to create mathematical problems by modifying a given problem and considering its mathematical and didactic aspects.

— To show that problem posing is a means of contributing to the development of teachers' didactic and mathematical competences.

Regarding the methodology, since the research relates to creativity, we have chosen a qualitative methodology that includes a strategy, observations and case studies.

The first step was choosing a topic and designing some easy and motivating problems as starting points to pose new problems.

At the beginning, the problem posing experiences were performed with pre-service primary school teachers as part of the mathematics course in the Faculty of Education of the Pontificia Universidad Católica del Perú. These did not have a context of a specially designed strategy. The positive experiences of both the individual and group works were the basis for designing problem posing workshops, which are summarized below.

A PROBLEM POSING STRATEGY

We give a short presentation on problem posing, including some examples of problems created in previous workshops, in which we emphasize the importance of creating problems that favour learning and mathematical thinking. We present a previously elaborated problem to the workshop participants considering the context of a concrete class episode of Teacher P. In this episode, some of the students' reactions when solving the problem are described briefly. We ask participants to: i) solve the given problem; ii) pose problems by modifying the given problem to make the solution easier and to help clarify students' reactions (these problems are called 'pre-problems'); iii) pose problems by modifying the given problem so as to challenge Teacher P's students beyond correctly solving the given problem (these problems are called 'post-problems'). The problem posing must be carried out individually at first and then in groups with the help of the instructor of the workshop. Moreover, the problems created by a group are also solved by other groups. There is also a socialization phase with all the participants. In this phase, the participants share the rationale behind the individually or collaboratively created problems. In addition, the problem solved by a group (which is not the author group of the problem) is exposed and commented critically. The purpose of this is that the discussion with the authors of the problem as well as the participants' and instructor's comments help to enhance the capacity of posing problems with mathematical and didactic potentialities.

This is the strategy we have followed in several workshops, especially with current secondary teachers. It should be mentioned that the experiences we show and examine in this article are about percentages.

CASES OBSERVED

After explaining and exemplifying some ways for varying a given problem, we applied the described strategy in a workshop with 15 current secondary school teachers. We proposed the following class episode and we asked the teachers to do the tasks (i)–(iii) related to the episode problem. Both, the interaction of the teachers with the instructor and the socialization phase, were very important for obtaining information about the rationale behind the created problems.

The episode

In a class of mathematics, teacher Sánchez asks his students to solve the following problem:

The first week of July, a store called ALFA sold all products at full price; the second week, the store discounted all items 20%; and the third week, the store applied an additional discount of 15% that was announced as the "GREAT DISCOUNT OF 20%+15% ON ALL THE PRODUCTS".

You have to decide whether or not during the third week of July ALFA sold its products at prices 35% less than during the first week of July.

After a few minutes:

— Most of the pupils say "Yes, they did."

— Juan and Carla say "No, because in the third week the discount was less than 35%."

— Maria says that in the third week the discount was 68%.

Some pre-problems posed by teachers in the workshop

Pre-problem 1 (Individual work)

Rosa bought a $100 blouse that was discounted 20% because of end of season sale and with an additional discount of 10% for having the store credit card. What was the total percentage discount that Rosa received?

The author's idea when she posed the problem was to set a price with percentages that are easy to calculate in order to help students focus their attention on the total discount.

Pre-problem 2 (Individual work)

In a clearance sale, a shop applies a 50% discount on all its textiles during a week, and the following week it applies an additional discount of 50%. What is the total percentage discount applied during the second week?

The author of this problem was interested in showing the students that the total discount is not a simple sum of percentages. In order to achieve this objective, the author had chosen discounts of 50% because a total discount of 100% is not intuitive.

Pre-problem 3 (Group work)
(The author of Pre-problem 1 joined this group)

Rosa bought a $100 blouse that was discounted 20% because of end of season sale and with an additional discount of 10% for having the store credit card.

a) *How much did Rosa pay for the blouse?*

b) *What percentage of the blouse´s original price did Rosa pay for the blouse?*

c) *What is the blouse's total percentage discount?*

The author group thought this problem would help students distinguish between the amount paid and the discount. This seemed to be the confusion of student Maria in the episode. Apparently she had done the calculations well, but she did not distinguish between the amount paid (68% of the initial price) and the total discount (100%–68% = 32%). Some of the participants commented that considering $100 as the initial price should be used as a counterexample for illustrating the wrong answers, and they said to be careful because it could reinforce a simplified and not deeply reasoned way of generalizing a particular case.

Some post-problems posed by teachers in the workshop

At the beginning of creating post-problems, the participating teachers created problems that were very similar to the given problem, some with other prices and, in other cases, considering three successive discounts; essentially, they were the same problems as the original, but with quantitative modifications in the information. However, little by little they carried out more creative modifications when they formulated post-problems:

Post-problem 1 (Group work)
Pedro and Juan each bought a shirt. Pedro bought a shirt with a discount of 20% plus an additional discount of 20%. Juan bought one with a discount of 30% plus an additional discount of 10%. Who received a greater discount?

The author group thought this problem would reinforce the fact that the total discount is not a simple sum. When it was solved by other group and socialized, the participating teachers appreciated that there were no specific initial prices for the shirts.

One of the teachers said that he created a similar post-problem considering percentage wage increments: for worker A, 5% in 2011 and 4% in 2012; and for worker B, 6% in 2011 and 3% in 2012. The problem was to determine which of the workers received a better percentage increment in the two years; or did they receive the same percentage increment? In his group, this problem was considered easier than the episode problem and, for that reason, it was not used as a group post-problem.

Post-problem 2 (Group work)
There is a store where you can pay 30 days later, but there is a 10% surcharge. And if you want to pay after 31 days but before 35 days, there is an additional 5% surcharge. If Julio bought something on August 20th and paid on September 23rd, what total percentage surcharge did he pay?

The author group thought it was interesting to pose situations about cumulative percentage, considering surcharges and not only discounts. As in Post-problem 1, initial amounts are not specified in this problem and its solution requires a better understanding of the percentage concept. In the socialization it was commented that this problem was easier than the problem of the percentage wage increments.

Post-problem 3 (Group work)
Celia bought a dress for $ 125.46. If the dress was 15% off with an additional discount of 18%, what was the original price of the dress?

The author group said its intention was to motivate students to use algebra to solve percentage problems. Indeed, the group that solved the problem used the equation:

$$(0{,}82)(0{,}85)x = 125.46$$

One of the comments during the discussion was that all problems maintained an extra mathematical context. However, we also need to create problems with an intra- mathematical context. Generalization allows us to work in this context. The following problem was created with this idea.

Post-problem 4 (Group work)
If the shop called BETA has an end of season discount of p%, plus an additional discount of q%, what is the total percentage discount in relation to the original price?

The problem allowed us to illustrate the total discount using a composition of linear functions that express the sale price of a product whose original price is *x* and has a discount of *r*%. That means, functions of the form $f(x) = (1 - \frac{r}{100})$.

COMMENTS

The studied cases show that the proposed strategy contributes to the development of the competency of problem posing and to thinking about problems didactically (e.g., Pre-problem 2, the comments about Pre-problem 3 and Post-problem 1).

In most cases, the primary modification of the initial problem is about its information and its requirement. Post-problem 4 did not arise spontaneously. However, the alterations are not only quantitative (e.g., different percentages), but also qualitative (e.g., the problem deals not only with discounts, but also with increases), relational (e.g., the information is given in a way that it makes it easier to reflect on possible wrong answers, as in Pre-problem 2) and, in some cases, a piece of information is added or a requirement is extended (e.g., Pre-problem 3)

The percentage theme favours the creation of problems in an extra mathematical context and we highlight the great diversity of imagined situations in problem posing workshops. The problems exposed in this paper are only a part of those imagined by the participants in this workshop and there are many others created in other workshops. The processes of reflecting on these diverse new problems – individually, in a small group and with all the workshop participants – contribute to the teachers' advances in knowledge of the mathematical object, in the observation of their reality and in elaborating tasks to go deeper into the subject to solve the problem created. We underscore the importance of working individually at first and the richness of working in groups later. All work is strongly enriched with socialization, in which arguments, opinions and comments that reflect involvement in creating problems arise.

Problem posing has to be designed to promote students' learning or to develop their mathematical thinking. We have studied (Malaspina & Vallejo, 2014) that problem-posing workshops related to one concrete theme allow participants to go deeper into the subject matter and to make mathematical connections. In the present research, Post-problem 4 shows the connection between percentages and linear functions, which was unknown for most of the participating teachers.

Examining the quality of the created problems is not the intention of this article, but we can appreciate that problems created by groups have a higher mathematical and didactic potential – above all if they have been created from episodes in classes. Certainly, the better the teachers' mathematics background and teaching experience, the higher the quality of the problems they create.

FINAL CONSIDERATIONS

We have developed problem posing activities with pre service teachers and current ones. We have considered individual work and group work in dealing with a given problem during an episode, which happens under certain characteristics. The analysis of these activities – in particular the cases considered in this paper, with current secondary teachers – show that they contribute to the development of didactic and mathematical competencies. Problem posing provides opportunities in which the two competencies have the possibility to interact in a creative way. The results are didactically valuable suggestions for their students as well as advances in teacher training.

It is important to break the "enculturation process of accepting problems that others create as those which need to be solved" (Ellerton, 2013, p. 87). We will contribute to this breaking by giving pre-service and in-service teachers good opportunities and orientations for creating problems. When teachers have experiences with didactic analysis and mathematical connections through processes of creating new problems, they improve their mathematical and didactic competencies, and they could induce their pupils to create their own problems.

This article is part of a wider area of research in which problem posing is also considered from a given situation. The corresponding activities have been considered in a next phase. We have interesting didactic experiences, some of which are explained in Malaspina and colleagues (2012) and Malaspina (2013a).

The problem posing strategy exposed and commented in this paper could be a good methodological tool for teacher training. Certainly, from a research point of view, it would be interesting to test it in relation to different mathematics topics at various teaching levels and in different countries.

As a part of the challenges posed by this research on creating math problems in mathematics education contexts, we invite readers to consider the following questions:

How do we measure the influence of the problem posing competency development for teachers on their performance in class with students?

How do we verify or reject the conjecture that assuming the challenge of creating math problems on a given topic activates new learning processes that favour intra mathematical connections with other fields of knowledge and reality?

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