Classification of Special Curves in the Space of Cubic Polynomials

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by

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Abstract. — We describe all special curves in the parameter space of complex cubic polynomials, that is all algebraic irreducible curves containing infinitely many post-critically finite polynomials. This solves in a strong form a conjecture by Baker and DeMarco for cubic polynomials.

Let $\mathrm{Per}_m(\lambda)$ be the algebraic curve consisting of those cubic polynomials that admit an orbit of period m and multiplier λ. We also prove that an irreducible component of $\mathrm{Per}_m(\lambda)$ is special if and only if $\lambda = 0$.

Contents

1. Introduction

The space Poly_d of complex polynomials of degree $d \geq 2$ modulo affine conjugacy forms a complex analytic space that admits a ramified parameterization by the affine space $\mathbb{A}^{d-1}_{\mathbb{C}}$. The study of the set of degree d polynomials with special dynamical features forms the core of the modern theory of holomorphic dynamics. We shall be concerned here with the distribution of the set of post-critically finite (PCF) polynomials for which all critical points have a finite orbit under iteration. This set is a countable union of points defined over a number field, see e.g. [**I**, Corollary 3]. It was proved in [**L**] in degree $d = 2$ that any sequence of Galois-invariant finite subsets of PCF polynomials converges in the sense of measures to the so-called bifurcation measure in Poly_2. This was generalized in [**FG**] in any degree under a mild assumption and further explored in [**GV**]. The support of this measure has been characterized in several ways in a series of works [**DF, Du1, Du2, G3**], and it was shown by the second author [**G1**] to have maximal Hausdorff dimension $2(d-1)$.

In a beautiful recent paper [**BDM**], Baker and DeMarco have proposed a way to describe the distribution of PCF polynomials from the point of view of the Zariski topology.

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They defined *special* algebraic subvarieties as those subvarieties $Z \subset \mathrm{Poly}_d$ admitting a Zariski-dense subset formed by PCF polynomials, and asked about the classification of such varieties. More precisely, they offered a quite general conjecture [**BDM**, Conjecture 1.4] inspired by the André-Oort conjecture in arithmetic geometry stating that any polynomial[(1)] lying in a special (proper) subvariety should admit a symmetry (possibly of degree ≥ 2). They gave a proof of a stronger version of this conjecture in the case the subvariety was isomorphic to an affine line.

Our objective is to give the list of all special curves in the case $d = 3$, thereby proving Baker-DeMarco's conjecture for cubic polynomials. To do so, following the seminal work [**BH**] of Branner and Hubbard, we shall use the parameterization $(c, a) \mapsto P_{c,a}$ of the parameter space by the affine plane with

$$P_{c,a}(z) := \frac{1}{3}z^3 - \frac{c}{2}z^2 + a^3,\ z \in \mathbb{C}, (c, a) \in \mathbb{C}^2\ .$$

Observe that $P_{c,a}$ then admits two critical points $c_0 := 0$ and $c_1 := c$ and that this map defines a finite branched cover of the moduli space Poly_3 of cubic polynomials with marked critical points.

Here is our main result.

Theorem A. — *An irreducible curve C in the space Poly_3 is special if and only if one of the following holds.*

1. *One of the two critical points is persistently pre-periodic on C, i.e. there exist integers $m > 0$ and $k \geq 0$ such that $P_{c,a}^{m+k}(c_0) = P_{c,a}^k(c_0)$ or $P_{c,a}^{m+k}(c_1) = P_{c,a}^k(c_1)$ for all $(c, a) \in C$.*
2. *There is a persistent collision of the two critical orbits on C, i.e. there exist $(m, k) \in \mathbb{N}^2 \setminus \{(1, 1)\}$ such that $P_{c,a}^m(c_1) = P_{c,a}^k(c_0)$ for all $(c, a) \in C$.*
3. *The curve C is given by the equation $\{(c, a),\ 12a^3 - c^3 - 6c = 0\}$, and coincides with the set of cubic polynomials having a non-trivial symmetry, i.e. the set of parameters (c, a) for which $Q_c(z) := -z + c$ commutes with $P_{c,a}$.*

Recall that for any integer $m \geq 1$ and any complex number $\lambda \in \mathbb{C}$ the set $\mathrm{Per}_m(\lambda)$ consisting of all polynomials $P_{c,a} \in \mathrm{Poly}_3$ that admits at least one periodic orbit of period m and multiplier λ is an algebraic curve (see §5 for a more precise description).

The geometry of these curves has been explored by several authors, especially when $\lambda = 0$. The irreducible components of $\mathrm{Per}_m(0)$ has been proven to be smooth by Milnor [**Mi3**], and the escape components of these curves have been described in terms of Puiseux series by Bonifant, Kiwi and Milnor [**BKM**] (see also [**K**, §7]). On the other hand, DeMarco and Schiff [**DMS**] have given an algorithm to compute its Euler characterisitc.

From the point of view of pluripotential theory, the distribution of the sequence of curves $(\mathrm{Per}_m(\lambda))_{m\geq 1}$ has been completely described by Bassanelli and Berteloot in [**BB2**] in the case $|\lambda| \leq 1$ (see also [**G2**] for the case $|\lambda| > 1$ and [**BB1**] for the case of quadratic rational maps).

Inspired by a similar result from Baker and DeMarco, see [**BDM**, Theorem 1.1] we also give a characterization of those $\mathrm{Per}_m(\lambda)$ that contain infinitely many PCF polynomials, and prove

(1) the conjecture is actually stated for any rational maps, and a stronger conjecture related to Pink and Zilber's conjectures can be found in [**De2**].

Theorem B. — *For any $m \geq 1$, the curve $\mathrm{Per}_m(\lambda)$ contains infinitely many post-critically finite polynomials if and only if $\lambda = 0$.*

The general strategy of the proof of these two theorems was set up by Baker and DeMarco. They start with an irreducible algebraic curve $C \subset \mathrm{Poly}_3$ containing infinitely many PCF polynomials (in Theorem B the curve C is a component of some $\mathrm{Per}_m(\lambda)$). We observe however that they used at several key points their assumption that the curve C has a single branch at infinity. To remove this restriction we had to include three new ingredients:

- we construct a one parameter family of heights for which Yuan-Thuillier's equidistribution theorem applies;
- we investigate systematically the arithmetic properties of the coefficients of the expansion of the Böttcher coordinates and its dependence on the parameters c, a;
- to build the symmetry we rely on a recent algebraization result of Xie [**X**] that gives a criterion for when a formal curve in the affine plane is a branch of an algebraic curve.

A characteristic feature of our proofs is to look at the dynamics induced by cubic polynomials over various fields: over the complex numbers and over p-adic fields (see e.g. §4.1), over the field of Laurent series (see the proof of Proposition 3.6 and §5), and over a number field (see §3). We use at one point the universality theorem of McMullen [**McM3**] which is a purely Archimedean statement. Moreover the work of Kiwi [**K**] on non-Archimedean cubic polynomials over a field of residual characteristic zero plays a key role in the proof of Theorem B.

Let us describe in more detail how we proceed, and so pick an irreducible algebraic curve $C \subset \mathrm{Poly}_3$ containing infinitely many PCF polynomials. We may suppose that neither c_0 nor c_1 are persistently pre-periodic on C. By a theorem of McMullen [**McM1**, Lemma 2.1] this is equivalent to say that both critical points exhibit bifurcations at some (possibly different) points in C. There is a more quantitative way to describe the set of bifurcations using the Green function $g_{c,a}(z) := \lim_{n\to\infty} \frac{1}{3^n} \log \max\{1, |P^n(z)|\}$. Indeed both functions $g_0(c,a) := g_{c,a}(c_0)$, $g_1(c,a) := g_{c,a}(c_1)$ are non-negative and pluri-subharmonic, and it is a fact [**De1**, §5] that the support of the positive measure $\Delta g_0|_C$ (resp. $\Delta g_1|_C$) is equal to the set of parameters where c_0 (resp. c_1) is unstable.

The first step consists in proving that $g_0|_C$ and $g_1|_C$ are proportional, and this conclusion is obtained by applying an equidistribution result of points of small height due to Yuan [**Y**] and Thuillier [**T**]. We first observe that C is necessarily defined over a number field $\mathbb{K}$ since it contains infinitely many PCF polynomials, so that we may introduce the functions $g_{0,v}, g_{1,v}$ for all (not necessarily Archimedean) places v over $\mathbb{K}$. These functions can now be used to build a one-parameter family of heights on C by setting

$$h_s(p) := \frac{1}{\deg(p)} \sum_{q,v} \max\{s_0 g_{0,v}(q), s_1 g_{1,v}(q)\}$$

where $s = (s_0, s_1) \in \mathbb{R}_+^2$, and the sum ranges over all Galois conjugates q of p and over all places v over $\mathbb{K}$. When s_0 and s_1 are positive integers, then we prove in §3 that the height h_s is induced by a continuous semi-positive adelic metrization in the sense of Zhang on a suitable line bundle over C of positive degree, so that Yuan-Thuillier's theorem applies. This gives us sufficiently many restrictions on g_0 and g_1 which force their proportionality. The key arguments are Proposition 3.6 that is close in spirit to [**BDM**, Proposition 2.1

(3)], and the fact that the function $\max\{g_{0,v}, g_{1,v}\}$ is a proper continuous function on Poly_3 for any place v.

From the proportionality of g_0 and g_1 on a special curve, we are actually able to conclude the proof of Theorem B. This step is done in §5. We suppose by contradiction that our special curve C is an irreducible component of some $\mathrm{Per}_m(\lambda)$ with $\lambda \neq 0$. Then each branch at infinity of C defines a cubic polynomial over the complete field of Laurent series $\mathbb{C}((t))$. And we show that except when c_0 or c_1 is persistently periodic in C the multipliers of all periodic points are exploding on that branch by [**K**]. We then analyze the situation of a unicritical[(2)] polynomial in C and computing the norm of the multiplier of its periodic points in a suitable field of residual characteristic 3, we are able to get the required contradiction.

Let us come back to the proof of Theorem A. At this point, we have an irreducible algebraic curve C defined over a number field $\mathbb{K}$ and such that $g_{0,v} = g_{1,v}$ at any place v over $\mathbb{K}$. Recall that for any polynomial $P_{c,a}$ there exists an analytic isomorphism near infinity conjugating the polynomial to the cubic monomial map. This isomorphism is referred to as the Böttcher coordinate $\varphi_{c,a}$ of $P_{c,a}$. We prove that when c, a are defined over a number field then $\varphi_{c,a}$ is a power series with coefficients in a number field whose domain of convergence is positive at any place, see Lemma 2.2 and Proposition 2.3.

Building on an argument of Baker and DeMarco, we then show that outside a compact subset of the analytification of C (for any completion of $\mathbb{K}$) the values of the Böttcher coordinates at c_0 and c_1 are proportional up to a root of unity (Theorem 4.1 (2)).

The proof now takes a slight twist as we fix any polynomial $P := P_{c,a}$ that is *not* post-critically finite and for which (c, a) belongs to $C(\mathbb{L})$ for some finite field extension $\mathbb{L}$ of $\mathbb{K}$. We prove that any such polynomial admits a weak form of symmetry in the sense that there exists an irreducible curve $Z_P \subset \mathbb{P}^1 \times \mathbb{P}^1$ that is stable by the map (P, P). The arguments used by Baker and DeMarco in [**BDM**, §5.6] break down here and we apply [**X**, Theorem 1.5] as a replacement for them. In order to get a polynomial that commutes with P instead of a correspondence, we proceed as Baker and DeMarco and use Medvedev-Scanlon's result [**MS**, Theorem 6.24] (see [**P**, Theorem 4.9] for another proof of this result).

At this point we have proved the following result that we feel is of independent interest.

Theorem C. — *Pick any irreducible complex algebraic curve $C \subset \mathrm{Poly}_3$. Then the following assertions are equivalent:*

1. *the curve C is special,*
2. *for any critical point that is not persistenly pre-periodic on C, the set of PCF polynomials lying in C is equal to the set where this critical point is pre-periodic;*
3. *the curve C is defined over a number field $\mathbb{K}$ and there exist integers $(s_0, s_1) \in \mathbb{N}^2 \setminus \{(0,0)\}$ such that for any place $v \in M_{\mathbb{K}}$, we have*
$$s_0 \cdot g_{0,v} = s_1 \cdot g_{1,v} \ ,$$
on the analytification of C over the completion of $\mathbb{K}$ w.r.t. the v-adic norm;
4. *for any sequence $X_k \subset C$ of Galois-invariant finite sets of PCF polynomials with $X_k \neq X_l$ for $l \neq k$, the probability measures μ_k equidistributed on X_k converge towards (a multiple of) the bifurcation measure $T_{\mathrm{bif}} \wedge [C]$ as $k \to \infty$;*

(2) i.e. has a single critical point

5. *there exists a root of unity* ζ*, and integers* $q, m \geq 0$ *such that the polynomial* $Q_{c,a}(z) := \zeta P_{c,a}^m(z) + (1-\zeta)\frac{c}{2}$ *commutes with any iterate* $P_{c,a}^k$ *such that* $\zeta^{3^k} = \zeta$*, and* $Q_{c,a}(P_{c,a}^q(c_i)) = P_{c,a}^q(c_j)$ *for some* $i, j \in \{0,1\}$ *and all* $(c,a) \in C$*.*

In (4) the current $T_{\rm bif}$ is defined as the dd^c of the plurisubharmonic function $g_0 + g_1$. Its support in $\mathbb{C}^2$ is known to be equal to the set of unstable parameters, see e.g. [**DF**, §3]. Notice that for any curve C there exists a critical point which is not persistently preperiodic on C since by [**BH**] the set $\{g_0 = 0\} \cap \{g_1 = 0\}$ is *compact* in $\mathbb{A}^2_{\mathbb{C}}$. In particular, the assertion (2) is consistent.

To complete the proof of Theorem A, we analyze in more detail the possibilities for a cubic polynomial to satisfy the condition (5) in the previous theorem. Namely, we prove that the set of parameters admitting a non-trivial symmetry of degree $3^m > 1$ is actually *finite*. Theorems A and C are proved in §7.

We have deliberately chosen to write the entire paper for cubic polynomials only. This simplifies the exposition but many parts of the proof actually extend to a larger context. Let us briefly discuss the possible extensions and the limitations of our approach.

All ingredients are present to prove Baker-DeMarco's conjecture for a *curve* in the space of polynomials of any degree $d \geq 2$. It is however not clear to the authors how to obtain the more precise classification of special curves in the same vein as in Theorem A.

We note that there are serious difficulties that lie beyond the methods presented here to handle higher dimensional special varieties V in Poly_d. The main issue is the following. To apply Yuan's equidistribution theorem of points of small heights it is necessary to have a continuous semi-positive adelic metrics on an ample line bundle on a compactification of V, and we are at the moment very far from being able to check any of the three underlined conditions.

Trying to understand special curves in the space of quadratic maps requires much more delicate estimates than in the case of polynomials. A first important step has been done by DeMarco, Wang and Ye in a recent paper [**DMWY**].

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While finishing the writing of this paper we have learned that Dragos Ghioca and Hexi Ye have independently obtained a proof of our Theorem A. Their approach differs from ours in the sense that they directly prove the continuity of the metrizations induced by the functions $g_{0,v}$ and $g_{1,v}$. We get around this problem by considering metrizations induced by $\max\{s_0 g_{0,v}, s_1 g_{1,v}\}$ for positive s_0, s_1 instead. We warmly thank Ghioca and Ye for sharing with us their preprint.

2. The Böttcher coordinate of a polynomial

In this section, K is any complete metrized field of characteristic zero containing a square-root λ of $\frac{1}{3}$. It may or may not be endowed with a non-Archimedean norm.

If X is an algebraic variety over K, then $X^{\rm an}$ denotes its analytification as a real-analytic or a complex variety if K is Archimedean, and as a Berkovich analytic space when K is non-Archimedean (see e.g. [**B**, §3.4-5]).

2.1. Basics

As in the introduction, we denote by $\mathrm{Poly}_3 \simeq \mathbb{A}^2$ the space of cubic polynomials defined by

$$P_{c,a}(z) := \frac{1}{3}z^3 - \frac{c}{2}z^2 + a^3 \ . \tag{1}$$

It is a branched cover of the parameter space of cubic polynomials with marked critical points. The critical points of $P_{c,a}$ are given by $c_0 := c$ and $c_1 := 0$.

For a fixed $(c,a) \in K^2$ the function $\frac{1}{3}\log^+|P_{c,a}(z)| - \log^+|z|$ is bounded on $\mathbb{A}^{1,\mathrm{an}}_K$ so that the sequence $\frac{1}{3^n}\log^+|P^n_{c,a}(z)|$ converges uniformly to a continuous sub-harmonic function $g_{c,a}(z)$ that is called the Green function of $P_{c,a}$.

We shall write $g_0(c,a) := g_{c,a}(c_0)$, $g_1(c,a) := g_{c,a}(c_1)$, and

$$G(c,a) := \max\{g_0(c,a), g_1(c,a)\} \ .$$

Proposition 2.1. — *There exists a constant $C = C(K) > 0$ such that*

$$\sup_{\mathbb{A}^{2,\mathrm{an}}_K} \left|G(c,a) - \log^+ \max\{|a|,|c|\}\right| \leq C \ ,$$

and this constant vanishes when the residual characteristic of K is at least 5.

A proof of this fact is given in [**BH**, §4] (see also [**DF**, §6] for a more detailed proof) in the Archimedean case and in [**FG**, §2] in general.

2.2. Expansion of the Böttcher coordinate

For any cubic polynomial $P \in K[z]$, we let the *Böttcher coordinate* of P be the only formal power series φ satisfying the equation

$$\varphi \circ P(z) = \varphi(z)^3 \tag{2}$$

which is of the form

$$\varphi(z) = \lambda z + \alpha + \sum_{k\geq 1} a_k z^{-k} \ , \tag{3}$$

with $\alpha, a_k \in K$ for all $k \geq 1$.

Lemma 2.2. — *Given any $(c,a) \in K \times K$, the Böttcher coordinate $\varphi_{c,a}(z)$ of the degree 3 polynomial $P_{c,a} := \frac{z^3}{3} - \frac{c}{2}z^2 + a^3$ exists, is unique, and satisfies*

$$\varphi_{c,a}(z) = \lambda\left(z - \frac{c}{2}\right) + \sum_{k\geq 1} a_k(c,a) z^{-k},$$

where

$$a_k(c,a) \in \mathbb{Z}\left[\lambda, \frac{1}{2}\right][c,a] \ \textit{with} \ \deg(a_k) = k+1 \ . \tag{4}$$

Moreover the 2-adic (resp. 3-adic) norm of the coefficients of a_k are bounded from above by 2^{k+1} (resp. $3^{k/2}$).

Proof. — The defining equation (2) reads as follows:

$$\left(\lambda\left(z-\frac{c}{2}\right)+\sum_{k\geq 1}a_k(c,a)z^{-k}\right)^3 = \lambda\left(\frac{z^3}{3}-\frac{c}{2}z^2+a^3-\frac{c}{2}\right)+\sum_{k\geq 1}\frac{3^k a_k(c,a)}{z^{3k}\,(1-\frac{3c}{2z}+\frac{3a^3}{z^3})^k}$$

An immediate check shows that terms in z^3 and z^2 are identical on both sides of the equation. Identifying terms in z yields

$$3\lambda^3(c^2/4)+3\lambda^2 a_1=0,\ \text{so that } a_1=-\frac{\lambda}{4}c^2,$$

whereas identifying constant terms, we get

$$3\lambda^2 a_2+6\lambda^2(-c/2)a_1+\lambda^3(-c^3/8)=\lambda(a^3-c/2)$$

hence

$$a_2=-\frac{5\lambda}{24}c^3+\frac{1}{3\lambda}(a^3-\frac{c}{2})\ .$$

This shows (4) for $k=1,2$, since $\lambda^{-1}=3\lambda$.

We now proceed by induction. Suppose (4) has been proven for k. Identifying terms in $z^{-(k-1)}$ in the equation above, we get

$$3\lambda^2 a_{k+1}-3c\lambda^2 a_k+\frac{3\lambda^2}{4}c^2a_{k-1}+ +\lambda\sum_{i+j=k}a_ia_j-\lambda\frac{c}{2}\sum_{i+j=k-1}a_ia_j+\sum_{i+j+l=k+1}a_ia_ja_l = \sum_{l\geq 1}3^l a_l\left[\left(1+\frac{3c}{2z}+\frac{a^3}{z^3}\right)^{-l}\right]_{k+1-3l}$$

where $\left[\left(1+\frac{3c}{2z}+\frac{a^3}{z^3}\right)^{-l}\right]_j$ denotes the coefficient in z^{-j} of the expansion of $(1+\frac{3c}{2z}+\frac{a^3}{z^3})^{-l}$ in power of z^{-1}. Observe that this coefficient belongs to $\mathbb{Z}[\frac{1}{2}][c,a]$ and has 2-adic norm $\leq 2^l$ and is a polynomial in c,a of degree at most j. It follows that the polynomial

$$a_l(c,a)\left[\left(1+\frac{3c}{2z}+\frac{a^3}{z^3}\right)^{-l}\right]_{k+1-3l}$$

is of degree at most $k+1-3l+l+1=k+2-2l<k+1$. The induction step is then easy to complete using again $\lambda^{-1}=3\lambda$. □

2.3. Extending the Böttcher coordinate

Recall that $G(c,a)=\max\{g_0(c,a),g_1(c,a)\}$.

Proposition 2.3. — *There exists a constant $\rho=\rho(K)\geq 0$ such that the Böttcher coordinate of $P_{c,a}$ is converging in $\{z,\ \log|z|>\rho+G(c,a)\}$.*

There exists another constant $\tau = \tau(K) \geq 0$ such that the map $(c,a,z) \mapsto \varphi_{c,a}(z)$ extends as an analytic map on the open set

$$\{(c,a,z) \in \mathbb{A}^{2,\mathrm{an}}_K \times \mathbb{A}^{1,\mathrm{an}}_K,\, g_{c,a}(z) > G(c,a) + \tau\}\ ,$$

and $\varphi_{c,a}$ defines an analytic isomorphism from $U_{c,a} := \{g_{c,a} > G(c,a)+\tau\}$ to $\mathbb{A}^{1,\mathrm{an}}_K \setminus \overline{\mathbb{D}(0, e^{G(c,a)+\tau})}$ satisfying the equation (2) on $U_{c,a}$. We have

$$g_{c,a}(z) = \log |\varphi_{c,a}(z)|_K \quad \text{on } U_{c,a}\ . \tag{5}$$

Finally, $\tau = 0$ except if the residual characteristic of K is equal to 2 or 3.

We shall use the following lemma which follows easily from e.g. [**FG**, Proposition 2.3].

Lemma 2.4. — *There exists a constant $\theta = \theta(K) \geq 0$*

$$\sup_{\mathbb{A}^{1,\mathrm{an}}_K} |g_{c,a}(z) - \log^+ |z|| \leq \theta\ .$$

Moreover, θ is equal to 0 except if the norm on K is Archimedean or the residual characteristic of K is equal to 2 or 3.

Proof of Proposition 2.3. — Assume first that K is Archimedean, and set $\tau = 0$. In that case most of the statements are proved in [**DH**] (see also [**BH**, §1]). In particular, $\varphi_{c,a}(z)$ is analytic in a neighborhood of ∞ and extends to $U_{c,a}$ by invariance and defines an isomorphism between the claimed domains. It is moreover analytic in c, a, z.

To estimate more precisely the radius of convergence of the power series (3), we rely on [**BH**, §4] as formulated in [**DF**, §6]. First choose $C = C_K > 0$ such that $G(c,a) > \log^+ \max\{|a|,|c|\} - C$. Then $\log|z| > C + G(c,a)$ implies $|z - \frac{c}{2}| > \max\{1,|a|,|c|\} - |\frac{c}{2}| \geq \frac{1}{2}\max\{1,|a|,|c|\}$ hence $\log|z-\frac{c}{2}| > G(c,a) - \log 2$, so that $g_{c,a}(z) > \log|z-\frac{c}{2}| - \log 4 > G(c,a)$, and φ converges in $\{z,\, \log|z| > G(c,a) + \rho\}$ with $\rho := C$ as required.

From now on, we assume that the norm on K is non-Archimedean.

When the residual characteristic of K is different from 2 and 3, then (4) implies $|a_k| \leq \max\{1,|c|,|a|\}^{k+1}$ so that φ converges for $|z| > \max\{1,|c|,|a|\}$, and $\log|\varphi(z)| = \log|z|$. Recall that that we have $G(c,a) = \log\max\{1,|c|,|a|\}$ by Proposition 2.1 so that one can take $\rho = 0$. Pick any z such that $g(z) > G(c,a)$, and observe that $|P^n(z)| \to \infty$. Then we get

$$g_{c,a}(z) = \lim_{n\to\infty} \frac{1}{3^n} \log|P^n(z)| = \lim_{n\to\infty} \frac{1}{3^n} \log|\varphi(P^n(z))| = \log|\varphi(z)| = \log|z|\ . \tag{6}$$

In particular the set $\{g > G(c,a)\}$ is equal to $\mathbb{A}^{1,\mathrm{an}}_K \setminus \overline{\mathbb{D}(0,e^{G(c,a)})}$, and φ is an analytic map from that open set onto itself. It is an isomorphism since $\log|\varphi(z)| = \log|z|$ as soon as $g(z) > G(c,a)$. The proposition is thus proved in this case with $\tau = 0$.

In residual characteristic 2, $|a_k| \leq (2\max\{1,|c|,|a|\})^{k+1}$ whence φ converges for $|z| > 2\max\{1,|c|,|a|\}$, and as above $\log|\varphi| = \log|z|$ in that range. Recall that $G(c,a) - \log^+\max\{|c|,|a|\} \geq C = C(K)$, so that $\log|z| > G(c,a) + \log 2 - C_K$ implies $|z| > 2\max\{1,|c|,|a|\}$, which proves that the power series (3) converges for $\log|z| > G(c,a) + \rho$ with $\rho = \log 2 - C$. Set $\tau := \rho + \theta$ where θ is the constant given by Lemma 2.4. Using $\log|\varphi(z)| = \log|z|$ as above, we get that $\varphi_{c,a}$ defines an analytic isomorphism from $U_{c,a} := \{g_{c,a} > G(c,a)+\tau\}$ to $\mathbb{A}^{1,\mathrm{an}}_K \setminus \overline{\mathbb{D}(0,e^{G(c,a)+\tau})}$.

In residual characteristic 3, $|a_k| \leq (3^{1/2}\max\{1,|c|,|a|\})^{k+1}$ whence φ converges $|z| > 3^{1/2}\max\{1,|c|,|a|\}$, and $\log|\varphi| = \log|\lambda z|$ in that range. Recall that $G(c,a) -$

$\log^+ \max\{|c|,|a|\} \geq C = C(K)$, so that $\log|z| > G(c,a) + \log 3^{1/2} - C_K$ implies $|z| > 3^{1/2}\max\{1,|c|,|a|\}$, which proves that the power series (3) converges for $\log|z| > G(c,a) + \rho$ with $\rho = \log 3^{1/2} - C$. We conclude the proof putting $\tau := \rho + \theta$ as before. □

Remark. — It is possible to argue that $\tau = 0$ also in residual characteristic 2. Although we do not know the optimal constant τ in residual characteristic 3 the Böttcher coordinate is likely not to induce an isomorphism from $\{g_{c,a} > G(c,a)\}$ to $\mathbb{A}^{1,\mathrm{an}}_K \setminus \overline{\mathbb{D}(0, e^{G(c,a)})}$.

3. Curves in Poly_3

In this section we fix a number field $\mathbb{K}$ containing a square-root λ of $\frac{1}{3}$ and take an irreducible curve C in Poly_3 that is defined over $\mathbb{K}$. Our aim is to build suitable height functions on C for which the distribution of points of small height can be described using Thuillier-Yuan's theorem. Our main statement is Theorem 3.9 below.

Recall that given any finite set S of places of $\mathbb{K}$ containing all Archimedean places, $\mathcal{O}_{\mathbb{K},S}$ denotes the ring of S-integers in $\mathbb{K}$ that is of elements of $\mathbb{K}$ of v-norm ≤ 1 for all $v \notin S$. We also write $\mathbb{K}_v$ for the completion of $\mathbb{K}$ w.r.t. the v-adic norm.

3.1. Adelic series

A formal power series $\sum_n a_n z^n$ is said to be *adelic on* $\mathbb{K}$ if its coefficients belong to $\mathcal{O}_{\mathbb{K},S}$ where $\mathbb{K}$ is a number field, and S a finite set of places on $\mathbb{K}$; and for each place v on $\mathbb{K}$ the series has a positive radius of convergence $r_v := \limsup_{n\to\infty} |a_n|_v^{-1/n} > 0$. Observe that $r_v = 1$ for all but finitely many places.

Lemma 3.1. — *Suppose $\alpha(t) = \sum_n a_n t^n$ is an adelic series with $a_0 = 0$ and $a_1 \neq 0$. Then there exists an adelic series β such that $\beta \circ \alpha(t) = t$.*

Proof. — Suppose $a_n \in \mathcal{O}_{\mathbb{K},S}$ for all n, and write $\beta(t) = \sum_n b_n t^n$. The equation $\beta\circ\alpha(t) = t$ amounts to $b_0 = 0$, $b_1 = a_1^{-1}$, and the relations

$$b_n a_1^n + \sum_{1\leq k\leq n-1} b_k \left[\left(\sum_{j\leq n} a_j t^j\right)^k\right]_n = 0 \ ,$$

for any $n \geq 2$ where $[\cdot]_n$ denotes the coefficient in t^n of the power series inside the brackets. It follows that $b_n \in \mathcal{O}_{\mathbb{K},S'}$ for all n where S' is the union of S and all places v for which $|a_1|_v > 1$. The convergence of the series follows from Cauchy-Kowalewskaia's method of majorant series or from the analytic implicit function theorem, see [**C**] and [**Se**, p. 73]. □

Lemma 3.2. — *Pick $k \in \mathbb{Z}$, and suppose $\alpha(t) = \sum_{n\geq k} a_n t^n$ is an adelic series with $a_k \neq 0$. Then there exists an adelic series β such that $\beta(t)^k = \alpha(t)$.*

Proof. — As in the previous proof, suppose $a_n \in \mathcal{O}_{\mathbb{K},S}$ for all n, and write $\beta(t) = b_1 t + \sum_{n\geq 2} b_n t^n$. We get $b_1^k = a_1$, and for all $n \geq 2$

$$a_n = k b_1^{k-1} b_{n-k} + P_n(b_1, \ldots, b_{n-k-1}) \ ,$$

where P_k is a polynomial with integral coefficients. This time all coefficients b_n belong to a finite extension of $\mathbb{K}$ containing a fixed k-th root of a_1, and S' is the union of S and all

places v such that $|kb_1^{k-1}|_v < 1$. The analyticity of the series is handled as in the previous proof. □

Lemma 3.3. — *Pick $k \in \mathbb{Z}$, and suppose $\alpha(t) = \sum_{n\geq k} a_n t^n$ is an adelic series with $a_k \neq 0$ Then there exists an adelic series β such that $\alpha \circ \beta(t) = t^k$.*

Proof. — The equation $\alpha \circ \beta(t) = t^k$ is equivalent to

$$\left(1 + \sum_{j\geq 2} a_j t^{j-1}\right)^k \left(1 + \sum_{l\geq 1} \alpha_l \left(t + \sum_{i\geq 2} a_i t^i\right)^l\right) = 1 \ .$$

Identifying terms of order t^n, one obtains

$$ka_{n+1} + \left[\left(1 + \sum_{2\leq j\leq n} a_j t^{j-1}\right)^k \left(1 + \sum_{l\geq 1} \alpha_l \left(t + \sum_{1\leq i\leq n} a_i t^i\right)^l\right)\right]_n = 0$$

which shows that β is unique, has coefficients in $\mathcal{O}_{\mathbb{L},S'}$ where S' contains S and all places at which $|k|_v < 1$. The fact that β is analytic at all places is a consequence of the inverse function theorem and the fact that the power series $t \mapsto (1+t)^{1/n} := 1 + \frac{1}{n}t + \frac{(1/n)(1/n-1)}{2}t^2 + O(t^3)$ has a positive radius of convergence. □

We shall also deal with *adelic series at infinity* which we define to be series of the form $\alpha(z) = \sum_{k\leq N} b_k z^k + \sum_{k\geq 1} \frac{a_k}{z^k}$ with $N \in \mathbb{N}$, $b_k, a_k \in \mathcal{O}_{\mathbb{K},S}$ and $\sum_{k\geq 1} a_k t^k$ is an adelic series. Observe that this is equivalent to assume that $\alpha(t^{-1})^{-1}$ is an adelic series.

3.2. Puiseux expansions

We shall need the following facts on the Puiseux parameterizations of a curve defined over $\mathbb{K}$. These are probably well-known but we include a proof for the convenience of the reader.

Proposition 3.4. — *Suppose $P \in \mathbb{K}[x,y]$ is a polynomial such that $P(0,0) = 0$ and $P(0,y)$ is not identically zero. Denote by $\mathsf{n} : \hat{D} \to D := \{P = 0\}$ the normalization map, and pick any branch $\mathsf{c} \in \mathsf{n}^{-1}(0)$ of D at the origin.*

Then one can find a finite extension $\mathbb{L}$ of $\mathbb{K}$, a finite set of places S of $\mathbb{L}$, a positive integer $n > 0$, and an adelic series $\beta(t) \in \mathcal{O}_{\mathbb{L},S}[[t]]$ such that

1. *there is an isomorphism of complete local rings $\widehat{\mathcal{O}_{\hat{D},\mathsf{c}}} \simeq \mathbb{L}[[t]]$;*
2. *the formal map $t \mapsto (t^n, \beta(t))$ parameterizes the branch c in the sense that $x(\mathsf{n}(t)) = t^n$, and $y(\mathsf{n}(t)) = \beta(t)$.*

Proof. — We first reduce the situation to the case D is smooth at 0. To do so we blow-up the origin $X_1 \to \mathbb{A}^2$ and let D_1 be the strict transform of D. Since $\hat{D}$ is normal the map n lifts to a map $\mathsf{n}_1 : \hat{D} \to D_1$, and we let p_1 be the image of c in D_1.

In the coordinates $(x,y) = (x', x'y')$ (or $(x'y', y')$) the point p_1 has coordinates $(0, y_1)$ where y_1 is the solution of a polynomial with values in $\mathbb{K}$ hence belongs to an algebraic extension of this field. We may thus choose charts $(x,y) = (x', (x'+c)y')$ (or $(x'(y'+c), x')$) with $c \in \bar{\mathbb{K}}$ such that c is now a branch of $D_1 = \{P_1 = 0\}$ at the origin, and $P_1 \in \bar{\mathbb{K}}[x', y']$.

We iterate this process of blowing-up to build a sequence of proper birational morphisms between smooth surfaces $X_{i+1} \to X_i$, $i = 1, \ldots, N$ until we arrive at the following situation

for $X := X_N$: the strict transform C of D by $\pi : X \to \mathbb{A}^2$ is smooth at a point $p \in \pi^{-1}(0)$ and intersects transversally the exceptional locus of π.

The normalization map $\mathsf{n} : \hat{D} \to D$ lifts to a map $\mathsf{m} : \hat{D} \to C$ and the image of c by m is equal to p. Finally there exist coordinates z, w centered at p such that $(x, y) = \pi(z, w) = (A(z, w), B(z, w))$ with $A, B \in \bar{\mathbb{K}}[z, w]$, the exceptional locus of π contains $\{z = 0\}$, and $C = \{R(z, w) := w - za(z) - wQ(z, w) = 0\}$ where $a \in \bar{\mathbb{K}}[z]$, $Q \in \bar{\mathbb{K}}[z, w]$ and $Q(0, 0) = 0$.

Fix an algebraic extension $\mathbb{L}$ of $\mathbb{K}$ and S finitely many places of $\mathbb{L}$ such that A, B, R have their coefficients in $\mathcal{O}_{\mathbb{L},S}$.

We now look for a power series $\gamma(t) = \sum_{k\geq 1} \gamma_k t^k$ such that $R(t, \gamma(t)) = 0$. Its coefficients satisfy the relations

$$\gamma_k = [t^2 a(t)]_k + \left[\left(\sum_{j=1}^{k-1} \gamma_j t^j \right) Q \left(t, \sum_{j=1}^{k-1} \gamma_j t^j \right) \right]_k$$

which implies that γ exists, is unique, and all its coefficients belongs to $\mathcal{O}_{\mathbb{L},S}$. It follows from the analytic implicit function theorem, that γ is also analytic as a power series in $\mathbb{L}_v[[t]]$ for any place v.

Let us now consider the two power series $(\alpha(t), \delta(t)) := \pi(t, \gamma(t))$. They both belong to $\mathcal{O}_{\mathbb{L},S}$, are analytic at any place, and we have $P(\alpha(t), \delta(t)) = 0$. Since $P(0, y)$ is not identically zero, we may write $\alpha(t) = t^n(a + \sum_{k\geq 1} \alpha_k t^k)$ for some $n > 0$ and $a \neq 0$. Replacing $\mathbb{L}$ by a suitable finite extension, and t by $a't$ for a suitable a' we may suppose that $a = 1$ and $\alpha_k \in \mathcal{O}_{\mathbb{L},S}$ for all k.

By Lemma 3.3, there exists an invertible power series $\hat{a}(t) = t + \sum_{k\geq 2} a_k t^k$ that is analytic at all places with coefficients $a_k \in \mathcal{O}_{\mathbb{L},S}$ and such that $\alpha \circ \hat{a}(t) = t^n$. Once this claim is proved one sets $\beta(t) := \delta \circ \hat{a}(t)$, so that $\pi(\hat{a}(t), \gamma(\hat{a}(t))) = (t^n, \beta(t))$.

Since m is injective and maps the smooth point $\mathsf{c} \in \hat{D}$ to the smooth point $p \in C$, it induces an isomorphism of complete local rings $\widehat{\mathcal{O}_{C,p}} \simeq \widehat{\mathcal{O}_{\hat{D},\mathsf{c}}}$. Observe that the complete local ring $\widehat{\mathcal{O}_{C,p}} = \mathbb{L}[[z, w]]/\langle R \rangle$ is isomorphic to $\mathbb{L}[[t]]$ by sending the class of a formal series Φ to $\Phi(t, \gamma(t)))$. Composing with the isomorphism of $\mathbb{L}[[t]]$ sending t to $\hat{a}(t)$, we get an isomorphism $\widehat{\mathcal{O}_{\hat{D},\mathsf{c}}} \simeq \mathbb{L}[[t]]$ such that $(x(\mathsf{n}(t)), y(\mathsf{n}(t))) = \pi(\mathsf{n}(t)) = \pi(\hat{a}(t), \gamma(\hat{a}(t))) = (t^n, \beta(t))$ as required. □

3.3. Branches at infinity of a curve in Poly_3

Consider an irreducible affine curve $C \subset \mathrm{Poly}_3$ defined over a number field $\mathbb{K}$. We denote by $\overline{\mathrm{Poly}_3} \simeq \mathbb{P}^2$ the natural completion of $\mathrm{Poly}_3 \simeq \mathbb{A}^2$ using the affine coordinates (c, a). Let $\bar{C}$ be the Zariski closure of the curve C in $\overline{\mathrm{Poly}_3}$, and $\mathsf{n} : \hat{C} \to \bar{C}$ be its normalization. A *branch at infinity* of C is a point in $\hat{C}$ lying over $\bar{C} \setminus C$.

Proposition 3.5. — *There exists a finite extension $\mathbb{L}$ of $\mathbb{K}$ and a finite set of places S such that the following holds.*

For any branch c of C at infinity there is an isomorphism of complete local rings $\widehat{\mathcal{O}_{\hat{C},\mathsf{c}}} \simeq \mathbb{L}[[t]]$ such that $c(\mathsf{n}(t)), a(\mathsf{n}(t))$ are adelic series at infinity.

Proof. — Pick a branch at infinity c of C. Let p_* be the image of c in $\overline{\mathrm{Poly}_3} \simeq \mathbb{P}^2$. It is given in homogeneous coordinates by $p_* = [c_* : a_* : 0]$ and since C is defined over $\mathbb{K}$ we may assume c_*, a_* are algebraic over $\mathbb{K}$. To simplify the discussion we shall assume that $c_* = 1$ so that $p_* = [1 : a_* : 0]$ (otherwise $p_* = [0 : 1 : 0]$ and the arguments are completely

analoguous). Let d be the degree of a defining equation $P \in \mathbb{K}[c,a]$ of C. Observe that $Q(\tau,\alpha) := \tau^d P(\frac{1}{\tau}, \frac{\alpha}{\tau} - a_*)$ is a polynomial vanishing at $(0,0)$ such that $Q(0,\alpha)$ is not identically zero. Note that $\{Q=0\}$ can be identified to an open Zariski subset of the completion of $\{P=0\}$ in $\overline{\mathrm{Poly}_3}$, and $\mathfrak{c}$ with a branch of $\{Q=0\}$ at the origin.

Apply Proposition 3.4 to this branch $\mathfrak{c}$. We get a finite extension $\mathbb{L}$, a finite set of places S of $\mathbb{L}$ containing all archimedean ones, a positive integer n, an isomorphism of complete local ring $\mathcal{O}_{\hat{C},\mathfrak{c}} \simeq \mathbb{L}[[t]]$, and a power series $\beta \in \mathcal{O}_{\mathbb{L},S}[[t]]$ that is analytic at all places such that $\alpha(\mathsf{n}(t)) = \beta(t)$ and $\tau(\mathsf{n}(t)) = t^n$. It follows that $c(\mathsf{n}(t)) = t^{-n}$, and $a(\mathsf{n}(t)) = t^{-n}\beta(t) - a_* \in \mathcal{O}_{\mathbb{L},S}[[t]]$. □

3.4. Estimates for the Green functions on a curve in Poly_3

In this section, we fix an irreducible curve C in Poly_3 and let $\mathbb{L}$ be a number field for which the previous proposition applies. Fix a place v of $\mathbb{L}$, and let $g_{0,v}(c,a)$ be the function g_0 evaluated at c,a in the completion $\mathbb{L}_v$ of $\mathbb{L}$ with respect to the v-adic norm.

By [**DF**] and [**FG**, Proposition 2.4], the function g_0 is the uniform limit on compact sets of $\frac{1}{3^n}\log^+|P^n_{c,a}(c_0)|$. It follows that its lift to the normalization of C is sub-harmonic (in the classical sense when v is Archimedean and in the sense of Thuillier [**T**] when v is non-Archimedean).

To simplify notations, we also write $g_{0,v}(t) := g_{0,v}(c(\mathsf{n}(t)), a(\mathsf{n}(t)))$ where the adelic series at infinity $c(\mathsf{n}(t))$ and $a(\mathsf{n}(t))$ are given as above.

Proposition 3.6. — *For each branch $\mathfrak{c}$ of C at infinity, there exists a finite set of place S of $\mathbb{L}$ such that one of the following two situations occur.*

1. *For any place v of $\mathbb{L}$, the function $g_{0,v}(t)$ extends as a locally bounded subharmonic function through $\mathfrak{c}$.*
2. *There exist two constants $a(\mathfrak{c}) \in \mathbb{Q}^*_+$ and $b(\mathfrak{c}) \in \mathcal{O}_{\mathbb{L},S}$ such that $g_{0,v}(t) = a(\mathfrak{c})\log|t|_v^{-1} + \log|b(\mathfrak{c})|_v + o(1)$ for any place v on $\mathbb{L}$.*

Remark. — This key result is very similar to [**BDM**, Proposition 2.1]. Ghioca and He have proved that $g_{0,v}(t)$ actually extends to a *continuous* function at $t=0$ in case 1. We also refer to [**De2**, Proposition 3.1] for a generalization of this result to the case of rational maps.

Notation. — We endow the field $\mathbb{L}((t))$ with the t-adic norm so that for any Laurent series $Q = \sum a_k t^k$ we have $|Q|_t := e^{-\mathrm{ord}_t(Q)}$ with $\mathrm{ord}_t(Q) = \min\{k,\, a_k \neq 0\}$. The resulting valued field is complete.

In order to avoid confusion, we denote by $\mathsf{P}(z) \in \mathbb{L}((t))[z]$ the cubic polynomial induced by the family $(P_{c(\mathsf{n}(t)),a(\mathsf{n}(t))})_t$. Observe that the critical points of P are given by c_0 and c_1 which correspond to the adelic series at infinity 0 and $c(\mathsf{n}(t))$ respectively.

Proof of Proposition 3.6. — For each $q \in \mathbb{N}^*$, we set $e_q := |\mathsf{P}^q(\mathsf{c}_0)|_t$, so that either e_q is bounded (i.e. c_0 belongs to the filled-in Julia set of P) or $e_q \to \infty$ (exponentially fast).

Suppose we are in the former case, and consider the sequence of subharmonic functions $\frac{1}{3^q}\log^+|P^q_t(0)|_v$ defined on a punctured disk $\mathbb{D}^*_v$ centered at 0 in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}_v}$. Since $\frac{1}{3^q}\log^+|P^q_t(0)|_v = \frac{\log^+ e_q}{3^q}\log|t|_v^{-1} + O(1)$, the function

$$h_q := \frac{1}{3^q}\log^+|P^q_t(0)|_v - \frac{\log^+ e_q}{3^q}\log|t|_v^{-1}$$

extends as a subharmonic function to $\mathbb{D}_v$. Since $\frac{1}{3^q}\log^+|P_t^q(0)|_v$ converges uniformly on compact subsets in $\mathbb{D}_v^*$ to $g_{0,v}$, h_q is uniformly bounded from above on its boundary, hence everywhere by the maximum principle. It follows from Hartog's theorem that h_q converges (in L^1_{loc} in the Archimedean case, and pointwise at any non-rigid point in the non-Archimedean case) to a subharmonic function, hence $g_{0,v}$ is subharmonic on $\mathbb{D}_v$. But $g_{0,v}$ is non-negative so that (1) holds.

Suppose that $\log e_q \to \infty$. Recall that $c(\mathsf{n}(t))$ and $a(\mathsf{n}(t))$ are adelic series at infinity that belong to $t^{-n}\mathcal{O}_{\mathbb{L},S}[[t]]$ for a suitable integer $n \geq 1$. Write $\varphi_t := \varphi_{P_{c(\mathsf{n}(t)),a(\mathsf{n}(t))}}$.

Lemma 3.7. — *There exists an integer $q \geq 1$ such that for any place v of $\mathbb{L}$, there exists $\epsilon > 0$ such that $P_t^q(c_0)$ belongs to the domain of convergence of φ_t for any $|t|_v < \epsilon$.*

Proof. — Indeed $P_t^q(c_0)$ is an adelic series at infinity having a pole of order $\log e_q$. On the other hand, we have

$$\begin{aligned} G(t) := G(c(\mathsf{n}(t)), a(\mathsf{n}(t))) &\leq \log\max\{|c(\mathsf{n}(t))|, |a(\mathsf{n}(t))|\} + C \\ &\leq n\log|t|^{-1} + O(1) \end{aligned}$$

by Proposition 2.1. By assumption we may take $\log e_q$ to be as large as we want so that $\log|P_t^q(c_0)|_v - G(t) \to \infty$ for any fixed place v when $|t|_v \to 0$. We conclude by Proposition 2.3. □

Our objective is to estimate $\varphi_t(P_t^q(c_0))$. Recall from Lemma 2.2 that

$$\varphi_{c,a}(z) = \lambda\left(z - \frac{c}{2}\right) + \sum_{k\geq 1} a_k(c,a)z^{-k},$$

with $a_k \in \mathbb{Z}\left[\lambda, \frac{1}{2}\right][c,a]$ of degree $\leq k+1$. It follows that

$$\mathsf{a}_k := a_k(c(\mathsf{n}(t)), a(\mathsf{n}(t))) \in t^{-n(k+1)}\,\mathcal{O}_{\mathbb{L},S}[[t]]\ ,$$

so that one can define

$$\varphi_{\mathsf{P}}(z) := \varphi_{c(\mathsf{n}(t)),a(\mathsf{n}(t))}(z) = \lambda\left(z - \frac{c(\mathsf{n}(t))}{2}\right) + \sum_{k\geq 1}\mathsf{a}_k z^{-k}$$

as an element of the ring $t^{-n}z\,\mathcal{O}_{\mathbb{L},S}((t))[[(t^n z)^{-1}]]$.

On the other hand, $P_{c,a}^q(c_0)$ is a polynomial in c, a of degree $\leq 3^q$ with coefficients in $\mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$ hence, if

$$\mathsf{c}_0 := c_0(\mathsf{n}(t)) \text{ and } \mathsf{P}^q(z) := P^q_{c(\mathsf{n}(t)),a(\mathsf{n}(t))}(z)\ ,$$

we have $\mathsf{P}^q(\mathsf{c}_0) \in t^{-3^q n}\,\mathcal{O}_{\mathbb{L},S}[[t]]$, so that

$$\frac{\mathsf{a}_k}{(\mathsf{P}^q(\mathsf{c}_0))^k} \in t^{3^q nk - n(k+1)}\,\mathcal{O}_{\mathbb{L},S}[[t]] \subset t^{nk}\,\mathcal{O}_{\mathbb{L},S}[[t]]\ . \tag{7}$$

It follows that $\Theta := \sum_{k\geq 1}\frac{\mathsf{a}_k}{(\mathsf{P}^q(\mathsf{c}_0))^k}$ converges as a formal power series and belongs to $t^n\,\mathcal{O}_{\mathbb{L},S}[[t]]$. Observe that Lemma 3.7 shows that Θ is convergent at all places hence defines an adelic power series.

Fix a place v of $\mathbb{L}$ and choose $|t|_v$ small enough. Then we get

$$\begin{aligned} \varphi_t(P_t^q(c_0)) &= \lambda\left(P_t^q(0) - \frac{c(\mathsf{n}(t))}{2}\right) + \Theta(t) \\ &= \lambda\left(P_t^q(0) - \frac{c(\mathsf{n}(t))}{2}\right) + o(1)\ . \end{aligned} \tag{8}$$

By (8), for $|t|_v$ small enough, one obtains

$$\begin{aligned} g_{0,v}(t) &= \frac{1}{3^q}\log|\varphi_t(P_t^q(0))|_v \\ &= \frac{1}{3^q}\log\left|\lambda\left(P_t^q(0)-\frac{1}{2t^n}\right)\right|_v + o(1) \\ &= \frac{1}{3^q}\log\left|\sum_{0\le k\le n_0}\frac{b_{k,0}}{t^k}\right|_v + o(1) = \\ &= \frac{n_0}{3^q}\log|t|_v^{-1} + \log\left|\sum_{0\le k\le n_0} b_{k,0}t^{n_0-k}\right|_v + o(1) \end{aligned}$$

where $b_{k,0}\in\mathcal{O}_{\mathbb{L},S}$, and $b_{n_0,0}\neq 0$. And the proof is complete with $a(\mathfrak{c}) := \frac{n_0}{3^q}$, and $b(\mathfrak{c}) = b_{n_0,0}$. □

Proposition 3.8. — *Fix any two positive integers $s := (s_0, s_1)$, and for any place v define*

$$g_{s,v}(c,a) := \max\{s_0 g_{0,v}(c,a), s_1 g_{1,v}(c,a)\}\ . \tag{9}$$

Then there exists an integer $q\geq 1$ such that

$$g_{s,v}(c,a) = \frac{1}{3^q}\ \max\left\{s_0\log^+|P_{c,a}^q(c_0)|, s_1\log^+|P_{c,a}^q(c_1)|\right\} \tag{10}$$

for all but finitely many places.

Proof. — During the proof S is a finite set of places on $\mathbb{L}$ that contains all Archimedean places and all places of residual characteristic 2 and 3. Pick any $v\notin S$, and recall from [**FG**, Proposition 2.5] that $G_v(c,a) = \log^+\max\{|c|_v, |a|_v\}$.

Suppose first that $g_{s,v}(c,a) = 0$. Then $g_{c,a,v}(c_0) = g_{c,a,v}(c_1) = 0$ and $G_v(c,a) = 0$ so that $\frac{1}{3^q}\log^+|P_{c,a}^q(c_0)|_v = \frac{1}{3^q}\log^+|P_{c,a}^q(c_1)|_v = 0$ for all q, and (9) holds in that case.

Pick q large enough such that $3^q > \max\{\frac{s_1}{s_0}, \frac{s_0}{s_1}\}$. Suppose now that $0 < g_{s,v}(c,a) = s_0 g_{0,v}(c,a)$ so that $s_0 g_{0,v}(c,a)\geq s_1 g_{1,v}(c,a)$. Then

$$g_{c,a,v}(P_{c,a}^q(c_0)) = 3^q g_{0,v}(c,a) \geq$$
$$\geq 3^q\min\left\{\frac{s_1}{s_0}, 1\right\}\max\{g_{0,v}(c,a), g_{1,v}(c,a)\} > G_v(c,a)\ .$$

By (6), we get

$$g_{s,v}(c,a) = s_0 g_{0,v}(c,a) = \frac{s_0}{3^q}g_{0,v}(P_{c,a}^q(c_0)) = \frac{s_0}{3^q}\log^+|P_{c,a}^q(c_0)|_v\ .$$

Now observe that either $P_{c,a}^q(c_1)$ falls into the domain of definition of $\varphi_{c,a}$ i.e. $\log|P_{c,a}^q(c_1)|_v > G_v(c,a)$ and $g_{1,v}(c,a) = \frac{1}{3^q}\log^+|P_{c,a}^q(c_1)|_v$, so that

$$\begin{aligned} g_{s,v}(c,a) &= \max\{s_0 g_{0,v}(c,a), s_1 g_{1,v}(c,a)\} \\ &= \frac{1}{3^q}\ \max\left\{s_0\log^+|P_{c,a}^q(c_0)|_v, s_1\log^+|P_{c,a}^q(c_1)|_v\right\}\ , \end{aligned}$$

as required. Or we have

$$\frac{s_1}{3^q}\log^+|P_{c,a}^q(c_1)|_v \leq \frac{s_1}{3^q}\log^+\max\{|a|_v, |c|_v\} \leq s_0 g_{0,v}(c,a)\ ,$$

and again (9) holds.

We complete the proof by arguing in the same way when $g_{s,v}(c,a) = s_1 g_{1,v}(c,a)$. □

3.5. Adelic semi-positive metrics on curves in $\mathbf{Poly}_3$

We fix a number field $\mathbb{L}$ and finitely many places S of this field that contains all Archimedean places and all places of residual characteristic 2 and 3. We also assume that Propositions 3.5, 3.6 and 3.8 are all valid for these choices.

Fix any pair of positive integers $s_0, s_1 \in \mathbb{N}^*$. For each place v, introduce the function

$$g_{s,v}(c,a) := \max\{s_0 \cdot g_{0,v}(c,a), s_1 \cdot g_{1,v}(c,a)\} \ ,$$

as in the previous section.

Pick a branch at infinity $\mathfrak{c}$ and choose parameterizations such that Proposition 3.6 is valid for $g_{0,v}$ and $g_{1,v}$. Observe that

$$G_v(t) = \max\{g_{0,v}(t), g_{1,v}(t)\} \to \infty$$

as $t \to 0$ by Proposition 2.1 so that either $g_{0,v}$ or $g_{1,v}$ tends to infinity near $t = 0$. Since n_0 and n_1 are both positive, we conclude to the existence of $a(\mathfrak{c}) \in \mathbb{Q}_+^*$ and $b(\mathfrak{c}) \in \mathcal{O}_{\mathbb{L},S}$ such that

$$g_{s,v}(t) = a(\mathfrak{c}) \log |t|_v^{-1} + \log |b(\mathfrak{c})|_v + o(1) \ . \tag{11}$$

We replace the integers s_0, s_1 by suitable multiples such that the constants $a(\mathfrak{c})$ become integral (for all branch $\mathfrak{c}$), and we introduce the divisor $\mathsf{D} := \sum a(\mathfrak{c})\,[\mathfrak{c}]$ on $\hat{C}$ where the sum is taken over all branches at infinity of C.

Pick a place v, an open subset $U \subset \hat{C}^{\mathrm{an},v}$ and a section σ of the line bundle $\mathcal{O}_{\hat{C}}(\mathsf{D})$ over U. By definition σ is a meromorphic function on U whose divisor of poles and zeroes satisfy $\mathrm{div}(\sigma) + \mathsf{D} \geq 0$. We set $|\sigma|_{s,v} := |\sigma|_v e^{-g_{s,v}}$.

By (11), $|\cdot|_{s,v}$ defines a continuous metrization on the line bundle $\mathcal{O}_{\hat{C}}(\mathsf{D})$ at any place. Since $g_{s,v}$ is subharmonic on $C^{v,\mathrm{an}}$, the metrization is semi-positive in the sense of Zhang, see [**CL**]. Finally this metrization is adelic thanks to Proposition 3.8 and [**FG**, §2.3].

Observe that the curvature form of this metrization (see [**CL**]) is given by $\Delta(g_{s,v} \circ \mathsf{n})$. We have thus obtained

Theorem 3.9. — *Pick any positive integers $s_0, s_1 > 0$. Then there exists a positive integer $t \geq 1$, and a non-zero effective and integral divisor D on $\hat{C}$ such that the collection of subharmonic functions*

$$g_{s,v}(c,a) := \max\{ts_0 \cdot g_{0,v}(c,a), ts_1 \cdot g_{1,v}(c,a)\} \ , \ (c,a) \in C^{v,\mathrm{an}}$$

induces a semi-positive adelic metrization on the line bundle $\mathcal{O}_{\hat{C}}(\mathsf{D})$.

Remark. — The line bundle $\mathcal{O}_{\hat{C}}(\mathsf{D})$ is defined over the same number field as C.

Remark. — It is likely that $g_{s,v}$ defines a semi-positive adelic metrization on an ample line bundle over a suitable compactification of Poly_3, but this seems quite delicate to prove.

4. Green functions on special curves

This section is devoted to the proof of the following result. If $\mathbb{K}$ is a number field, and v a place of $\mathbb{K}$, recall the definition of $\tau_v = \tau(\mathbb{K}_v)$ from Proposition 2.3, and that $\tau_v = 0$ if the residual characteristic of $\mathbb{K}$ is larger than 5.

Theorem 4.1. — *Let C be an irreducible curve in the space Poly_3 of complex cubic polynomials parameterized as in (1). Suppose that C contains infinitely many post-critically finite parameters and that neither c_0 nor c_1 is persistently pre-periodic. Then the following holds.*

1. *The curve C is defined over a number field $\mathbb{K}$ and there exist positive integers s_0, s_1 such that for any place v of $\mathbb{K}$*
$$s_0\, g_{0,v}(c,a) = s_1\, g_{1,v}(c,a) \text{ for all } (c,a) \in C^{v,\mathrm{an}}\ .$$
2. *For any branch $\mathfrak{c}$ of C at infinity, there exists an integer $q \geq 1$ and a root of unity ζ such that for any place v of $\mathbb{K}$, one has*
$$\big(\varphi_{c,a}(P^q_{c,a}(c_0))\big)^{s_0} = \zeta \cdot \big(\varphi_{c,a}(P^q_{c,a}(c_1))\big)^{s_1} \tag{12}$$
on the connected component of $\{g_{0,v} > \tau_v/s_0\} = \{g_{1,v} > \tau_v/s_1\}$ in $C^{v,\mathrm{an}}$ clustering at $\mathfrak{c}$.

A remark is in order about the second assertion of the theorem.

Remark. — We shall prove that for any parameter on the connected component $\{g_{0,v} > \tau_v/s_0\} = \{g_{1,v} > \tau_v/s_1\}$ in $C^{v,\mathrm{an}}$ clustering at $\mathfrak{c}$, the two points $P^q_{c,a}(c_0)$ and $P^q_{c,a}(c_1)$ belongs to the domain of definition of the Böttcher coordinate $\varphi_{c,a}$ for q large so that (12) is consistent. We also insist on the fact that (12) holds as an equality of adelic series at infinity.

4.1. Green functions are proportional

Recall first that the set of post-critically finite polynomials is a countable union of varieties
$$V_{n,m} := \{P^{n_0+m_0}_{c,a}(c_0) = P^{n_0}_{c,a}(c_0)\} \cap \{P^{n_1+m_1}_{c,a}(c_1) = P^{n_1}_{c,a}(c_1)\}$$
with $n_0, n_1 \in \mathbb{N}$ and $m_0, m_1 \in \mathbb{N}^*$, and each of which is cut out by two polynomial equations with coefficients in $\mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$. Since $V_{n,m}(\mathbb{C})$ are all contained in a fixed compact set by e.g. [**DF**, Proposition 6.2], it is a finite set, hence all its solutions are defined over a number field.

It follows that C is an irreducible curve containing infinitely many algebraic points (c_n, a_n). Let $Q \in \mathbb{C}[c,a]$ be a defining equation for C with at least one coefficient equal to 1 and pick σ an element of the Galois group of $\mathbb{C}$ over the algebraic closure of $\mathbb{Q}$. Then $Q \circ \sigma$ vanishes also on $\{(c_n, a_n)\}$ hence everywhere on C, and therefore $Q \circ \sigma = \lambda Q$ for some $\lambda \in \mathbb{C}^*$. Since one coefficient of Q is 1, we get $\lambda = 1$ and $Q \in \mathbb{K}[c,a]$ for a number field $\mathbb{K}$.

Recall that we denote by $\mathsf{n} : \hat{C} \to \bar{C}$ the normalization of the completion $\bar{C}$ of C in $\overline{\mathrm{Poly}_3}$. Pick any pair of positive integers $s = (s_0, s_1)$ and scale them such that apply Theorem 3.9. This gives us a non-zero effective divisor D_s supported on $\hat{C} \setminus \mathsf{n}^{-1}(C)$. Replacing s by a suitable multiple, we may suppose that it is very ample and pick a rational function ϕ on $\hat{C}$ whose whose divisor of poles and zeroes is greater or equal to $-\mathsf{D}_s$. Observe in particular that ϕ is a regular function on $\mathsf{n}^{-1}(C)$ that vanishes at finitely many points.

Consider the height h_s induced by the semi-positive adelic metrics given by $g_{s,v}$, see Theorem 3.9. If (c,a) is a point in $\mathsf{n}^{-1}(C)$ that is defined over a finite extension $\mathbb{K}$, denote by $\mathsf{O}(c,a)$ its orbit under the action of the absolute Galois group of $\mathbb{K}$, and by $\deg(c,a)$ the cardinality of this orbit. Fix a rational function ϕ as above that is not vanishing at c,a (this exists since $-\mathsf{D}_s$ is very ample). Let $M_{\mathbb{K}}$ be the set of places of $\mathbb{K}$. By [**CL**, §3.1.3], since $\phi(c,a)\neq 0$ we have

$$\begin{aligned} h_s(c,a) &= \frac{1}{\deg(c,a)}\sum_{\mathsf{O}(c,a)}\sum_{v\in M_{\mathbb{K}}} -\log|\phi|_v(c',a') \\ &= \frac{1}{\deg(c,a)}\sum_{\mathsf{O}(c,a)}\sum_{v\in M_{\mathbb{K}}} (g_{s,v}-\log|\phi|_v)(c',a') \\ &= \frac{1}{\deg(c,a)}\sum_{\mathsf{O}(c,a)}\sum_{v\in M_{\mathbb{K}}} g_{s,v}(c',a') \geq 0 \end{aligned}$$

where the last equality follows from the product formula.

We now estimate the total height of the curve $\hat{C}$ using freely [**CL**]. Choose any two meromorphic functions ϕ_0,ϕ_1 such that $\mathrm{div}(\phi_0)+\mathsf{D}_s$ and $\mathrm{div}(\phi_1)+\mathsf{D}_s$ are both effective with disjoint support included in $\mathsf{n}^{-1}(C)$. Let σ_0 and σ_1 be the associated sections of $\mathcal{O}_{\hat{C}}(\mathsf{D}_s)$. Let $\sum n_i[c_i,a_i]$ be the divisor of zeroes of σ_0, and $\sum n'_j[c'_j,a'_j]$ be the divisor of zeroes of σ_1. Then

$$\begin{aligned} h_s(\hat{C}) &= \sum_{v\in M_{\mathbb{K}}} (\widehat{\mathrm{div}}(\sigma_0)\cdot\widehat{\mathrm{div}}(\sigma_1)|\hat{C})_v \\ &= \sum_i n_i h_s(c_i,a_i) - \sum_{v\in M_{\mathbb{K}}} \log|\sigma_0|_{s,v}\,\Delta g_{s,v} \\ &= \sum_i n_i h_s(c_i,a_i) - \sum_j n'_j h_s(c'_j,a'_j) + \sum_{v\in M_{\mathbb{K}}} g_{s,v}\,\Delta g_{s,v} \\ &\geq \sum_i n_i h_s(c_i,a_i) - \sum_j n'_j h_s(c'_j,a'_j)\ . \end{aligned}$$

Reversing the order of the two sections, we get $h_s(\hat{C})\geq 0$.

The formula for the height of a closed point implies that for all post-critically finite polynomials P_{c_n,a_n} we have $h_s(c_n,a_n)=0$. By the arithmetic Hilbert-Samuel theorem (see [**Z**, Theorem 5.2]), we get $h_s(\hat{C})=0$ hence we may apply Thuillier-Yuan's theorem (see [**T, Y**]). It follows that the sequence of probability measures $\mu_{n,v}$ that are equidistributed on $\mathsf{O}(c_n,a_n)$ in $\hat{C}^{v,\mathrm{an}}$ converges to a probability measure $\mu_{\infty,v}$ that is proportional to $\Delta g_{s,v}$. We may thus write $\mu_{\infty,v}=w(s)\,\Delta g_{s,v}$ where $w(s)\in\mathbb{R}_+^*$ is equal to the inverse of the mass of $\Delta g_{s,v}$, i.e. to $\deg(\mathsf{D}_s)^{-1}$.

We now observe that $g_{s,v}$ is 1-homogeneous in s, and continuous with respect to this parameter. It follows that $w(s)$ is also continuous on $(\mathbb{R}_+^*)^2$, and $\mu_{\infty,v}=w(s)\,\Delta g_{s,v}$ for all $s\in(\mathbb{R}_+^*)^2$.

From now on we fix an Archimedean place v. We shall treat the non-Archimedean case latter. We work in $\mathsf{n}^{-1}(C^{v,\mathrm{an}})$ which is the complement of finitely many points in the analytification of the smooth projective curve $\hat{C}^{v,\mathrm{an}}$. To simplify notation we write $g_{0,v},g_{1,v}$ instead of $g_{0,v}\circ\mathsf{n},g_{1,v}\circ\mathsf{n}$. Recall that by [**DF**, Theorem 2.5] (see also [**McM1**, Theorem 2.2] or [**De2**, Theorem 1.1]) the equality $g_{0,v}=0$ on $\mathsf{n}^{-1}(C^{v,\mathrm{an}})$ implies c_0 to be

persistently pre-periodic. We may thus assume that both c_0 and c_1 are not persistently pre-periodic so that $g_{0,v}$ and $g_{1,v}$ are not identically zero on $\mathsf{n}^{-1}(C^{v,\mathrm{an}})$.

Recall also that $g_{0,v}$ is harmonic where it is positive and that the support of $\Delta g_{0,v}$ is exactly the boundary of $\{g_{0,v}=0\}$ (see e.g. [**DF**, Proposition 6.7]). In particular $\Delta g_{0,v}$ is a non-zero positive measure, and its mass is finite by Proposition 3.6. Observe now that $g_{s,v}\to g_{0,v}$ uniformly on compact sets when s tends to $(1,0)$, hence $\Delta g_{s,v}\to\Delta g_{0,v}$ and $\Delta g_{0,v}=t_0\mu_{\infty,v}$ for some positive t_0. In the same way, we get $\Delta g_{s,v}\to\Delta g_{1,v}$ as $s\to(0,1)$ which implies that the three positive measures $\mu_{\infty,v}$, $\Delta g_{0,v}$ and $\Delta g_{1,v}$ are proportional. We may thus find $s_0,s_1>0$ such that the function $H_v:=s_0g_{0,v}-s_1g_{1,v}$ is harmonic on $\mathsf{n}^{-1}(C^{v,\mathrm{an}})$.

Recall from [**McM2**] that the bifurcation locus of the family $P_{c,a}$ parameterized by $(c,a)\in\mathsf{n}^{-1}(C^{v,\mathrm{an}})$ is defined as the set where either c_0 or c_1 is unstable (or active in the terminology of [**DF**]). It follows from [**DF**] that the bifurcation locus is equal to the union of the support of $\Delta g_{0,v}$ and $\Delta g_{1,v}$, hence to the support of $\mu_{\infty,v}$.

Suppose now that H_v is not identically zero. Then this support is included in the locus $\{H_v=0\}$ which is real-analytic. This is impossible by McMullen's universality theorem, since the Hausdorff dimension of the bifurcation locus of any one-dimensional analytic family is equal to 2, see [**McM3**, Corollary 1.6].

We have proved that $s_0g_{0,v}=s_1g_{1,v}$ on $\mathsf{n}^{-1}(C^{v,\mathrm{an}})$ hence on $C^{v,\mathrm{an}}$ for some positive real numbers $s_0,s_1>0$.

Since $g_{0,v}$ and $g_{1,v}$ are proportional, and $G_v=\max\{g_{0,v},g_{1,v}\}$ is proper on $C^{v,\mathrm{an}}$, it follows that $g_{0,v}$ is unbounded near any branch at infinity. By Proposition 3.6, $g_{0,v}$ admits an expansion of the form $g_{0,v}(t)=a(\mathfrak{c})\log|t|^{-1}+O(1)$ with $a(\mathfrak{c})\in\mathbb{Q}_+^*$ on the branch $\mathfrak{c}$ hence is locally superharmonic on that branch.

It follows that $\Delta g_{0,v}$ is a signed measure in $\hat{C}^{\mathrm{an},v}$ whose negative part is a divisor D_0 with *rational* coefficients supported on $\hat{C}\setminus\mathsf{n}^{-1}(C)$. The same being true for $\Delta g_{1,v}$, we conclude to the equality of divisors $s_0\mathsf{D}_0=s_1\mathsf{D}_1$. This implies that s_0/s_1 is rational, and we can assume s_0 and s_1 to be integers. This ends the proof of the first statement in the case the place is Archimedean.

Assume now that v is non-Archimedean. One cannot copy the proof we gave in the Archimedean setting since we used arguments that specifically relied on the Archimedean assumption at two places.

Instead we apply Proposition 3.6. For each $s'=(s'_0,s'_1)$ the function $g_{s',v}$ extends near any branch $\mathfrak{c}$ at infinity as an upper-semicontinuous function $\widehat{g_{s',v}}$ whose Laplacian puts some non-positive mass at $\mathfrak{c}$. When $s'_0,s'_1\neq 0$ then $g_{s',v}$ defines a positive continuous metric on $\mathcal{O}_{\hat{C}}(\mathsf{D}_s)$ hence $\Delta\widehat{g_{s',v}}\{\mathfrak{c}\}=-\operatorname{ord}_{\mathfrak{c}}(\mathsf{D}_s)<0$. This mass is in particular independent on the place. We get that

$$-\Delta\widehat{g_{0,v}}\{\mathfrak{c}\}\geq\lim_{s\to(1,0)}-\Delta\widehat{g_{s,v}}\{\mathfrak{c}\}=\operatorname{ord}_{\mathfrak{c}}(\mathsf{D}_0)>0\ .$$

We infer that the mass of $\Delta g_{0,v}$ is equal to the degree of D_0 hence is non-zero.

We may now argue as in the Archimedean case, and prove that $\Delta g_{0,v}$ and $\Delta g_{1,v}$ are proportional. The coefficient of proportionality is the only $t>0$ such that $\mathsf{D}_0=t\mathsf{D}_1$ hence $t=s_0/s_1$. Then $H_v:=s_0g_{0,v}-s_1g_{1,v}$ is harmonic on C and bounded near any branch at infinity by Proposition 3.6, hence defines a harmonic function on the compact curve $\hat{C}^{\mathrm{an},v}$. It follows H_v is a constant (in the non-Archimedean case by [**T**, Proposition 2.3.2]) which is necessarily zero since it is zero at all post-critically finite parameters.

We mention here the following result that follows from the previous argument.

Corollary 4.2. — *Let C be an irreducible curve in* Poly_3 *defined over a number field* $\mathbb{K}$ *that contains infinitely many post-critically finite parameters and that neither c_0 nor c_1 is persistently pre-periodic. Pick an Archimedean place v.*

Pick any sequence $X_n \subset C(\bar{\mathbb{K}})$ of Galois-invariant finite sets of postcritically finite parameters such that $X_n \neq X_m$ for $m \neq n$. Let μ_n be the measure equidistributed on $X_n \subset C^{v,\mathrm{an}}$.

Then the sequence μ_n converges weakly to (a multiple of) $T_{\mathrm{bif}} \wedge [C]$ as $n \to \infty$.

Recall that T_{bif} is defined as the dd^c of the plurisubharmonic function $g_0 + g_1$, and $[C]$ is the current of integration over the analytic curve $C^{v,\mathrm{an}}$.

Proof. — Let $s_0, s_1 > 0$ be given by Theorem 4.1. As seen above, the sequence μ_n converges weakly towards $dd^c \max\{s_0 \cdot g_0, s_1 \cdot g_1\} = s_0 \cdot dd^c g_0$ on $C^{v,\mathrm{an}}$. It thus only remains to prove that $dd^c(g_0|_{C^{v,\mathrm{an}}}) = \kappa \cdot T_{\mathrm{bif}} \wedge [C]$ for some $\kappa > 0$. Recall that $T_{\mathrm{bif}} = dd^c(g_0 + g_1)$. By Theorem 4.1, on C,

$$g_0 + g_1 = g_0 + \frac{s_1}{s_0} \cdot g_0 = \left(1 + \frac{s_0}{s_1}\right) g_0.$$

Let $\kappa := 1 + \frac{s_0}{s_1}$. We thus have $dd^c(g_0|_{C^{v,\mathrm{an}}}) = \kappa^{-1} \cdot dd^c((g_0 + g_1)|_{C^{v,\mathrm{an}}})$. Finally, since $g_0 + g_1$ is continuous, we have $T_{\mathrm{bif}} \wedge [C] = dd^c((g_0 + g_1)|_{C^{v,\mathrm{an}}})$, which ends the proof. □

4.2. Values of the Böttcher coordinates at critical points are proportional near infinity

Let us fix a branch at infinity $\mathfrak{c}$ of an irreducible curve C containing infinitely many PCF polynomials, and an isomorphism of complete local rings $\mathcal{O}_{\hat{C},\mathfrak{c}} \simeq \mathbb{L}[[t]]$, such that $c(\mathsf{n}(t)) = t^{-n}$, and $a(\mathsf{n}(t)) \in \mathcal{O}_{\mathbb{L},S}((t))$ is an adelic series. Write $P_t = P_{c(\mathsf{n}(t)),a(\mathsf{n}(t))}$, and $\varphi_t = \varphi_{P_t}$.

By Lemma 3.7 there exists an integer $q \geq 1$ large enough such that $P_t^q(c_0)$ and $P_t^q(c_1)$ both lie in the domain of convergence of the Böttcher coordinate φ_t for t small enough, and (8) holds, i.e.

$$\varphi_t(P_t^q(c_\varepsilon)) = \lambda \left(P_t^q(c_\varepsilon) - \frac{c(\mathsf{n}(t))}{2}\right) + \Theta(t)$$

where Θ is an adelic series vanishing at 0.

We now fix a place v and compute using Proposition 2.3 for $|t|_v \ll 1$. We get

$$\begin{aligned} \frac{|\varphi_t\left(P_0^q(t)\right)|_v^{s_0}}{|\varphi_t\left(P_1^q(t)\right)|_v^{s_1}} &= \frac{\exp\left(s_0 \cdot g_{c(\mathsf{n}(t)),a(\mathsf{n}(t))}(P_0^q(t))\right)}{\exp\left(s_1 \cdot g_{c(\mathsf{n}(t)),a(\mathsf{n}(t))}(P_1^q(t))\right)} \\ &= \frac{\exp\left(3^q s_0 \cdot g_{0,v}(c(\mathsf{n}(t)), a(\mathsf{n}(t)))\right)}{\exp\left(3^q s_1 \cdot g_{1,v}(c(\mathsf{n}(t)), a(\mathsf{n}(t))\right)} = 1\ . \qquad (\star) \end{aligned}$$

Applying $(\star)$ in the case of an Archimedean place, we see that the complex analytic map

$$t \longmapsto \frac{\left(\varphi_t\left(P_0^q(t)\right)\right)^{s_0}}{\left(\varphi_t\left(P_1^q(t)\right)\right)^{s_1}}$$

has a modulus constant equal to 1, hence is a constant, say ζ. Since both power series $\varphi_t(P_0^q(t))$ and $\varphi_t(P_1^q(t))$ have their coefficients in $\mathcal{O}_{\mathbb{L},S}$, we conclude that $\zeta \in \mathcal{O}_{\mathbb{L},S}$. But $|\zeta|_v = 1$ for all place v over $\mathbb{L}$ by $(\star)$ hence it is a root of unity.

Note also that the equality $\varphi_t\left(P_0^q(t)\right)^{s_0} = \zeta\, \varphi_t\left(P_1^q(t)\right)^{s_1}$ holds as equality between adelic series, so that it is also true for analytic functions at any place.

To conclude the proof of Theorem 4.1, pick a place v of $\mathbb{L}$ and consider the connected component U of $\{g_{0,v} > \tau_v/s_0\} = \{g_{1,v} > \tau_v/s_1\}$ in $C^{v,\mathrm{an}}$ clustering at $\mathfrak{c}$. We need to argue that $P^q_{c,a}(c_0)$ and $P^q_{c,a}(c_1)$ belongs to the domain of convergence of the Böttcher coordinate $\varphi_{v,c,a}$ for any $c, a \in U$.

Recall that $s_0 g_{0,v}(c,a) = s_1 g_{1,v}(c,a)$ for some positive integers s_0, s_1. It follows that

$$\min\{g_{c,a}(P^q_{c,a}(c_0)), g_{c,a}(P^q_{c,a}(c_1))\} = 3^q \min\{g_{c,a}(c_0), g_{c,a}(c_1)\} >$$
$$3^q \min\left\{\frac{s_0}{s_1}, \frac{s_1}{s_0}\right\} \max\{g_{c,a}(c_0), g_{c,a}(c_1)\} >$$
$$G(c,a) + \max\{s_0 g_{c,a}(c_0), s_1 g_{c,a}(c_1)\} > G(c,a) + \tau_v$$

for q large enough and we conclude by Proposition 2.3.

5. Special curves having a periodic orbit with a constant multiplier

In this section, we prove Theorem B.

Pick an integer $m \geq 1$ and a complex number $\lambda \in \mathbb{C}$, and consider the set of polynomials $P_{c,a}$ that admits a periodic orbits of period m and multiplier λ. It follows from [**Si**, p. 225] that this set is an algebraic curve in Poly_3 (see also [**Mi1**, Appendix D], [**BB2**, Theorem 2.1] or [**FG**, §6.2]). Let us be more precise:

Theorem 5.1 (Silverman). — *For any integer $m \geq 1$, there exists a polynomial $p_n \in \mathbb{Q}[c,a,\lambda]$ with the following properties.*

1. *For any $\lambda \in \mathbb{C} \setminus \{1\}$, $p_m(c,a,\lambda) = 0$ if and only if $P_{c,a}$ has a cycle of exact period m and multiplier λ.*
2. *When $\lambda = 1$, then $p_m(c,a,1) = 0$ if and only if there exists an integer k dividing m such that $P_{c,a}$ has a cycle of exact period k whose multiplier is a primitive m/k-th root of unity.*

We now come to the proof of Theorem B.

One implication is easy. For any integer $m \geq 1$, the curve $\mathrm{Per}_m(0)$ is contained in the union of the two curves $\{(c,a) \in \mathbb{C}^2 ;\ P^m_{c,a}(c_0) = c_0\}$ and $\{(c,a) \in \mathbb{C}^2 ;\ P^m_{c,a}(c_1) = c_1\}$. According to lemma 5.2 below, it contains infinitely many post-critically finite parameters.

Lemma 5.2. — *Pick $n \geq 0$, $k > 0$ and $i \in \{0,1\}$. Any irreducible component C of the set $\{(c,a),\ P^{n+k}_{c,a}(c_i) = P^n_{c,a}(c_i)\}$ contains infinitely many post-critically finite parameters.*

Proof. — We argue over the complex numbers, and use the terminology and results from [**DF**]. In particular, a critical point c_i, $i = 0,1$ is said to be active at a parameter (c,a) if the family of analytic functions $P^n_{c,a}(c_i)$ is normal in a neighborhood of (c,a).

Suppose that C is an irreducible component of the set

$$\{(c,a),\ P^{n+k}_{c,a}(c_i) = P^n_{c,a}(c_i)\}$$

where $n \geq 0$, $k > 0$ and $i \in \{0,1\}$. To fix notation we suppose $i = 0$. Observe that $g_{c,a}(c_0) = 0$ on C, and since $G(c,a) = \max\{g_{c,a}(c_0), g_{c,a}(c_1)\}$ is a proper function on Poly_3 (see Proposition 2.1) it follows that $g_{c,a}(c_1)$ is also proper on C. In particular, c_1 has an unbounded orbit when $c, a \in C$ is close enough to infinity in Poly_3. It follows from e.g. [**DF**, Theorem 2.5] (which builds on [**McM1**, Theorem 2.2]) that c_1 is active at at least one point (c_0, a_0) on C. The arguments of [**DF**, Lemma 2.3] based on Montel's

theorem show that (c_0, a_0) is accumulated by parameters for which c_1 is pre-periodic to a repelling cycle, hence by post-critically finite polynomials. In particular, it contains infinitely many post-critically finite parameters. □

For the converse implication, we proceed by contradiction and suppose that we can find a complex number $\lambda \neq 0$, an integer $m \geq 1$, and an irreducible component C of $\mathrm{Per}_m(\lambda)$ containing infinitely many post-critically finite polynomials.

Observe that, whenever $0 < |\lambda| \leq 1$, any parameter $(c, a) \in C \subset \mathrm{Per}_m(\lambda)$ has a non-repelling cycle which is not super-attracting. In particular, at least one of its critical points has an infinite forward orbit (see e.g.[**Mi2**]). It follows that $\mathrm{Per}_m(\lambda)$ contains no post-critically finite parameter when $0 < |\lambda| \leq 1$. This argument is however not sufficient to conclude in general. But we shall see that a combination of this argument applied at a place of residual characteristic 3 together with the study of the explosion of multipliers on a branch at infinity of C gives a contradiction.

Proposition 5.3. — *Suppose C is an irreducible component of $\mathrm{Per}_m(\lambda)$ with $\lambda \in \mathbb{C}^*$ and $m \geq 1$ containing infinitely many post-critically finite polynomials. Then one of the two critical points is persistently periodic on C and λ is equal to the multiplier of a repelling periodic orbit of a quadratic polynomial whose critical point is periodic.*

We may thus assume that the curve C is included in $\mathrm{Per}_m(0)$. Observe that the equation $P^m_{c,a}(c_0) = c_0$ (resp. $P^m_{c,a}(c_1) = c_1$) is equivalent to the vanishing of a polynomial of the form $\sqrt{3}^{1-3^m} a^{3^m} + \text{l.o.t}$ (resp. $\sqrt{3}^{1-3^m}(a^3 - \frac{c^3}{6})^{3^m} + \text{l.o.t}$). It follows that the closure of C in $\overline{\mathrm{Poly}_3}$ intersects the line at infinity in a set included in $\{[1:0:0], [\zeta:1:0]\}$ with $\zeta^3 = 6$ (see also [**BB2**, Theorem 4.2]).

Consider the curve of unicritical polynomials $c_0 = c_1$, which is defined by the equation $c = 0$. It intersects the line at infinity at $[0:1:0]$, so that Bezout' theorem implies the existence of one parameter $(c, a) \in C$ which is unicritical.

We conjugate $P_{c,a}$ by a suitable affine map to a polynomial $Q(z) = z^3 + t$. This unicritical polynomial has a periodic critical orbit. By the previous proposition, it also has a periodic orbit whose multiplier is equal to the multiplier of a repelling orbit of a quadratic polynomial having a periodic critical point. The next two lemmas yield a contradiction, and the proof of Theorem B is complete.

Lemma 5.4. — *Suppose that the unicritical polynomial $Q(z) = z^3 + t$ is postcritically finite and admits a periodic cycle with a multiplier $\lambda \neq 0$.*

Then λ belongs to a number field $\mathbb{K}$, and we have $|\lambda|_v < 1$ for any place v of $\mathbb{K}$ of residual characteristic 3.

Lemma 5.5. — *Suppose $\lambda \neq 0$ is equal to the multiplier of a periodic orbit of a quadratic polynomial having a periodic critical point.*

Then λ belongs to a number field $\mathbb{K}$, and $|\lambda|_v = 1$ for any non-Archimedean place v of $\mathbb{K}$ of residual characteristic $\neq 2$.

Proof of Lemma 5.4. — Since Q is post-critcally finite, t satisfies a polynomial equation with integral coefficients hence belongs to a number field. Its periodic point are solutions of a polynomial of the form $Q^n(z) - z$ so that the periodic points of Q and their multipliers also belongs to a number field.

We may thus fix a number field containing t, and λ, and fix a place v of $\mathbb{K}$ of residual characteristic 3. Observe that the completion of its algebraic closure of the completion

of $\mathbb{K}$ with respect to the norm induced by v is a complete algebraically closed normed field isometric to the 3-adic field $\mathbb{C}_3$. We consider the action of Q on the Berkovich analytification of the affine plane over that field. To simplify notation we denote by $|\cdot|$ the norm on $\mathbb{C}_3$.

Suppose that $|t|>1$. Then we have $|Q(0)|=|t|>1$, and thus $|Q^n(0)|=|Q^{n-1}(0)|^3=|t|^{3^n}\to\infty$ by an immediate induction. This would imply the critical point to have an infinite orbit contradicting our assumption that Q is post-critically finite.

We thus have $|t|\leq 1$. This implies that any point having a bounded orbit lies in the closed unit ball $\{z,|z|\leq 1\}$. Indeed the same induction as before yields $|Q(z)|=|z|^3$ and $|Q^n(z)|=|Q^{n-1}(z)|^3=|z|^{3^n}\to\infty$ for any $|z|>1$.

Pick any periodic point w of period k with multiplier $\lambda=(Q^k)'(w)$. Observe that $Q'(z)=3z^2$, and $|Q^j(w)|\leq 1$ for all $j\geq 0$ by what precedes. We thus have

$$|\lambda|=\prod_{j=0}^{k-1}|Q'(Q^j(w))|=\prod_{j=0}^{k-1}|3(Q^j(z))|^2\leq 3^{-k}<1\ ,$$

which concludes the proof. □

Proof of Lemma 5.5. — Suppose z_0 is a periodic point of period k of a quadratic polynomial $P_c(z)=z^2+c$ such that the critical point 0 is periodic of period l.

When 0 is fixed, we have $c=0$ and $P_c^n(z)=z^{2^n}$, hence the multiplier of z_0 is a power of 2 (or equal to 0) so that the lemma holds in this case. From now on we assume $c\neq 0$.

Observe that c satisfies the polynomial equation with integer coefficients $P_c^l(0)=0$ hence is an algebraic number. The polynomial $P^k(z)-z$ has also algebraic coefficients hence z_0 is algebraic too. Let $\mathbb{K}$ be a number field containing both z_0 and c.

Choose any non-Archimedean place v on $\mathbb{K}$. We claim that $|c|_v\leq 1$. Indeed, when $|c|_v>1$ one has $|P_c(0)|=|c|>1$, so that $|P_c^n(0)|=|c|^{3^{n-1}}\to\infty$ by an immediate induction, and this contradicts our assumption that 0 is periodic.

It follows that $|z_0|_v\leq 1$ since otherwise $|P_c^n(z_0)|=|z_0|^{3^n}\to\infty$.

Since 0 is periodic of exact period l, we may write $P_c^l(z)=\sum_{i=0}^{2^l}a_iz^i$ with $a_0=a_1=0$. The bound $|c|_v\leq 1$ forces $|a_i|_v\leq 1$ for all i. It follows that $|P_c^{ln}(z)|_v\leq|P_c^{l(n-1)}(z)|_v^2\to 0$ as soon as $|z_v|<1$. This implies that the orbit of z_0 is included in the annulus $\{|z|_v=1\}$.

Finally let us estimate the multiplier λ of z_0. By what precedes, we have

$$|\lambda|_v=\prod_{i=0}^{l-1}|P'(P^i(z_0))|_v=\prod_{i=0}^{l-1}|2\,P^i(z_0)|_v=|2|_v^l=1$$

where the last equality holds when v has residual characteristic $\neq 2$. □

Proof of Proposition 5.3. — Since C contains infinitely many post-critically finite polynomials we may assume it is defined over a number field $\mathbb{K}$. Let $\hat{C}$ be the normalization of the completion of C in $\overline{\mathrm{Poly}_3}$. Pick any branch $\mathfrak{c}$ of C at infinity (i.e. a point in $\hat{C}$ which projects to the line at infinity in $\overline{\mathrm{Poly}_3}$). By Proposition 3.5 we may choose an isomorphism of complete local rings $\widehat{\mathcal{O}_{\hat{C},\mathfrak{c}}}\simeq\mathbb{L}[[t]]$ such that $c(\mathsf{n}(t)),a(\mathsf{n}(t))$ are adelic series, i.e. formal Laurent series with coefficients in $\mathcal{O}_{\mathbb{L},S}((t))$ that are analytic at all places.

In the remaining of the proof, we fix an Archimedean place, and embed $\mathbb{L}$ into the field of complex numbers (endowed with its standard norm). We may suppose $c(\mathsf{n}(t)),a(\mathsf{n}(t))$ are holomorphic in $0<|t|<\epsilon$ for some ϵ, and meromorphic at 0. We get a one-parameter

family of cubic polynomials $P_t := P_{c(\mathsf{n}(t)),a(\mathsf{n}(t))}$ parameterized by the punctured disk $\mathbb{D}^*_\epsilon = \{0 < |t| < \epsilon\}$.

Consider the subvariety $Z := \{(z,t),\, P_t^m(z) = z\} \subset \mathbb{C} \times \mathbb{D}^*_\epsilon$. The projection map $Z \to \mathbb{D}^*_\epsilon$ is a finite cover which is unramified if ϵ is chosen small enough. By reducing ϵ if necessary, and replacing t by t^N, we may thus assume that $Z \to \mathbb{D}^*_\epsilon$ is a trivial cover. In other words, there exists a meromorphic function $t \mapsto p(t)$ such that $P_t^m(p(t)) = p(t)$ and $(P_t^m)'(p(t)) = \lambda$.

As in Section 3, we denote by $\mathsf{P}(z) \in \mathbb{C}((t))[z]$ the cubic polynomial induced by the family P_t. It induces a continuous map on the analytification $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((t))}$, for which the point $\mathsf{p} \in \mathbb{A}^1(\mathbb{C}((t)))$ corresponding to $p(t)$ is periodic of period m with multiplier $(\mathsf{P}^m)'(\mathsf{p}) = \lambda$. Observe that P has two critical points c_0 and c_1 corresponding to the meromorphic functions 0 and $c(\mathsf{n}(t))$ respectively.

Lemma 5.6. — *If c_0 is not pre-periodic for P, then $|\mathsf{P}^q(\mathsf{c}_0)|_t$ tends to infinity when $q \to \infty$.*

Proof. — Observe that our assumption is equivalent to the fact that c_0 is not persistently pre-periodic on C.

We claim that $g_0(t) := g_{P_t}(c_0)$ tends to infinity when $t \to 0$.

Suppose first that c_1 is persistently pre-periodic on C. Then the function g_1 is identically zero on C, so that $G|_C = \max\{g_0, g_1\}|_C = g_0$. Since G is proper by Proposition 2.1, and $(c(\mathsf{n}(t)), a(\mathsf{n}(t)))$ tends to infinity in $\overline{\mathrm{Poly}_3}$ when $t \to 0$, we conclude that $g_0(t) \to \infty$.

When c_1 is not persistently pre-periodic on C, the two functions $g_0(t)$ and $g_1(t) := g_{P_t}(c_1)$ are proportional on $\mathfrak{c}$ by Theorem 4.1 (1). As before $\max\{g_0, g_1\} \to \infty$ as $t \to 0$ so that again $g_0(t) \to \infty$.

By Proposition 3.6, we can find $a > 0$ such that $g_0(P_t) = a \log|t|^{-1} + O(1)$. And [**DF**, Lemma 6.4] implies[(3)] the existence of a constant $C > 0$ such that $g_{P_t}(z) \leq \log\max\{|z|, |c(\mathsf{n}(t))|, |a(\mathsf{n}(t))|\} + C$ for all t. Since $g_{P_t} \circ P_t = 3g_{P_t}$, we conclude that for all $q \geq 1$

$$\log\max\{|P_t^q(c_0)|, |c(\mathsf{n}(t))|, |a(\mathsf{n}(t))|\} \geq 3^q g_t(0) - C = 3^q a \log|t|^{-1} + O(1)\ .$$

This implies $|P_t^q(c_0)|_t \geq 3^q a|t|_t \to \infty$ when $q \to \infty$ as required. □

We continue the proof of Proposition 5.3. Suppose neither c_0 nor c_1 is persistently pre-periodic so that the previous lemma applies to both critical points. Translating its conclusion over the non-Archimedean field $\mathbb{C}((t))$, we get that $\mathsf{P}^q(\mathsf{c}_0)$ and $\mathsf{P}^q(\mathsf{c}_1)$ both tend to infinity when $q \to \infty$. We may thus apply [**K**, Theorem 1.1 (ii)], and [**K**, Corollary 1.4] (which is directly inspired from a result of Bezivin). We conclude that all periodic cycles of P are repelling so that $|(\mathsf{P}^m)'(\mathsf{p})|_t > 1$. This contradicts $|\lambda|_t = 1$.

Suppose next that c_0 is persistently strictly pre-periodic (which implies c_1 not to be persistently pre-periodic). Then c_0 is strictly pre-periodic whereas c_1 escapes to infinity by the previous lemma, and we may now apply [**K**, Theorem 1.1 (iii)(a)] to P. As before [**K**, Corollary 1.4] implies that all cycles of P are repelling which gives a contradiction.

We are thus reduced to the case c_0 is persistently periodic on C, say of (exact) period n. We shall prove that λ is equal to the multiplier of a (repelling) periodic orbit of some complex quadratic polynomial having a periodic critical orbit.

(3) observe that the statement of the lemma is incorrectly stated in [**DF**], and the constant C is actually *independent* on P.

By [**K**, Theorem 1.1 (iii)(a)], the periodic critical point $\mathsf{c}_0(=0)$ is contained in a unique closed ball $B = \{\mathsf{z} \in \mathbb{C}((t)),\ |\mathsf{z}|_t \leq r\}$ for some positive $r > 0$ that is also periodic of some period m dividing n. Since B is fixed by the polynomial $\mathsf{P}^n = \sum_{j\geq 2} b_j \mathsf{z}^j$ with coefficients $b_j \in \mathbb{C}((t))$, the radius r satisfies an equation of the form $|b_j| r^j = r$ for some j hence $r = |t|_t^l$ for some $l \in \mathbb{Z}$. To simplify the discussion to follow we conjugate P by the automorphism $\mathsf{z} \mapsto t^{-l}\mathsf{z}$ so that B becomes the closed unit ball. Observe that 0 remains a critical point of P after this conjugacy.

Recall that the closed unit ball B defines the Gauss point $\mathsf{x}_g \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((t))}$ for which we have

$$|\mathsf{Q}(z)(\mathsf{x}_g)| := \sup_{z\in B} |\mathsf{Q}(z)|_t = \max |q_i| \text{ for all } \mathsf{Q} = \sum q_i z^i \in \mathbb{C}((t))[z]\ .$$

Since B is fixed by P^m, it follows that x_g is also fixed by P^m. This is equivalent to say that P^m can be written as $\mathsf{P}^m(z) = \sum_{i=1}^{3^m} a_i z^i$ where $\max |a_i| = 1$.

For any $\mathsf{z} \in \mathbb{C}((t))$ of norm 1, denote by $\tilde{z}$ the unique complex number such that $|\mathsf{z} - \tilde{z}|_t < 1$.

Lemma 5.7. — *We have $a_1 = 0$, $|a_0| \leq |a_2| = 1$, and $|a_i| < 1$ for all $i \geq 3$; and the complex quadratic polynomial $\tilde{P}(z) := \widetilde{a_2} z^2 + \widetilde{a_0}$ has a periodic critical orbit.*

Lemma 5.8. — *The orbit of the periodic point p intersects the ball B.*

Replacing p by its image by a suitable iterate of P we may suppose that it belongs to B, i.e. $|\mathsf{p}|_t \leq 1$. In fact we have $|\mathsf{P}^i(\mathsf{p})|_t = 1$ for all $i \geq 0$ since any point in the open unit ball converges to the fixed critical point 0 under P^n. Also the period of p is necessarily a multiple of m, say mk with $k \geq 1$.

To render the computation of the multiplier of p easier, we conjugate P^m by $z \mapsto a_2 z$. Since $|a_2| = 1$, we still have $|\mathsf{p}|_t = 1$, and the equality $a_2 = 1$ is now satisfied.

By Lemma 5.7, we get $\sup_B |\mathsf{Q}| < 1$ with $\mathsf{Q} := \mathsf{P}^m - \widetilde{P}$, so that

$$(\mathsf{P}^{mk})'(\mathsf{p}) = \prod_{i=0}^{k-1} (\mathsf{P}^m)'(\mathsf{P}^{mi}(\mathsf{p})) = (\widetilde{P}^k)'(\widetilde{p})\ .$$

But the multiplier of p is equal to $\lambda \in \mathbb{C}$. Hence it is equal to the multiplier of a repelling periodic orbit of some quadratic polynomial (namely $\tilde{P}$) having a periodic critical orbit, as was to be shown. □

Proof of Lemma 5.7. — The point 0 is critical for P hence $a_1 = 0$.

Since the Gauss point is fixed by P^m, we have $\max_{i\geq 2} |a_i| = 1$. Let $d \geq 2$ be the maximum over all integers i such that $|a_i| = 1$. The number of critical points of P^n lying in the closed unit ball (counted with multiplicity) is precisely equal to $d-1$. But the exact period of 0 is n, and no points $\mathsf{P}^i(0)$, $1 \leq i \leq n-1$ can collide with the other critical point since the latter escapes to infinity. It follows that $d = 2$, and $|a_2| = 1 > \max_{i\geq 3} |a_i|$.

Finally 0 is fixed by P^n where n is a multiple of m, hence the complex quadratic polynomial $\widetilde{P^n}$ has a periodic critical orbit of period m/n. □

Proof of Lemma 5.8. — Since the multiplier of p is $\lambda \in \mathbb{C}$, its t-adic norm is 1, hence a small ball U centered at p of positive radius is included in the filled-in Julia set of P. By [**K**, Corollary 4.8], U is eventually mapped into B, hence the claim. □

6. A polynomial on a special curve admits a symmetry

We fix $\mathbb{K}$ a number field, and s_0, s_1 three positive integers such that s_0 and s_1 are coprime. We shall say that a cubic polynomial $P := P_{c,a}$ with c, a in a finite extension $\mathbb{L}$ of $\mathbb{K}$ satisfies the condition $(\mathfrak{P})$ if the following holds:

($\mathfrak{P}1$) For any place v of $\mathbb{L}$, we have $s_0 g_{P,v}(c_0) = s_1 g_{P,v}(c_1)$.

($\mathfrak{P}2$) Given any place v of $\mathbb{L}$, if $\min\{g_{P,v}(P^n(c_0)), g_{P,v}(P^n(c_1))\} > G_v(P) + \tau_v$ for some integer $n \geq 1$, then

$$\frac{\varphi_{P,v}(P^n(c_0))^{s_0}}{\varphi_{P,v}(P^n(c_1))^{s_1}}$$

is a root of unity lying in $\mathbb{K}$.

Recall the definition of the constant $\tau_v := \tau(\mathbb{L}_v)$ from Proposition 2.3.

Observe that if the condition in ($\mathfrak{P}2$) never occurs, then the normalized heights by P of both sequence of points $P^n(c_0)$ and $P^n(c_1)$ are bounded, hence P is post-critically finite. We prove here the following

Theorem 6.1. — *Suppose $P = P_{c,a}$ is a cubic polynomial defined over a number field $\mathbb{L}$ satisfying the assumptions $(\mathfrak{P})$ which is not post-critically finite and such that $\min\{g_{P,v}(P^q(c_0)), g_{P,v}(P^q(c_1))\} > G_v(P) + \tau_v$ for some integer q and some place v of $\mathbb{L}$.*

Then there exists a root of unity $\zeta \in \mathbb{K}$, an integer $q' \leq C(\mathbb{K}, q)$, and an integer $m \geq 0$ such that the polynomial $Q(z) := \zeta P^m(z) + (1-\zeta)\frac{c}{2}$ commutes with all iterates P^k such that $\zeta^{3^k} = \zeta$, and either $Q(P^{q'}(c_0)) = P^{q'}(c_1)$, or $Q(P^{q'}(c_1)) = P^{q'}(c_0)$.

Remark. — We shall prove along the way that there exists an integer $k \geq 1$ with $\zeta^{3^k} = \zeta$ so that the commutativity statement is non empty.

6.1. Algebraization of adelic branches at infinity

The material of this section is taken from [**X**]. Let $\mathbb{K}$ be a number field. For any place v on $\mathbb{K}$, denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$ w.r.t. the v-adic norm.

We cover the line at infinity H_∞ of the compactification of the affine space $\mathbb{A}^2_{\mathbb{K}} = \operatorname{Spec}\mathbb{K}[x,y]$ by $\mathbb{P}^2_{\mathbb{K}}$ by charts $U_\alpha = \operatorname{Spec}\mathbb{K}[x_\alpha, y_\alpha]$ centered at $\alpha \in H_\infty(\mathbb{K})$ such that $\alpha = \{(x_\alpha, y_\alpha) = (0,0)\}$, $H_\infty \cap U_\alpha = \{x_\alpha = 0\}$, and $x_\alpha = 1/x$, $y_\alpha = y/x + c$ for some $c \in \mathbb{K}$ (or $x_\alpha = 1/y$, $y_\alpha = x/y$).

Fix S a finite set of places of $\mathbb{K}$. By definition, an adelic branch $\mathfrak{s}$ at infinity defined over the ring $\mathcal{O}_{\mathbb{K},S}$ is a formal branch based at a point $\alpha \in H_\infty(\mathbb{K})$ given in coordinates x_α, y_α as above by a formal Puiseux series

$$y_\alpha = \sum_{j \geq 1} a_j x_\alpha^{j/m} \in \mathcal{O}_{\mathbb{K},S}[[x_\alpha^{1/m}]]$$

such that $\sum_{j\geq 1} a_j x^j$ is an adelic series.

Observe that for any place $v \notin S$, then the radius of convergence is a least 1. In the sequel, we set $r_{Z,\alpha,v}$ to be the minimum between 1 and the radius of convergence over $\mathbb{K}_v$ of this Puiseux series. Any adelic branch $\mathfrak{s}$ based at α at infinity thus defines an analytic curve in an (unbounded) open subset of $\mathbb{A}^{2,\mathrm{an}}_v$:

$$Z^v(\mathfrak{s}) := \left\{ (x_\alpha, y_\alpha) \in U_\alpha(\mathbb{K}_v)\,;\; y_\alpha^m = \sum_{j\geq 1} a_j x_\alpha^j \;,\; 0 < |x_\alpha|_v < \min\{r_{Z,\alpha,v}, 1\} \right\} .$$

Theorem 6.2 (Xie). — *Suppose $\mathfrak{s}_1, \dots, \mathfrak{s}_l$ are adelic branches at infinity, and let $\{B_v\}_{v\in M_{\mathbb{K}}}$ be a set of positive real numbers such that $B_v = 1$ for all but finitely many places.*

Assume that there exists a sequence of distinct points $p_n = (x_n, y_n) \in \mathbb{A}^2(\mathbb{K})$ such that for all n and for each place $v \in M_{\mathbb{K}}$ then either we have $\max\{|x_n|_v, |y_n|_v\} \leq B_v$ or $p_n \in \cup_{i=1}^l Z^v(\mathfrak{s}_i)$.

Then there exists an irreducible algebraic curve Z defined over $\mathbb{K}$ such that any branch of Z at infinity is contained in the set $\{\mathfrak{s}_1, \dots, \mathfrak{s}_l\}$ and p_n belongs to $Z(\mathbb{K})$ for all n large enough.

6.2. Construction of an invariant correspondence

Our aim is to prove the following statement.

Theorem 6.3. — *Suppose $P = P_{c,a}$ is a cubic polynomial satisfying the assumptions $(\mathfrak{P})$. Then there exists a (possibly reducible) algebraic curve $Z_P \subset \mathbb{A}^1 \times \mathbb{A}^1$ such that:*

1. *$\phi(Z_P) = Z_P$ with $\phi(x, y) := (P(x), P(y))$;*
2. *for all n large enough, we have $(P^n(c_0), P^n(c_1)) \in Z_P$;*
3. *any branch at infinity of Z_P is given by an equation $\varphi_P(x)^{s_0} = \zeta \cdot \varphi_P(y)^{s_1}$ for some root of unity $\zeta \in \mathbb{K}$.*

Proof. — The proof is a direct application of Xie's theorem. Recall that the set $U_{\mathbb{K}}$ of roots of unity that is contained in the number field $\mathbb{K}$ is finite.

Recall that for each place v over $\mathbb{L}$, we let $g_{P,v} := \lim_n \frac{1}{3^n} \log^+ |P^n|_v$ be the Green function of P, and write $G_v(P) = \max\{g_{P,v}(c_0), g_{P,v}(c_1)\}$.

Lemma 6.4. — *For any $\zeta \in U$, there exists an adelic branch $\mathfrak{c}_\zeta$ based at a point $q \in H_\infty(\mathbb{L})$ such that for any place v the analytic curve $Z^v(\mathfrak{c}_\zeta)$ is defined by the equation $\{\varphi_{P,v}(x)^{s_0} = \zeta \cdot \varphi_{P,v}(y)^{s_1}\}$ in the range $\min\{|x|_v, |y|_v\} > \exp(G_v(P) + \tau_v)$.*

Define $(x_n, y_n) := (P^n(c_0), P^n(c_1)) \in \mathbb{A}^2(\mathbb{L})$, and consider the family of all adelic curves $\mathfrak{c}_\zeta$ given by Lemma 6.4 for all $\zeta \in U_{\mathbb{K}}$. We shall first check that all hypothesis of Xie's theorem are satisfied.

To do so pick any integer n and any place v on $\mathbb{L}$. Suppose first that $g_{P,v}(c_0) = 0$. Since $g_{P,v}(P^n(c_0)) = 3^n g_{P,v}(c_0) = 0$, we get $|x_n|_v \leq e^{C_v} =: B_v$ by Lemma 2.4. The same upper bound applies to $|y_n|_v$ since $g_{P,v}(c_1) = 0$ by $(\mathfrak{P}1)$ so that $\max\{|x_n|_v, |y_n|_v\} \leq B_v$ in this case.

Suppose now that $g_{P,v}(c_0) > 0$ so that $g_{P,v}(c_1) > 0$ by $(\mathfrak{P}1)$. Fix N large enough such that $g_{P,v}(P^N(c_0)) > G_v(P) + \tau_v$ and $g_{P,v}(P^N(c_1)) > G_v(P) + \tau_v$. Then $P^N(c_0)$ and $P^N(c_1)$ lie in the domain of definition of the Böttcher coordinate by Proposition 2.3. Since

$$g_{P,v}(P^n(c_0)) = 3^{n-N} g_{P,v}(P^N(c_0)) \geq g_{P,v}(P^N(c_0)) > G_v(P) \ ,$$

we may also evaluate φ_P at x_n for all $n \geq N$. The same holds for y_n and we get

$$\frac{\varphi_P(x_n)^{s_0}}{\varphi_P(y_n)^{s_1}}$$

is a root of unity $\zeta \in \mathbb{K}$ by $(\mathfrak{P}2)$ hence (x_n, y_n) belongs to $Z^v(\mathfrak{c}_\zeta)$ for all $n \geq N$.

Xie's theorem thus applies to the sequence $\{(x_n, y_n)\}_{n\geq N}$, and we get an irreducible curve $Z \subset \mathbb{A}^1 \times \mathbb{A}^1$ that contains infinitely many points (x_n, y_n) and such that each of its branch at infinity is equal to $\mathfrak{c}_\zeta$ for some $\zeta \in U_{\mathbb{K}}$.

Recall that $\phi(x,y) = (P(x), P(y))$, and pick any integer $n \geq 1$. Then $\phi^n(Z)$ is an irreducible curve defined over $\mathbb{L}$ whose branches at infinity are the images under ϕ^n of the branches at infinity of Z.

Fix $\zeta \in U_{\mathbb{K}}$ and pick $(x,y) \in Z^v(\mathfrak{c}_\zeta)$. Then $(x',y') = (P(x), P(y))$ satisfies

$$\frac{\varphi_P(x')^{s_0}}{\varphi_P(y')^{s_1}} = \frac{\varphi_P(x)^{3s_0}}{\varphi_P(y)^{3s_1}} = \zeta^3 \ ,$$

hence $\phi(\mathfrak{c}_\zeta) = \mathfrak{c}_{\zeta^3}$. We conclude that any branch at infinity of $\phi^n(Z)$ is of the form $\mathfrak{c}_\zeta$ for some $\zeta \in U_{\mathbb{K}}$. Since two irreducible curves having a branch at infinity in common are equal, we see that Z is pre-periodic for the morphism ϕ so that $\phi^{l+k}(Z) = \phi^k(Z)$ for some $l,k > 0$. Setting $Z_P := \cup_{j=k}^{l+k-1} \phi^j(Z)$, we obtain a (possibily reducible) curve defined over $\mathbb{L}$ such that $\phi(Z_P) = Z_P$ and $(x_n, y_n) \in Z_P$ for all $n \geq k$. This concludes the proof of the theorem. □

Proof of Lemma 6.4. — Recall from Lemma 2.2 that

$$\varphi_P(z) = \lambda \left(z - \frac{c}{2} \right) + \sum_{k \geq 1} \frac{a_k}{z^k},$$

is an adelic series at infinity in the sense of §3.1 , and therefore

$$\varphi_P^{-1}(z) = \frac{1}{\lambda} z + \frac{c}{2} + \sum_{k \geq 1} \frac{b_k}{z^k},$$

too by Lemma 3.1. We may assume that $a_k, b_k \in \mathcal{O}_{\mathbb{K},S}$. Recall from Proposition 2.3 that $\varphi_{P,v}$ induces an analytic isomorphism between $\{z,\, g_{P,v}(z) > G_v(P) + \tau_v\}$ and $\{z',\, |z'|_v > \exp(G_v(P) + \tau_v)\}$. By Lemma 3.1 the formal map φ_P^{-1} defines an adelic series at infinity in the terminology of §3.1. For each place v, this series coincides with the inverse map of φ_P on the complement of the closed disk of radius $\exp(G_v(P) + \tau_v)$ hence its domain of convergence is exactly $\{z',\, |z'|_v > \exp(G_v(P) + \tau_v)\}$.

It follows that

$$Z_v := \{(x,y),\, \varphi_P(x)^{s_0} = \zeta\, \varphi_P(y)^{s_1}\}$$

defines an analytic curve in the domain $\min\{g_{P,v}(x), g_{P,v}(y)\} > G_v(P) + \tau_v$, whose image under the isomorphism $(x',y') := (\varphi_{P,v}(x), \varphi_{P,v}(y))$ is given by

$$Z_v' := \{(x',y'),\, (x')^{s_0} = \zeta\, (y')^{s_1}\}$$

where $\min\{|x'|_v, |y'|_v\} > \exp(G_v(P) + \tau_v)$.

Pick any $\xi \in \bar{\mathbb{Q}}$ such that $\xi^{s_1}\zeta = 1$. Let $\mathfrak{c}_\zeta$ be the adelic branch at infinity defined by the formal Laurent series $(\varphi_P^{-1}(t^{-s_1}), \varphi_P^{-1}(\xi\, t^{-s_0}))$. For any place v, the analytic curve $Z^v(\mathfrak{c}_\zeta)$ is included in Z_v. Since s_0 and s_1 are coprime, for any pair (x',y') with $(x')^{s_0} = \zeta\, (y')^{s_1}$ and $\min\{|x'|_v, |y'|_v\} > \exp(G_v(P) + \tau_v)$, there exists $0 < |t|_v < \exp\left(-\frac{G_v(P) + \tau_v}{\min\{s_0,s_1\}} \right)$ such that $x' = t^{-s_1}$ and $y' = \xi t^{-s_0}$.

This proves that $Z^v(\mathfrak{c}_\zeta) = Z_v$ for all place as required. □

6.3. Invariant correspondences are graphs

Let $Z_0, \ldots, Z_{p-1}$ be the irreducible components of Z_P such that $\phi(Z_i) = Z_{i+1}$ (the index computed modulo p). Apply [**MS**, Theorem 6.24] (or [**P**, Theorem 4.9]) to the component Z_0 of Z_P that is ϕ^p-invariant. It implies that after exchanging x and y if necessary, the curve Z_0 is the graph of a polynomial map, i.e. $Z_0 = \{(Q(t), t)\}$ for some $Q \in \mathbb{L}[t]$ such

that $Q \circ P^p = P^p \circ Q$. Observe that by [**J**] the two polynomials P and Q share a common iterate since we assumed P not to be post-critically finite.

We now work at an Archimedean place. Recall that the branch at infinity of Z_0 is of the form $\varphi_P(x)^{s_0} = \zeta\varphi_P(y)^{s_1}$ for some $\zeta \in U_{\mathbb{K}}$. Since s_0 and s_1 are coprime, it follows that $s_0 = 1$ and $s_1 = \deg(Q)$, and therefore s_1 is a power of 3, say $s_1 = 3^m$. We get

$$\varphi_P(Q(t)) = \zeta\varphi_P(t)^{3^m} = \zeta\varphi_P(P^m(t)) \ . \tag{13}$$

for all t of large enough norm. By Lemma 2.2, we get that $\varphi_P(t) = \lambda\left(t - \frac{c}{2}\right) + o(1)$ so that

$$\lambda\left(Q(t) - \frac{c}{2}\right) = \lambda\zeta\left(P^m(t) - \frac{c}{2}\right) + o(1) \tag{14}$$

which implies $Q(t) := \zeta P^m(t) + (1-\zeta)\frac{c}{2}$ since a polynomial which tends to 0 at infinity is identically zero.

At this point, recall our assumption that $\min\{g_{P,v}(P^q(c_0)), g_{P,v}(P^q(c_1))\} > G_v(P) + \tau_v$ for some integer q and some place v of $\mathbb{L}$. Then by $(\mathfrak{P}2)$ $\varphi_P(P^q(c_0))^{s_0} = \xi\varphi_P(P^q(c_1))^{s_1}$, for some root of unity $\xi \in \mathbb{K}$ which implies $\varphi_P(P^{q+n}(c_0)) = \xi^{3^n}\varphi_P(P^{q+n}(c_1))^{3^m}$. Since for some n large enough the point $(P^{q+n}(c_0), P^{q+n}(c_1))$ belongs to Z_0, we get $\xi^{3^n} = \zeta$. Now observe that the least integer n such that $\xi^{3^n} = \zeta$ is less that the cardinality of $U_{\mathbb{K}}$. We get the existence of $q' \leq C(\mathbb{K}, q)$ such that $\varphi_P(P^{q'}(c_0)) = \zeta\varphi_P(P^{q'}(c_1))^{3^m}$. Since φ_P is injective, the equation (13) shows that $P^{q'}(c_0) = Q(P^{q'}(c_1))$.

Observe that $\zeta^{3^p} = \zeta$. Indeed, since Z_0 is ϕ^p-invariant and since $\phi(\mathfrak{c}_\zeta) = \mathfrak{c}_{\zeta^3}$, we get $\mathfrak{c}_{\zeta^{3^p}} = \mathfrak{c}_\zeta$, hence $\zeta^{3^p} = \zeta$.

We now pick *any* integer $k \geq 1$ such that $\zeta^{3^k} = \zeta$. Then for all t large enough, we have

$$\varphi_P(Q \circ P^k(t)) = \zeta\varphi_P(P^k(t))^{3^m} = \zeta\varphi_P^{3^{m+k}}(t)$$

whereas

$$\varphi_P(P^k \circ Q(t)) = \varphi_P(Q(t))^{3^k} = \zeta^{3^k}\varphi_P^{3^{k+m}}(t) \ .$$

Since φ_P is injective on a neighborhood of ∞, and since $\zeta^{3^k} = \zeta$ by assumption, we conclude that $Q \circ P^k = P^k \circ Q$.

This concludes the proof of Theorem 6.3.

7. Classification of special curves

In this section, we prove Theorems C and A.

Before starting the proofs, let us introduce some notation. Pick $q, m \geq 0$ and ζ a root of unity. We let $Z(q, m, \zeta)$ be the algebraic set of those $(c, a) \in \mathbb{A}^2$ such that the polynomial $Q_{c,a} := \zeta P_{c,a}^m + (1-\zeta)\frac{c}{2}$ commutes with all iterates $P_{c,a}^k$ of $P_{c,a}$ such that $\zeta^{3^k} = \zeta$, and either $Q_{c,a}(P_{c,a}^q(c_0)) = P_{c,a}^q(c_1)$, or $Q_{c,a}(P_{c,a}^q(c_1)) = P_{c,a}^q(c_0)$.

Observe that, when $m \geq 1$, Q has degree 3^m and when (c, a) belongs to a fixed normed field K then the Green function $g_Q := \lim_n \frac{1}{3^{nm}} \log^+ |Q^n|$ is equal to $g_{c,a}$. Indeed since Q and P^k commute they have the same filled-in Julia set, hence coincide with the filled-in Julia set $\mathcal{K}_P$ of P. And g_Q (resp. g_P) is the unique continuous sub-harmonic function g on $\mathbb{A}_K^{1,\mathrm{an}}$ that is zero on $\mathcal{K}_P$, harmonic outside with a logarithmic growth at infinity $g_P(z) = \log|z| + O(1)$ (resp. $g_Q(z) = \log|z| + O(1)$). As $\mathcal{K}_P = \mathcal{K}_Q$, this gives $g_P = g_Q$.

7.1. Proof of Theorem C

The implication (1) $\Rightarrow$ (3) is exactly point 1. of Theorem 4.1.

The implication (3) $\Rightarrow$ (4) follows from Corollary 4.2 when s_0 and s_1 are both nonzero. When $s_1 = 0$, then $g_{0,v} \equiv 0$ on C at all places. By [**DF**, Theorem 2.5] there exists $n > 0$ and $k \geq 0$ such that C is an irreducible component of $\{(c,a) \in \mathbb{A}^2 ;\ P_{c,a}^{n+k}(c_0) = P_{c,a}^k(c_0)\}$. By Theorem 3.9 (applied to arbitrary weights) the family of functions $\{g_{1,v}\}_{v \in M_{\mathbb{K}}}$ induces a semi-positive adelic metric on some ample line bundle on the normalization of the completion of C so that Thuillier-Yuan's theorem applies. This gives (4) by observing that $g_0 + g_1 = g_1$. The case $s_0 = 0$ is treated similarly.

The implication (4) $\Rightarrow$ (1) follows from the fact that the positive closed $(1,1)$ current T_{bif} is the dd^c of a continuous function (namely $g_0 + g_1$) hence the measure $T_{\text{bif}} \wedge [C]$ cannot charge any point. Since μ_k is a sequence of positive measures supported on the set of PCF polynomials lying in C that converges to $T_{\text{bif}} \wedge [C]$, the curve C is necessarily special.

To prove (2) $\Rightarrow$ (1), we observe that if c_0 is not persistently pre-periodic on C then it is active at at least one parameter by [**DF**, Theorem 2.5] and that the set of parameters for which it is pre-periodic is infinite by e.g. [**DF**, Lemma 2.3].

We now prove (3) $\Rightarrow$ (2). We suppose c_0 is not persistently pre-periodic on C. Pick some parameter $(c,a) \in C$ and suppose c_0 is pre-periodic. We need to show that $P_{c,a}$ is post-critically finite.

Since c_0 is not persistently pre-periodic on C we have $s_1 \neq 0$ (again by [**DF**, Theorem 2.5] applied at any Archimedean place). In the case $s_0 = 0$ then c_1 is persistently pre-periodic and $P_{c,a}$ is clearly post-critically finite. We may thus assume that s_0 and s_1 are both non-zero and the functions $g_{0,v}, g_{1,v}$ vanish on the same set in $C^{v,\text{an}}$ for any place v of $\mathbb{K}$. Observe that c_0 being pre-periodic implies (c,a) to be defined over a number field. It follows that for all the Galois-conjugates (c',a') of (c,a) (over the defining field $\mathbb{K}$ of the curve C) we have $G_v(c',a') := \max\{g_{0,v}(c',a'), g_{1,v}(c',a')\} = 0$. It follows from [**I**] or [**FG**, Theorem 3.2] that $P_{c,a}$ is post-critically finite.

Let us now prove (3) $\Rightarrow$ (5). We suppose C is special. If either c_0 or c_1 is persistently pre-periodic in C, the assertion (5) holds true with $\zeta = 1$ and $i = j$ by [**DF**, Theorem 2.5].

Assume from now on that we are not in this case. Replacing $\mathbb{K}$ by a finite extension we may assume that all roots of unity ζ appearing in Theorem 6.1 2. belongs to $\mathbb{K}$, since there are at most the number of branches at infinity of C of such roots of unity.

Let $\mathcal{B}$ be the set of all $(c,a) \in C(\mathbb{L})$ where $\mathbb{L}$ is a finite extension of $\mathbb{K}$ such that $P_{c,a}$ is not post-critically finite. Given a place v of $\mathbb{K}$ we also define the subset $\mathcal{B}_v$ of $\mathcal{B}$ of parameters c, a such that $g_{0,v}(c,a) > 0$. This set is infinite since post-critically finite polynomials form a bounded set in $C^{v,\text{an}}$.

Lemma 7.1. — *For any $(c,a) \in \mathcal{B}$, the polynomial $P_{c,a}$ satisfies* ($\mathfrak{P}1$) *and* ($\mathfrak{P}2$).

Pick q large enough such that $3^q > \max\{s_0/s_1, s_1/s_0\}$, and fix a place v of residual characteristic ≥ 5. Now choose any $(c,a) \in \mathcal{B}_v$. Then $g_{1,v}(c,a)$ is also positive and $\min\{g_{c,a,v}(P^q(c_0)), g_{c,a,v}(P^q(c_1))\} > G_v(c,a)$ so that Theorem 6.1 applies by the previous lemma. We get a positive integer q' (bounded from above by a constant C depending only on $\mathbb{K}$ and q), a root of unity $\zeta \in \mathbb{K}$ and an integer $m \geq 0$ such that $(c,a) \in Z(q', m, \zeta)$.

Since $g_Q = g_P$, and $Q(P^{q'}(c_0)) = P^{q'}(c_1)$ we have

$$3^m g_{P,v}(P^{q'}(c_0)) = g_{P,v}(Q(P^{q'}(c_0))) = g_{P,v}(P^{q'}(c_1))$$

so that $3^m = \frac{s_0}{s_1}$. We conclude that the algebraic set consisting of the union of the curves $Z(q', m, \zeta)$ with $3^m = \frac{s_0}{s_1}$, $q' \leq C$ and ζ ranging over all roots of unity lying in $\mathbb{K}$ contains $\mathcal{B}_v$.

It follows that C is an irreducible component of one of these curves.

To end the proof of the theorem, we are left with proving (5) $\Rightarrow$ (3). Suppose that C is an irreducible component of $Z(m, q, \zeta)$ for some $m \geq 0$ and $q \geq 0$ and some root of unity ζ. Observe that $Z(q, m, \zeta)$ hence C are defined over a number field say $\mathbb{K}$.

When $m \geq 1$, then for all place v of that number field we have $g_{Q_{c,a},v} = g_{P_{c,a},v}$ for all $(c, a) \in C(\mathbb{L})$ for some finite extension $\mathbb{L}$ of $\mathbb{K}$. In particular $Q_{c,a}(P^q_{c,a}(c_i)) = P^q_{c,a}(c_j)$ implies $3^m g_{i,v}(c, a) = g_{j,v}(c, a)$ which proves (3) (with $s_0 = 0$ or $s_1 = 0$ if $i = j$). When $m = 0$ and $\zeta \neq 1$ and $C \neq \{c = 0\}$, then $Q_{c,a}(c_0) = (1 - \zeta)c/2 \neq 0$ hence $i \neq j$. It follows that $g_{P_{c,a},v} \circ Q_{c,a} = g_{P_{c,a},v}$ hence $g_{0,v} = g_{1,v}$. When $C = \{c = 0\}$, then $c_0 = c_1$ so that again $g_{0,v} = g_{1,v}$. Finally when $m = 0$ and $\zeta = 1$, then $i \neq j$ and $P^q_{c,a}(c_0) = P^q_{c,a}(c_1)$ hence $g_{0,v} = g_{1,v}$ at all places.

Proof of Lemma 7.1. — Pick $(c, a) \in \mathcal{B}$. By Theorem 4.1 1., $P_{c,a}$ satisfies ($\mathfrak{P}1$). To check ($\mathfrak{P}2$), we need to introduce a few sets. Fix any place v of $\mathbb{K}$, and for any integer $n \geq 0$, define the open subset of $C^{v,\mathrm{an}}$

$$\Omega_{n,v} := \{(c', a'),\, \min\{g_{c',a',v}(P^n_{c',a'}(c_0)), g_{c',a',v}(P^n_{c',a'}(c_1))\} > G_v(c', a') + \tau_v\} \ .$$

On $\Omega_{n,v}$ one can define the analytic map

$$M_n(c', a') := \frac{\varphi_{c',a'}(P^n_{c',a'}(c_0))^{s_0}}{\varphi_{c',a'}(P^n_{c',a'}(c_1))^{s_1}} \ .$$

Observe that $\Omega_{n+1,v} \subset \Omega_{n,v}$, and $M_{k+l}(c', a') = M_k(c', a')^{3^l}$ on $\Omega_{k,v}$ for all integers $k, l \geq 0$. We also define the increasing sequence of open sets

$$U_{n,v} := \left\{(c', a'),\, G_v(c', a') > \frac{\tau_v}{3^n - 1}\right\} \subset C^{v,\mathrm{an}} \ .$$

Since G_v is subharmonic and proper on $C^{v,\mathrm{an}}$, the set $U_{n,v}$ contains no bounded component by the maximum principle.

Lemma 7.2. — *Suppose $3^r \geq \max\{s_0/s_1, s_1/s_0\}$. Then we have $\Omega_{n,v} \subset U_{n,v}$ and $U_{n,v} \subset \Omega_{n+r,v}$.*

By Theorem 4.1 2., one can find two integers $q \geq 1$ and $N \geq 1$ such that M_q is well-defined and constant equal to a root of unity lying in $\mathbb{K}$ in each component of $U_{N,v}$.

Let V be the connected component of $\Omega_{n,v}$ containing (c, a). This open set might or might not be bounded. By the previous lemma, if $n \geq \max\{r + N, q\}$, then $U_{n-r,v} \subset \Omega_{n,v}$ so that M_n is well-defined on $U_{n-r,v}$. Since all components of $U_{n-r,v}$ are unbounded, and $M_n = M_q^{3^{n-q}}$ in $U_{N,r}$, we conclude that M_n is locally constant in $U_{n-r,v}$ (hence on V) equal to a root of unity lying in $\mathbb{K}$.

When $n \leq n_0 = \max\{r + N, q\}$, then $(M_n)^{3^{n_0-n}} = M_{n_0}$ which we know is constant in V equal to a root of unity lying in $\mathbb{K}$. We conclude that M_n is constant on V equal to a root of unity lying in a fixed extension $\mathbb{K}'$ of $\mathbb{K}$ that only depends on the constants r, N and q. Since these constants are in turn independent of the place v, we conclude the proof of the lemma replacing $\mathbb{K}$ by $\mathbb{K}'$. □

Proof of Lemma 7.2. — Pick $(c',a') \in \Omega_{n,v}$. We may suppose that $G_v(c',a') = g_{c',a',v}(c_0)$ so that $G_v(c',a') = g_{c',a',v}(c_0) = \frac{1}{3^n} g_{c',a',v}(P^n_{c',a'}(c_0)) > \frac{1}{3^n}(G_v(c',a') + \tau_v)$ which implies $(c',a') \in U_{n,v}$.

Conversely suppose $(c',a') \in U_{n,v}$. As before we may suppose that $G_v(c',a') = g_{c',a',v}(c_0)$ so that

$$g_{c',a',v}(P^{n+r}(c_0)) \geq g_{c',a',v}(P^n(c_0)) = 3^n g_{c',a',v}(c_0) = \\ 3^n G_v(c',a') = G_v(c',a') + (3^n-1)G_v(c',a') > G_v(c',a') + \tau_v \ .$$

Similarly we have

$$g_{c',a',v}(P^{n+r}(c_1)) = 3^{n+r}\frac{s_0}{s_1} g_{c',a',v}(c_1) \geq 3^n g_{c',a',v}(c_0) > G_v(c',a') + \tau_v$$

hence $(c',a') \in \Omega_{n+r,v}$. □

7.2. Proof of Theorem A

According to the implication (1) ⇒ (5) of Theorem C, any irreducible algebraic curve C of Poly_3 containing infinitely many post-critically finite polynomials is a component of some $Z(q,m,\zeta)$ so that Theorem A reduces to the following.

Proposition 7.3. —

1. *The set $Z(q,m,1)$ is equal to the union $\{P^{m+q}_{c,a}(c_1) = P^q_{c,a}(c_0)\} \cup \{P^{m+q}_{c,a}(c_0) = P^q_{c,a}(c_1)\}$, hence contains an algebraic curve. Moreover, one has $Z(1,0,1) = Z(0,0,1)$.*
2. *The set $Z(q,m,-1)$ is infinite if and only if $m = 0$, and we have $Z(q,0,-1) = \{12a^3 - c^3 - 6c = 0\}$ for any $q \geq 0$.*
3. *if $\zeta^2 \neq 1$, the set $Z(q,m,\zeta)$ is finite.*

We shall rely on the following observation. Denote by $\mathrm{Crit}(P)$ the set of critical points of the polynomial P.

Lemma 7.4. — *Pick any $(c,a) \in Z(q,m,\zeta)$, and suppose that*

$$Q_{c,a} = \zeta P^m_{c,a} + (1-\zeta)\frac{c}{2}$$

is a polynomial that commutes with $P^k_{c,a}$ and ζ is a (3^k-1)-root of unity. Then we have

$$Q_{c,a}(\mathrm{Crit}(P^{k+m}_{c,a})) = Q_{c,a}(\mathrm{Crit}(P^m_{c,a})) \cup \mathrm{Crit}(P^k_{c,a}) \ .$$

Proof. — Write $P = P_{c,a}$ and $Q = Q_{c,a}$. Differentiate the equality $P^k \circ Q = Q \circ P^k$. Since $Q' = \zeta \cdot (P^m)'$, we get

$$\mathrm{Crit}(Q \circ P^k) = P^{-k}(\mathrm{Crit}(Q)) \cup \mathrm{Crit}(P^k) = \\ P^{-k}(\mathrm{Crit}(P^m)) \cup \mathrm{Crit}(P^k) = \mathrm{Crit}(P^{k+m}) \ ,$$

and therefore

$$\mathrm{Crit}(P^{k+m}) = \mathrm{Crit}(P^k \circ Q) = \mathrm{Crit}(P^m) \cup Q^{-1}(\mathrm{Crit}(P^k)) \ ,$$

and we conclude taking the image of both sides by Q. □

We now come to the proof of the Proposition.

Proof of Proposition 7.3. — We may and shall assume that all objects are defined over the field of complex numbers.

1. Suppose $Z(q,0,\zeta)$ contains an irreducible curve C. We shall prove that either $\zeta=\pm 1$, or $C=\{c=0\}$.

Observe that for any $(c,a)\in Z(q,0,\zeta)$, the polynomial $Q_{c,a}$ is an affine map which commutes with $P_{c,a}^k$, hence $g_{c,a}(Q_{c,a}(z))=g_{c,a}(z)$ for all $z\in\mathbb{C}$. Without loss of generality, we may suppose that $Q_{c,a}(P_{c,a}^q(c_0))=P_{c,a}^q(c_1)$, hence $G(c,a)=g_0(c,a)=g_1(c,a)$.

Suppose that $Z(q,0,\zeta)$ contains an irreducible curve C. If g_0 vanishes identically on C then g_1 also, and this implies both critical points to be persistently preperiodic so that all polynomials in C are post-critically finite. This cannot happen, so that we can find an open set U in C such that $G(c,a)>0$ for all $(c,a)\in U$.

Pick any parameter (c,a) in U. We have $Q_{c,a}(\mathrm{Crit}(P_{c,a}^k))=\mathrm{Crit}(P_{c,a}^k)$ by Lemma 7.4, so that $Q_{c,a}(c_0),Q_{c,a}(c_1)\in\mathrm{Crit}(P_{c,a}^k)$. Since

$$\mathrm{Crit}(P_{c,a}^k)=\bigcup_{0\le j\le k-1}P_{c,a}^{-j}(\mathrm{Crit}(P_{c,a}))$$

we get $g_{c,a}(\alpha)=3^{-j}g_{c,a}(c_0)<g_{c,a}(c_0)=G(c,a)$ for any α lying in $\mathrm{Crit}(P_{c,a}^k)$ but not in $\mathrm{Crit}(P_{c,a})$. However $g_{c,a}(Q_{c,a}(c_0))=g_{c,a}(Q_{c,a}(c_0))=G(c,a)$, therefore we have $Q_{c,a}(c_0),Q_{c,a}(c_1)\in\mathrm{Crit}(P_{c,a})=\{c_0,c_1\}$.

This implies either $(1-\zeta)c=0$, or $(1+\zeta)c=0$, hence $\zeta=\pm1$ or $C=\{c=0\}$ as required.

2. Suppose now that C is an irreducible curve included in $Z(q,m,\zeta)$ with $m>0$. We claim that either $\zeta=1$, or $C=\{c=0\}$ as above.

We proceed similarly as in the previous case. We suppose that $Z(q,m,\zeta)$ is infinite. For any $(c,a)\in Z(q,m,\zeta)$, the polynomial $Q_{c,a}$ commutes with $P_{c,a}^k$ for some k, and has degree $3^m>1$. In particular we have equality of Green functions $g_{Q_{c,a}}=g_{c,a}$. Without loss of generality we may (and shall) assume $Q_{c,a}(P_{c,a}^q(c_0))=P_{c,a}^q(c_1)$, which implies $g_0(c,a)=3^m g_1(c,a)$.

Assume now by contradiction that $\zeta\neq 1$. Proceeding as in the previous case, we can find an open set $U\subset C$ such that $G(c,a)=g_0(c,a)>0$ for all $(c,a)\in U$. Pick now $(c,a)\in U$.

Lemma 7.5. — *For any $\alpha\in P_{c,a}^{-m}\{c_0\}$, we have $Q_{c,a}(\alpha)\in\{c_0,Q_{c,a}(c_1)\}$.*

Observe that

$$Q_{c,a}(\alpha)=\zeta P_{c,a}^m(\alpha)+(1-\zeta)\frac{c}{2}=\zeta c_0+(1-\zeta)\frac{c}{2}=(1-\zeta)\frac{c}{2}.$$

The equality $Q_{c,a}(c_1)=Q_{c,a}(\alpha)$ therefore gives $Q_{c,a}(c_1)=\zeta P_{c,a}^m(c_1)+(1-\zeta)\frac{c}{2}=(1-\zeta)\frac{c}{2}$, and we find $P_{c,a}^m(c_1)=0=c_0$ so that C is a component of $Z(1,m,1)$.

The equality $c_0=Q_{c,a}(\alpha)$, implies $(1-\zeta)\frac{c}{2}=0$ so that either $\zeta=1$, or C equals $\{c=0\}$.

3. We have $Z(q,0,-1)=Z(0,0,-1)$ for all $q\ge 0$.

Fix $q\ge 0$, and pick any $(c,a)\in Z(q,0,-1)$. Observe that $(-1)^3=-1$ hence $Q_{c,a}(z)=-z+c$ commutes with $P_{c,a}$ by definition. A direct computation shows that this happens if and only if (c,a) belongs to the curve $D_1:=\{12a^3-c^3-6c=0\}$.

One can also check that $Q_{c,a}(c_0)=c_1$ for any parameter on D_1, and this implies $(Q_{c,a}\circ P_{c,a}^q)(c_0)=P_{c,a}^q(Q_{c,a}(c_0))=P_{c,a}^q(c_1)$ for any $q\ge 0$. This implies the claim.

4. The irreducible curve $D_0 = \{c = 0\}$ is included in $Z(q, m, \zeta)$ if and only if $m = 0$ and $\zeta = 1$.

Observe that any polynomial $P := P_{0,a}$ in D_0 is unicritical with a single critical point at 0, so that D_0 is included in $Z(q, 0, 1)$ for all $q \geq 0$. Observe also that $g_0 = g_1 > 0$ on a non-empty open subset of D_0.

Suppose that D_0 is included in $Z(q, m, \zeta)$ for some positive integer $m > 0$. Then the Green function of $Q := Q_{0,a}$ is equal to g_P and the equation $Q(P^q(c_0)) = P^q(c_1)$ implies $g_P(c_0) = 3^m g_P(c_1) > g_P(c_1)$ at least when P is close enough to infinity. This is absurd.

Suppose now that D_0 is included in $Z(q, 0, \zeta)$ with $\zeta \neq 1$ so that $Q(z) = \zeta z$. One checks by induction that for any integer $k \geq 1$ one has

$$P^k(z) = q_k z^{3^k} + s_k a^3 z^{3^k - 3} + \text{l.o.t}$$

with $q_k, s_k \in \mathbb{Q}_+^*$. Choose k minimal such that $\zeta^{3^k} = \zeta$. We get

$$Q^{-1} \circ P^k \circ Q(z) = q_k z^{3^k} + s_k a^3 \zeta^{3^k - 4} z^{3^k - 3} + \text{l.o.t} \neq P^k$$

which yields a contradiction, and concludes the proof of our claim.

5. We may now prove the proposition. The first statement follows from the definition of $Z(q, m, 1)$, since in that case we have $Q = P^m$ which always commutes with P. Moreover, the curve $Z(1, 0, 1)$ is given by the equation

$$0 = P_{c,a}(c_0) - P_{c,a}(c_1) = a^3 - \left(a^3 - \frac{c^3}{6}\right) = c\frac{c^3}{6} \ ,$$

whence $Z(1, 0, 1) = \{c = 0\} = Z(0, 0, 1)$.

For the second statement, suppose first that $Z(q, m, -1)$ is infinite. By the second step, we have $m = 0$, or $D_0 = \{c = 0\}$ is included in $Z(q, m, -1)$. The fourth step rules out the latter possibility so that $m = 0$. Conversely if $m = 0$ we may apply the third step to conclude that $Z(q, 0, -1)$ is a curve equal to $D_1 = \{12a^3 - c^3 - 6c = 0\}$.

For the third statement, pick $\zeta \neq \pm 1$ and suppose by contradiction that $Z(q, m, \zeta)$ is infinite. The first and second step imply that $Z(q, m, \zeta)$ contains D_0 which is impossible by Step 4.

This concludes the proof of the proposition. □

Proof of Lemma 7.5. — Take $\alpha \in P_{c,a}^{-m}\{c_0\}$, and observe that $\alpha \in \mathrm{Crit}(P_{c,a}^{k+m})$. According to Lemma 7.4, we have

$$Q_{c,a}(\alpha) \in \mathrm{Crit}(P_{c,a}^k) \cup Q_{c,a}(\mathrm{Crit}(P_{c,a}^m))$$

and $g_{c,a}(Q_{c,a}(\alpha)) = 3^m g_{c,a}(\alpha) = 3^m \cdot 3^{-m} g_{c,a}(c_0) = g_{c,a}(c_0) = G(c, a) > 0$.

Pick any point $z \in \mathrm{Crit}(P_{c,a}^k) \cup Q_{c,a}(\mathrm{Crit}(P_{c,a}^m))$, and suppose it is equal to neither c_0 nor $Q_{c,a}(c_1)$. Then we are in one of the following (excluding) cases:

1. z is a preimage of c_0 under $P_{c,a}^j$ for some $1 \leq j \leq k-1$, and $g_{c,a}(z) < g_P(c_0)$;
2. z is a preimage of c_1 under $P_{c,a}^j$ for some $0 \leq j \leq k-1$, in which case $g_{c,a}(z) \leq g_{c,a}(c_1) < g_{c,a}(c_0)$;
3. $z \in Q_{c,a}(\mathrm{Crit}(P_{c,a}^m))$, so that $g_{c,a}(z) = 3^m g_{c,a}(w)$ for some point $w \in \mathrm{Crit}(P_{c,a}^m) = \bigcup_{0 \leq j \leq m-1} P_{c,a}^{-j}(\mathrm{Crit}(P_{c,a}))$.

In the last case two sub-cases arise. When w is a preimage of c_0, we find

$$g_{c,a}(z) = 3^m g_{c,a}(w) \geq 3^m \frac{1}{3^{m-1}} g_{c,a}(c_0) > g_{c,a}(c_0) \ .$$

Otherwise w is a preimage of c_1 distinct from c_1 since $z \neq Q_{c,a}(c_1)$. And we find $g_{c,a}(z) = 3^m g_{c,a}(w) \leq 3^{m-1} g_{c,a}(c_1) < g_{c,a}(c_0)$.

Since $g_{c,a}(Q_{c,a}(\alpha)) = g_{c,a}(c_0)$ we conclude that $z \neq Q_{c,a}(\alpha)$ as required. □

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