HAL open science

Angular parametrization of rectangular paraunitary matrices

Didier Pinchon, Pierre Siohan

▶ To cite this version:

Didier Pinchon, Pierre Siohan. Angular parametrization of rectangular paraunitary matrices. 2016. hal-01289570

HAL Id: hal-01289570
https://hal.science/hal-01289570v1

Preprint submitted on 17 Mar 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Angular parametrization of rectangular paraunitary matrices

Didier Pinchon and Pierre Siohan

March 17, 2016

Contents

1 Introduction **2**

2 Paraunitary matrices **4**
2.1 Definitions and examples . . . 4
2.2 Real algebraic sets . . . 5
2.3 McMillan valuation and degree . . . 6

3 Givens sets **7**
3.1 Definition of a Givens set . . . 7
3.2 Connected Givens sets . . . 8
3.3 Dimension of a Givens set . . . 10
3.4 McMillan valuation and degree of a Givens set . . . 11
3.5 Algebraic Givens sets . . . 13
3.6 Some representations of $SO(N)$ as a Givens set . . . 13

4 Main theorems **17**
4.1 SVD and order reduction in $\mathcal{O}_{M,N}^{(m)}$ . . . 17
4.2 Technical lemmas . . . 19
4.3 The Givens sets $\mathcal{U}_{M,N,m,d}$ . . . 20
4.4 The case $M = N$ . . . 23
4.5 The case $N \geq 2M$ . . . 25
4.6 The case $M < N < 2M$ . . . 26

5 Conclusion **27**

6 Acknowledgements **28**

References **29**

A Counterexamples for Theorem 1 and Lemma 2 in [1] **31**

B Generic properties **34**

C Proof of Theorem 4.9 (b) **36**

Nomenclature **42**

Abstract

In this preprint a new approach is proposed to find efficient angular parametrizations of rectangular paraunitary matrices (PU). As for square PU matrices, the problem is stated as a matrix factorization, the efficiency of which is ensured by the fact that the proposed representations are complete, non redundant and lead to a minimal number of delays. As in our previous publication, dedicated to the case of square real coefficients PU matrices, our framework is based on an *ensemblist* approach. The angular parametrization sets, named Givens sets, are analyzed according to three different possibilities related to the matrix size. Thus, for a $N \times M$ PU matrix, different properties of Givens sets are established corresponding to the cases where $N = M$, $N \geq 2M$ and $M < N < 2M$.

Keywords: Cosine-Sine Decomposition (CSD),Paraunitary matrix, polynomial matrix, QR matrix decomposition, singular value decomposition (SVD), $SO(N)$ group.

1 Introduction

Paraunitary matrices play an essential role in the design and implementation of filter bank and wavelet based systems either for source coding or transmission purposes. Indeed such devices are actively investigated for subband coding [2], multicarrier modulation [3, 4], channel error correction [5] and spatial multiplexing in wireless communications [6]. Therefore it is of paramount importance to get a deep knowledge of the fundamental properties of paraunitary matrices (PU). Whatever the application at hand, a key problem is to get an efficient factorization of the PU matrices associated to the polyphase matrix of a given multirate system. First of all, it is important to distinguish the case of square PU matrices, i.e. the ones associated to critically sampled systems, from the case of rectangular PU matrices, i.e. associated to oversampled systems. In the signal processing community, the factorization of PU matrices has been initially studied to build perfect reconstruction critically decimated filter banks, see e.g. [7] to get a general overview. In [8], it is shown that the order one factorization (OOF), initially described in [9], leads to a complete parametrization. Furthermore, Gao et al. [8] have introduced a new method, namely the Cosine-Sine Decomposition (CSD), thus reducing around half the number of parameters. The CSD is both complete and minimal, i.e. it uses the smallest possible number of delays, but it involves redundant parameters. The post-filtering based method by Gan and Ma [10] permits an improvement with respect to [8], reducing the necessary number of parameters to represent a *given* unitary square matrix.

In [3], the authors summarize all these results presenting the number of angular parameters used to represent square PU matrices containing either real or complex coefficients. In the case of real coefficients square $N \times N$ PU matrix of polynomial degree m we get the following figures for the number of angular parameters.

- For the OOF structure (rewritten from [3, equation (47)])

$$\mu_1 = (m+1)\frac{N(N-1)}{2}; \tag{1.1}$$

- For the CSD, one gets (equivalent to equation (54) in [3])

$$\mu_2 = \begin{cases} m\frac{N^2}{4} + \frac{N(N-1)}{2} & \text{if } N \text{ is even,} \\ m\frac{N^2-1}{4} + \frac{N(N-1)}{2} & \text{if } N \text{ is odd;} \end{cases} \tag{1.2}$$

- Finally, for the post-filtered based method [10], the number of rotations when parameters $r_k, 1 \leq k \leq m$ have the same value r with $1 \leq r \leq \lfloor \frac{N}{2} \rfloor$ is equal to

$$\mu_3 = mr(N-r) + \frac{N(N-1)}{2}. \tag{1.3}$$

  This is formula (59) in [3], in a simplified form.

Up to now, most often the factorization of rectangular paraunitary matrices has been considered as a direct extension with respect to the factorization of square ones. Thus, for oversampled M-band filter banks characterized by rectangular $N \times M$ PU matrices with $N > M$, the authors in [11] consider that such a rectangular matrix can be embedded in a larger $N \times N$ matrix. More recently, in a similar way, looking now at the dual case, i.e. an oversampled transmultiplexer, Rahimi and Champagne [3] also propose to adapt the OOF, CSD and prefiltered based method to the case of rectangular matrices. Contrary to [11], [3], in [12] our study directly handles the factorization of rectangular PU matrices as being a specific case. Furthermore, we did not only focus on a *given* matrix or factorization method but, using an *ensemblist* approach, we explore the whole sets of possible factorizations of the PU matrices associated to Discrete Fourier Transform (DFT) oversampled filter banks of given length.
In this preprint, in the same line of idea, our aim is to characterize the whole sets of real rectangular PU matrices leading to complete, minimal and non redundant factorizations. As in [8], [10], our decomposition is based on products of rotation and delay matrices but, in our case, those are non-square matrices and associated to matrices sets. Denoting by $\mathcal{U}_{M,N,m,d}$ these sets, we call Givens sets, m being their degree and d the total number of delays, we establish a criterion to find out to which $\mathcal{U}_{M,N,m,d}$ set a given real PU matrix belongs. Knowing the polyphase matrix coefficients, we can get such a result using, in addition to the classical McMillan degree, the McMillan valuation parameter. For $N = M$, Gan and Ma have already found complete, minimal and non redundant factorizations, we therefore arrive at the same number of rotations in the case of square matrices but, using our approach, no post processing is required. In addition, using the concept of dimension of a Givens set we provide a rigorous proof of the non-redundancy of the proposed parametrization. To summarize our analysis in terms of Givens sets leads to following results

- When $N = M$, the Givens sets are disjoint and their minimality can be directly deduced from the determinants of their constituent matrices;
- When $N \geq 2M$, all paraunitary matrices with polynomial degree less than m are members of an unique Givens set;
- When $M < N < 2M$, maximal Givens sets for inclusion intersect and they can nevertheless be characterized using their McMillan valuation and degree.

Our preprint is organized as follows. Section 2 contains the basic elements related to paraunitary matrices, real algebraic sets and introduces the notions of McMillan valuation and degree of a rectangular polynomial matrix. Section 3 introduces Givens sets, some of their basic properties and provides several representations of the $SO(N)$ group of matrices as Givens sets. In Section 4, using the Singular Value Decomposition (SVD) for rectangular matrices, our main theorems give a description of the set of paraunitary matrices with bounded polynomial degree as an union of maximal Givens sets for inclusion, for different ranges of values of N with respect to M. Finally, we end in Section 5 with a conclusion stating our achievements together with open problems issued from our work.

2 Paraunitary matrices

2.1 Definitions and examples

In the rest of this paper the row and column indexes of matrices will be denoted starting with the index 0 and matrices appear in bold characters while sets of matrices are written with standard characters.
Given two positive integers M and N with $1 \leq M \leq N$, a matrix $\boldsymbol{A}(X)$ with N rows and M columns for which elements are polynomial in X with real coefficients is said to be *paraunitary* if

$$\boldsymbol{A}(1/X)^T \boldsymbol{A}(X) = \boldsymbol{I}_M, \tag{2.1}$$

where $(.)^T$ is the transposition operator and $\boldsymbol{I}_M$ is the identity matrix of size M. Note that the X variable corresponds to a delay element and is denoted z^{-1} in the signal processing literature.

It is straightforward to verify that the product $\boldsymbol{AB}$ of a $N \times M$ paraunitary matrix $\boldsymbol{A}$ by a $M \times M$ paraunitary matrix $\boldsymbol{B}$ is a $N \times M$ paraunitary matrix.

Definition 2.1. . *Let $s : [0, 1, \ldots, M-1] \to [0, 1, \ldots, N-1]$ be an injective mapping and $\varepsilon \in \{-1, 1\}^M$ a map from $[0, 1, \ldots, M-1]$ to $\{-1, 1\}$. We denote by $\boldsymbol{E}[s, \varepsilon]$ the constant $N \times M$ orthogonal matrix such that $\boldsymbol{E}[s, \varepsilon]_{s(c),c} = \varepsilon(c),\ 0 \leq c \leq M-1$ and $\boldsymbol{E}[s, \varepsilon]_{r,c} = 0$ otherwise. Such matrix is said to be an* elementary paraunitary matrix.

As an example, for $M = 4, N = 5$, s and ε defined by

$$\begin{aligned} &s(0) = 3,\ s(1) = 2,\ s(2) = 1,\ s(3) = 4, \\ &\varepsilon(0) = \varepsilon(2) = 1,\ \varepsilon(1) = \varepsilon(3) = -1, \end{aligned}$$

we get

$$\boldsymbol{E}[s, \varepsilon] \quad = \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

In the following ε_0 denotes the function such that $\varepsilon_0(c) = 1,\ 0 \leq c \leq M-1$ and $\boldsymbol{E}[s, \varepsilon_0]$ will be denoted by $\boldsymbol{E}[s]$.
We note $s_0[0, 1, \ldots, M-1] \to [0, 1, \ldots, N-1]$ the mapping such that $s_0(c) = c,\ 0 \leq c \leq M-1$, and the matrix $\boldsymbol{E}[s_0]$ is also denoted $\boldsymbol{I}_{N,M}$.

For $N \geq 2$ and $i, j, 0 \leq i, j < N;\ i \neq j$ and $\theta \in \mathbb{R}$, the square $N \times N$ orthogonal matrix $\boldsymbol{R}_{i,j}(\theta)$ defined by $[\boldsymbol{R}_{i,j}(\theta)]_{i,i} = [\boldsymbol{R}_{i,j}(\theta)]_{j,j} = \cos\theta$, $[\boldsymbol{R}_{i,j}(\theta)]_{j,i} = -[\boldsymbol{R}_{i,j}(\theta)]_{i,j} = \sin\theta$, $[\boldsymbol{R}_{i,j}(\theta)]_{r,r} = 1,\ r \neq i,\ r \neq j$ and $[\boldsymbol{R}_{i,j}(\theta)]_{r,c} = 0$ otherwise, is called an *elementary rotation matrix*, or a *Givens matrix*, of indexes i, j.
As an example, for $N = 5, i = 0, j = 2$, we get

$$\boldsymbol{R}_{0,2}(\theta) \quad = \quad \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For $N \geq 2$ and $0 \leq r < N$ the paraunitary diagonal square matrix $\boldsymbol{Z}_r$ of dimension N, such that $[\boldsymbol{Z}_r]_{r,r} = X$ and $[\boldsymbol{Z}_r]_{k,k} = 1$ if $k \neq r$ is called the *shift matrix* on row r, since it indeed introduces a shift in time, i.e. a delay, when being implemented in practice.
For $I \subset \{0, 1, \ldots, N-1\}$, $\boldsymbol{Z}_I = \prod_{r \in I} \boldsymbol{Z}_r$ and $\boldsymbol{Z}_{r_1..r_2}$ with $0 \leq r_1 \leq r_2$ stands for $\boldsymbol{Z}_I$ with $I = \{r_1, \ldots, r_2\}$.

The set of all orthogonal $N \times N$ matrices, which is a group, is denoted by $O(N)$, and its subgroup of orthogonal matrices with determinant 1 is denoted by $SO(N)$. $SO(N)$ may be identified to the set of direct rotations in $\mathbb{R}^N$ [13].
When N is given, if $I \subset \{0, \ldots, N-1\}$ is a set of indexes, the subset of matrices $\boldsymbol{A}$ in $SO(N)$ such that $\boldsymbol{A}_{c,c} = 1,\ c \in \{0, \ldots, N-1\} \setminus I$ will be denoted by $SO(I)$. $SO(I)$ is the subgroup of $SO(N)$ of rotations acting only on coordinates in I. When $I = \{i_1, i_2, \ldots, i_K\}$ for $2 \leq K < N$, $SO(I)$ is also denoted by $SO(i_1, i_2, \ldots, i_K)$ instead of $SO(\{i_1, i_2, \ldots, i_K\})$.

Let m be a nonnegative integer. The set of $M \times N$ matrices $\boldsymbol{A}(X)$, the coefficients of which are real polynomial in X of degree less or equal to m, is a real vector space denoted by $\mathcal{M}_{M,N}^{(m)}$.
Any matrix $\boldsymbol{A}(X) \in \mathcal{M}_{M,N}^{(m)}$ may be written

$$\boldsymbol{A}(X) = \sum_{k=0}^{m} \boldsymbol{A}_k X^k, \tag{2.2}$$

where $\boldsymbol{A}_k,\ 0 \leq k \leq m$ are constant $N \times M$ matrices.
The maximum value of m for which $\boldsymbol{A}_m$ is not null is called the *order* or the *polynomial degree* of $\boldsymbol{A}(X)$ to distinguish it from its McMillan degree that will be defined below.

For $0 \leq r < N, 0 \leq c < M,\ 0 \leq k \leq m$, let us define the matrix $\boldsymbol{E}_{r,c,k}$ by $[\boldsymbol{E}_{r,c,k}]_{r,c} = X^k$ and 0 otherwise.
In this way, $\boldsymbol{A}(X)$ given by (2.2) may be written

$$\boldsymbol{A}(X) = \sum_{k=0}^{m} \sum_{r=0}^{N-1} \sum_{c=0}^{M-1} a_{r,c,k} \boldsymbol{E}_{r,c,k}, \tag{2.3}$$

where $a_{r,c,k} = [\boldsymbol{A}_k]_{r,c}$ are called the coefficients of $\boldsymbol{A}(X)$.
So the $MN(m+1)$ matrices $\boldsymbol{E}_{r,c,k}$ form a basis of $\mathcal{M}_{M,N}^{(m)}$ which is called its *canonical basis* and the linear application ψ from $\mathcal{M}_{M,N}^{(m)}$ to $\mathbb{R}^{MN(m+1)}$ such that $\psi(\boldsymbol{E}_{r,c,k}) = e_n$ with $n = kMN + rM + c$, where $e_n, 0 \leq n < MN(m+1)$ is the canonical basis of $\mathbb{R}^{MN(m+1)}$, is an isomorphism of vector spaces.

We denote by $\mathcal{O}_{M,N}^{(m)}$ the subset of all paraunitary matrices in $\mathcal{M}_{M,N}^{(m)}$.
Given a $N \times M$ matrix $\boldsymbol{A}(X)$ with $1 < M \leq N$, $\Pi_{M,N}\boldsymbol{A}(X)$ denotes the $N \times (M-1)$ matrix obtained by suppressing the last column in $\boldsymbol{A}(X)$ and for $\boldsymbol{A}(X) \in \mathcal{O}_{M,N}^{(m)}$, $\sigma_m \boldsymbol{A}(X)$ is the matrix $X^m \boldsymbol{A}(1/X)$.
Operators Π and σ_m readily extend to sets of matrices.

2.2 Real algebraic sets

For the definitions and main properties of algebraic sets, we refer to online notes of Andreas Gathmann [14].

A subset A of $\mathcal{M}_{M,N}^{(m)}$ is a *real algebraic set* of $\mathcal{M}_{M,N}^{(m)}$ if there exists a finite family of polynomials, $P_1(x), P_2(x), \ldots, P_l(x)$ in $\mathbb{R}[x]$ with $x = \{x_{r,c,k},\ 0 \leq r \leq N-1,\ 0 \leq c \leq M-1,\ 0 \leq k \leq m\}$ such that $\boldsymbol{A} \in A$ if and only if the coefficients of $\boldsymbol{A}$, $a_{r,c,k}$, $0 \leq r \leq N-1$, $0 \leq c \leq M-1$, satisfy $P_1(a) = P_2(a) = \cdots = P_l(a) = 0$.
Remark that the finite family $P_1(x), P_2(x), \ldots, P_l(x)$ may be replaced by the unique polynomial $\sum_{i=1}^{l} P_i^2(x)$.

The set of all algebraic subsets of a real algebraic set A is closed by finite union and intersection. Thus it can be considered as the set of closed subsets of A for a topology which is called the Zariski topology. A may also be endowed by the induced topology of the usual topology on $\mathbb{R}^{MN(m+1)}$, which will be called the *Euclidean topology* on A.
The Zariski topology on A is coarser than the Euclidean topology on A meaning that an Euclidean closed set in A is not necessarily a Zariski closed set, that is a real algebraic subset of A.
If $E \subset A$, the closure of E for the Euclidean topology, is denoted by $\bar{E}$ and its closure for the Zariski topology, i.e. the smallest real algebraic set containing E, is denoted by $\bar{E}^Z$. So $\bar{E} \subset \bar{E}^Z$ and in general the inclusion is strict.

The set $\mathcal{O}_{M,N}^{(m)}$ is a real algebraic set because the paraunitary relation (2.1) is equivalent to a set of polynomial equations on the coefficients of $\boldsymbol{A} \in \mathcal{M}_{M,N}^{(m)}$.

A real algebraic set A is said to be *reducible* if it can be written as $A = A_1 \cup A_2$ where A_1 and A_2 are real algebraic sets strictly included in A. Otherwise A is said to be *irreducible*. It can be proved ([14, Proposition 2.15]) that a real algebraic set A can be written in an unique way as a finite union of irreducible real algebraic sets, called the *irreducible components* of A, A_i, $1 \leq i \leq K$, satisfying $A_i \not\subset A_j$ if $i \neq j$.

The following property shows an important peculiarity of the Zarisky topology : if A is an irreducible algebraic real set, then any non-empty subset U of A is dense in A for the Zarisky topology, i.e. $\overline{U}^Z = A$.

2.3 McMillan valuation and degree

Given a polynomial $p(X) = \sum_{n \geq 0} p_n X^n$ in variable X with real coefficients, the degree of p, denoted by $\deg(p)$, is defined by $\deg(p) = -\infty$ if $p = 0$ and, if $p \neq 0$, is the maximum value of n such that $p_n \neq 0$. The valuation of p, denoted by $\mathrm{val}(p)$, is defined by $\mathrm{val}(p) = +\infty$ if $p = 0$ and , if $p \neq 0$, is the minimum value of n such that $p_n \neq 0$.

For given positive integers $M \leq N$ and a non negative integer m, let us consider a matrix $\boldsymbol{A} \in \mathcal{M}_{M,N}^{(m)}$.
For $I \subset \{0, 1, \ldots, N-1\}$ and $J \subset \{0, 1, \ldots, M-1\}$ with $\mathrm{Card}(I) = \mathrm{Card}(J) = n$, we denote by $\boldsymbol{A}_{I,J}$ the $n \times n$ matrix of elements of $\boldsymbol{A}$ with row indexes in I and column indexes in J. The determinant $\mathrm{Det}(\boldsymbol{A}_{I,J})$ is the *minor* of $\boldsymbol{A}$ relative to subsets I and J. Any minor $\mathrm{Det}(\boldsymbol{A}_{I,J})$ is a polynomial in X with coefficients that can be expressed as polynomials in the coefficients of $\boldsymbol{A}$.

Definition 2.2. *The number*

$$\mathrm{val}(\boldsymbol{A}) = \min\left\{\mathrm{val}(\det(\boldsymbol{A}_{I,J})),\ \mathrm{Card}(I) = M,\ J = \{0, \ldots, M-1\}\right\}, \tag{2.4}$$

is called the McMillan valuation *of* $\boldsymbol{A}$ *and the number*

$$\deg(\boldsymbol{A}) = \max\left\{\deg(\det(\boldsymbol{A}_{I,J})),\ \mathrm{Card}(I) = M,\ J = \{0, \ldots, M-1\}\right\}, \tag{2.5}$$

is called the McMillan degree *of* $\boldsymbol{A}$.

For $0 \leq v \leq d \leq mM$, let us define now the subsets $A_{M,N,m,v,d}$ and $B_{M,N,m,v,d}$ of $\mathcal{O}_{M,N}^{(m)}$ by :

$$A_{M,N,m,v,d} = \{\boldsymbol{A} \in \mathcal{O}_{M,N}^{(m)},\ v \leq \mathrm{val}(\boldsymbol{A}) \leq \deg(\boldsymbol{A}) \leq d\}, \tag{2.6}$$

$$B_{M,N,m,v,d} = \{\boldsymbol{A} \in \mathcal{O}_{M,N}^{(m)},\ \mathrm{val}(\boldsymbol{A}) = v,\ \deg(\boldsymbol{A}) = d\}. \tag{2.7}$$

In a context where M, N and m are fixed, $A_{M,N,m,v,d}$ and $B_{M,N,m,v,d}$ will be simply denoted by $A_{v,d}$ and $B_{v,d}$.
It is quite evident that $\{B_{v,d},\ 0 \leq v \leq d\}$ is a partition of $\mathcal{O}_{M,N}^{(m)}$, and that

$$A_{v,d} = \bigcup \{B_{v',d'},\ v \leq v' \leq d' \leq d\}. \tag{2.8}$$

Lemma 2.3. *For given M and N such that $1 \leq M \leq N$, $m \geq 1$, and $0 \leq d \leq mM$, $A_{d,d} \neq \emptyset$.*

Proof.– We may write $d = am + b$ with $0 \leq a \leq M$ and $0 \leq b \leq m-1$. If $a = M$ then $b = 0$ because $d \leq mM$. Let us consider now the $N \times N$ matrix $\boldsymbol{A}(X)$ defined by $[\boldsymbol{A}(X)]_{c,c} = X^m$ for $0 \leq c \leq a-1$, $[\boldsymbol{A}(X)]_{a,a} = X^b$, $[\boldsymbol{A}(X)]_{c,c} = 1$ for $a+1 \leq c \leq M-1$ and 0 elsewhere. It is quite straightforward that $\boldsymbol{A}(X) \in \mathcal{O}_{M,N}^{(m)}$ and that $\mathrm{val}(\boldsymbol{A}) = \deg(\boldsymbol{A}) = am + b = d$. □

The structure of $\mathcal{O}_{M,N}^{(m)}$ is studied in detail in the next paragraph and it will be shown, using Givens sets, that

(a) For given M and N such that $1 \leq M \leq N$, $m \geq 1$, and $0 \leq d \leq mM$, $A_{d,d}$ is irreducible,

(b) When $M = N$, $\mathcal{O}_{M,N}^{(m)} = \bigcup_{d=0}^{mM} A_{d,d}$ is the decomposition of $\mathcal{O}_{M,N}^{(m)}$ in irreducible disjoint algebraic components,

(c) When $N \geq 2M$, $\mathcal{O}_{M,N}^{(m)}$ is an irreducible algebraic set.

The case where $M < N < 2M$ is not still completely understood, unless the following conjecture is true.

Conjecture 2.4. *The following properties are verified :*

(a) For $M < N$, $m \geq 1$, let v and d such that $0 \leq v < d \leq mM$, if $B_{v,d} \neq \emptyset$ then $A_{v,d}$ is an irreducible algebraic set and $\overline{B_{v,d}}^{Z} = A_{v,d}$,

(b) For $M < N < 2M$, $m \geq 1$, $\mathcal{O}_{M,N}^{(m)} = \bigcup_{d=m(N-M)}^{mM} A_{d-m(N-M),d}$ is the decomposition of $\mathcal{O}_{M,N}^{(m)}$ in irreducible algebraic components.

3 Givens sets

3.1 Definition of a Givens set

Angles are considered in this paper as real numbers modulo 2π, that is members of $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, the one-dimensional torus. Endowed with the quotient topoloy, $\mathbb{T}$ is a compact space. $\mathbb{T}^n$, for $n \geq 1$, is a compact real C^∞ manifold of dimension n ([15, 5.3]).

Notations 3.1. *Let $N \geq 2$.*
For $i, j, 0 \leq i, j \leq N-1$; $i \neq j$, we denote by $R_{i,j}$ the set $R_{i,j} = \{\boldsymbol{R}_{i,j}(\theta),\ \theta \in \mathbb{T}\}$. $R_{i,j}$ is called a rotation set.
For $0 \leq r \leq N-1$, Z_r, $I \subset \{0, 1, \ldots, N-1\}$ or $0 \leq r_1 \leq r_2 \leq N-1$, Z_r, Z_I and $Z_{r_1..r_2}$ denote the one element sets $\{\boldsymbol{Z}_r\}$, $\{\boldsymbol{Z}_I\}$ or $\{\boldsymbol{Z}_{r_1..r_2}\}$, respectively, and they are called shift sets.
Let M and N be two integers with $N \geq 2$, s an injective function from $[0, 1, \ldots, M-1]$ to $[0, 1, \ldots, N-1]$ and $\varepsilon \in \{-1, 1\}^M$. $E[s, \varepsilon]$ denotes the set that contains the unique elementary $N \times M$ paraunitary matrix $\boldsymbol{E}[s, \varepsilon]$ and is called an elementary set.

If A and B designate two sets of matrices with compatible dimensions, we denote by AB the set of matrix products $\boldsymbol{AB}$ when $\boldsymbol{A} \in A$ and $\boldsymbol{B} \in B$.

The following definition is fundamental.

Definition 3.2. *Let M and N be two integers with $N \geq 2$ and $1 \leq M \leq N$. A* Givens set *with parameters M, N is a set G of $N \times M$ paraunitary matrices such that there exists a sequence $\phi = [T_1, T_2, \ldots T_{n_T}, E[s, \varepsilon]]$ where*

- *For each k, $1 \leq k \leq n_T$, T_k is a $N \times N$ rotation set $R_{i,j}$ or a shift set Z_r with $0 \leq r \leq N-1$,*
- *$E[s, \varepsilon]$ is an elementary set,*

such that G that may be written as the product $G = T_1 T_2 \ldots T_{n_T} E[s, \varepsilon]$.
ϕ is called a parametric representation *of G or simply a* representation *of G and will be denoted by*

$$\phi : G = T_1 T_2 \ldots T_{n_T} E[s, \varepsilon]. \tag{3.1}$$

When G is represented by two representations ϕ_1 and ϕ_2, ϕ_1 and ϕ_2 are said to be equivalent representations *of G.*

In most cases there exist several different parametric representations for a Givens set. A simple example is given, for $M = N = 3$, by the set $SO(3)$ of rotations of $\mathbb{R}^3$ with determinant 1 that can be represented by $SO(3) = R_{0,1} R_{0,2} R_{0,1}$ or $SO(3) = R_{0,2} R_{0,1} R_{1,2}$. Many other examples for $N = M$ and the set $SO(N)$ will be given later in 3.6.

If there exist n_R sets of rotations in the sequence $[T_i,\ 1 \leq i \leq n_T]$ of a representation ϕ of a Givens set G, let $i_1 < i_2 < \cdots < i_{n_R}$ denote their indexes in this sequence. If $\theta_l,\ 1 \leq l \leq n_R$ are n_R given real numbers, then we can choose the rotation of angle θ_l in the set of rotations T_{i_l} for $1 \leq l \leq n_R$. Thus, we get an application from $\mathbb{T}^{n_R}$ to G, which is also called the *parametric representation* of G and also denoted by ϕ.
The matrix $\phi(\theta) \in G$, $\theta = (\theta_1, \ldots, \theta_{n_R})$ is denoted by $\boldsymbol{G}_\phi(\theta)$ or more briefly $\boldsymbol{G}(\theta)$ when the parametric representation ϕ is fixed in a given context.

3.2 Connected Givens sets

For a given value of $N > 2$, let us consider a set of rotation sets $\mathcal{G} = \{R_{i,j},\ (i,j) \in I$ where $I = \{(i,j),\ 0 \leq i, j \leq N-1,\ i \neq j\}$ and the following relation $\mathcal{R}$ on the set of row indexes $\{0, 1, \ldots, N-1\}$: for $r_1 \mathcal{R} r_2$ if $r_1 = r_2$ or if there exists a sequence of rotation sets in $\mathcal{G}$, R_{i_l, j_l}, $l = 1, \ldots, L-1$ such that $r_1 \in \{i_1, j_1\}$, $r_2 \in \{i_L, j_L\}$ and for any l, $1 \leq l \leq L-1$, $\{i_l, j_l\} \cap \{i_{l+1}, j_{l+1}\} \neq \emptyset$. Obviously, $\mathcal{R}$ is an equivalence relation.

Definition 3.3. *$\mathcal{G}$ is said to be* connected *if there is only one equivalence class for the corresponding relation $\mathcal{R}$. A Given set $G = T_1T_2\dots T_KE[s,\varepsilon]$ is said to be connected if the set of rotation sets in the sequence $T_1,\dots,T_K$ is connected.*

For a given value of $N \geq 3$ and a given row index i , let us define by $\Delta[i]$ the diagonal $N \times N$ matrix with -1 on row i and 1 elsewhere on the diagonal.

Lemma 3.4. *Let $2 \leq M \leq N$ with $N \geq 3$. Let $G = T_1T_2\dots T_{n_T}E[s,\varepsilon]$ be a connected Givens set of $N \times M$ matrices. Then for any $r_1, r_2 \in \{0,1,\dots,N-1\}$, $r_1 \neq r_2$,*

$$T_1T_2\dots T_{n_K}\Delta[r_1] = T_1T_2\dots T_{n_K}\Delta[r_2]. \tag{3.2}$$

Proof.– As G is connected there exists a sequence of rotation sets in $\{T_1,T_2,\dots,T_K\}$, R_{i_l,j_l}, $l = 1,\dots,L-1$ such that $r_1 \in \{i_1,j_1\}$, $r_2 \in \{i_L,j_L\}$ and for any l, $1 \leq l \leq L-1$, $\{i_l,j_l\}\cap\{i_{l+1},j_{l+1}\} \neq \emptyset$. Let us denote by a_1 the row index such that $\{i_1,j_1\} = \{r_1,a_1\}$, and by a_l the row index such that $\{i_l,j_l\} \cap \{i_{l+1},j_{l+1}\} = \{a_l\}$, $2 \leq l \leq L-2$. It follows that $\{i_L,j_L\} = \{a_{L_2},r_2\}$. Now let k_1 be the index such that $R_{r_1,a_1} = T_{k_1}$. For $k_1 < k \leq K$, we get $T_k\Delta[r_1] = \Delta[r_1]T_k$ because one of these cases is verified

- T_k is a set Z_r that commutes as a diagonal matrix with $\Delta[r_1]$,
- T_k is a rotation set $R_{i,j}$ with $i \neq r_1$ and $j \neq r_1$ for which any matrix commutes with $\Delta[r_1]$,
- T_k is a rotation set $R_{i,j}$ with $i = r_1$ (or $j = r_1$ which is equivalent because $R_{i,j} = R_{j,i}$). In this case, applying the relation

$$\boldsymbol{R}(\theta)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{R}(-\theta), \text{ with } \boldsymbol{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{3.3}$$

  we can see that again $T_k\Delta[r_1] = \Delta[r_1]T_k$.

Then we use the relation

$$\boldsymbol{R}(\theta)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{R}(\theta+\pi)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.4}$$

to get $R_{r_1,a_1}\Delta[r_1] = R_{r_1,a_1}\Delta[a_1]$. Using now relations $\Delta[a_1]T_k = T_k\Delta[a_1]$ for $k_1 < k \leq K$, we obtain the relation

$$T_1T_2\dots T_K\Delta[r_1] = T_1T_2\dots T_K\Delta[a_1]. \tag{3.5}$$

Using the same arguments, we prove that $\Delta[a_1]$ may be replaced by $\Delta[a_l]$, $l = 2,\dots,L-2$ in (3.5) and finally by $\Delta[r_2]$, which proves the lemma. □

Theorem 3.5. *Let $2 \leq M < N$ and $G = T_1T_2\dots T_KE[s,\varepsilon]$ be a connected Givens set of $N \times M$ matrices. Then $G = T_1T_2\dots T_KE[s,\varepsilon_0]$.*

Proof.– The matrix $\boldsymbol{E}[s,\varepsilon]$ may be written as

$$\boldsymbol{E}[s,\varepsilon] = \boldsymbol{D}\boldsymbol{E}[s,\varepsilon_0], \tag{3.6}$$

where $\boldsymbol{D}$ is a $N \times N$ diagonal matrix with element $\varepsilon(r(c))$ on row $r(c), 0 \leq c \leq M-1$ and elements equal to -1 or $+1$ on the other rows (that exist because $M < N$), in such a way that the number

of -1 is even. So there exists a sequence of couple of row indices $(r_{2i}, r_{2i+1}),\ 0 < i \leq L-1$ with $2L \leq N$ such that $\boldsymbol{D}$ be written as

$$\boldsymbol{D} = \prod_{i=0}^{L-1} \Delta[r_{2i}]\Delta[r_{2i+1}]. \tag{3.7}$$

Using lemma 3.4, we get, for $0 < i \leq L-1$

$$T_1 T_2 \dots T_K \Delta[r_{2i}] = T_1 T_2 \dots T_K \Delta[r_{2i+1}], \tag{3.8}$$

because G is connected and

$$G = T_1 T_2 \dots T_K \prod_{i=0}^{L-1} \Delta[r_{2i+1}]^2 E[s, \varepsilon_0] = T_1 T_2 \dots T_K E[s, \varepsilon_0]. \tag{3.9}$$

□

3.3 Dimension of a Givens set

Theorem and Definition 3.6. *(a) For a Givens set G with a parametric representation ϕ containing n_R rotations, the rank of the Jacobian matrix of its parametric representation ϕ_G reaches its maximum value, denoted* $\dim_\phi(G)$*, except on an open dense set in $\mathbb{T}^{n_R}$ of Lebesgue null measure.*

(b) The number $\dim_\phi(G)$ *is the same for all representations of G and their common value is called the* dimension *of G and is denoted by* $\dim(G)$*.*

Proof.– (a) Let us consider ϕ a parametric representation of a Givens set $G = T_1 T_2 \dots T_{n_T} I_{N,M}$ from $\mathbb{T}^{n_R}$ to G with n_R rotation sets. The coefficients of $\boldsymbol{G}(\theta)$, $\theta = (\theta_1, \theta_2, \dots, \theta_{n_R})$, are polynomial functions of $\cos\theta_l$ and $\sin\theta_l,\ 1 \leq l \leq n_R$.
Denoting by R_{i_l,j_l} the l-th rotation set in G, the Jacobian matrix element $\frac{\partial \phi_G}{\partial \theta_l}$ is obtained by replacing R_{i_l,j_l} in G by $D_{i_l,j_l} = \{\boldsymbol{D}_{i_l,j_l}(\theta_l),\ \theta_l \in \mathbb{T}\}$ where $\boldsymbol{D}_{i_l,j_l}(\theta_l)$ is defined by (C.10).
So the elements of the Jacobian matrix $\boldsymbol{J}(\theta)$, are also polynomials in $\cos\theta_l$ and $\sin\theta_l,\ 1 \leq l \leq n_R$.
If the maximum rank of the Jacobian is equal to $d = \dim_\phi$, then all minors of $\boldsymbol{J}(\theta)$ of order $r, r > d$ are identically null while there exists a minor $m(\theta)$ of order d of the Jacobian matrix and a value of $\theta^{(m)}$ with $m(\theta^{(m)}) \neq 0$.
Because any minor of $\boldsymbol{J}(\theta))$ is a polynomial function of the elements of $\boldsymbol{J}(\theta))$, and thus a polynomial function of $\cos\theta_l$ and $\sin\theta_l,\ 1 \leq l \leq n_R$, $m(\theta)$ is a continuous function of θ. Therefore the set V_m of all θ such that $m(\theta) \neq 0$ is an open set. In another hand, applying Theorem B.1, $m(\theta)$ is non null except on a set of θ with null measure, which implies that V_m is dense in $\mathbb{T}^{n_R}$.
We deduce that the subset $U_\phi \in \mathbb{T}^{n_R}$ of θ such that there exists a non null minor of order d of the Jacobian matrix is also an open dense set, because the number of minors of order d is finite.
As $G = \phi(\mathbb{T}^{n_R})$, $\phi(U_\phi)$ is a dense open set in G for the Euclidean topology.

(b) Let ϕ_1 and ϕ_2 be two representations of a Givens set $G \subset \mathcal{M}^{(m)}_{M,N}$. $\phi_1(U_{\phi_1})$ and $\phi_2(U_{\phi_2})$, as constructed in (a), are open dense sets in G. Let V be their intersection. $\boldsymbol{A} \in V$ is also a dense open set in G. So there exists W, open set in $\mathcal{M}^{(m)}_{M,N}$, such that $V = W \cap G$. There exists $\theta_1 \in \mathbb{T}^{n_1}$, where n_1 is the number of rotation sets in ϕ_1, such that $\phi_1(\theta_1) = \boldsymbol{A}$ with an open neighborhood U_1 of θ_1 in $\mathbb{T}^{n_1}$ such that the rank of ϕ_1 is equal to $\dim_{\phi_1}$ in U_1.

From the Constant Rank Theorem ([15, Theorem 11.1]), there exists a neigbourhood of $\boldsymbol{A} = \phi_1(\theta_1)$ in G and a C^∞-diffeormorphism ψ_1 from an open set $V_1 \subset G$ to an open set in $\mathbb{R}^{\dim_{\phi_1}}$ such that $\psi_1(\boldsymbol{A}) = 0 \in \mathbb{R}^{\dim_{\phi_1}}$.

In the same way, there exists a C^∞-diffeormorphism from an open neighbourhood V_2 of $\boldsymbol{A}$, $V_2 \subset G$ to an open set in $\mathbb{R}^{\dim_{\phi_2}}$ such that $\psi_2(\boldsymbol{A}) = 0 \in \mathbb{R}^{\dim_{\phi_2}}$.

$V = V_1 \cap V_2$ is an open neighborhood of $\boldsymbol{A}$ in G, and $\psi_1(V)$ and $\psi_2(V)$ are open sets in $\mathbb{R}^{\dim_{\phi_1}}$ and $\mathbb{R}^{\dim_{\phi_2}}$, respectively, C^∞-diffeomorphic to V, thus between themselves, and so $\dim_{\phi_1} = \dim_{\phi_2}$. □

A value of θ for which the rank of the Jacobian matrix is equal to $\dim(G)$ is called a *regular value* of ϕ and $\phi(\theta)$ a *regular point* of G for ϕ. Otherwise θ is said to be a *singular value* of ϕ and $\phi(\theta)$ a *singular point* of G for ϕ.

If the dimension of G with representation ϕ is equal to n_R then, for any point θ of $\mathbb{T}^{n_R}$ where the rank of the Jacobian matrix is exactly equal to n_R, there exists a neighborhood U of θ where the rank is still equal to n_R and the restriction of ϕ to U is injective being more precisely a diffeomorphism of U onto $\phi(U)$. ϕ is said to be a *locally injective* representation of G.

Let us remark that the set of regular points of a Givens set is dependent of the parametric representation because two of its representations have generally different singular points.

Example 3.7. *For $M = N = 3$, $SO(3)$ can be represented in two ways, up to a permutation of indexes applied simultaneously to rows and columns, by*

$$\phi_1 \quad : \quad SO(3) = R_{0,1}R_{0,2}R_{0,1}. \tag{3.10}$$

$$\phi_2 \quad : \quad SO(3) = R_{0,2}R_{0,1}R_{1,2}. \tag{3.11}$$

Table 3.7 gives the singular values and the singular points for each of these two representations.

	Singular values	Singular points
ϕ_1	$\theta_2 = 0$ or $\theta_2 = \pi$	$\{\boldsymbol{A}, \boldsymbol{A} \in SO(3), [\boldsymbol{A}]_{2,2} = \pm 1\}$
ϕ_2	$\theta_2 = \pm\pi/2$	$\{\boldsymbol{A}, \boldsymbol{A} \in SO(3), [\boldsymbol{A}]_{1,0} = \pm 1\}$

Table 1: Singular points for two Givens representations of $SO(3)$ (θ_2 is the angle of the second rotation).

3.4 McMillan valuation and degree of a Givens set

Definition 3.8. *For a Givens set G the maximum* $\max\{\deg(\boldsymbol{A}),\ \boldsymbol{A} \in G\}$*, denoted* $\deg(G)$*, is called the* McMillan degree *of G and the minimum* $\min\{\mathrm{val}(\boldsymbol{A}),\ \boldsymbol{A} \in G\}$*, denoted* $\mathrm{val}(G)$*, is called the* McMillan valuation *of G.*

When $M = N$, the McMillan degree and the McMillan valuation of G are both equal to the number d of shifts sets in any representation of G because $\mathrm{Det}(\boldsymbol{G}(\theta)) = X^d$ for any $\theta \in \mathbb{T}^{n_R}$. When $M < N$, $\deg(\mathrm{Det}(\boldsymbol{G}(\theta)_{I,J}) = \deg(G)$ except on a set of null Lebesgue measure and $\mathrm{val}(\mathrm{Det}(\boldsymbol{G}(\theta)_{I,J}) = \mathrm{val}(G)$ except on a set of null Lebesgue measure as shown in Appendix B.

Theorem 3.9. *Let M, N be positive integers with $M \leq N$ and let G be a Givens set $G = T_1 T_2 \dots T_{n_T} I_{N,M}$ that can be written*

$$G = R_1 Z_{I_1} R_2 Z_{I_2} \dots R_m Z_{I_m} R_{m+1} I_{N,M}, \tag{3.12}$$

where R_k, $1 \leq k \leq m+1$ are products of rotation sets and I_k, $1 \leq k \leq m$ are subsets of $\{0, \dots, N-1\}$ such that $1 \leq \mathrm{Card}(I_k) < N$. The number of shifts in the sequence of transformations of G, $d = \sum_{k=1}^{m} \mathrm{Card}(I_k)$ verifies $m \leq d \leq m(N-1)$.
Then

(a) For every matrix $\boldsymbol{A} \in G$ and for every $I \subset \{0, \dots, N-1\}$, $J \subset \{0, \dots, M-1\}$ such that $n = \mathrm{Card}(I) = \mathrm{Card}(J) \leq M$

$$\max(0, d - m(N-n)) \leq \mathrm{val}(\mathrm{Det}(\boldsymbol{A}_{I,J})) \leq \deg(\mathrm{Det}(\boldsymbol{A}_{I,J})) \leq d. \tag{3.13}$$

(b) $G \subset A_{d-m(N-M),d}$ and for a.e. $\theta \in \mathbb{T}^{n_R}$, $\boldsymbol{G}(\theta) \in B_{d-m(N-M),d}$.

Proof.– The proof of $\deg(\mathrm{Det}(\boldsymbol{A}(_{I,J})) \leq d$ is done by recurrence following the sequence of transformations of G from right to left. The matrices $R_{m+1} I_{N,M}$ are constant matrices and thus if $\boldsymbol{B} \in R_{m+1} I_{N,M}$, $\mathrm{Det}(\boldsymbol{B}_{I,J}) = 0$ or $\deg(\mathrm{Det}(\boldsymbol{B}_{I,J})) = 0$. Now suppose that $G_l = T_l T_{l+1} \dots T_{n_T} I_{M,N}$, $l > 1$ verifies $\deg(\mathrm{Det}(\boldsymbol{B}_{I,J})) \leq d'$ for every $\boldsymbol{B} \in G_l$ for every I and J where d' is the number of shift sets in the sequence $T_l, T_{l+1}, \dots, T_{n_T}$. T_{l-1} is either a rotation or a shift set.
Let $I = \{i_0, \dots, i_{n-1}\}$ with $i_0 < i_1 < \dots < i_n$ and $J = \{j_0, \dots, j_{n-1}\}$ with $j_1 < j_2 < \dots < j_n$.
Suppose that T_{l-1} is a shift set Z_r and let $\boldsymbol{C} = \boldsymbol{Z}_r \boldsymbol{B}$. If $r \notin I$, $\boldsymbol{C}_{I,J} = \boldsymbol{B}_{I,J}$ and $\mathrm{Det}(\boldsymbol{C}_{I,J}) = \mathrm{Det}(\boldsymbol{A}_{I,J})$. If $r \in I$, $r = i_k$, $0 \leq k < n$ and $\boldsymbol{C}_{I,J} = \bar{\boldsymbol{Z}}_k B$ where $\bar{\boldsymbol{Z}}_k$ is a $n \times n$ shift matrix. Thus $\mathrm{Det}(\boldsymbol{C}_{I,J}) = \mathrm{Det}(\boldsymbol{A}_{I,J}) + 1$
If T_{l-1} is a rotation set $R_{i,j}$, $i \neq j$, let $\boldsymbol{C} = \boldsymbol{R}_{i,j}(\theta) \boldsymbol{B}$.

- if $i \notin I$ and $j \notin i$, then $\boldsymbol{C}_{I,J} = \boldsymbol{B}_{I,J}$ and $\mathrm{Det}(\boldsymbol{C}_{I,J}) = \mathrm{Det}(\boldsymbol{A}_{I,J})$.
- if $i \in I$ and $j \in I$, then $i = i_k, j = i_l, 0 \leq k, l < n$, $k \neq l$ and $\boldsymbol{C}_{I,J} = \bar{\boldsymbol{R}}_{k,l}(\theta) \boldsymbol{B}_{I,J}$ where $\bar{\boldsymbol{R}}_{k,l}(\theta)$ is a $n \times n$ rotation matrix. Therefore $\mathrm{Det}(\boldsymbol{C}_{I,J}) = \mathrm{Det}(\boldsymbol{B}_{I,J})$
- if $i \in I$ and $j \notin I$, $i = i_r$, $0 \leq r < n$ and $\boldsymbol{C}_{I,J} = \cos\theta \boldsymbol{B}_{I,J} - \sin\theta \boldsymbol{B'}_{I',J}$ where $\boldsymbol{B'}$ is obtained from $\boldsymbol{B}$ by permuting the rows of indices i and j and $I' = \{i_0, \dots, i_{r-1}, j, i_{r+1}, \dots, i_{n-1}\}$.

  By linearity of the determinant with respect to a row of fixed index, we deduced that $\mathrm{Det}(\boldsymbol{C}_{I,J}) = \cos\theta \ \mathrm{Det}(\boldsymbol{B}_{I,J}) - \sin\theta \ \mathrm{Det}(\boldsymbol{B'}_{I,J})$. As $\mathrm{Det}(\boldsymbol{B'}_{I,J})$ is equal, up to a sign, to a determinant of a submatrix of $\boldsymbol{B}$ of size n, its degree is less or equal to d', and the same is true for $\mathrm{Det}(\boldsymbol{C}_{I,J})$.

So any matrix $\boldsymbol{C} \in T_{l-1} T_l \dots T_{n_T} I_{M,N}$ satisfies $\deg(\mathrm{Det}(\boldsymbol{C}_{I,J})) \leq d'$ where d' is the number of shift sets in the sequence $T_{l-1}, T_l, \dots, T_{n_T}$, which proves the recurrence argument.

From (3.12) we deduce that $G \subset \mathcal{O}_{N,M}^{(m)}$ and that the set $\{\sigma_m \boldsymbol{A},\ \boldsymbol{A} \in G\}$, denoted $\sigma_m G$ is also a Givens set that may be written

$$\sigma_m G = R_1 Z_{I_1^c} R_2 Z_{I_2^c} \dots R_m Z_{I_m^c} R_{m+1} I_{N,M}, \tag{3.14}$$

where $I_i^c = \{0, 1, \dots, N-1\} \setminus I_i$, $1 \leq i \leq m$. The number of shift sets in $\sigma_m G$ is equal to $mN - d$. We deduce, using the first part of the proof, that for any $I \subset \{0, 1, \dots, N-1\}$, any $J \subset \{0, 1, \dots, M-1\}$ with $\mathrm{Card}(I) = \mathrm{Card}(J)$, and any $\sigma \boldsymbol{A} \in \sigma_m G$

$$\deg(\mathrm{Det}([\sigma_m \boldsymbol{A}]_{I,J})) \leq mN - d. \tag{3.15}$$

Now using $[\boldsymbol{A}(1/X)]_{I,J} = \boldsymbol{A}_{I,J}(1/X)$ and $\mathrm{Det}(X^m \boldsymbol{A}_{I,J}(1/X)) = X^{mn}\mathrm{Det}(\boldsymbol{A}_{I,J}(1/X))$, we get $\deg(\mathrm{Det}([\sigma_m A]_{I,J})) = mn - \mathrm{val}(\mathrm{Det}(\boldsymbol{A}_{I,J}))$. So $mn - \mathrm{val}(\mathrm{Det}(\boldsymbol{A}_{I,J})) \leq nN - d$, which proves $\mathrm{val}(\mathrm{Det}(\boldsymbol{A}_{I,J})) \geq d - m(N-n)$ and thus (a).

The proof of (b) follows from the definitions of $\deg(G)$ and $\mathrm{val}(G)$. Because the coefficients of $\boldsymbol{G}(\theta)$, $\theta \in \mathbb{T}^{n_R}$ are polynomials in $\cos\theta_i, \sin\theta_i$, $1 \leq i \leq n_R$, each coefficient of $\mathrm{Det}(\boldsymbol{G}_{I,J}(\theta))$ has the same property, for any I and J such that $\mathrm{Card}(I) = \mathrm{Card}(J) = M$. Thus $\deg(\boldsymbol{G}(\theta)) = \deg(G)$ for a.e. $\theta \in \mathbb{T}^{n_R}$ is a consequence of Theorem B.1, (b). The same argument holds to prove that $\mathrm{val}(\boldsymbol{G}(\theta)) = \mathrm{val}(G)$ for a.e. $\theta \in \mathbb{T}^{n_R}$. □

3.5 Algebraic Givens sets

Theorem 3.10. *Let $G = T_1 T_2 \ldots T_{n_T} I_{M,N} \in \mathcal{O}^{(m)}_{M,N}$ be a Givens set with parametric representation ϕ such as defined in (3.2) and A the smallest real algebraic set that contains G, i.e. its Zariski closure. Then A is irreducible.*

Proof.– Let us denote by $\phi_G(\theta)$, $\theta = (\theta_1, \ldots, \theta_{n_R})$, the parametric representation of G where n_R is the number of rotation sets in the sequence $T_1, T_2, \ldots, T_{n_T}$.
Let us now suppose that $A = A_1 \cup A_2$ where A_1 and A_2 are real algebraic sets with $A_1 \subsetneq A$ and $A_2 \subsetneq A$.
There exists $\theta_0 \in \mathbb{T}^{n_R}$ such that $x = \phi_G(\theta_0) \in A_1 \setminus A_2$, which is an open set for the ordinary topology on A. If not, G would be included in A_2 and A would not be the Zariski closure of G.
Let V be an open neighborhood of x in $A_1 \setminus A_2$. The intersection $V \cap G$ is an open neighborhood of x in G. As ϕ_G is continuous from $\mathbb{T}^{n_R}$ onto G, $U(\theta_0) = \phi_G^{-1}(V)$ is an open set that contains θ_0. Now let $P_{A_1} \in \mathbb{R}[a]$, $a = (a_1, a_2, \ldots, a_{MN(m+1)})$ be a polynomial such that $A_1 = \mathrm{Zer}(P_{A_1})$, the set of real zeroes of P_{A_1} in $\mathcal{O}^{(m)}_{M,N} \subset \mathbb{R}^{MN(m+1)}$. We get $P_{A_1}(\phi_G(\theta)) = 0$ for any $\theta \in U(\theta_0)$.

As the coordinates $\phi_G(\theta)_i$, $1 \leq i \leq MN(m+1)$ are polynomials in $\cos\theta_l, \sin\theta_l$ $1 \leq l \leq n_R$, $Q(\theta) = P_{A_1}(\phi_G(\theta))$ too. From Theorem B.1 and because $Q(\theta) = 0$ on $U(\theta_0)$, $Q(\theta)$ is identically null. That proves $G \subset A_1$, which is a contradiction, and thus A is irreducible. □

Corollary 3.11. *If a Givens set $G \subset \mathcal{O}^{(m)}_{M,N}$ is a real algebraic set, then G is irreducible.*

Remark 3.12. *As shown in Proposition A.1 there exist Givens sets that are not algebraic sets.*

3.6 Some representations of $SO(N)$ as a Givens set

Let us recall that $SO(N)$ denotes the set of all $N \times N$ orthogonal real matrices with determinant 1.

Notations 3.13. *For $N \geq 2$ and $0 \leq a < b < N$, $A_{a..b}$ denotes the matrix set*

$$A_{a..b} = \prod_{i=a}^{b-1} \prod_{j=i+1}^{b} R_{i,j}. \tag{3.16}$$

while for $0 \leq a < N$, $A_{a..a} = I_N$ by convention. For $N \geq 2$ and $0 \leq a \leq b < c \leq d < N$, $B_{a..b,c..d}$ denotes the matrix set

$$B_{a..b,c..d} = \prod_{i=a}^{b} \prod_{j=c}^{d} R_{i,j}. \tag{3.17}$$

Remark. Matrices in a set $B_{a..b,c..d}$ correspond to CSD blocks in [8].

Using the commutation relation $R_{i,j}R_{k,l} = R_{k,l}R_{i,j}$ if $\{i,j\} \cap \{k,l\} = \emptyset$, it is straightforward to prove that

$$B_{a..b,c..d} = \prod_{j=c}^{d} \prod_{i=a}^{b} R_{i,j}. \tag{3.18}$$

Theorem 3.15 gives several representations of $SO(N)$ as a Givens set that result from variations of the QR algorithm and will be used in the rest of this paper. An elementary lemma will be used below that allows to define the notion of *extraction of a rotation* from a set of orthogonal matrices.

Lemma 3.14. *([16, algorithm 5.1.8, page 240]) Given two real numbers a and b, there exists an angle θ and a nonnegative real number c such that*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Let us consider now a set A of orthogonal $N \times N$ matrices such that there exist two row indexes i and j with the property that for any column index c, $\boldsymbol{A}_{i,c} = 0$ for any $\boldsymbol{A} \in A$ if and only if $\boldsymbol{A}_{j,c} = 0$ for any $\boldsymbol{A} \in A$, and there exists a column index k such that $\boldsymbol{A}_{i,k}^2 + \boldsymbol{A}_{j,k}^2 \neq 0$ for any $\boldsymbol{A} \in A$. From Lemma 3.14 we deduce that for any $\boldsymbol{A} \in A$ there exists an angle θ and an orthogonal $N \times N$ matrix $\boldsymbol{B}$ such that $\boldsymbol{R}_{i,j}(\theta)\boldsymbol{B} = \boldsymbol{A}$ with $\boldsymbol{B}_{j,k} = 0$. So the set A may be written $A = R_{i,j}B$ where $\boldsymbol{B}_{j,k} = 0$ for any $\boldsymbol{B} \in B$.
This operation is called the *rotation extraction* from the set A with the same non null elements on row of indexes i and j. It is quite evident that B is again a set of orthogonal matrices.

Theorem 3.15. *For $N \geq 3$, $SO(N)$ has the following representations as a Givens set.*

(a)

$$SO(N) = \prod_{i=0}^{N-2} \prod_{j=i+1}^{N-1} R_{i,j} = A_{0..N-1}. \tag{3.19}$$

(b)

$$SO(N) = \prod_{i=0}^{N-2} \prod_{j=N-1(-1)}^{i+1} R_{j-1,j}. \tag{3.20}$$

(c) For $1 \leq M < N$,

$$SO(N) = A_{0..M-1}B_{0..M-1,M..N-1}A_{M..N-1}. \tag{3.21}$$

(d) For $0 \leq M_1 < M_2 < N-1$,

$$\begin{aligned} SO(N) &= A_{0..M_2-1}B_{0..M_1-1,M_2..N-1}B_{M_1..M_2-1,M_2..N-1}A_{M_2..N-1} && (3.22) \\ &= A_{0..M_1-1}B_{0..M_1-1,M_1..M_2-1}B_{0..M_1-1,M_2..N-1}A_{M_1..N-1}. && (3.23) \end{aligned}$$

Proof.– **(a)** For a real $N \times N$ matrix $\boldsymbol{A}$, the QR algorithm ([16, from page 252]) first introduces, from left to right, rotation matrices between rows of indices 0 and j for j varying from 1 to $N-1$ to cancel the values in column 0 and rows j for $1 \leq j \leq N-1$. Then rotations are chosen between

rows 1 and $j, 1 \leq j < N$ to cancel the elements in column 1 and rows $j = 2, \ldots, N-1$ and so on until $i = N-2$ and $j = N-1$ to cancel the element at row $N-1$ and column $N-2$.
The result is an expression

$$\boldsymbol{A} = \prod_{i=0}^{N-2} \prod_{j=i+1}^{N-1} \boldsymbol{R}_{i,j}\, \boldsymbol{T}, \tag{3.24}$$

where $\boldsymbol{T}$ is an upper triangular matrix. In this process we may choose the rotations to ensure that $\boldsymbol{T}_{i,i} \geq 0$ for $0 \leq i \leq N-2$. If $\boldsymbol{A}$ belongs to $SO(N)$, $\boldsymbol{T}$ too and from orthogonality of its column vectors we deduce that $\boldsymbol{T}$ is a diagonal matrix. Because the column vectors are unitary vectors, it comes that $\boldsymbol{T}_{i,i} = 1,\ 0 \leq i \leq N-2$ and $\boldsymbol{T}_{N-1,N-1} = \pm 1$. Finally, because the determinant of $\boldsymbol{T}$ is equal to 1 and is the product of diagonal elements, we get $\boldsymbol{T}_{N-1,N-1} = 1$.
We deduce from this algorithm the classical result (3.19) (see [17, 18]).

(b) To cancel the elements $\boldsymbol{A}_{i,j}, j < i$, we can proceed with a different order to get the expression (3.22) by introducing from left to right rotations to cancel elements in the lower triangular part of the matrix. In this method, rotations have always consecutive row indexes and are also extracted from matrices with the same non null elements on the row indexes.

(c) Another way to cancel the lower triangular part of the matrix is to introduce the rotations of

1. $A_{0..M-1}$ to cancel elements at places $S_1 = \{(i,j),\ 1 \leq i \leq M-1,\ 0 \leq j < i\}$,
2. $B_{0..M-1,M..N-1}$ to cancel elements at places $S_2 = \{(i,j),\ M \leq i \leq N-1,\ 0 \leq j \leq M-1\}$,
3. $A_{M..N-1}$ to cancel elements at places $S_3 = \{(i,j),\ M+1 \leq i \leq N-1,\ M \leq j < i\}$.

This proves identity (3.21).

(d) The identity (3.22) is obtained by cancelling the elements $\boldsymbol{A}_{i,j},\ j < i$ by introducing from left to right the rotations of

1. $A_{0..M_2-1}$ to cancel elements at places $S_1 = \{(i,j),\ 1 \leq i \leq M_2-1,\ 0 \leq j < i\}$,
2. $B_{0..M_1-1,M_2..N-1}$ to cancel elements at places $S_2 = \{(i,j),\ M_2 \leq i \leq N-1,\ 0 \leq j \leq M_1-1\}$,
3. $B_{M_1..M_2-1,M_1..N-1}$ to cancel elements at places $S_3 = \{(i,j), M_2 \leq i \leq N-1,\ M_1 \leq j \leq N-1\}$,
4. $A_{M_2..N-1}$ to cancel elements at places $S_4 = \{(i,j),\ M_2+1 \leq i \leq N-1,\ M_2 \leq j < i\}$.

It is easy to see that each rotation $\boldsymbol{R}_{i,j}$ in this procedure is introduced from matrices with rows of indices i and j with null elements at the same places, as sketched in the left part of Figure 1.

The proof of identity (3.23) follows the same method by cancelling the elements $\boldsymbol{A}_{i,j},\ j < i$ by introducing from left to right the rotations of

1. $A_{0..M_1-1}$ to cancel elements at places $S_1 = \{(i,j),\ 1 \leq i \leq M_1-1,\ 1 \leq j < i\}$,
2. $B_{0..M_1-1,M_1..M_2-1}$ to cancel elements at places $S_2 = \{(i,j),\ M_1 \leq i \leq M_2-1,\ 0 \leq j \leq M_1-1\}$,
3. $B_{0..M_1-1,M_2..N-1}$ to cancel elements at places $S_3 = \{(i,j), M_2 \leq i \leq N-1,\ 0 \leq j \leq M_1-1\}$,
4. $A_{M_1..N-1}$ to cancel elements at places $S_4 = \{(i,j),\ M_1 \leq i \leq N-1,\ M_1 \leq j < i\}$.

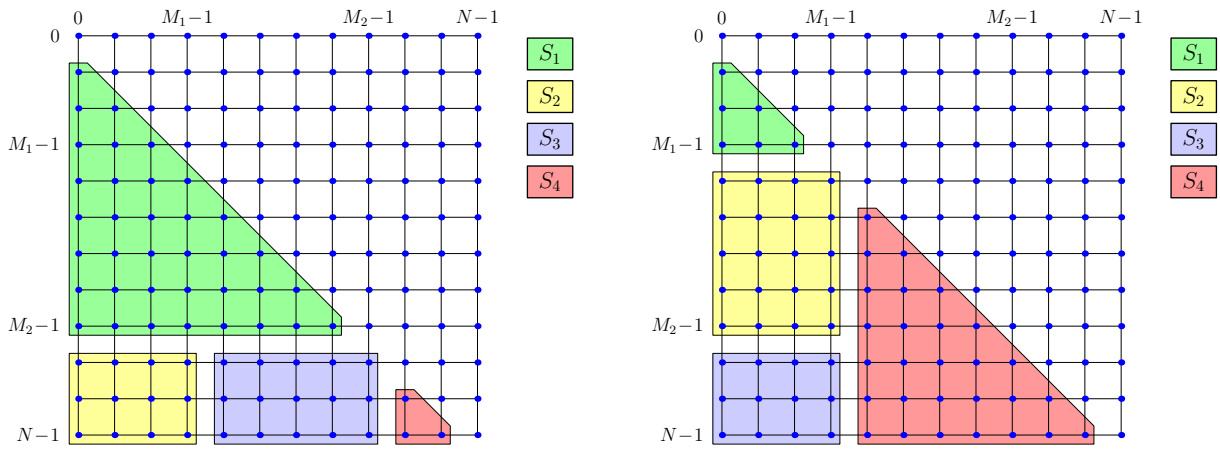

Figure 1: The sets $S_i, 1 \leq i \leq 4$ at left for relation (3.22), at right for (3.23).

This method is illustrated in the right part of Figure 1. □

Relation (3.21) may be strengthened by showing that $A_{0..M-1}, B_{0..M-1,M..N-1}$ and $A_{M..N-1}$ are globally commuting sets.

Theorem 3.16. *Let M, N such that $2 \leq M < N$.*

(a) For a with $0 \leq a < M-1$,

$$B_{0..M-1,M..N-1} R_{a,a+1} = R_{a,a+1} B_{0..M-1,M..N-1}. \tag{3.25}$$

(b) The following set commutation relations are verified

$$A_{M..N-1} A_{0..M-1} = A_{0..M-1} A_{M..N-1}, \tag{3.26}$$

$$B_{0..M-1,M..N-1} A_{0..M-1} = A_{0..M-1} B_{0..M-1,M..N-1}, \tag{3.27}$$

$$B_{0..M-1,M..N-1} A_{M..N-1} = A_{M..N-1} B_{0..M-1,M..N-1}. \tag{3.28}$$

Proof.– As

$$B_{0..M-1,M..N-1} = \prod_{i=0}^{M-1} \prod_{j=M}^{N-1} R_{i,j}, \tag{3.29}$$

and because $R_{a,a+1}$ commute with each rotation set in $\prod_{j=M}^{N-1} R_{i,j}$ for $i \neq a$ and $i \neq a+1$, it is sufficient to show that

$$\prod_{j=M}^{N-1} R_{a,j} \prod_{j=M}^{N-1} R_{a+1,j}\, R_{a,a+1} = R_{a,a+1} \prod_{j=M}^{N-1} R_{a,j} \prod_{j=M}^{N-1} R_{a+1,j}. \tag{3.30}$$

Now, we may write,

$$\prod_{j=M}^{N-1} R_{a,j} \prod_{j=M}^{N-1} R_{a+1,j} = \prod_{j=M}^{N-1} \left(R_{a,j} R_{a+1,j} \right), \tag{3.31}$$

and then, for every $j, M \leq j \leq N-1$,

$$R_{a,j}R_{a+1,j}R_{a,a+1} = R_{a,a+1}R_{a,j}R_{a+1,j}, \tag{3.32}$$

because this set equality amounts to use two factorizations of $SO(3)$ as acting on rows $a, a+1, j$. Using (3.32) for j decreasing from $N-1$ to M in the right member of (3.31) proves (3.30) and thus part (a) of Theorem 3.16.

Relation (3.26) is evident because any rotation in $A_{0..M-1}$ and any rotation in $A_{M..N-1}$ have no common indexes. Relation (3.28) may be easily deduced from relation (3.27) by applying a permutation of indexes simultaneously on rows and columns.

To prove (3.27) remark that $A_{0..M-1}$ is $SO(M)$ acting on rows $\{0, 1, \ldots, M-1\}$ and thus, thanks to relation (3.20), may be written

$$A_{0..M-1} = \prod_{i=0}^{M-2} \prod_{a=M-1(-1)}^{i+1} R_{a-1,a}. \tag{3.33}$$

Then, applying (a) for each of the rotation set in the right member of (3.33) from left to right, proves (3.27).

□

4 Main theorems

4.1 SVD and order reduction in $\mathcal{O}^{(m)}_{M,N}$

The singular value decomposition (SVD) of a square matrix (cf. for example [16, 2.4, page 76], such as used in [8], may be formulated as follows for a rectangular matrix.

Theorem 4.1. *(SVD decomposition) Let M, N be integers such that $M \leq N$ and $\boldsymbol{A}$ a $N \times M$ real matrix with rank ρ. The singular value decomposition of $\boldsymbol{A}$ is given by*

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T, \tag{4.1}$$

where

- *$\boldsymbol{\Sigma}$ is a $\rho \times \rho$ diagonal matrix with strictly positive diagonal elements (the singular values),*
- *$\boldsymbol{U}$ is a $N \times \rho$ orthogonal matrix,*
- *$\boldsymbol{V}$ is a $M \times \rho$ orthogonal matrix.*

So $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}_r$.

The order reduction method in [8] may then be extended in a straightforward manner to the case of a rectangular paraunitary matrix.

Theorem 4.2. *Let $M \leq N$ and $\boldsymbol{A}(X) \in \mathcal{O}^{(m)}_{M,N}$. There exist $m+1$ orthogonal matrices $\boldsymbol{W}_k$, $0 \leq k \leq m$, with $\boldsymbol{W}_k \in SO(N)$ $1 \leq k \leq m$ and $\boldsymbol{W}_0 \in O(N)$, and r_k, $0 < r_k < N$, $1 \leq k \leq m$ such that*

$$\boldsymbol{A}(X) = \boldsymbol{W}_m\boldsymbol{Z}_{r_m..N-1}\boldsymbol{W}_{m-1}\boldsymbol{Z}_{r_{m-1}..N-1}\ldots\boldsymbol{W}_1\boldsymbol{Z}_{r_1..N-1}\boldsymbol{W}_0\boldsymbol{I}_{N,M}. \tag{4.2}$$

When $M < N$, $\boldsymbol{W}_0$ may be also chosen in $SO(N)$.

Proof.– A matrix $\boldsymbol{A}(X) \in \mathcal{O}_{M,N}^{(m)}$ with order m can be expressed as

$$\boldsymbol{A}(X) = \boldsymbol{A}_0 + \boldsymbol{A}_1 X + \ldots \boldsymbol{A}_m X^m, \tag{4.3}$$

where $\boldsymbol{A}_k, 0 \leq k \leq m$ are constant $N \times M$ matrices. From the relation $\boldsymbol{A}^T(1/X)\boldsymbol{A}(X)$ it results that

$$\boldsymbol{A}_m^T \boldsymbol{A}_0 = \boldsymbol{0}_M, \tag{4.4}$$

where $\boldsymbol{0}_M$ is the null $M \times M$ matrix.
This equation implies that $\rho_0 + \rho_m \leq N$ where ρ_0 and ρ_m are the ranks of $\boldsymbol{A}_0$ and $\boldsymbol{A}_m$ respectively The SVD of $\boldsymbol{A}_0$ and $\boldsymbol{A}_m$ are, following Theorem 4.1,

$$\boldsymbol{A}_0 = \boldsymbol{U}_0 \boldsymbol{\Sigma}_0 \boldsymbol{V}_0^T, \tag{4.5}$$
$$\boldsymbol{A}_m = \boldsymbol{U}_m \boldsymbol{\Sigma}_m \boldsymbol{V}_m^T,. \tag{4.6}$$

Plugging (4.5) and (4.6) in (4.4) gives

$$\boldsymbol{V}_m \boldsymbol{\Sigma}_m \boldsymbol{U}_m^T \boldsymbol{U}_0 \boldsymbol{\Sigma}_0 \boldsymbol{V}_0^T = \boldsymbol{0}_M. \tag{4.7}$$

Multiplying each member of this equality by $\boldsymbol{\Sigma}_m^{-1}\boldsymbol{V}_m^T$ at left and by $\boldsymbol{V}_0\boldsymbol{\Sigma}_0^{-1}$ at right gives $\boldsymbol{U}_m^T\boldsymbol{U}_0 = \boldsymbol{0}_{\rho_m,\rho_0}$, which means that column vectors of $\boldsymbol{U}_0$ are orthogonal to column vectors of $\boldsymbol{U}_m$.
As in [8], we can construct a $N \times (N - \rho_0 - \rho_m)$ orthogonal matrix $\boldsymbol{Q}_\perp$ with column vectors of $\boldsymbol{U}_0$ and $\boldsymbol{U}_m$ to construct a $N \times N$ orthogonal matrix $\boldsymbol{W}_m$

$$\boldsymbol{W}_m = [\boldsymbol{U}_0 \; \boldsymbol{Q}_\perp \; \boldsymbol{U}_m]. \tag{4.8}$$

When $M < N$, $N - \rho_0 - \rho_m > 0$ and $\boldsymbol{Q}_\perp$ may be chosen to ensure that $\det(\boldsymbol{W}_m) = 1$, i.e. $\boldsymbol{W}_m \in SO(N)$. If $M = N$ and $\rho_0 + \rho_m = N$, there is no matrix $\boldsymbol{Q}_\perp$ but, when $\det(\boldsymbol{W}_m) = -1$, we may change the first diagonal element of $\boldsymbol{\Sigma_0}$ and the first column vector of $\boldsymbol{U_0}$ by their opposites to ensure that $\boldsymbol{W}_m \in SO(N)$.
Now $\boldsymbol{A}_0$ and $\boldsymbol{A}_m$ may be written

$$\boldsymbol{A}_0 = \boldsymbol{W}_m \begin{bmatrix} \boldsymbol{\Sigma_0}\boldsymbol{V}_0^T \\ \boldsymbol{0}_{N-\rho_0,M} \end{bmatrix}, \quad \boldsymbol{A}_m = \boldsymbol{W}_m \begin{bmatrix} \boldsymbol{0}_{N-\rho_m,M} \\ \boldsymbol{\Sigma}_m\boldsymbol{V}_m^T \end{bmatrix}, \tag{4.9}$$

The end of the proof is strictly the same as in [8]. From (4.3) we get

$$A(X) = \boldsymbol{W}_m \left(\begin{bmatrix} \boldsymbol{\Sigma_0}\boldsymbol{V}_0^T \\ \boldsymbol{0}_{N-\rho_0,M} \end{bmatrix} + \sum_{k=1}^{m-1} \boldsymbol{W}_m^{-1}\boldsymbol{A}_k X^k + \begin{bmatrix} \boldsymbol{0}_{N-\rho_m,M} \\ \boldsymbol{\Sigma}_m\boldsymbol{V}_m^T \end{bmatrix} X^m \right) \tag{4.10}$$
$$= \boldsymbol{W}_m \boldsymbol{Z}_{r_m..N-1} \hat{\boldsymbol{A}}(X), \tag{4.11}$$

where the order of $\hat{\boldsymbol{A}}(X)$ is equal to $m-1$ and $\rho_0 \leq r_m \leq N - \rho_m$. This order reduction method can be repeated until we get a constant $N \times M$ orthogonal matrix that may be written as $\boldsymbol{W}_0\boldsymbol{I}_{N,M}$ with $\boldsymbol{W}_0 \in O(N)$ when $M = N$ and $\boldsymbol{W}_0 \in SO(N)$ when $M < N$.

□

4.2 Technical lemmas

Lemma 4.3. *(Lemma 2 in [1]) For any $I \subset \{0, \dots, N-1\}$ with* $\mathrm{Card}(I) = l < N$,

$$SO(N) Z_I SO(N) = SO(N) Z_{0..l-1} SO(N). \tag{4.12}$$

Proof.– Let $I = \{i_1, i_2, \dots, i_l\}$ be a set of distinct positive integers in $\{0, \dots, N-1\}$ and s be any permutation of $\{0, \dots, N-1\}$ such that $s(\{0, \dots, l-1\}) = I$ and such that the permutation matrix $\boldsymbol{P}_s$ of size n defined by $[\boldsymbol{P}_s]_{i,s(i)} = 1$ and 0 elsewhere has a determinant equal to 1.
Of course, with $P_s = \{\boldsymbol{P}_s\}$,

$$SO(N) = SO(N) P_s = P_s^T SO(N),$$

and then

$$SO(N) Z_I SO(N) = SO(N) Z_I SO(N) = SO(N) P_s Z_I P_s^T SO(N).$$

Since $P_s Z_I P_s^T = Z_{0..l-1}$, the result follows. □

Lemma 4.4. *(Lemma 3 in [1])*
For $l_1 < N-1$ et $l_2 < N-1$, the following set equality holds

$$SO(N) Z_{0..l_2-1} SO(N) Z_{0..l_1-1} SO(N) = SO(N) Z_{0..l_1-1} SO(N) Z_{0..l_2-1} SO(N). \tag{4.13}$$

Proof.– We may suppose without restriction that $l_2 < l_1$. Using (3.22)

$$SO(N) = A_{0..l_1-1} B_{0..l_2-1, l_1..N-1} B_{l_2..l_1-1, l_1..N-1} A_{l_1..N-1}. \tag{4.14}$$

with $M_1 = l_2$ and $M_2 = l_1$, we get

$$\begin{aligned} & SO(N) Z_{0..l_1-1} SO(N) Z_{0..l_2-1} SO(N) \\ &= SO(N) Z_{0..l_1-1} A_{0..l_1-1} B_{0..l_2-1, l_1..N-1} B_{l_2..l_1-1, l_1..N-1} A_{l_1..N-1} Z_{0..l_2-1} SO(N), \\ &= SO(N) Z_{1..l_1} B_{1..l_2, l_1+1..n} Z_{1..l_2} SO(N), \end{aligned} \tag{4.15}$$

because $A_{0..l_1-1}$ commutes with $Z_{0..l_1-1}$ and $B_{l_2..l_1-1, l_1..N-1} A_{l_1..N-1}$ commutes with $Z_{0..l_2-1}$.

Now $Z_{0..l_1-1} = Z_{0..l_2-1} Z_{l_2..l_1-1}$ and because $Z_{l_2..l_1-1}$ commutes with $B_{0..l_2-1, l_1..N-1}$, we get

$$\begin{aligned} SO(N) Z_{0..l_1-1} B_{0..l_2-1, l_1..N-1} Z_{0..l_2-1} SO(N) &= SO(N) Z_{0..l_2-1} B_{0..l_2-1, l_1..N-1} Z_{0..l_1-1} SO(N) \\ &\subset SO(N) Z_{0..l_2-1} B_{0..l_2-1, l_1..N-1} Z_{0..l_1-1} SO(N). \end{aligned}$$

and thus

$$SO(N) Z_{0..l_1-1} SO(N) Z_{0..l_2-1} SO(N) \subset SO(N) Z_{0..l_2-1} SO(N) Z_{0..l_1-1} SO(N), \tag{4.16}$$

After replacing X by $1/X$ in (4.15) and multiplying by X^2, we get

$$X^2 SO(N) Z_{0..l_1-1}^{-1} SO(N) Z_{0..l_2-1}^{-1} SO(N) \subset X^2 SO(N) Z_{0..l_2-1}^{-1} SO(N) Z_{0..l_1-1}^{-1} SO(N), \tag{4.17}$$

and because

$$\begin{aligned} X^2 SO(N) Z_{0..l_1-1}^{-1} SO(N) Z_{0..l_2-1}^{-1} SO(N) &= SO(N) Z_{1_1..N-1} SO(N) Z_{l_2..N-1} SO(N) \\ &= SO(N) Z_{0..n-l_1-1} SO(N) Z_{0..n-l_2-1}, \end{aligned}$$

using Lemma 4.3, and in the same way

$$\begin{aligned} X^2 SO(N) Z_{0..l_2-1}^{-1} SO(N) Z_{0..l_1-1}^{-1} SO(N) &= SO(N) Z_{1_2..N-1} SO(N) Z_{l_1..N-1} SO(N) \\ &= SO(N) Z_{0..n-l_2-1} SO(N) Z_{0..n-l_1-1}, \end{aligned}$$

we get from (4.12)

$$SO(N) Z_{0..n-l_1-1} SO(N) Z_{0..n-l_2-1} \subset SO(N) Z_{0..n-l_2-1} SO(N) Z_{0..n-l_1-1} SO(N). \tag{4.18}$$

Replacing $n - l_1$ by l_2 and $n - l_2$ by l_1 in (4.18), which is allowed because $n - l_1 < n - l_2$, if follows that

$$SO(N) Z_{0..l_2-1} SO(N) Z_{0..l_1-1} SO(N) \subset SO(N) Z_{0..l_1-1} SO(N) Z_{0..l_2-1} SO(N). \tag{4.19}$$

The lemma is proved from (4.15) and (4.19).

□

Lemma 4.5. *(Lemma 4 in [1]) For $l_1 < N - 1$, $l_2 < N - 1$ and $l_1 > l_2 + 1$,*

$$SO(N) Z_{0..l_2-1} SO(N) Z_{0..l_1-1} SO(N) \subset SO(N) Z_{0..l_1-2} SO(N) Z_{0..l_2} SO(N). \tag{4.20}$$

Proof.– The proof uses the same argument as in the previous lemma. From equality (4.15), as $Z_{0..l_1-1} = Z_{0..l_1-2} Z_{l_1-1}$ and observing that Z_{l_1-1} commutes with $B_{0..l_2-1,l_1..N-1}$, we get

$$\begin{aligned} & SO(N) Z_{0..l_1-1} B_{0..l_2-1,l_1..N-1} Z_{0..l_2-1} SO(N) \\ &= SO(N) Z_{0..l_1-2} B_{0..l_2-1,l_1..N-1} Z_{l_1-1} Z_{0..l_2-1} SO(N), \end{aligned} \tag{4.21}$$

and therefore

$$SO(N) Z_{0..l_1-1} SO(N) Z_{0..l_2-1} SO(N) \subset SO(N) Z_{0..l_1-2} SO(N) Z_{l_1-1} Z_{0..l_2-1} SO(N).$$

Because, from Lemma 4.3,

$$SO(N) Z_{0..l_1-2} SO(N) Z_{l_1-1} Z_{0..l_2-1} SO(N) = SO(N) Z_{0..l_1-2} SO(N) Z_{0..l_2} SO(N), \tag{4.22}$$

this achieves the proof. □

4.3 The Givens sets $\mathcal{U}_{M,N,m,d}$

Definition 4.6. *Let M and N be two positive integers such that $1 \leq M \leq N$ and $N \geq 2$ when $M = 1$. For $m \geq 1$ and $d, m \leq d \leq m(N-1)$ let us define k and l by $d = (l-1)m + k$ with $1 \leq k \leq m$, that is*

$$k = [(d-1) \mod m] + 1, \; l = [(d-1) \text{ quo } m] + 1. \tag{4.23}$$

The set $\mathcal{U}_{M,N,m,d}$ is defined for $m \geq 1$ by

$$\mathcal{U}_{M,N,m,d} = SO(N) \left[Z_{0..l-1} SO(N) \right]^k \left[Z_{0..l-2} SO(N) \right]^{m-k} I_{N,M}. \tag{4.24}$$

Lemma 4.7. *For given $M, N, 1 \leq M \leq N$, $N \geq 2$ when $M = 1$, $m \geq 1$ and a sequence of m subsets $I_i, 1 \leq i \leq m$ of $\{0, \ldots, N-1\}$ with $1 \leq \mathrm{Card}(I_i) \leq N-1$, the following set inclusion is satisfied*

$$SO(N)Z_{I_1}SO(N)Z_{I_2}\ldots SO(N)Z_{I_m}SO(N)I_{N,M} \subset \mathcal{U}_{M,N,m,d}, \tag{4.25}$$

where $d = \sum_{i=1}^{m} \mathrm{Card}(I_i)$.

Proof.– As $\mathcal{U}_{M,N,m,d} = \mathcal{U}_{N,N,m,d}I_{N,M}$, it is sufficient to prove the lemma for $N = M$.
Using Lemma 4.3, for $1 \leq k \leq m$, we may replace Z_{I_k} in (4.7) by $Z_{0..r_k-1}$ where $r_k = \mathrm{Card}(I_k)$ and then, without changing the set $SO(N)Z_{I_1}SO(N)Z_{I_2}\ldots SO(N)Z_{I_m}SO(N)$, we may assume that r_k, $1 \leq k \leq m$ is a increasing sequence thanks to Lemma 4.4. Of course, $d = \sum_{k=1}^{m} r_k$. Let us denote by $U(r_1, r_2, \ldots, r_m)$ the set

$$U(r_1, r_2, \ldots, r_m) = SO(N)Z_{0..r_1-1}SO(N)Z_{0..r_2-1}SO(N)\ldots Z_{0..r_m-1}SO(N). \tag{4.26}$$

Now, for a such that $r_m \leq a \leq r_1$, let n_a be the number of r_k, $1 \leq k \leq m$ equal to a. Of course $n_{r_m} > 0$, $n_{r_1} > 0$, $n_a \geq 0$ for $r_m < a < r_1$ and $\sum_{a=r_m}^{r_1} n_a = m$, $\sum_{a=r_m}^{r_1} an_a = d$. When $r_1 - r_m \leq 2$, let us consider the two steps following transformation :

1. Move the last factor $Z_{0..r_m-1}$ in the expression of $U(r_1, r_2, \ldots, r_m)$ to the second place at left, without changing this set thanks to Lemma 4.4, i.e. $U(r_1, r_2, \ldots, r_m) = U(r_1, r_m, r_2, \ldots, r_{m-1})$.

2. Using Lemma 4.5, we get $U(r_1, r_m, \ldots, r_{m-1}) \subset U(r_1-1, r_m+1, \ldots, r_{m-1}) = U(r'_1, r'_2, \ldots, r'_m)$ where $(r'_1, r'_2, \ldots, r'_m)$ is the reordering of $(r_1 - 1, r_m + 1, \ldots, r_{m-1})$ in an increasing sequence. Let us denote again by $(r_1, r_2, \ldots, r_m)$ the sequence $(r'_1, r'_2, \ldots, r'_m)$.

The values of n_{r_1} and n_{r_m} for the new sequence are thus decreased by 1. So the transformation may be iterated until $n_{r_1} = 0$ or $n_{r_m} = 0$, in which case the value of $r_1 - r_m$ is decreased by 1 or 2, while $U(r_1, \ldots, r_m)$ is increased for set inclusion.
This process may be continued until $r_1 - r_m = 0$ or $r_1 - r_m = 1$ and provides an increasing sequence $U(r_1, \ldots, r_m)$ for set inclusion. The value of $\sum_{i=1}^{m} r_i$ is kept invariant, equal to d.

If $r_1 - r_m = 0$, $r_i = r_1$ for $1 \leq i \leq m$ and denotes by l their common value. We thus get $d = ml = m(l-1) + k$ with $k = m$.
If $r_1 - r_m = 1$, let us denote by l the number r_1 and by k the number of r_i's equal to l. So $m - k$ is the number of r_i equal to $l - 1$ and $d = kl + (m-k)(l-1) = m(l-1) + k$ with $1 \leq k < m$.
Therefore in both cases $U(r_1, r_2, \ldots, r_m) = \mathcal{U}_{M,N,m,d}$. □

Lemma 4.8. *Let M and N be two positive integers such that $1 \leq M \leq N$ and $N \geq 2$ when $M = 1$. For $m \geq 1$ and $d, m \leq d \leq m(N-1)$,*

$$\sigma_m \mathcal{U}_{M,N,m,d} = \mathcal{U}_{M,N,m,mN-d}. \tag{4.27}$$

Proof.– From

$$X^m \mathcal{U}_{M,N,m,d}(1/X) = SO(N)\left[XZ_{0..l-1}(1/X)SO(N)\right]^k \left[XZ_{0..l-2}(1/X)SO(N)\right]^{m-k} I_{N,M}, \tag{4.28}$$

and

$$XZ_{0..l-1}(1/X) \quad = \quad Z_{0..N-1}\, Z_{0..l-1}(1/X) = Z_{l..N-1}, \tag{4.29}$$

we get, using Lemma 4.3

$$X^m \mathcal{U}_{M,N,m,d}(1/X) \quad = \quad SO(N)\,[Z_{0..N-l-1}SO(N)]^k\ [Z_{0..N-l}SO(N)]^{m-k}\ I_{N,M}, \tag{4.30}$$

and then, using Lemma 4.4

$$X^m \mathcal{U}_{M,N,m,d}(1/X) \quad = \quad SO(N)\,[Z_{0..N-l}SO(N)]^{m-k}\ [Z_{0..N-l-1}SO(N)]^k\ I_{N,M}. \tag{4.31}$$

Therefore $\sigma_m \mathcal{U}_{M,N,m,d} = \mathcal{U}_{M,N,m,d'}$ where $d' = (m-k)(N-l+1) + k(N-l) = mN - m(l-1) - k$. As $d = m(l-1)+k$, we get $d' = mN - d$ and this proves (4.8). □

The sets of matrices defined by (4.6) may be expressed as Givens sets as shown in the following theorem.

Theorem 4.9. *Let M and N be two positive integers such that $1 \le M \le N$ and $N \ge 2$ when $M = 1$, $m \ge 1$, $d, m \le d \le m(N-1)$ and k, l defined by equation (4.23). Then*

(a) The set $\mathcal{U}_{M,N,m,d}$ has the following representation as a Givens set:

$$\begin{aligned} &\mathcal{U}_{M,N,m,d} \\ &\quad = \quad (B_{0..l-1,l..N-1}Z_{0..l-1})^k\,(B_{0..l-2,l-1..N-1}Z_{0..l-2})^{m-k}\,B_{0..M-1,M..N-1}A_{0..M-1}I_{N,M}. \end{aligned} \tag{4.32}$$

(b) If d satisfies $m \le d \le mM$, then the dimension of (4.32) is equal to

$$f(M,N,m,d) = kl(N-l) + (m-k)(l-1)(N-l+1) + M(N-M) + \frac{M(M-1)}{2}, \tag{4.33}$$

and the representation of $\mathcal{U}_{M,N,m,d}$ given by (4.32) is thus locally injective.

Proof.– From equation (4.23) it follows that $1 \le l \le M$.
Using (3.21) and (3.27), we get

$$SO(N)Z_{0..l-1}SO(N) = B_{0..l-1,l..N-1}A_{0..l-1}A_{l..N-1}Z_{0..l-1}SO(N) = B_{0..l-1,l..N-1}Z_{0..l-1}SO(N),$$

because $A_{l..N-1}$ and $A_{0..l-1}$ commute with $Z_{0..l-1}$. In a similar way

$$SO(N)Z_{0..l-2}SO(N) \quad = \quad B_{0..l-2,l-1..N-1}Z_{0..l-2}SO(N). \tag{4.34}$$

Furthermore

$$SO(N)I_{N,M} = B_{0..M-1,M..N-1}A_{0..M-1}A_{M..N-1}I_{N,M} = B_{0..M-1,M..N-1}A_{0..M-1}I_{N,M}. \tag{4.35}$$

This achieves the proof of (a).

Because of its length, a detailed proof of (b) has been moved in Appendix C. □

Denoting by $\mathcal{U}_{M,N,m}$ the set of $N \times M$ paraunitary matrices that belong to a set given by (4.25), it results from Lemma 4.7 that

$$\mathcal{U}_{M,N,m} = \bigcup_{d=m}^{m(N-1)} \mathcal{U}_{M,N,m,d}. \tag{4.36}$$

Depending on the values of M, N and m, this expression can be simplified as proved in the following Theorems 4.13, 4.14 and 4.10.

4.4 The case $M = N$

Theorem 4.10. *[1] Let $N \geq 2$. For $m \geq 1$ and $m \leq d \leq m(N-1)$, let us define k and l defined by (4.23). Then*

(a) For $m \leq d \leq (N-1)m$

$$\mathcal{U}_{N,N,m,d} = \left(B_{0..l-1,l..N-1}Z_{0..l-1}\right)^k \left(B_{0..l-2,l-1..N-1}Z_{0..l-2}\right)^{m-k} A_{0..N-1}, \tag{4.37}$$

and this angular parametrization is locally injective with dimension $f(N,N,m,d)$.

(b) Thus

$$\mathcal{U}_{N,N,m} = \bigcup_{d=m}^{m(N-1)} \mathcal{U}_{N,N,m,d}. \tag{4.38}$$

The $m(N-2)+1$ sets $\mathcal{U}_{M,N,m,d}$ form a partition of $\mathcal{U}_{N,N,m}$ and are the real algebraic components of the algebraic decomposition of $\mathcal{U}_{N,N,m}$ as well as its connected components.

Proof.– A proof is given in [1] using the lemmas of 4.2. The real algebraic property and the partition property of the $\mathcal{U}_{M,N,m,d}$ result from the fact that

$$\mathcal{U}_{N,N,m,d} = \{\boldsymbol{A},\ A \in \mathcal{U}_{N,N,m},\ \mathrm{val}(A) = \deg(\boldsymbol{A}) = d\}. \tag{4.39}$$

A detailed proof of $\dim(\mathcal{U}_{N,N,m,d}) = f(N,N,m,d)$, which is not given in [1], is given in Appendix C where only $M \leq N$ is used, not $M < N$. □

Remark 4.11. *For $d = rm$, $1 \leq r \leq N-1$, we get $f(N,N,m,rm) = mr(N-r) + \frac{N(N-1)}{2}$, which is the number μ_3 of angular parameters for this particular case given by formula (1.3).*

It is now possible to prove the following result that completes Lemma 2.3

Theorem 4.12. *For given M and N such that $1 \leq M \leq N$, $m \geq 1$, and $0 \leq d \leq m(N-1)$, $A_{d,d}$ is an irreducible algebraic Givens set.*

Proof.– For $N = M$, the result is already known from Theorem 4.10.
For $M < N$, because the proof uses matrices of sizes $N \times M$ and $M \times M$, we will denote by $A_{M,N,d,d}$ the set $A_{d,d}$ relative to parameters M, N, m and by $A_{M,M,d,d}$ the set $A_{d,d}$ relative to parameters M, M, m.

Let $\boldsymbol{A} \in A_{M,N,d,d}$. For every set $J \subset \{0,1,\ldots,N-1\}$ with $\mathrm{Card}(J) = M$ and $I = \{0,1,\ldots,M-1\}$, we get $\mathrm{Det}(A_{J,I}) = \alpha_J X^d$ where at least one α_J is not null. So there exists a $N \times N$ permutation matrix $\boldsymbol{\Pi}$ such that $\boldsymbol{B} = \boldsymbol{\Pi}\boldsymbol{A}$ satisfies $\mathrm{Det}(\boldsymbol{B}_{I,I}) = \beta_0 X^d$ with $\beta_0 \neq 0$. Now introducing the subsets $J_k = \{k, 1, \ldots, M-1\}$, $M \leq k \leq N-1$, $\mathrm{Det}(\boldsymbol{B}_{J_k,I}) = \beta_k X^d$.
For $k = M$, there exists an angle $\theta_M \in \mathbb{T}$ such that

$$\begin{bmatrix} \cos\theta_M & -\sin\theta_M \\ \sin\theta_M & \cos\theta_M \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_M \end{bmatrix} = \begin{bmatrix} \gamma_M \\ 0 \end{bmatrix},$$

with $\gamma_M > 0$, $\gamma_M^2 = \beta_0^2 + \beta_k^2$. Let $\boldsymbol{B}_M = \boldsymbol{R}_{0,M}(\theta_M)\boldsymbol{B}$. The row of index 0 of $\boldsymbol{B}_M$ is given by

$$[\boldsymbol{B}_M]_{0,j} = \cos\theta_M \boldsymbol{B}_{0,j} - \sin\theta_M \boldsymbol{B}_{M,j},\ 0 \leq j \leq M-1,$$

the row of index M is given by

$$[\boldsymbol{B}_M]_{M,j} = \sin\theta_M \boldsymbol{B}_{0,j} + \cos\theta_M \boldsymbol{B}_{M,j},\ 0 \le j \le M-1,$$

while the others rows of $\boldsymbol{B}_M$ are equal to those of $\boldsymbol{B}$. It follows that

$$\begin{aligned} \mathrm{Det}([\boldsymbol{B}_M]_{I,I}) &= \cos\theta_M \mathrm{Det}(\boldsymbol{B}_{I,I}) - \sin\theta_M \mathrm{Det}(\boldsymbol{B}_{J_k,I}) = \gamma_M, \\ \mathrm{Det}([\boldsymbol{B}_M]_{J_M,I}) &= \sin\theta_M \mathrm{Det}(\boldsymbol{B}_{I,I}) + \cos\theta_M \mathrm{Det}(\boldsymbol{B}_{J_k,I}) = 0. \end{aligned}$$

In the same way, for successive k from $M+1$ to $N-1$, angles $\theta_{M+1}, \dots, \theta_{N_1}$ can be found such that the matrix

$$\boldsymbol{C} = \boldsymbol{R}_{0,N-1}(\theta_{N-1}) \dots \boldsymbol{R}_{0,M}(\theta_M)\boldsymbol{B}, \tag{4.40}$$

satisfies

$$\mathrm{Det}(\boldsymbol{C}_{I,I}) = \gamma_{N_1} X^d,\ \gamma_{N-1} > 0 \text{ and } \mathrm{Det}(\boldsymbol{C}_{J_k,I}) = 0,\ M \le k \le N-1. \tag{4.41}$$

For $0 \le r \le N-1$, the row of $\boldsymbol{C}$ of index r is denoted by v_r. Because $\mathrm{Det}(\boldsymbol{C}_{I,I}) \neq 0$, $v_1, v_2, \dots, v_{M-1}$ are independent polynomials, while for $M \le k \le N-1$, $\mathrm{Det}(\boldsymbol{C}_{J_k,I}) = 0$ and v_k is thus a linear combination of $v_1, v_2, \dots, v_{M-1}$. From this we may express $\boldsymbol{C}$ as $\boldsymbol{C} = \boldsymbol{K}\boldsymbol{D}$ where $\boldsymbol{D} = \boldsymbol{C}_{I,I}$ and $\boldsymbol{K}$ is a $N \times M$ constant matrix such that

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{I}_M \\ \boldsymbol{L} \end{bmatrix}, \tag{4.42}$$

where $\boldsymbol{L}$ is a constant $(N-M) \times M$ matrix.

Using the QR factorization ([16, Theorem 5.2.1]), there exists an orthogonal $N \times N$ matrix $\boldsymbol{Q}$ and an upper triangular $N \times M$ matrix $\boldsymbol{R}$ with positive diagonal entries such that $\boldsymbol{K} = \boldsymbol{Q}\ \boldsymbol{R}$. Now defining $\boldsymbol{E} = \boldsymbol{R}_{I,I}\boldsymbol{D}$, we get $\boldsymbol{C} = \boldsymbol{Q}\ \boldsymbol{E}$.

Because $\boldsymbol{C}$ is a paraunitary matrix, because $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ defined by (4.40) are paraunitary, we get

$$\boldsymbol{I}_M = \boldsymbol{C}^T(1/X)\boldsymbol{C}(X) = \boldsymbol{E}^T(1/X)\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{E}(X) = \boldsymbol{E}^T(1/X)\boldsymbol{E}(X), \tag{4.43}$$

and so $\boldsymbol{E}$ is a $M \times M$ paraunitary matrix such that $\mathrm{Det}(\boldsymbol{E}) = \mathrm{Det}(\boldsymbol{R}_{I,I})\mathrm{Det}(\boldsymbol{D}) = \gamma X^d$ with $\gamma > 0$ and thus from (4.43), $\gamma = 1$ and $\boldsymbol{E}$ is paraunitary and $\boldsymbol{E} \in A_{M,M,d,d}$.

Now from Theorem 4.10, $A_{M,M,d,d}$ is a Givens set and $\boldsymbol{E}$ may be written, as any matrix of $\mathcal{U}_{M,M,m,d}$, using the representation (4.37).
To any $M \times M$ paraunitary matrix $\boldsymbol{M}$, we may associate the $N \times M$ paraunitary matrix $\tilde{\boldsymbol{M}}$ such that $\tilde{\boldsymbol{M}}_{I,I} = \boldsymbol{M}$ and $\tilde{\boldsymbol{M}}_{I,\{0,\dots N-1\}\setminus I} = \boldsymbol{O}_{N-M,M}$ where $\boldsymbol{O}_{N-M,M}$ is the null $(N-M) \times M$ matrix. Moreover if G is a Givens set of $M \times M$ paraunitary matrices, then $\tilde{G} = \{\tilde{\boldsymbol{M}},\ \boldsymbol{M} \in G\}$ is a Givens set of $N \times M$ paraunitary matrix because if $G = T_1 T_2 \dots T_{n_T} E[s, \varepsilon]$ then

$$\tilde{G} = \hat{T}_1 \hat{T}_2 \dots \hat{T}_{n_T} \tilde{\boldsymbol{E}}[s, \epsilon], \tag{4.44}$$

where the $M \times M$ matrices in $T_1, T_2, \dots T_{n_T}$ sets are extended to similar matrices of size $N \times N$, shifts on rows r with $0 \le r \le M-1$ or rotations with row indexes i, j with $0 \le i < j \le M-1$. $\tilde{\boldsymbol{E}}[s, \epsilon]$ is also an elementary paraunitary matrix $\boldsymbol{E}[\tilde{s}, \varepsilon]$ where $\tilde{s}$ from $\{0, 1, \dots, M-1\}$ to $\{0, 1, \dots, N-1\}$ is defined by $\tilde{s}c = s(c),\ 0 \le c \le M-1$.

So we have proved that $\boldsymbol{A}$ may be written as a $\boldsymbol{A} = \boldsymbol{U}\tilde{\boldsymbol{D}}$ with $\boldsymbol{U} \in SO(N)$ and $A_{M,N,d,d} = SO(N)\tilde{A}_{M,M,d,d}$ is a Givens set. Because $A_{M,N,d,d}$ is an algebraic set, it is an irreducible Givens set from Theorem 3.10.

□

4.5 The case $N \geq 2M$

Theorem 4.13. *Let $M \geq 1$ and $N \geq 2M$. Then, for $m \geq 1$ and k, l defined by (4.23),*

(a) For $m \leq d < mM$, $\mathcal{U}_{M,N,m,d} \subsetneq \mathcal{U}_{M,N,m,d+1}$,

(b) For $m(N-M) \leq d < m(N-1)-1$, $\mathcal{U}_{M,N,m,d} \supsetneq \mathcal{U}_{M,N,m,d+1}$,

(c) For $mM \leq d \leq m(N-M)$, $\mathcal{U}_{M,N,m,d} = \mathcal{U}_{M,N,m,mM}$

(d)

$$\mathcal{U}_{M,N,m} = \left(B_{0..M-1,M..N-1}Z_{0..M-1}\right)^m B_{0..M-1,M..N-1}A_{0..M-1}I_{N,M}, \tag{4.45}$$

which is a locally injective Givens set with dimension $(m+1)M(N-M)+\frac{M(M-1)}{2}$.

Proof.– For $m=1$ and $M \leq d \leq N-2$, from (4.23) we get $k=1$, $l=d$ and

$$\mathcal{U}_{M,N,1,d} = SO(N)Z_{0..d-1}SO(N)I_{N,M}, \tag{4.46}$$
$$\mathcal{U}_{M,N,1,d+1} = SO(N)Z_{0..d}SO(N)I_{N,M}. \tag{4.47}$$

Using (3.23) with $M_1 = d$ and $M_2 = d+1$ gives

$$\mathcal{U}_{M,N,1,d+1} = SO(N)Z_{0..d}A_{0..d-1}B_{0..d-1,d..d}B_{0..d-1,d+1..N-1}A_{d..N-1}I_{N,M}. \tag{4.48}$$

Because $A_{0..d-1}$ and $B_{0..d-1,d..d}$ commute with $Z_{0..d}$ and $A_{d..N-1}I_{N,M} = I_{N,M}$, it follows that

$$\mathcal{U}_{M,N,1,d+1} = SO(N)Z_{0..d}B_{0..d-1,d+1..N-1}I_{N,M}. \tag{4.49}$$

Now $Z_{0..d} = Z_{0..d-1}Z_d$, Z_d commutes with $B_{0..d-1,d+1..N-1}$ and $Z_dI_{N,M} = I_{N,M}$. So

$$\mathcal{U}_{M,N,1,d+1} = SO(N)Z_{0..d-1}B_{0..d-1,d+1..N-1}I_{N,M}. \tag{4.50}$$

Using relation (3.21) with $M = d$ gives

$$\mathcal{U}_{M,N,1,d} = SO(N)Z_{0..d-1}A_{0..d-1}B_{0..d-1,d..N-1}A_{d..N-1}I_{N,M} \tag{4.51}$$
$$= SO(N)Z_{0..d-1}B_{0..d-1,d..N-1}I_{N,M}. \tag{4.52}$$

Because $B_{0..d-1,d+1..N-1} \subset B_{0..d-1,d..N-1}$, we get $\mathcal{U}_{M,N,1,d+1} \subset \mathcal{U}_{M,N,1,d}$.
Thus the sequence $\mathcal{U}_{M,N,1,d}$, $M \leq d \leq N-1$ is a decreasing sequence of d for set inclusion. Using that $\sigma_1\mathcal{U}_{M,N,1,d} = \mathcal{U}_{M,N,1,N-d}$, it follows that the sequence $\mathcal{U}_{M,N,1,d}$, $1 \leq d \leq N-M$ is an increasing sequence for inclusion and therefore a constant sequence for $M \leq d \leq N-M$ with value $\mathcal{U}_{M,N,1,M}$.

When $m > 1$ and $mM \leq d < m(N-1)$, then there exists one and only one distinct factor in the factorization of $\mathcal{U}_{M,N,1,d}$ and $\mathcal{U}_{M,N,1,d+1}$, $Z_{0..l-1}$ for $\mathcal{U}_{M,N,1,d}$ and $Z_{0..l}$ for $\mathcal{U}_{M,N,1,d+1}$ with $M \leq l < N-1$.
Using Lemma 4.5, these two factors can be moved to the right of the factorization and using the previous result for $m=1$, we deduce that the sequence $\mathcal{U}_{M,N,m,d}$, $mM \leq d \leq m(N-1)$ is a decreasing sequence for inclusion. Using the symmetry relation (4.27), it follows that the sequence $\mathcal{U}_{M,N,m,d}$, $m \leq d \leq m(N-M)$, is an increasing sequence for inclusion. This achieves the proof of parts (a), (b) and (c) of the theorem but not for the strict inclusions in (a) and (b).

For $m \leq d \leq m(N-1)$, let us denote here by $f_l(d)$ and $f_k(d)$ the numbers defined by (4.23). For $d = lm$, $1 \leq l \leq N-1$, we obtain $f_l(d) = l$ and $f_k(d) = m$, that gives $f(M,N,m,lm) =$

$ml(N-l)+M(N-M)+\frac{M(M-1}{2}$ where the function f is given by (4.33). So, as a function of l, $f(M,N,m,lm)$ is a second degree polynomial, which is a strictly increasing function for $1 \le l \le \lfloor \frac{N}{2} \rfloor$. Now for a fixed value of l and for $1 \ge k \le m$, we get from (4.23), $f_l(lm+k)=l+1$ and $f_k(lm+k)=k$. Thus, because $f(M,N,m,d)$ is a linear function of k for a fixed value of l, we deduce that $f(M,N,m,d)$ is linear function of d in each integer interval $[lm,(l+1)m]$, $1 \le l < N-1$. This implies that $f(M,N,m,d)$ is a strictly increasing function for d, $m \le d \le m\frac{N}{2}$.
As $M \le \frac{N}{2}$, we obtain that the dimension of $\mathcal{U}_{M,N,m,d}$ is a strictly increasing function of d for $m \le d \le mM$ and thus the inclusions in (a) are strict inclusions.
Thanks to the symmetry relation (4.27), this implies that inclusion in (b) are strict inclusions.

From (a), (b) and (c), it results that $\mathcal{U}_{M,N,m} = \mathcal{U}_{M,N,m,mM}$ and (d) is a direct consequence of Theorem 4.9.

□

4.6 The case $M < N < 2M$

Theorem 4.14. *Let $M \ge 2$ and $M < N < 2M$. Then, for $m \ge 1$ and k, l defined by (4.23),*

(a) For $m \le d \le (N-1)m$, and k,l such that $d=(l-1)m+k$,

$$\mathcal{U}_{M,N,m,d} = \left(B_{0..l-1,l..N-1}Z_{0..l-1}\right)^k \left(B_{0..l-2,l-1..N-1}Z_{0..l-2}\right)^{m-k} A_{0..M-1}B_{0..M-1,M..N-1}I_{N,M}, \tag{4.53}$$

and this angular parametrization is locally injective with dimension $f(M,N,m,d)$.

(b) For $m \le d < m(N-M)$, $\mathcal{U}_{M,N,m,d} \subsetneq \mathcal{U}_{M,N,m,d+1}$,

(c) For $mM \le d < m(N-1)-1$, $\mathcal{U}_{M,N,m,d} \supsetneq \mathcal{U}_{M,N,m,d+1}$,

(d) Thus

$$\mathcal{U}_{M,N,m} = \bigcup_{d=m(N-M)}^{mM} \mathcal{U}_{M,N,m,d}. \tag{4.54}$$

Each of the $m(2M-N)+1$ sets $\mathcal{U}_{M,N,m,d}$ is not included in the union of the others.

Proof.– (a) From Theorem 4.9, it comes that (a) is true for $m \le d \le mM$. Using the symmetry relation (4.27), (a) is also true for $m(N-M) \le d \le m(N-1)$. Because $N < 2M$, $m(N-M) < mM$ and thus (a) is true for every d, $m < d < m(N-1)$.

(b) The set inclusion is proved as in Theorem 4.13 and the strict inclusion results from the fact that $f(M,N,m,d)$ is a strictly increasing function of d for $m \le d \le m\frac{N}{2}$.

(c) Using symmetry relation (4.27), (c) is equivalent to (b).

(d) For d such that $m(N-M) \le d \le mM$, let $G_d = \mathcal{U}_{M,N,m,d}$. From Theorem (3.9), we get $G_d \subset A_{d-m(N-M),d}$. In (4.53), setting all angles to 0 gives $\boldsymbol{G}_d(0) = (\boldsymbol{Z}_{0..l-1})^k\ (\boldsymbol{Z}_{0..l-2})^{m-k}\ \boldsymbol{I}_{N,M}$ Because $l=[(d-1) \text{ quo } m]+1$, we get $N-M \le l \le M$ and for $I=J=\{0,\dots,M-1\}$, $[\boldsymbol{G}_d]_{I,J}$ is the $M \times M$ diagonal matrix for which the element of row r is X^m for $0 \le r \le l-2$, X^k for $r=l-1$, and 1 for $l \le r \le M-1$. Its determinant is equal to $X^{(l-1)m+k}=X^d$. Hence $\deg(G_d)=d$.

That $\mathrm{val}(G_d) = d - m(N-M)$ is proved by symmetry. As $\deg(\boldsymbol{G}_{mN-d}(0) = m(N-M) - d$, $\mathrm{val}(\sigma_m[\boldsymbol{G}_{m(N-M)-d}(0)]) = mM - (mN - d) = d - m(N-M)$. Because $\sigma_m G_{mN-d} = G_d$ and $\sigma_m[\boldsymbol{G}_{m(N-M)-d}(0)] \in G_d$, this proves that $\mathrm{val}(G_d) = d - m(N-M)$.

Let us consider the partition of $\mathcal{U}_{M,N,m}$ formed by the subsets $B_{v,d}, 0 \leq v \leq d \leq m(N-1)$ defined in subsection 2.3.
Let $d, m(N-M) \leq d \leq mM$ and $\boldsymbol{A} \in B_{d-m(N-M),d}$.
If $d < mM$, as $\mathrm{val}(\boldsymbol{B}) \leq d' - m(N-M) > d - m(N-M)$ for any $d' > d$ and $\boldsymbol{B} \in \mathcal{U}_{M,N,m,d'}$ from Theorem 3.9 (b), $\boldsymbol{A} \notin \mathcal{U}_{M,N,m,d'}$ for $d' > d$.
If $d > m(N-M)$, as $\deg(\boldsymbol{B}) \leq d' < d$ for any $d' < d$ and $\boldsymbol{B} \in \mathcal{U}_{M,N,m,d'}$ from Theorem 3.9 (b) again, $\boldsymbol{A} \notin \mathcal{U}_{M,N,m,d'}$ for $d' < d$.
Therefore $\boldsymbol{A} \in \mathcal{U}_{M,N,m} \setminus \bigcup_{d' \neq d} \mathcal{U}_{M,N,m,d'} = \mathcal{U}_{M,N,m,d} \setminus \bigcup_{d' \neq d} \mathcal{U}_{M,N,m,d'}$.

$B_{d-m(N-M),d}$ is thus a dense subset of $\mathcal{U}_{M,N,m,d}$, elements of which do not belong to $\bigcup_{d' \neq d} \mathcal{U}_{M,N,m,d'}$. This achieves the proof of Theorem 4.14. □

Remark 4.15. *It does not result from Theorem 4.14 (d) that (4.54) is the decomposition of $\mathcal{U}_{M,N,m}$ in irreducible algebraic sets because it is not proved that $\mathcal{U}_{M,N,m,d}$ is irreducible.*
However, from Theorem 3.10, it follows that $\overline{\mathcal{U}_{M,N,m,d}}^Z$ is irreducible.
If Conjecture 2.4 was true, then $\overline{B_{d-m(N-M),d}}^Z = A_{d-m(N-M),d}$ and because $B_{d-m(N-M),d} \subset \mathcal{U}_{M,N,m,d} \subset A_{d-m(N-M),d}$, $\overline{\mathcal{U}_{M,N,m,d}}^Z = A_{d-m(N-M),d}$ and the decomposition of $\mathcal{U}_{M,N,m}$ would be given by

$$\mathcal{U}_{M,N,m} = \bigcup_{d=m(N-M)}^{mM} A_{d-m(N-M),d}. \tag{4.55}$$

More generally, we are led to the following conjecture

Conjecture 4.16. *Let M, N such that $M < N$. For any v and d such that $v \leq d$, if $B_{v,d} \neq \emptyset$ then $A_{v,d}$ is an irreducible Givens set.*

5 Conclusion

An ensemblist approach has been presented to cover the different possibilities of getting new parametrization descriptions of the rectangular $N \times M$ paraunitary matrices sets composed of degree-m polynomials with real coefficients. The proposed parameterizations are complete, and for given minimum number of shifts, d, involve a minimum number of independent angular parameters. These new representations result from an in depth algebraic analysis of sets, named Givens sets, composed of rotation and shift (i.e. delay) matrices. We have shown that the mathematical properties of these Givens sets could be different according to the respective values of N and M. For three cases, corresponding either to $N = M$, $N \geq 2M$ and $M < N < 2M$, we have derived key algebraic properties of the associated Givens sets, together with the expression of their dimension as a function of N, M, m and d. Our analysis shows that the study PU matrices has strong connections with some algebraic geometry topics. However, if the investigation of the reducibility properties of polynomial matrices could help us to characterize Givens sets, there are still two essential conjectures to solve in order to completely characterize Givens sets for $M < N < 2M$ when being partitioned according to the McMillan degree and valuation of their representative matrices.

6 Acknowledgements

We would like to thank Jean Moulin Ollagnier who provides us with the first example of a non algebraic Givens set.

References

[1] D. Pinchon and P. Siohan. Angular parametrization of real paraunitary matrices. *IEEE Signal Processing Letters*, 15:353–356, 2008.

[2] J. Zhou, M. N. Do, and J. Kovačević. Special paraunitary matrices, Cayley transform, and multidimensional orthogonal filter banks. *IEEE Transactions on Image Processing*, 35(4):511–519, 2006.

[3] S. Rahimi and B. Champagne. Oversampled perfect reconstruction DFT modulated filter banks for multi-carrier transceiver systems. *Signal Processing*, 93(11):2942–2955, November 2013.

[4] D. Pinchon and P. Siohan. Design of perfect reconstruction oversampled DFT transmultiplexers with minimal dimension. In *Proc. EUSIPCO, Nice, France, 2015*, 2015.

[5] F. Labeau, J-C. Chiang, M. Kieffer, P. Duhamel, L. Vandendorpe, and B. Macq. Oversampled filter banks as error correcting codes: theory and impulse noise correction. *IEEE Transactions on Signal Processing*, 53(12):4619–4630, 2005.

[6] S. Redif, S. Weiss, and J. G. McWhirter. Sequential matrix diagonalisation algorithms for polynomial EVD of parahermitian matrices. *IEEE Transactions on Signal Processing*, 63(1):81–89, 2015.

[7] P. P. Vaidyanathan. *Multirate systems and filter banks*. Prentice Hall, Englewood Cliffs, 1993. ISBN 0-13-605718-7.

[8] X. Gao, T.Q. Nguyen, and G. Strang. On factorization of m-channel paraunitary filterbanks. *IEEE Trans. Sig. Proc.*, 49(7):1433–1446, 2001.

[9] P. P. Vaidyanathan. Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M having the perfect-reconstruction property. *IEEE Trans. Acoust. Speech Sig. Proc.*., 35(4):476–492, 1987.

[10] L. Gan and K.K. Ma. On simplified order-one factorization of paraunitary filterbanks. *IEEE Trans. Sig. Proc.*, 52(3):674–686, 2004.

[11] Z. Cvetkovic and M. Vetterli. Oversampled filter banks. *IEEE Transactions on Signal Processing*, 46(5):1245–1255, May 1998.

[12] D. Pinchon and P. Siohan. Oversampled Paraunitary DFT Filter Banks: A General construction Algorithm and Some Specific Solutions. *IEEE Trans. Sig. Proc.*, 59(7):3058–3071, 2011.

[13] J.J. Duistermaat and J.A.C. Kolk. *Lie Groups*. Springer, 2000.

[14] A. Gathmann. *Algebraic Geometry*. Class Notes TU Kaiserslautern 2014, 133 pages, available on line at www.mathematik.uni-kl.de/∼gathmann/class/alggeom-2014/main.pdf.

[15] L.W. Tu. *An introduction to manifolds*. Springer, 2008.

[16] G.H. Golub and C.F. Van Loan. *Matrix computations*. North Oxford Academic Publishing Co. Ltd., Oxford, 4th edition, 2013.

[17] R.C. Raffeneti and K. Ruedenberg. Parametrization of an orthogonal matrix in terms of generalized Eulerian angles. *J. Quantum Chemistry*, 4(III S):625–634, 1969.

[18] D.K. Hoffman, R.C. Raffeneti, and K. Ruedenberg. Generalization of Euler angles to n-dimensional orthogonal matrices. *J. Math. Physics*, 13:528–533, 1972.

[19] C. Lawrence, J.L. Zhou, and A.L. Tits. User's guide for CFSQP version 2.5: A C code for solving (large scale) constrained nonlinear (minimax) optimization problems, generating iterates satisfying all inequality constraints. Technical Report TR-94-16r1, University of Maryland, 1998.

[20] R.J. Caron and T. Traynor. The zero set of a polynomial. Technical Report WSMR 05-03, Department of Mathematics and Statistics, University of Windsor, Windsor, ON Canada, May 2005. www1.uwindsor.ca/math/sites/uwindsor.ca.math/files/05-03.pdf.

A Counterexamples for Theorem 1 and Lemma 2 in [1]

Let us consider, in $SO(4,\mathbb{R})$, the $6! = 720$ Givens sets obtained as a product in any order of the six rotation sets $R_{0,1}, R_{0,2}, R_{0,3}, R_{0,1}, R_{0,2}, R_{1,2}$, and look for their property of being algebraic sets. Those sets belong to two disjoint subsets

1. Those for which the two first rotations sets starting from left have a common index, and their algebraic property is not modified when we apply a simultaneous permutation on rows and columns. So we may assume that the two first rotation sets at left are $R_{0,1}$ et $R_{0,2}$. The number of cases to study is then reduced to $4! = 24$.

2. Those for which the two first rotation sets at left have no common indexes and this reduces also the number of cases to study to 24, assuming that the two rotation sets starting from left are $R_{0,1}$ et $R_{1,2}$. Observing that the transposition of a Givens set is also a Givens set because $R_{i,j}^T = R_{i,j} = R_{j,i}$ and preserves the algebraic property, we may suppress from this second set those Givens sets for which the two rotation sets at right have a common index and thus for which their transposed Givens set are already considered in the first subset. Then there remains only 8 Givens sets in this subset.

For remaining Givens sets, we may then use

- the commutation rule between two rotation sets when their indexes are disjoints sets;
- the different expressions of $SO(3,\mathbb{R})$ as a product of three rotations sets when the set of indexes of three consecutive rotations sets has only three elements;
- a permutation of indexes followed by a transposition.

Then there remains 5 Givens sets to study

$$\begin{aligned}
G_1 &= R_{0,1}\ R_{0,2}\ R_{0,3}\ R_{1,2}\ R_{1,3}\ R_{2,3}\\
G_2 &= R_{0,1}\ R_{0,2}\ R_{1,3}\ R_{0,3}\ R_{1,2}\ R_{2,3}\\
G_3 &= R_{0,1}\ R_{0,2}\ R_{1,3}\ R_{2,3}\ R_{0,3}\ R_{1,2}\\
G_4 &= R_{0,1}\ R_{0,2}\ R_{2,3}\ R_{1,3}\ R_{0,3}\ R_{1,2}\\
G_5 &= R_{0,1}\ R_{2,3}\ R_{0,2}\ R_{1,3}\ R_{0,3}\ R_{1,2}
\end{aligned}$$

It is known by 3.19 that $G_1 = SO(4,\mathbb{R})$ and because $G_i, i = 2,\ldots,5$ have also a dimension equal to 6, we try to know if they are also equal to $SO(4,\mathbb{R})$.

A numerical computation allows to have a first idea of the result. For $i \in 1,2,3,4$, we choose at random a point $G_1(\alpha)$ in G_1 where $\alpha = (\alpha_1,\ldots,\alpha_6)$ and we try to know if there exists $\theta = (\theta_1,\ldots,\theta_6)$ such that $G_2(\theta) = G_1(\alpha)$ by computing the minimum of the function

$$F(\alpha) = \min\{\sum_{r=0}^{3}\sum_{c=0}^{3} |[G_2(\theta) - G_1(\alpha)]_{r,c}|^2\,,\ \theta \in \mathbb{R}^6\}. \tag{A.1}$$

With a global optimization software (CFSQP, [19]), we compute $F(\alpha)$ for M random choices of α and a fixed maximum number of random initial values for θ. The optimization process is stopped when the value of $F(\alpha)$ is less to a given threshold $\varepsilon > 0$. In our tests, we first choose $M = 10$, $N = 2000$, $\varepsilon = 10^{-13}$ and if we obtained $F(\alpha) < \epsilon$, we then chose $M = 100000$. Numerical results

allows us to believe that $G_3 = G_4 = G_5 = SO(4,\mathbb{R})$ but that the measure of $SO(4,\mathbb{R}) \setminus G_2$ is strictly positive.

We can, after that experience, prove the following proposition.

Proposition A.1. *G_2 is strictly included in $SO(4,\mathbb{R})$ and thus G_2 is not a real algebraic set.*

Proof.– Indexes of angles being decreasing from left to right, with

$$\boldsymbol{G}_2(\theta) = \boldsymbol{R}_{0,1}(\theta_6)\ \boldsymbol{R}_{0,2}(\theta_5)\ \boldsymbol{R}_{1,3}(\theta_4)\ \boldsymbol{R}_{0,3}(\theta_3)\ \boldsymbol{R}_{1,2}(\theta_2)\ \boldsymbol{R}_{2,3}(\theta_1), \tag{A.2}$$

we get, with notations $c_i = \cos\theta_i,\ s_i = \sin\theta_i,\ 1 \le i \le 6$

$$\begin{aligned} g_{2,0} = \boldsymbol{G}_2(\theta)_{2,0} = c_3 s_5, \quad g_{2,1} = \boldsymbol{G}_2(\theta)_{2,1} = s_2 c_5, \\ g_{3,0} = \boldsymbol{G}_2(\theta)_{3,0} = s_3 c_4, \quad g_{3,1} = \boldsymbol{G}_2(\theta)_{3,1} = c_2 s_4. \end{aligned}$$

From Schwarz's inequality, it follows that

$$|g_{2,0}| + |g_{2,1}| + |g_{3,0}| + |g_{3,1}| \le (s_5^2 + c_5^2 + c_4^2 + s_4^2)^{\frac{1}{2}}\ (c_3^2 + s_2^2 + s_3^2 + c_3^2)^{\frac{1}{2}} = 2. \tag{A.3}$$

A maximum equal to 2 is moreover reached for $\theta_i = \frac{\pi}{4},\ 2 \le i \le 5$.

On the other hand, let us consider the orthogonal matrix $\boldsymbol{A} = \boldsymbol{G}_1(\alpha)$ with $\alpha_i = \frac{\pi}{3},\ 1 \le i \le 6$,

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{8} & -\frac{3}{8} - \frac{1}{8}\sqrt{3} & \frac{9}{16} + \frac{1}{8}\sqrt{3} & -\frac{5}{16}\sqrt{3} + \frac{3}{8} \\ \frac{1}{8}\sqrt{3} & -\frac{3}{8}\sqrt{3} + \frac{1}{8} & \frac{1}{16}\sqrt{3} - \frac{3}{8} & \frac{9}{16} + \frac{1}{8}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & -\frac{1}{4}\sqrt{3} & -\frac{3}{8}\sqrt{3} + \frac{1}{8} & -\frac{3}{8} - \frac{1}{8}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{4}\sqrt{3} & \frac{1}{8}\sqrt{3} & \frac{1}{8} \end{bmatrix}.$$

As $|\boldsymbol{A}_{2,0}| + |\boldsymbol{A}_{2,1}| + |\boldsymbol{A}_{3,0}| + |\boldsymbol{A}_{3,1}| = \frac{5\sqrt{3}}{4} > 2$, A does not belong to G_2, and therefore also all the points in a neighborhood of A in $SO(4,\mathbb{R})$.

Let now $\boldsymbol{A} \in G_2$ and V an open neighbouhood of $\boldsymbol{A} \subset G_2$ for the Euclidean topology. As for any $\boldsymbol{X} \in SO(A)$, there exists $\boldsymbol{Y} \in SO(4)$ such that $\boldsymbol{Y}\boldsymbol{A} = \boldsymbol{X}$ (namely $\boldsymbol{Y} = \boldsymbol{X}_;\boldsymbol{A}^{-1}$) , $SO(4) = \cup_{Y \in SO(4)} \boldsymbol{Y}V$ and each $\boldsymbol{Y}V$ is an open set in $SO(4)$ and because $SO(4)$ is compact, we can extract a finite covering $SO(4) = \cup_{i \in I} \boldsymbol{Y}_{i \in I} V = \cup_{i_i n I} \boldsymbol{Y}_i G_2$. If G_2 were irreductible, then each $\boldsymbol{Y}_i G_2$, $i \in I$ too, and because $SO(4)$ is irreductible, this would imply that $\boldsymbol{Y}_i G_2 = SO(4)$ for all $i \in I$, Then $G_2 = \boldsymbol{Y}_i^{-1} SO(4) = SO(4)$, which is not true. Thus G_2 is not an algebraic set.

□

It is because the conclusion of Lemma 2 is false that we obtained a wrong proof of Theorem 1, which is also false, in the appendix of [1].

Indeed let us consider the Givens set

$$G = R_{0,3} G_2 = R_{0,3}\ R_{0,1}\ R_{0,2}\ R_{1,3}\ R_{0,3}\ R_{1,2}\ R_{2,3}. \tag{A.4}$$

Setting to 0 the angle of rotation $R_{0,3}$, we get that G is included in

$$G' = R_{0,3}\ R_{0,1}\ R_{0,2}\ R_{1,3}\ R_{0,3}\ R_{1,2}\ R_{2,3}. \tag{A.5}$$

Let us now introduce the permutation on indexes $\sigma : (1,2,3,4) \rightarrow (1,3,4,5)$ and let us apply this permutation on both rows and columns of G'. The following Givens set G" is obtained

$$G" = R_{0,1}\ R_{0,2}\ R_{0,3}\ R_{1,2}\ R_{2,3}\ R_{1,3}. \tag{A.6}$$

So $G" = \Pi_\sigma\ G'\Pi_\sigma^T$ where Π_σ is the permutation matrix of σ. The product $R_{1,2}\ R_{2,3}\ R_{1,3}$ is the set of all rotations acting on rows and columns with indexes 2,3,4, and can also be written as $R_{1,2}\ R_{1,3}\ R_{2,3}$ because Theorem 1 is true for $n = 3$. Therefore we get

$$G" = R_{0,1}\ R_{0,2}\ R_{0,3}\ R_{1,2}\ R_{1,3}\ R_{2,3} = SO(4,\mathbb{R}) \tag{A.7}$$

We deduce that $G' = \Pi_\sigma^T\ SO(4,\mathbb{R})\ \Pi_\sigma = SO(4,\mathbb{R})$, and thus $R_{0,3}G_2$ is such that $\dim(R_{0,3}G_2) = \dim(G_2) = 6$, but because G_2 is strictly included in $R_{0,3}G_2 = SO(4,\mathbb{R})$ from the previous result, we obtain a counterexample for Lemma 2.

B Generic properties

The following theorem is well known and often used in the literature, but we could not find a precise reference to a published book or paper. As its proof is an immediate consequence of Tonelli's theorem first published in 1909, we are still looking for a good old reference. For the sake of completeness, a proof is given below.

Theorem B.1. *(a) Let $n \geq 1$ and $p(x),\ x \in \mathbb{R}^n$ a polynomial in n variables. Then p is identically null or the set of its zeroes has a null measure in $\mathbb{R}^n$.*

(b) Let $p(\theta_1, \ldots, \theta_n)$ a polynomial in $\cos\theta_i, \sin\theta_i,\ 1 \leq i \leq n$ with real coefficients. Then p is identically null or the set of its zeroes has a null measure in $\mathbb{R}^n$.

Proof.– The proof of (a) is a classical application of Tonelli's theorem for Lebesgue measure on $\mathbb{R}^n$ and is done by recurrence on n. It is given here, adapting the elegant simple proof in [20], for the sake of completeness.
Here λ_n denotes the Lebesgue measure on $\mathbb{R}^n$. First remark that the set of zeroes of p is a measurable set in $\mathbb{R}^n$ for λ_n because it is a closed set.
The result for $n = 1$ is true because a non null polynomial has only a finite number of real roots. Let $n > 1$ and suppose that the result is true for $n - 1$. $p(x), x = (x_1, x_2, \ldots, x_n)$ may be written as a polynomial in $\bar{x} = (x_2, \ldots, x_n)$ for which coefficients are polynomials in x_1:

$$p(x) = \sum_j p_j(x_1)\bar{x}^j,\ j = (j_2, \ldots, j_n),\ \bar{x}^j = x_2^{j_2} \ldots x_n^{j_n}. \tag{B.1}$$

As p is not identically null, there exists a multi-indice j_0 such that p_{j_0} is not identically null, and thus has a finite number of zeros in x_1: $N_1 = \{x_1,\ p_{j_0}(x_1) = 0\}$.
Now, for every real x_1 consider the polynomial $r_{x_1}(\bar{x}) = \sum_j p_j(x_1)\bar{x}^j$ with $n - 1$ variables $\bar{x}$. As the result is true pour $n - 1$ variables, two cases can occur

1. r_{x_1} is identically null, that is $p_j(x_1) = 0$ for every j and thus $p_{j_0}(x_1) = 0$. Therefore $N_2 = \{x_1,\ r_{x_1} \equiv 0\} \subset N_1$ is finite and has measure 0.
2. For $x_1 \in \mathbb{R} \setminus N_2$, $\lambda_{n-1}(\{\bar{x}, r_{x_1}(\bar{x}) = p(x_1, \bar{x}) = 0\}) = 0$.

Using Tonelli theorem

$$\begin{aligned}
\lambda_n(\{x,\ p(x) = 0\}) &= \int \lambda_{n-1}(\{\bar{x}\ p(x_1, \bar{x}) = 0\})\ dx_1 \\
&= \int_{N_2} \lambda_{n-1}(\{\bar{x}\ p(x_1, \bar{x}) = 0\})\ dx_1 + \int_{\mathbb{R}\setminus N_2} 0\ dx_1 \\
&= 0.
\end{aligned} \tag{B.2}$$

For any $i,\ 1 \leq i \leq n$, by excluding the set $\{(2k+1)\pi,\ k \in \mathbb{Z}\}$ of measure 0, $\cos\theta_i$ and $\sin\theta_i$ may be expressed as rational functions of $t_i = \tan\frac{\theta_i}{2}$: $\cos\theta_i = \frac{1-t_i^2}{1+t_i^2}$, $\sin\theta_i = \frac{2t_i}{1+t_i^2}$. So $p(\theta_1, \ldots, \theta_n)$ may be expressed as a rational function $\frac{N(t_1,\ldots,t_n)}{D(t_1,\ldots,t_n)}$ of $t_1, \ldots, t_n$. If p is not identically null, the polynomial $N(t_1, \ldots, t_n)$ is not null too. Following (a), the set of zeroes of N has measure 0, and the same property holds for the zeroes $(\theta_1, \ldots, \theta_n)$ of p.

□

Several applications of Theorem B.1 to Givens sets are used in the paper.

As in 3.2, let us consider a Givens set $G = T_1 T_2 \dots T_{n_T} I_{N,M}$ with n_R rotation sets and ϕ_G its parametric representation from $\mathbb{R}^{n_R}$ to G. Elements in G are polynomial functions of $\cos\theta_l$ and $\sin\theta_l,\ 1 \le l \le n_R$.
Denoting by R_{i_l,j_l} the l-th rotation set in G, the Jacobian matrix element $\frac{\partial \phi_G}{\partial \theta_l}$ is obtained by replacing R_{i_l,j_l} in G by $D_{i_l,j_l} = \{\boldsymbol{D}_{i_l,j_l}(\theta_l),\ \theta_l \in \mathbb{R}\}$ where $\boldsymbol{D}_{i_l,j_l}(\theta_l)$ is defined by (C.10). So the elements of the Jacobian matrix $\boldsymbol{J}(\theta)$, $\theta = (\theta_1, \dots, \theta_{n_R})$, are also polynomial in $\cos\theta_l$ and $\sin\theta_l,\ 1 \le l \le n_R$. If the dimension of G is equal to d, then all minors of $\boldsymbol{J}(\theta)$ of order $r, r > d$ are identically null while there exists a value of θ for which a minor of order d is non null. Because any minor of $\boldsymbol{J}(\theta))$ is a polynomial function of the elements of $\boldsymbol{J}(\theta))$, and thus a polynomial function of $\cos\theta_l$ and $\sin\theta_l,\ 1 \le l \le n_R$, applying Theorem B.1 this minor is non null except on a set of θ with null measure.

For any matrix $\boldsymbol{A} \in G$ and $I \subset \{0, \dots, N-1\}$, $J \subset \{0, \dots, M-1\}$ such that $n = \mathrm{Card}(I) = \mathrm{Card}(J) \le M$, the coefficients of elements of $\boldsymbol{A}_{I,J}$ defined in 2.3 are polynomials in $\cos\theta_l$ and $\sin\theta_l,\ 1 \le l \le n_R$ and so are the coefficients of its determinant $\mathrm{Det}(\boldsymbol{A}_{I,J})$.
$\deg(G)$ is the maximum degree in X of $\mathrm{Det}(\boldsymbol{A}_{I,J})$ for $\mathrm{Card}(I) = \mathrm{Card}(J) = M$. So there exists $\boldsymbol{A} \in G$, I_0, J_0 with $\mathrm{Card}(I_0) = \mathrm{Card}(J_0) = M$, such that the coefficients of order $d > \deg(G)$ of $\mathrm{Det}(\boldsymbol{A}_{I,J})$, $\mathrm{Card}(I) = \mathrm{Card}(J) = M$, are indentically null, while the coefficient or order $\deg(G)$ of $\mathrm{Det}(\boldsymbol{A}_{I_0,J_0})$ is non null. So, from Theorem B.1, the coefficient or order $\deg(G)$ of $\mathrm{Det}(\phi_G(\theta)_{I_0,J_0})$ is non null except on a set with null measure.

The same proof holds for the valuation of $\mathrm{Det}(\boldsymbol{A}_{I,J})$, i.e. the maximum order of a non null of its coefficients, and thus for $\mathrm{val}(G)$.

C Proof of Theorem 4.9 (b)

The number of rotation sets in (4.32) being equal to $f(M,N,m,d)$, where f is defined by (4.33), the dimension of $\mathcal{U}_{M,N,m,d}$ is less or equal to $f(M,N,m,d)$.

The parametric representation associated to expression (4.32) may be denoted

$$\begin{aligned}\boldsymbol{G}(\theta,\phi,\alpha,\beta) &= \prod_{a=1}^{k}\left(\prod_{i=0}^{l-1}\prod_{j=l}^{N-1}\boldsymbol{R}_{i,j}(\theta_{a,i,j})\,\boldsymbol{Z}_{0..l-1}\right)\prod_{b=1}^{m-k}\left(\prod_{i=0}^{l-2}\prod_{j=l-1}^{N-1}\boldsymbol{R}_{i,j}(\phi_{b,i,j})\,\boldsymbol{Z}_{0..l-2}\right)\\ &\quad \prod_{i=0}^{M-1}\prod_{j=M}^{N-1}\boldsymbol{R}_{i,j}(\alpha_{i,j})\prod_{i=0}^{M-2}\prod_{j=i+1}^{M-1}\boldsymbol{R}_{i,j}(\beta_{i,j})\,\boldsymbol{I}_{N,M} \qquad \text{(C.1)}\end{aligned}$$

where

$$\begin{aligned}\theta &= \{\theta_{n,i,j},\ 1\le n\le k,\ (i,j)\in I_1\}, && \text{(C.2)}\\ \phi &= \{\phi_{n,i,j},\ 1\le n\le m-k,\ (i,j)\in I_2\}, && \text{(C.3)}\\ \alpha &= \{\alpha_{i,j},\ (i,j)\in I_3\}, && \text{(C.4)}\\ \beta &= \{\beta_{i,j},\ (i,j)\in I_4\}, && \text{(C.5)}\end{aligned}$$

with

$$\begin{aligned}I_1 &= \{i,j),\ 0\le i\le l-1,\ l\le j\le N-1\}, && \text{(C.6)}\\ I_2 &= \{i,j),\ 0\le i\le l-2,\ l-1\le j\le N-1\}, && \text{(C.7)}\\ I_3 &= \{i,j),\ 0\le i\le M-1,\ M\le j\le N-1\}, && \text{(C.8)}\\ I_4 &= \{i,j),\ 0\le i\le M-2,\ i+1\le j\le M-1\}.. && \text{(C.9)}\end{aligned}$$

We denote by $\mathbf{0}$ the value of $(\theta,\phi,\alpha,\beta)$ where all angles are set to 0 and we prove below that $\mathbf{0}$ is a regular point, i.e. that the rank of the Jacobian matrix of $\boldsymbol{G}(\theta\phi,\alpha,\beta)$ at $\mathbf{0}$ is equal to $f(M,N,m,d)$.

The derivative of $\boldsymbol{R}_{i,j}(\theta)$ with respect to θ is the $N\times N$ matrix $\boldsymbol{D}_{i,j}(\theta)$ defined by

$$\begin{aligned}&[\boldsymbol{D}_{i,j}(\theta)]_{i,i}=[\boldsymbol{D}_{i,j}(\theta)]_{j,j}=-\sin\theta\\ &[\boldsymbol{D}_{i,j}(\theta)]_{j,i}=-[\boldsymbol{D}_{i,j}(\theta)]_{i,j}=\cos\theta \qquad \text{(C.10)}\\ &0\text{elsewhere.}\end{aligned}$$

So for $\theta=0$, we get the $N\times N$ matrix $\boldsymbol{D}_{i,j}(0)$, simply denoted by $\boldsymbol{D}_{i,j}$, such that $[\boldsymbol{D}_{i,j}]_{i,j}=-1$, $[\boldsymbol{D}_{i,j}]_{j,i}=1$ and 0 elsewhere.

Using the fact that $\boldsymbol{R}_{i,j}(0)=\boldsymbol{I}_N$, it is thus straightforward to see that the partial derivative of $\boldsymbol{G}(\theta,\phi,\alpha,\beta)$ at $\mathbf{0}$ with respect to any angle in a rotation with rows indexes i and j, $i<j$, has the form $\boldsymbol{\Delta}_1\boldsymbol{D}_{i,j}\boldsymbol{\Delta}_2\boldsymbol{I}_{N,M}$, where $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ are diagonal $N\times N$ matrices for which the diagonal elements are power of X. If $[\boldsymbol{\Delta}_1]_{r,r}=X^{a_r}$, $[\boldsymbol{\Delta}_2]_{r,r}=X^{b_r}$, $a_r,b_r\ge 0$, $0\le r\le N-1$, then the partial derivative is equal to $\boldsymbol{E}_{j,i,a_j+b_i}-\boldsymbol{E}_{i,j,a_i+b_j}$ when $i,j\le M-1$ and $\boldsymbol{E}_{j,i,a_j+b_i}$ when $0\le i\le M-1$, $M\le j\le N-1$.

Let us consider a couple of rows indices $(i,j)\in I_1\cup I_2\cup I_3\cup I_4=\{(i,j),\ 0\le i\le M-1,\ i+1\le j\le N-1\}$, and let us denote by $\mathcal{E}_{i,j}$ the linear subspace of $M\times N$ matrices generated by matrices $\boldsymbol{E}_{j,i,k}$, $0\le k\le m$ and matrices $\boldsymbol{E}_{i,j,k}$, $i<j\le M-1$, $0\le k\le m$. For any rotation with rows

indexes equal to i, j, the partial derivative of $\boldsymbol{G}(\theta, \phi, \alpha, \beta)$ at 0 for the corresponding angle variable belongs to $\mathcal{E}_{i,j}$.
As the different subspaces $\mathcal{E}_{i,j}$ are disjoint, the rank of the Jacobian matrix is equal to the sum of the rank of partial derivatives in each subspace $\mathcal{E}_{i,j}$. In particular, the Jacobian is equal to the total number of rotation sets if and only if it is the case in each (i, j) for rotations with row indexes i and j.
Depending on the values of parameters M, N, m and $d, m \leq d \leq mM$, several cases must be considered.
Case 1: $d = mM$ (Figure 2)

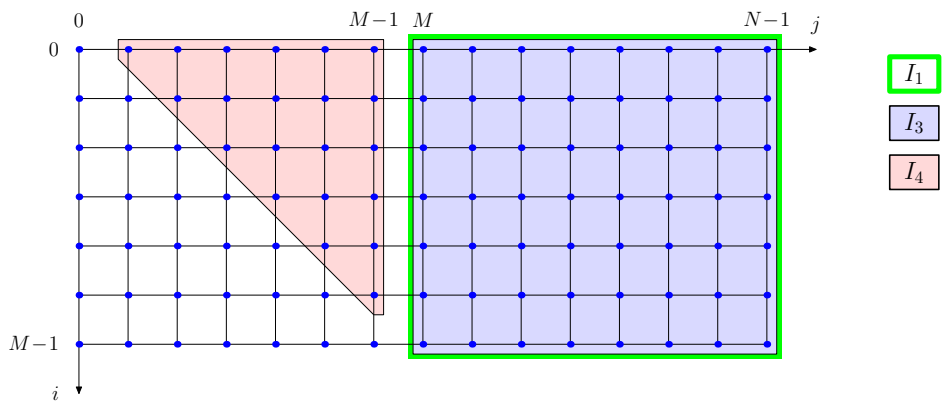

Figure 2: Rotations indexes for $d = mM$.

From equation (4.23), it follows that $l = M$, $k = M$, $I_2 = \emptyset$, $I_1 = I_3$ and $I_1 \cap I_4$.

As there is only one rotation set $R_{i,j}$ for $(i, j) \in I_4$ corresponding to angular variable $\beta_{i,j}$ and because

$$\frac{\partial \boldsymbol{G}}{\partial \beta_{i,j}}(\mathbf{0}) = \boldsymbol{D}_{i,j}\boldsymbol{Z}_{0..M-1}\boldsymbol{I}_{M,N} = \boldsymbol{E}_{j,i,m} - \boldsymbol{E}_{i,j,m} \neq 0, \tag{C.11}$$

the rank of the generated subspace is well 1.

For $(i, j) \in I_1$, there are $m + 1$ rotation sets with row indexes corresponding to variables $\theta_{n,i,j}, 1 \leq n \leq m$ and $\alpha_{i,j}$.
For rotation $\boldsymbol{R}_{i,j}(\alpha_{i,j})$, $a_j = 1$, $b_i = 1$ and therefore

$$\frac{\partial \boldsymbol{G}}{\partial \alpha_{i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,0}. \tag{C.12}$$

For $n, 1 \leq n \leq m$, we check that $a_j = 1$, $b_i = X^{m-n+1}$ and therefore

$$\frac{\partial \boldsymbol{G}}{\partial \theta_{n,i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,m-n+1}. \tag{C.13}$$

The partial derivatives in (C.11) and (C.12) are linearly independent and so the rank of the Jacobian matrix at $\mathbf{0}$ is equal to the number of rotations sets, i.e. $(m + 1)M(N - M) + \frac{M(M-1)}{2}$.

Case 2: $m > 1$ and $m(M - 1) < d < mM$ (Figure 3)
From (4.23), we get $l = M$ and $1 \leq k < m$, and $I_1 = I_3$.

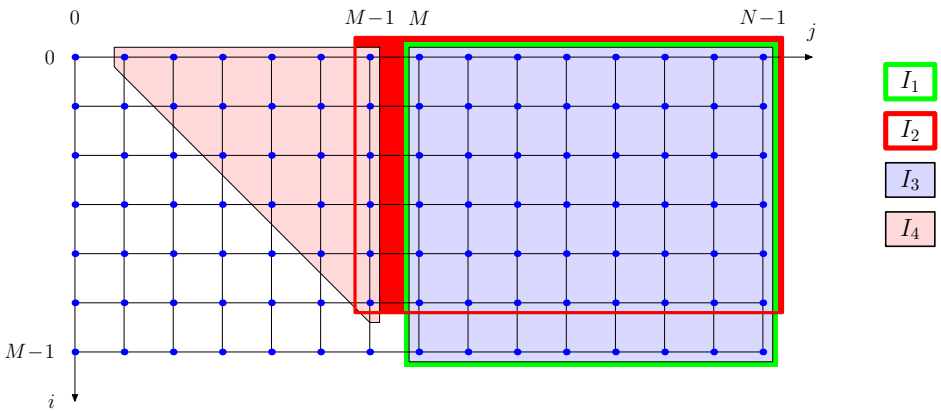

Figure 3: Rotations indexes for $m(M-1) < d < mM$.

For $(i,j) \in I_4 \setminus I_2$, i.e. $0 \leq i \leq M-3,\ i+1 \leq j \leq M-2$ there is only one rotation set $R_{i,j}$ with angle $\beta_{i,j}$ and

$$\frac{\partial \boldsymbol{G}}{\partial \beta_{i,j}}(\boldsymbol{0}) = \boldsymbol{E}_{j,i,m} - \boldsymbol{E}_{i,j,m} \neq 0, \tag{C.14}$$

For $(i,j) \in I_1 \setminus I_2$, i.e. $(i,j) = (M-1,j),\ M \leq j \leq N-1$, there are $k+1$ rotation sets $R_{M-1,j}$ for variables $\alpha_{M-1,j}$ and $\theta_{n,i,j},\ 1 \leq n \leq k$.
For variable $\alpha_{M-1,j}$, $\boldsymbol{D}_1 = \boldsymbol{Z}^k_{0..M-1}\boldsymbol{Z}^{m-k}_{0..M-2}$, $a_{M-1} = X^k$ and $b_j = 1$. So

$$\frac{\partial \boldsymbol{G}}{\partial \alpha_{M-1,j}}(\boldsymbol{0}) = \boldsymbol{E}_{j,M-1,0}. \tag{C.15}$$

For variable $\theta_{n,i,j}$, $\boldsymbol{D}_1 = \boldsymbol{Z}^{n-1}_{0..M-1}$ and $\boldsymbol{D}_2 = \boldsymbol{Z}^{k-n+1}_{0..M-1}\boldsymbol{Z}^{m-k}_{0..M-2}$, so $a_{M-1} = X^{k-n+1},\ a_j = 1,\ b_{M-1} =$ and $b_j = 1$. It follows that, for $1 \leq n \leq k$,

$$\frac{\partial \boldsymbol{G}}{\partial \theta_{n,M-1,j}}(\boldsymbol{0}) = \boldsymbol{E}_{j,M-1,k-n+1}. \tag{C.16}$$

It follows that the partial $k+1$ derivatives (C.15) and (C.16) are linearly independent.

For $(i,j) \in I_2 \cap I_4$, i.e. $(i,j) = (i,M-1),\ 0 \leq i \leq M-2$, there are $m-k+1$ rotation sets $R_{i,M-1}$ for variables $\phi_{n,i,M-1},\ 1 \leq n \leq m-k$ and $\beta_{i,M-1}$. For variable $\beta_{i,M-1}$, $\boldsymbol{D}_1 = \boldsymbol{Z}^k_{0..M-1}\boldsymbol{Z}^{m-k}_{0..M-2}$ and $\boldsymbol{D}_2 = \boldsymbol{I}_N$, and thus

$$\frac{\partial \boldsymbol{G}}{\partial \beta_{i,M-1}}(\boldsymbol{0}) = \boldsymbol{E}_{M-1,i,k} - \boldsymbol{E}_{i,M-1,m}, \tag{C.17}$$

and for variable $\phi_{n,i,M-1}$, $\boldsymbol{D}_1 = \boldsymbol{Z}^k_{0..M-1}\boldsymbol{Z}^{n-1}_{0..M-2}$ and $\boldsymbol{D}_2 = \boldsymbol{Z}^{m-k-n+1}_{0..M-2}$. It follows that

$$\frac{\partial \boldsymbol{G}}{\partial \phi_{n,i,M-1}}(\boldsymbol{0}) = \boldsymbol{E}_{M-1,i,m-n+1} - \boldsymbol{E}_{i,M-1,k+n-1}. \tag{C.18}$$

For $(i,j) \in I_1 \cap I_2$, i.e. $0 \leq i \leq M-2,\ M \leq j \leq N-1$, there are $m+1$ rotation sets $R_{i,j}$ for variables $\alpha_{i,j},\ \theta_{n,i,j},\ 1 \leq n \leq k$ and $\phi_{n,i,j},\ 1 \leq n \leq m-k$. Computing for each rotation, the matrices $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ give the results

$$\frac{\partial \boldsymbol{G}}{\partial \alpha_{i,j}}(\boldsymbol{0}) = \boldsymbol{E}_{j,i,0}, \tag{C.19}$$

for $1 \leq n \leq k$,

$$\frac{\partial \boldsymbol{G}}{\partial \theta_{n,i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,m-n+1}, \tag{C.20}$$

and for $1 \leq n \leq m-k$,

$$\frac{\partial \boldsymbol{G}}{\partial \phi_{n,i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,m-k-n+1}. \tag{C.21}$$

Case 3: $m > 1$ and $d = lm,\ 1 \leq l < M$ (Figure 4)

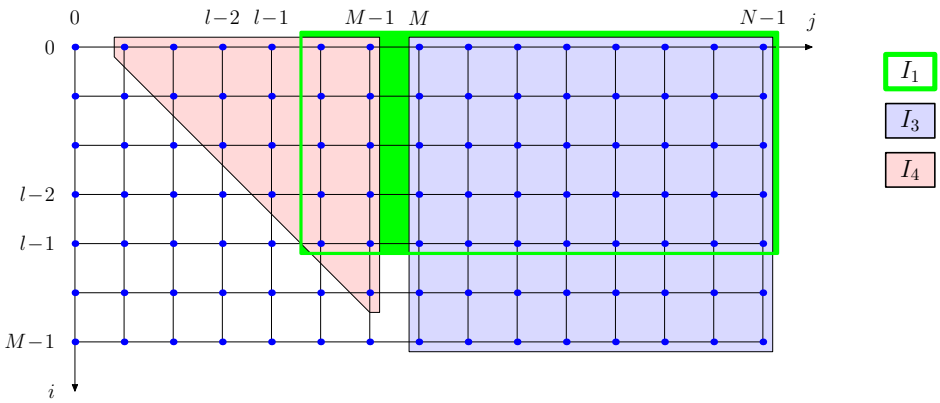

Figure 4: Rotations indexes for $d = lm,\ 1 \leq l < M$.

In this case $k = m$ and $I_2 = \emptyset$.
For $(i,j) \in I_4 \setminus I_1$, if $0 \leq i \leq l-2,\ i+1 \leq j \leq l-1$, there is only rotation with angle $\beta_{i,j}$ and

$$\frac{\partial \boldsymbol{G}}{\partial \beta_{i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,m} - \boldsymbol{E}_{i,j,m}, \tag{C.22}$$

or if $l \leq i \leq M-2,\ i+1 \leq j \leq M-1$,

$$\frac{\partial \boldsymbol{G}}{\partial \beta_{i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,0} - \boldsymbol{E}_{i,j,0}, \tag{C.23}$$

For $(i,j) \in I_3 \setminus I_1$, i.e. $l \leq i \leq M-1,\ M \leq j \leq N-1$, there is only one rotation for angle $\alpha_{i,j}$ and

$$\frac{\partial \boldsymbol{G}}{\partial \alpha_{i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,0}. \tag{C.24}$$

For $(i,j) \in I_1 \cap I_4$, i.e. $0 \leq i \leq l-1,\ l \leq j \leq M-1$, there are $m+1$ rotations sets with rotations of row indexes i and j for variables $\alpha_{i,j}$ and $\theta_{n,i,j},\ 1 \leq n \leq m$ and we get

$$\frac{\partial \boldsymbol{G}}{\partial \alpha_{i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,0} - \boldsymbol{E}_{i,j,m}, \tag{C.25}$$

and, for $1 \leq n \leq m$,

$$\frac{\partial \boldsymbol{G}}{\partial \theta_{n,i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,m-n+1} - \boldsymbol{E}_{i,j,n-1}. \tag{C.26}$$

For $(i,j) \in I_1 \cap I_3$, i.e. $0 \leq i \leq l-1,\ M \leq j \leq N-1$, there are $m+1$ rotations sets with rotations of row indexes i and j for variables $\alpha_{i,j}$ and $\theta_{n,i,j},\ 1 \leq n \leq m$ and we get

$$\frac{\partial \boldsymbol{G}}{\partial \alpha_{i,j}}(\boldsymbol{0}) = \boldsymbol{E}_{j,i,0}, \tag{C.27}$$

and, for $1 \leq n \leq m$,

$$\frac{\partial \boldsymbol{G}}{\partial \theta_{n,i,j}}(\boldsymbol{0}) = \boldsymbol{E}_{j,i,m-n+1}. \tag{C.28}$$

Case 4: $m > 1$ and $m < d < m(M-1)$, d not multiple of m (Figure 5)

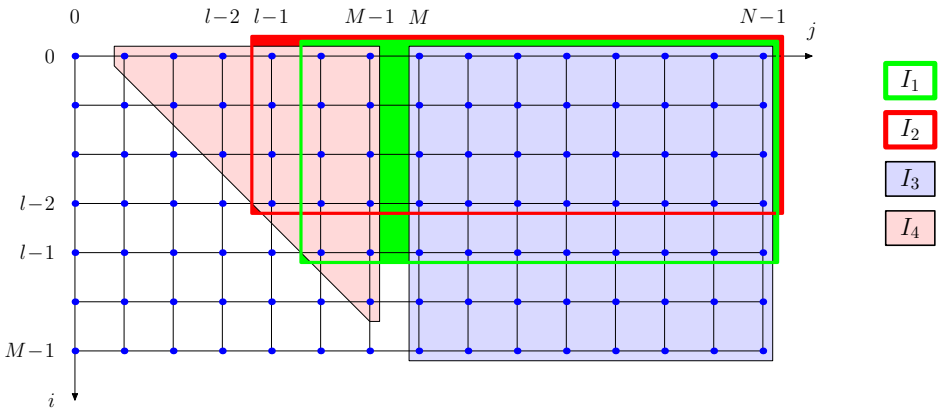

Figure 5: Rotations indexes for $m < d < m(M-1),\ d \neq lm$.

In this case $k < m$ and $l < M$. As shown in Figure 5, there are eight subcases to consider. There is only one rotation with indexes i and j:

- For $(i,j) \in I_4$ with $0 \leq i < l-2,\ i+1 \leq j < l-1$,

$$\frac{\partial \boldsymbol{G}}{\partial \beta_{i,j}}(\boldsymbol{0}) = \boldsymbol{E}_{j,i,m} - \boldsymbol{E}_{i,j,m}, \tag{C.29}$$

- For $(i,j) \in I_4$ with $l \leq i \leq M-2,\ i+1 \leq j \leq M-1$,

$$\frac{\partial \boldsymbol{G}}{\partial \beta_{i,j}}(\boldsymbol{0}) = \boldsymbol{E}_{j,i,0} - \boldsymbol{E}_{i,j,0}, \tag{C.30}$$

- For $(i,j) \in I_3 \setminus I_1$, i.e. $l \leq i \leq M-1,\ M \leq j \leq N-1$,

$$\frac{\partial \boldsymbol{G}}{\partial \alpha_{i,j}}(\boldsymbol{0}) = \boldsymbol{E}_{j,i,0}. \tag{C.31}$$

For $(i,j) \in (I_2 \cap I_4) \setminus I_1$, i.e $0 \leq i \leq l-2,\ j = l-1$, there are $m-k+1$ rotations with indexes i and $l-1$ with rotation angles $\beta_{i,l-1}$ and $\phi_{n,i,l-1},\ 1 \leq n \leq m-k$ such that

$$\frac{\partial \boldsymbol{G}}{\partial \beta_{i,l-1}}(\boldsymbol{0}) = \boldsymbol{E}_{l-1,i,k} - \boldsymbol{E}_{i,l-1,m}, \tag{C.32}$$

and for $1 \leq n \leq m-k$,

$$\frac{\partial \boldsymbol{G}}{\partial \phi_{n,i,l-1}}(\mathbf{0}) = \boldsymbol{E}_{l-1,i,m-n+1} - \boldsymbol{E}_{l-1,i,k+n-1}. \tag{C.33}$$

For $(i,j) \in (I_3 \cap I_1) \setminus I_2$, i.e. $i = l-1,\ M \leq j \leq N-1$, there are $k+1$ rotations with indexes $l-1$ and j with rotation angles $\alpha_{l-1,j}$ and $\theta_{n,l-1,j},\ 1 \leq n \leq k$ such that

$$\frac{\partial \boldsymbol{G}}{\partial \alpha_{l-1,j}}(\mathbf{0}) = \boldsymbol{E}_{j,l-1,0}, \tag{C.34}$$

and for $1 \leq n \leq k$,

$$\frac{\partial \boldsymbol{G}}{\partial \theta_{n,l-1,j}}(\mathbf{0}) = \boldsymbol{E}_{j,l-1,k-n+1}. \tag{C.35}$$

For $(i,j) \in (I_4 \cap I_1) \setminus I_2$, i.e. $i = l-1,\ l \leq j \leq M-1$, there are $k+1$ rotations with indexes $l-1$ and j with rotation angles $\beta_{l-1,j}$ and $\theta_{n,l-1,j},\ 1 \leq n \leq k$ such that

$$\frac{\partial \boldsymbol{G}}{\partial \beta_{l-1,j}}(\mathbf{0}) = \boldsymbol{E}_{j,l-1,0} - \boldsymbol{E}_{l-1,j,k}, \tag{C.36}$$

and for $1 \leq n \leq k$,

$$\frac{\partial \boldsymbol{G}}{\partial \theta_{n,l-1,j}}(\mathbf{0}) = \boldsymbol{E}_{j,l-1,k-n+1}. - \boldsymbol{E}_{l-1,j,n-1}. \tag{C.37}$$

For $(i,j) \in I_1 \cap I_2 \cap I_4$, i.e. $0 \leq i \leq l-2\ l \leq j \leq M-1$, there are $m+1$ rotations with indexes i and j with rotation angles $\beta_{i,j},\ \theta_{n,i,j},\ 1 \leq n \leq k$ and $\phi = n,i,j,\ 1 \leq n \leq m-k$ such that

$$\frac{\partial \boldsymbol{G}}{\partial \beta_{i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,0} - \boldsymbol{E}_{i,j,m}, \tag{C.38}$$

for $1 \leq n \leq k$,

$$\frac{\partial \boldsymbol{G}}{\partial \theta_{n,i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,m-n+1} - \boldsymbol{E}_{i,j,n-1}, \tag{C.39}$$

and for $1 \leq n \leq m-k$,

$$\frac{\partial \boldsymbol{G}}{\partial \phi_{n,i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,m-k-n+1} - \boldsymbol{E}_{i,j,k+n-1}. \tag{C.40}$$

For $(i,j) \in I_1 \cap I_2 \cap I_3$, i.e. $0 \leq i \leq l-2\ M \leq j \leq N-1$, there are $m+1$ rotations with indexes i and j with rotation angles $\alpha_{i,j},\ \theta_{n,i,j},\ 1 \leq n \leq k$ and $\phi = n,i,j,\ 1 \leq n \leq m-k$ such that

$$\frac{\partial \boldsymbol{G}}{\partial \alpha_{i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,0}, \tag{C.41}$$

for $1 \leq n \leq k$,

$$\frac{\partial \boldsymbol{G}}{\partial \theta_{n,i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,m-n+1}, \tag{C.42}$$

and for $1 \leq n \leq m-k$,

$$\frac{\partial \boldsymbol{G}}{\partial \phi_{n,i,j}}(\mathbf{0}) = \boldsymbol{E}_{j,i,m-k-n+1}.. \tag{C.43}$$

In all those eight subcases, the partial derivatives are linearly independent.

This achieves the proof of part (b) of Theorem 4.9.

□

Nomenclature

$\bar{E}$ closure of E for the Euclidean topology, page 6

$\bar{E}^Z$ closure of E for the Zariski topology, page 6

$\Delta[i]$ diagonal $N \times N$ matrix with -1 on row i and 1 elsewhere on the diagonal, page 9

$\dim(G)$ dimension of the Givens set G, page 10

$\mathbf{0}_M$ the null $M \times M$ matrix, page 18

$\boldsymbol{E}[s, \varepsilon]$ Elementary paraunitary matrix, page 4

$\boldsymbol{E}[s]$ equals $\boldsymbol{E}[s, \varepsilon_0]$, page 4

$\boldsymbol{E}_{r,c,k}$ canonical basis for $\mathcal{M}_{M,N}^{(m)}$, page 5

$\boldsymbol{I}_{N,M}$ equals $\boldsymbol{E}[s_0]$, page 4

$\boldsymbol{R}_{i,j}(\theta)$ elementary $N \times N$ rotation matrix or Givens matrix, page 4

$\boldsymbol{Z}_I$, $\boldsymbol{Z}_{r_1..r_2}$ shift matrices, page 5

$\boldsymbol{Z}_r$ shift $N \times N$ matrix, page 5

$\deg(\boldsymbol{A})$ McMillan degree of a polynomial $M \times N$ matrix, page 7

$\deg(G)$ McMillan degree of the Givens set G, page 11

$\deg(p)$ degree of a polynomial $p(X)$, page 6

$\mathcal{O}_{M,N}^{(m)}$ subset of paraunitary matrices in $\mathcal{M}_{M,N}^{(m)}$, page 5

$\phi : G = T_1 T_2 \ldots T_{n_T} E[s, \varepsilon]$ parametric representation of a Givens set, page 8

$\Pi_{M,N}$ projection operator from $\mathcal{O}_{M,N}^{(m)}$ to $\mathcal{O}_{M-1,N}^{(m)}$, page 5

$\mathcal{M}_{M,N}^{(m)}$ $M \times N$ polynomial matrices with polynomial degree less or equal to m, page 5

σ_m the symmetry operator in $\mathcal{O}_{M,N}^{(m)}$, page 5

$\mathrm{val}(\boldsymbol{A})$ McMillan valuation of a polynomial $M \times N$ matrix, page 6

$\mathrm{val}(G)$ McMillan valuation of the Givens set G, page 11

$\mathrm{val}(p)$ valuation of a polynomial $p(X)$, page 6

ε_0 $\varepsilon_0(c) = 1,\ 0 \leq c \leq M-1$, page 4

$A_{a..b}$ product of rotation sets, page 13

$A_{M,N,m,v,d}$, $B_{M,N,m,v,d}$ subsets of $\mathcal{O}_{M,N}^{(m)}$, page 7

$B_{a..b,c..d}$ another product of rotation sets, page 13

$E[s, \varepsilon]$ elementary set, page 8

$O(N)$ group of rotations in $\mathbb{R}^N$, page 5

$R_{i,j}$ rotation set, page 8

s_0 $s_0(c) = c,\ 0 \leq c \leq M-1$, page 4

$SO(I)$, $SO(i_1, i_2, \ldots, i_K)$ subgroups of $SO(N)$ acting on a subset of rows, page 5

$SO(N)$ group of direct rotations in $\mathbb{R}^N$, page 5

$Z_r,\ Z_I,\ Z_{r_1..r_2}$ shift sets, page 8

$\mathcal{U}_{M,N,m,d}$ Particular Givens set, page 20

$\mathcal{U}_{M,N,m,d}$ is defined by equation (4.24), page 20

$\mathcal{U}_{M,N,m}$ Union of $\mathcal{U}_{M,N,m,d},\ m \leq d \leq m(N-1)$, page 22