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Some characteristics of learning to notice students' mathematical understanding of the classification of quadrilaterals

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The goal of this study is to examine how prospective secondary school mathematics teachers learn to notice students' mathematical thinking about the process of classification of quadrilaterals. Findings point out that when prospective teachers identified the inclusive classification of quadrilaterals as a "key developmental understanding" (Simon, 2006), they modified what they considered evidence of secondary school students' understanding of quadrilaterals classification process.

Keywords: Noticing skill, classification of quadrilaterals, students' mathematical thinking, prospective teachers learning.

INTRODUCTION

Previous research focused on prospective teachers education has underlined the importance of learning to notice what it is happening in a classroom (an important teacher's skill) (Mason, 2002). Furthermore, research has shown that this skill could be developed in teacher education programs in some contexts designed ad hoc (Coles, 2002; Fernández, Llinares, & Valls, 2012; Sánchez-Matamoros, Fernández, & Llinares, 2014).

A particular focus of this skill is how prospective teachers notice students' mathematical thinking. Magiera, van den Kieboom and Moyer (2013) showed that prospective teachers demonstrated a limited ability to recognise and interpret the overall algebraic thinking exhibited by students in the context of one-to-one interviews. Furthermore, research has underlined the relationship between mathematical content knowledge (MKT, Ball, Thames, & Phelps, 2008) and

the process of recognising evidence of students' understanding (Sztajn, Confrey, Wilson, & Edgington, 2012). For instance, Bartell, Webel, Bowen and Dyson (2013) examined the role mathematical content knowledge plays in prospective teachers' ability to recognise evidence of children's conceptual understanding. After an instructional intervention (based on lessons where prospective teachers had to examine many examples of students thinking), their ability to analyse children's responses improved. Fernández, Llinares and Valls (2012) indicated that discriminating between proportional and non-proportional situations was a key element in the development of prospective mathematics teachers' abilities to identify evidence of different levels of students' mathematical understanding in the domain of proportionality. Sánchez-Matamoros, Fernández and Llinares (2014) examined the development of prospective teachers noticing skill of students' mathematical understanding of the derivative concept. This study indicated that a key element in this development was prospective teachers' progressive understanding of the mathematical elements that students use to solve problems in the domain of the derivative. However, difficulties in developing the skill of noticing students' mathematical understanding make more research in this line necessary, and particularly, in the different mathematical domains. In this study, we are going to identify key elements in the development of this skill in the domain of the classification of quadrilaterals.

Learning how to classify quadrilaterals causes difficulties for secondary school students. These difficulties are related to the relationship between inclusive and exclusive classifications of quadrilaterals since students recognise the different quadrilaterals by means of prototype examples without considering the

inclusion relations associated with the classification processes (De Villiers, 1994; Fujita, 2012). Inclusive classifications result when the application of a classifying criterion to a specific set creates subsets in which it is possible to establish an inclusion relation (hierarchical chain) among its elements. For example, in an inclusive classification of parallelograms, the square can be considered a special type of rhombus; while in an exclusive classification (partition) the square and the rhombus belong to separate groups. Understanding the role inclusive and exclusive classifications plays when classifying the quadrilaterals in order to define different types of quadrilateral (Usiskin & Griffin, 2008) is important in learning about students' mathematical understanding. In this context, understanding inclusive classifications and how they are related to the process of defining geometric figures can be considered as a *key developmental understanding* (KDU) (Simon, 2006) since the understanding of inclusive classifications implies a conceptual advance for students that enables them to understand inclusive definitions (for example, that a square is a special type of rhombus).

Taking these aspects into account, our research questions are: (1) how do prospective teachers use their knowledge of quadrilateral classification in order to identify evidence of students' understanding? (2) What teaching decisions do prospective teachers take in order to support the development of students' understanding?

METHOD

Participants and design principles

The participants of the study were six Spanish science graduates (mathematics and engineering) enrolled on an initial training programme that provided them the skills needed to teach mathematics in the secondary school (we will refer to the prospective teachers as PSTs). The programme included subjects such as school organisation, psychology of instruction, mathematics education and teaching practice in secondary schools. In the part corresponding to mathematics education, the prospective teachers were studying a subject aimed at learning the characteristics of secondary school students' mathematical understanding. This subject was taught for 4 hours a week, for 13 weeks, and focused particularly on students' understanding and how to select tasks that would promote a concep-

tual understanding. One of the topics was students' understanding of the classification of quadrilaterals.

The module focused on the classification of quadrilaterals consisted of 3 sessions of two hours, and the participation in a week-long online debate. The design of the module incorporated a socio-cultural perspective (the spiral of knowing, Wells, 2002) and considered four aspects: Experience, Information, Knowledge Building and Understanding. "Experience" is the prior knowledge that prospective teachers have constructed during their participation in learning and teaching situations. "Information" consists of our understanding (as a scientific community) of the quadrilateral classification processes (theoretical information) that we provided to prospective teachers. "Knowledge Building" is related to how prospective teachers engage in meaning-making with others in an attempt to extend and transform their understanding of a student's mathematical thinking and their own understanding of mathematics. Finally, "Understanding" constitutes the interpretative framework in terms of which prospective teachers make sense of new situations, that is, what they mobilise to identify students' mathematical thinking in order to anticipate and monitor student response, select and sequence tasks and make connections with students' responses.

In the module, firstly, PSTs had to answer a task where they had to anticipate, individually, the way in which students' answers to the classification problems reflected evidence of understanding and had to take decisions to promote students' understanding ("Experience"). Next, the teacher trainer presented information about the characteristics of the quadrilaterals classification process and about students' understanding of quadrilaterals classification (inclusive and exclusive classifications, De Villiers, 1994; Usiskin & Griffin, 2008), and discussed this with the

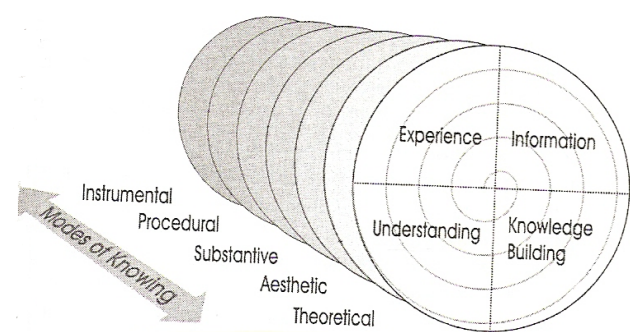


Figure 1: The spiral of knowing (Wells, 2002, p. 85)

PSTs (“Information” in the spiral of Wells). Finally, the PSTs compared their answers in pairs in order to explore differences and similarities in the way they recognised evidence of students’ understanding of the classification process (Knowledge building) and take teaching decisions to promote students’ understanding (“Understanding” in Wells’ spiral of knowledge).

The task (instrument)

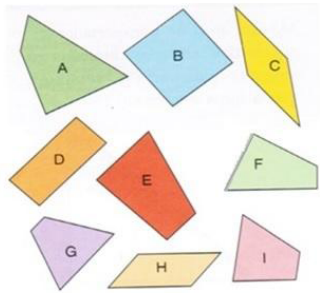
Prospective teachers had to answer a task consisted of two quadrilateral classification problems (ages 14–15) from secondary school textbooks (Figure 2), and six professional questions aimed at prompting prospective teachers to anticipate the response of students with different levels of conceptual understanding and propose tasks to improve their understanding:

A1. Indicate exactly what Maria, a 3rd year secondary school student (aged 14–15), would have to do and say in each problem in order to demonstrate that she has achieved the learning objective assigned for the problem (Classify the quadrilaterals according to different criteria).

A2. Explain which aspects of Maria’s answer to each problem make you think that she has understood the classification of quadrilaterals. Explain your answer.

B1. Indicate exactly what Pedro, another 3rd year secondary school student (aged 14–15), would have to do and say in each problem in order to demonstrate an understanding of certain elements of the classification of quadrilaterals while remaining unable to achieve the learning objective. Explain your answer.

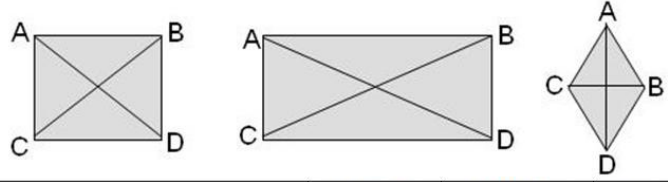
Problem 1
Which of these quadrilaterals



have the properties described below:

- Only have two parallel sides
- Have two parallel sides and two right angles
- Their sides are equal, but their angles are not

Problem 2
2.1 Look at the figures and complete the table below, putting YES or NO in the empty boxes:



	Square	Rectangle	Rhombus
The four sides are equal			
The four angles are equal			
The opposite sides are equal			
The diagonals are equal			
The diagonals intersect at the middle point			
The diagonals form right angles			

2.2. Indicate the similarities and differences of the diagonals when comparing:
 a) rectangle and square
 b) rhombus and rectangle
 c) rhombus and square

2.3. How are the parallelograms classified when using the diagonals as criteria? Justify the answer.

Figure 2: The two quadrilateral classification problems of the task

B2. Explain which aspects of Pedro's answer to each problem makes you think that he has not achieved the intended learning objective. Explain your answer.

C. If you were the teacher of these students,

How would you modify/extend the task in order to confirm that Maria has achieved the intended learning objective? Explain your answer.

How would you modify/extend the task so that Pedro achieves the intended learning objective? Explain your answer.

The first four questions refer to the teacher's ability to anticipate possible answers to the problems that reflect different levels of secondary school students' understanding of the process of classifying quadrilaterals. The last two questions (section C) are related to the teaching decisions; the decisions that the teacher should take in order to promote student progress.

The aim of problem 1 is to classify a set of nine quadrilaterals using three different criteria. The different sections of the problem could be solved by identifying the figures that met the criterion and grouping them together. The problem 2 requires the use of some elements of the geometric figures (sides, angles and diagonals of the square) to classify the parallelograms on the basis of the diagonals.

Analysis

We analysed PSTs answers to the individual task (Experience) and the modifications introduced when solving the task in pairs (knowledge building) on the basis of the theoretical information related to students' understanding of the classification process (Information). Initially, we focused on how the PSTs considered that the hypothetical answers they gave individually to problems 1 and 2 indicated different levels of students' understanding of the classification process. We then identified the decisions they took to help students consolidate or improve their understanding. In the analysis of the task resolution carried out in pairs we tried to identify how the previously discussed theoretical information modified what the PSTs understood as evidence of secondary students' understanding of the classification process (taking into account how PSTs considered the under-

standing of inclusive and exclusive classification as a key development understanding). We also identified modifications in the decisions taken to help students to improve or consolidate their understanding.

The data was analysed by four researchers creating categories. The initial categories were redefined as new data was added. Points of agreement and disagreement were discussed, with the aim of reaching a consensus on the inferences from the data by means of a process that looked for evidence that did or did not confirm the characteristics initially produced.

RESULTS

The results section is organised in two parts. In the first part, we identify the changes in how the PSTs characterised students' understanding and, in the second part, the changes in how they decided to support students understanding.

Changes in how the PSTs characterised students' understanding

The 6 PSTs, initially, considered the use of just one criterion as evidence of the understanding of the classification process: the one that generated exclusive classifications. For example classifications in which a square is a parallelogram but is not a special type of rectangle or rhombus. One PST considered as criteria whether the diagonals were congruent and if they formed a right angle (item 2.3 of problem 2). From this perspective the parallelograms formed three groups: squares (parallelograms with congruent and perpendicular diagonals), rectangles (parallelograms with congruent diagonals that do not form a right angle), and rhombuses (parallelograms with non-congruent diagonals).

On the other hand, PSTs considered evidence of an incomplete understanding of the classification process of parallelograms when the secondary school student was not capable of generating exclusive classifications. Anna, one of the PST, justified this fact by stating that:

(the student with an inadequate understanding of the classification process) does (would do) problem 1 correctly but not problem 2. The student's error comes (would come) from considering that the diagonals of the square do not intersect at right angles – perhaps because of a

printing error- (referring to Figure B in problem 1), which means that the classification of item 2.3 only creates two large groups: those that have equal diagonals (square and rectangles) and those that do not (rhombus).

After the discussion of the theoretical information ("Information" in the Wells' spiral of Knowing) focused on the relations between the inclusive and exclusive classifications and definitions, PSTs started to consider the inclusive relation as a key developmental understanding when they tried to identify evidence of students' understanding. Putting the relation between inclusive and exclusive classifications in the focus of their noticing allows them to accept that a square can be considered a special type of rhombus or rectangle and this understanding should be considered as an advance in the student understanding.

For example, a pair of PSTs anticipated using a criterion based on diagonals (item 2.3 in problem 2, Figure 1) and the possibility of generating an inclusive relation. They used the criterion "congruent diagonals", which leads to a classification in which the rectangles and squares are in the same set and the rhombuses and rhomboids in another. Then, in both sets they considered the criterion "the diagonals intersect at a right angle". From this way, they obtained an inclusive classification that allowed them to define, for example, that "*a square is a rectangle whose diagonals intersect at a right angle*". To demonstrate an inadequate understanding of the classification of parallelograms, they considered that the student was not capable of generating inclusive classifications that could allow him/her to get definitions where a square can be considered a rectangle. So, for example, they evidenced the difficulties involved in handling inclusive relations assuming that the student could define squares and rectangles with no relationship: a square is a parallelogram with equal diagonals that form a right angle; a rectangle is a parallelogram with equal diagonals that do not form a right angle.

Changes in the PSTs' teaching decisions to support students' understanding

The changes in teaching decisions were linked to how PSTs understood the inclusive classification as a key factor in the understanding of classification processes.

Initially, when PSTs considered that the understanding of the classification process was linked with exclusive classifications, they supported their teaching decisions in helping students to identify and apply classification criteria that generate singleton subsets. For example, to consolidate students' learning of the classification process, Anna proposed the following problem:

- a) In problem 1 find a square, a rectangle and a rhombus, and check the characteristics in the table (section 2.1. of problem 2).
- b) Analyse the characteristics in the table for all the figures in problem 1. What do you observe?
- c) Give definitions for the parallelograms

Furthermore, to help better understanding of the classification process, Anna tried that the student was capable of recognising a second criterion "diagonals intersect at right angle" that produce an exclusive classification with singleton subsets. Anna stated:

First I would ask the student about the differences between the square and the rectangle in their classification (observing the answers given in activity 2.1); and then I would make the student draw different squares with their diagonals and measure the angles with a protractor.

If it is possible, in a computer room, I would also make sure that the student correctly draws and measures the angles the diagonals form with enough squares so as to be sure that they fully understand that this characteristic is common in all squares.

After the discussion of the theoretical information ("Information" in the Wells' spiral of Knowing) the PSTs understood that inclusive classifications were a key factor in the understanding of classification processes and they took different teaching decisions. For example, to consolidate understanding of the classification process in students that accepted inclusive classifications, Anna and Robert proposed *extension* activities aimed at establishing the equivalence of different definitions.

- a) Analyse the characteristics in Table 2.1. in the figures in problem 1.

b) Give two different definitions of parallelograms

With students that hypothetically only generated exclusive classifications, they recognised that inclusive classifications are a key factor in helping them to understanding the classification of quadrilaterals. To this end, they designed tasks that serve, through analysis and reflection, to underline the possibility of seeing a figure as a special case within a larger group. For example, Anna and Robert modified sections 2.1 and 2.2 in problem 2, incorporating the figure of a rhomboid and asking *Classify Figure H in Table 2.1. Is it a parallelogram?* and *Classify the parallelograms into only two groups. Can you do it in another way?* In this way, in their view, the new problem 2 forces the student to choose classification criteria for the parallelograms in two groups and thereby creates an opportunity to generate an inclusive classification.

These changes in the PSTs' teaching decisions can be linked to the knowledge building (an aspect from the Wells' spiral of knowing) in which the inclusive classification was considered as a key developmental understanding in the task of attempting to recognise evidence of students understanding of quadrilaterals classification.

DISCUSSION

This research aims to examine some characteristics of PSTs learning. Our focus is how the changes in the way that PSTs understand the classification process of quadrilaterals influence on what they consider evidence of students' understanding, and on how they decide to assist students in their development. Initially, the PSTs only considered exclusive classifications, which implied that the definition of the different parallelograms was a succession of properties without the establishment of relationships between them. This meant that the way in which they were able to help students was to propose tasks in order to identify the largest possible number of characteristics in the figures, but without establishing relationships between them. The recognition of inclusive classifications of quadrilaterals as a key mathematical factor for conceptual development enabled them to modify the way in which they characterised students' understanding. This led the PSTs to focus their attention on helping students to identify relationships between the characteristics of figures as a means to generate

inclusive definitions. To some extent, the prospective teachers set the development of the understanding of inclusive classifications (a "key developmental understanding", Simon, 2006) as a learning objective. For example, to define a rhombus as a parallelogram with four congruent sides and the opposite angles congruent two by two, and to consequently be able to define a square as a rhombus with four congruent angles. This result is in line of previous studies which shown the importance of prospective teachers' mathematical knowledge when attempting to interpret students' understanding (Fernández et al., 2013; Magiera et al., 2013; Sánchez-Matamoros et al., 2014; Son, 2013). To summarise, participation in this module enabled prospective teachers to recognise the understanding of inclusive classification as a conceptual advance in the development of classification and definition of geometric figures. As a consequence, prospective teachers identified the understanding of inclusive classification as a learning objective, recognising it as a necessary qualitative transition in the ability of students to think about and perceive relationships between elements of geometric figures. This recognition was demonstrated in the way they posed new tasks to support the understanding of classifications and the process of defining geometric figures.

Although more research is still needed to help us to identify the factors that influence the development of prospective teachers' noticing of teaching and learning, our data provide characteristics of the learning of knowledge needed to teach and its use in noticing evidence of students' understanding. Although the intervention might be considered short in terms of time, these results provide ideas that can help in the design of sequences of learning activities for prospective teachers, aimed to make explicit what they considered "key knowledge" when notice students' understanding and make teaching decisions in order to support students in their learning.

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