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► **To cite this version:**

Pedro Nicolás. Structuralism and theories in mathematics education. CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education, Charles University in Prague, Faculty of Education; ERME, Feb 2015, Prague, Czech Republic. pp.2688-2694. hal-01289460

HAL Id: hal-01289460

<https://hal.science/hal-01289460>

Submitted on 16 Mar 2016

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Structuralism and theories in mathematics education

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We present the structuralist conception of scientific theories as a Deus ex Machina which allows to resolve the entanglements of theories in Mathematics Education. We illustrate with examples how this conception, which forms a solid and solvent body of knowledge in Philosophy of Science, provides us with tools to perform a careful analysis of a theory, both by itself and in connection with other theories.

Keywords: Praxeology, theory, model, law, networking.

RECONSTRUCTION OF SCIENTIFIC THEORIES

As it is the case in many other disciplines, in Mathematics Education there are several theories living together: Theory of Didactic Situations (Brousseau, 1997), Anthropological Theory of the Didactic or ATD (Chevallard, 1999; Bosch et al., 2011), APOS' theory (Dubinsky & McDonald, 2002), Onto-Semiotic Approach (Godino, Batanero, & Font, 2007), Theory of Abstraction in Context (Dreyfus, Hershkowitz, & Schwarz, 2001), Theory of Knowledge Objectification (Radford, 2003)... Whereas the cohabitation of theories is perfectly normal, efforts aiming to connect some of them, especially from the CERME working team "Theoretical perspectives and approaches in mathematics education research" (CERME8, 2013), are also very natural and desirable.

We defend in this work that, for a better understanding of the possibility of connection of two theories, we must reconstruct them by using the same language. The reconstruction of a theory can be carried out from different conceptions. When we speak of 'conceptions' we mean ways of giving an account of what a scientific theory is, and not of how a scientific theory (in particular, a scientific law) is constructed. Thus, a priori these conceptions do not pay attention to methodological aspects.

The one favoured here is the so-called *structuralist conception* (Balzer, Moulines, & Sneed, 1987). This is an elaboration of the *semantical conception* (initiated by Suppes and Adams in the 1970s), and it seems to reconcile the most important aspects of the *syntactical conception* (advocated by Reichenbach, Ramsey, Bridgman, Campbell, Carnap in several works from the 1920s to the 1950s) and the *historicist conception* (advocated by Kuhn, Lakatos, Laudan in several works in the 1960s), while it avoids their problems (Diez & Moulines, 1997).

Now we will give a brief explanation of the main points of the structuralist conception. For a more extensive treatment, see (Balzer et al., 1987).

According to the structuralist conception, a scientific theory is a net of many nodes (which will be called *elements of the theory* or *theory-elements*) connected in several via *specialization*, see Definition 2 below. Of course, such a net does not appear out of the blue, but it is developed little by little along the time. This is how the structuralist captures the diachronic character of a theory. The synchronic character of a theory appears in the description of the theory-elements.

Definition 1: To determine a *theory-element* one has to specify:

- 1) The portions of reality the theory-element conceptualizes, i. E. The portions of reality the theory can speak of, called *potential models*. These potential models are described as portions of reality which can be modelled by using a *structure* (that is to say, a tuple $(D_1, D_2, \dots, R_1, R_2, \dots)$ of sets D_i and relations R_j between these sets) and a list of properties applicable to the structures of the former type. We call M_p the set of potential models.

- 2) The *laws* with which the theory-element aims to enlighten reality. Each law is a property applicable to the structures of the specified type. The laws distinguish the so-called *actual models* among the potential models. We call M the set of actual models.
- 3) The *partial potential models*, which are these portions of reality which can be checked to be potential models without assuming the laws of the theory-element. Notice that to verify that a portion of reality is a potential model we check, in particular, that the relations R_j appearing in the type of structure are satisfied. In this checking we use some method and this method might, or might not, assume the laws of the theory-element. A relation R_j is *theoretical with respect to a theory element T* (or, in short, *T -theoretical*) if every method of determination of R_j assumes the validity of the laws of this theory-element. Thus, a partial potential model of a theory-element is nothing but a potential model in which we omit the theoretical relations. We call M_{pp} the set of partial potential models.
- 4) Those partial potential models that are expected to be actual models. These partial potential models are, after all, the *intended applications* of our theory-element. We call I the set of intended applications of our theory-element.

Thus a *theory-element* is an ordered pair $T = (K, I)$ where I is the set of intended applications and $K = (M_p, M_{pp}, M)$ is the *core*, formed by the set of potential models, the set of partial potential models, and the set of actual models.

The *empirical claim* of a theory-element is just the statement which asserts that the intended applications are actual models, $I \subseteq M$, that is to say, that in certain portions of reality, which can be detected without assuming the laws of the theory-element, these laws actually hold.

In the next section we will give several examples of theory-elements but, unfortunately, we will not point out a theoretical relation in any of them. It is an important open question whether there are theoretical relations in the current theories of Mathematics Education. In Classical Mechanics (CM), the relations of *position* or *time* are not CM-theoretical, since you

can determine them without assuming any proper law of Classical Mechanics. However, the relation *mass* is CM-theoretical, since any method of determination of the amount of mass of an object assumes a law proper of the CM. For examples in other disciplines see (Balzer et al., 1987).

NETWORKING THEORIES

In what follows we use the structuralist approach to present different kinds of possible connections between theory-elements.

Definition 2: A theory-element T' is a *specialization* of another theory-element T , and we write $T' \sqsubseteq T$, if:

1.
 - 1.1. $M'_p = M_p$, that is to say, both theory-elements conceptualize the world in the same way.
 - 1.2. $M'_{pp} = M_{pp}$, that is to say, both theory-elements consider the same theoretical relations.
 - 1.3. $M' \subseteq M$, that is to say, every law in T is also a law in T' .
2. $I' \subseteq I$, that is to say, every portion of reality aimed to be explained by T' is also a portion of reality aimed to be explained by T .

In short, to specialize consists of increasing the amount of laws without changing the conceptual architecture.

Definition 3: A *net-theory* is a pair $N = (\{T_i\}, \sigma)$ where $\{T_i\}$ is a non-empty set of theory-elements and σ is a specialization relation on $\{T_i\}$.

Next we are defining the notion of *theorization*, but first we need the following:

Definition 4: Given two structures (see Definition 1) $x = (D_1, \dots, D_m, R_1, \dots, R_n)$ and $y = (D'_1, \dots, D'_p, R'_1, \dots, R'_q)$, we say that y is a *substructure* of x if:

1. $p \leq m, q \leq n$.
2. Every D'_i is a subset of some D_j .
3. Every R'_i is a subset of some R_j .

Definition 5: A theory-element T' is a *theorization* of a theory-element T if:

1. Every intended application of T' admits an actual model of T as substructure.
2. There are potential models of T' which are not substructures of potential models of T (because they contemplate new domains and/or new relations).

The first condition says that every portion of reality T' aims to explain satisfies the laws of T . The second condition says that T' includes new (not necessarily T' -theoretical) concepts not contemplated by T .

Next I will show in examples some tentative structuralist descriptions of some elements of the ATD.

Example of theory-element: Our first example is inspired in the so-called *Herbartian scheme* (Chevallard, 2015), which is probably the most general structure proposed by the ATD to deal with situations of study. In this structure there are things like a task or question which requires some answer, a series of partial answers, and a final answer. Therefore, the structure corresponding to our theory-element T_1 will be the tuple $(\{1, \dots, n\}, P, s)$ where $\{1, \dots, n\}$ is the set of the first n natural numbers, P is a non-empty set P , and s is a map from $\{1, \dots, n\}$ to P . The image of 1 is said to be a *generating question*, the image of n is said to be a *final answer* and the other images are said to be *partial answers*. Since no law is stated, there is no distinction between potential, partial potential and actual models. Notice that every temporal sequence of n events fits in this structure, but, of course, not every such sequence is an intended application of T_1 . This is why it is important to explain which are our intended applications, namely, those sequences of events consisting in finding an answer to a question.

Example of theorization: If, moreover, in each of the partial answers of T_1 we distinguish between tasks, techniques and *logos* elements, that is, if we look at the constituent parts of the so-called *praxeologies* (Chevallard, 1999), we would have reached a theorization, T_2 , of T_1 . The structure corresponding to T_2 will be a tuple $(\{1, \dots, n\}, S_T, S_t, S_L, s)$ where S_T, S_t , and S_L are non-empty sets whose elements are called *tasks*, *techniques* and *logos-elements*, respectively, and s is a map from $\{1, \dots, n\}$ to $S_T \times S_t \times S_L$. Since no law is stated there

is no distinction between potential, partial potential and actual models. Now not every temporal sequence of n events fits in the structure of T_2 . Not even every temporal sequence of n events consisting in finding an answer to a question! In fact, our intended applications are temporal sequences of events consisting in finding an answer to a question such that in each of these events we find three components and such that, moreover,

- all the first components of the events are “of the same nature” (this is encoded in the fact that they belong to the same set), namely, tasks;
- all the second components of the events are of the same nature, namely, solutions to the task specified in the corresponding first component, and
- all the third components of the events are of the same nature, namely, explanations of why the corresponding second element solves the corresponding first element.

Example of theorization: If, moreover, we take into account the dynamics of each of these praxeologies, recognizing the so-called *study moments* (Chevallard, 1999), we would have a theorization, T_3 , of T_2 . The structure corresponding to T_3 will be a tuple $(\{1, \dots, n\}, S_T, S_t, S_L, \{0,1\}, \{*\}, s)$ where

- $\{1, \dots, n\}, S_T, S_t$ and S_L are as before;
- s is a map from $\{1, \dots, n\}$ to $S \times (S \cup \{*\}) \times (S \cup \{*\}) \times \{0, 1\} \times \{0, 1\} \times (\{0, 1\} \cup \{*\}) \times (\mathbb{N} \cup \{*\}) \times (\mathbb{N} \cup \{*\})$, where S is the union of S_T, S_t and S_L , called the *study sequence map*, and its images are called *events*.

The structure is now more complicated because it has to model more ambitious intended applications. Indeed, in the events of the sequence we still look at tasks, techniques and logos, but we also pay attention to the way they are related:

- The first (respectively, second, third, fourth and fifth) component of an event refers to the first (respectively, second, third, fourth and fifth) study moment (Chevallard, 1999).
- The last three components of an event refer to the sixth study moment, namely, to the evaluation moment. More precisely, the sixth component refers to

the scope of the technique (it is a bounded magnitude which reaches the value 1 if the technique covers all the possible cases of the task), the seventh component refers to its economy and the eighth component refers to its reliability (see Sierra, Bosch, & Gáscon, 2013)².

For example, an event which is an element of $S_T \times S_t \times S_L \times \dots$ is regarded as a task followed by an elaboration of a technique followed by an explanation of why this technique works, whereas an element of $S_t \times S_T \times \{*\} \times \dots$ is regarded as a technique followed by a task which is solved by the technique followed by no explanation of why the technique works. We use * to express absence of activity in the second, third, sixth, seventh and eighth components, and we use 0 (respectively, 1) to express absence (respectively, presence) of activity in the fourth and fifth components. We can add some axioms devoted to prevent us from considering impossible events, for example:

Axiom 1: There are not events starting with a task and continuing with a *logos* element.

Axiom 2: If an event starts with a technique, then it cannot continue with a *logos* element.

Axiom 3: In an event there are not two tasks, two techniques or two *logos* elements. Thus, for instance, there are not events which are elements of $S_T \times S_T \times \dots$

Axiom 4: If in an event there is no task, then the last three components of the event are (*, *, *).

Axiom 5: If the fourth component of an event is *, then the last three components are (*, *, *).

Examples of specialization: Imagine we create a new theory-element T_4 by adding the following law to the theory-element T_3 :

Law: The last three components of every event are (*, *, *).

The new theory-element T_4 is a specialization of T_3 . Indeed, there are actual models in T_3 which are not actual models in T_4 , namely, those study sequences having at least an event in which the last three components are not (*, *, *). After the axioms, it is clear that the former law holds for those study sequences in which each event $s_i = (s_{i1}, s_{i2}, s_{i3}, s_{i4}, s_{i5}, s_{i6}, s_{i7}, s_{i8})$ satisfies that none of the s_{ij} are a task or that $s_{i4} = *$. Hence,

those study sequences would be actual models of our theory-element T_4 .

The notion of *didactic contract* (Brousseau, 1986) is a good source of laws for theory-elements dealing with study sequences. Indeed, a didactic contract can be regarded as a special family of *clauses* or *conventions*, and, inspired in Lewis (1969), we could express a convention as a law stating that a certain regularity in the events of a study sequence holds (see for instance the law above).

Remark: In Chevallard (1988b), there is a sketch of the possible sets and relations of the structures an anthropological theory of the didactics would deal with. It would be interesting to compare them with the ones used in our examples above.

Remark: Brousseau (1986), inspired among others by Suppes (1969, 1976)³, used finite automata to give a structuralist formulation of the notion of *situation*. Our structuralist formulations of notions of the ATD are more in the spirit of the Stimulus-Sampling Theory (Estes & Suppes, 1959). It is worth noting that, as proved in Suppes (1969), given any finite connected automaton there is a stimulus-response model that asymptotically becomes isomorphic to it.

Finally, let us consider the relation of *reduction* between theory-elements.

Definition 6: A theory-element T is *reducible* to a theory-element T^* if there exists a relation $\rho \subseteq M_p(T) \times M_p(T^*)$ such that:

1. If $(x, x^*) \in \rho$ and $x^* \in M(T^*)$, then $x \in M(T)$.
2. If $y \in I(T) \cap M(T)$ then there exists $y^* \in I(T^*) \cap M(T^*)$ such that $(y, y^*) \in \rho$.

The underlying idea is to regard the elements (x, x^*) of ρ as pairs of portions of reality so that x^* is the T^* -version of x . The first condition says that the laws of T can be derived from those of T^* . The second condition says that all the successful applications of T admit T^* -versions which are also successful applications of T^* . In other words, the successes of T can be explained in virtue of those of T^* . Notice that, in contrast to what happened with the *theorization* (Definition 5), reduction does not require an increase in the conceptual map, that is to say, the kind of structures contemplated

as potential models. Indeed, the conceptual map of T^* might be completely different to the conceptual map T .

Examples of reduction: The *classical mathematics education* (see Gascón, 1998) explains certain phenomena with laws involving cognitive or motivational concepts. Indeed, these would be the kind of concepts used by the classical mathematics education to explain the kind of phenomena presented in IREM de Grenoble (1979). One can use Chevallard (1988a) to sketch how part of this classical mathematics education can be reduced to a theory-element including among the laws the clauses of the *didactic contract*. On the other hand, one can also use (Chevallard, Bosch, & Gascón, 1997) to reduce part of this classical mathematics education to a theory-element with laws stating the *incompleteness* of scholar study processes (this incompleteness can be expressed in terms of the study moments, for example, by saying that *the moment of the construction of the technological-theoretical frame* or *the moment of the work of the technique* is lost).

In Bikner-Ahsbabs and Prediger (2010) the following “networking strategies” are presented: to ignore other theories (as an extreme strategy of non-connection), to make your own theory and foreign theories understandable, to compare/contrast, to coordinate/combine, to integrate locally/synthesize and to unify globally (as an extreme strategy of total connection). Next, let us clarify this taxonomy by presenting, in a brief and simplified way, possible translations of these strategies to the structuralist language:

- To ignore other theories: not to consider the possibility of (even partial) specialization, theorization or reduction (see Definitions 2, 5 and 6) as a relation among two theory-elements.
- To make your own theory and foreign theories understandable: to accomplish this, as we said at the beginning of this paper, one need to translate both theories to the same language. What we suggest here is to use the structuralist language. So, in a sense, in the present work we take seriously this second networking strategy.
- To compare/contrast: to check which are the potential models shared by two theory-elements. Thus, when comparing/contrasting we could be performing a theorization.

- To coordinate/combine two theory-elements T and T' consists in saying that a common intended application is both an actual model of T and an actual model of T' . It is important to notice that, for this to happen, T and T' must share the partial potential models. This last sentence explains the meaning of the following statement of Bikner-Ahsbabs and Prediger (2010): “Whereas all theories can of course be compared or contrasted, the combination of (elements of) different theories risks becoming difficult when the theories are not compatible.”
- To integrate locally/synthesize two theory-elements T and T' : to find a third theory-element T'' to which we can reduce the theory-element derived from T when considered just some sub-structures z of the structures x of T , as well as the theory-element derived from T' when considered just some sub-structures z' of the structures x' of T' . Notice that the structures x'' of T'' should admit both z and z' as sub-structures.
- To unify globally: to find a theory to which any other theory could be reduced.

CONCLUSION

Here we suggest to use the structuralist formalization of scientific theories to the benefit of the questions about the theoretical status of different approaches in Mathematics Education. Needless to say, we do not mean one cannot work properly in theory unless this is formalized. For example, it is not reasonable to say that Newton was not doing Mechanics just because he did not have at hand a strict formalization. On the other hand, theories in Mathematics Education are still far from being formalizable, being this (even partial) formalization a long-term project in any case. Concerning this, it is important to point out that the degree of resistance of a theory to be formalized is inversely proportional to its degree of development. For example, if we cannot distinguish the actual models among the potential models, then we cannot identify any law of the theory (and at this point we should wonder whether this forces us to accept this theory is nonexistent...). Anyway, regardless of the difficulty of a complete formalization, we defend that:

- The framework offered by the structuralist conception of scientific theories is illuminating to

the extent that it provides us with high order tools which allow a better understanding of the theoretical scene in Mathematics Education.

- Even if we were not interested in networking theories, the attempt to formalize a theory in the structuralist way forces us to consider extremely interesting questions about this theory. For instance: which are the underlying structures? Which are the laws? Which are the theoretical relations?

Among many other things, it is still an open question which are the links between our structuralist approach, the definition of *theory* by Radford (2008) and the notion of *research praxeology* by Artigue, Bosch, and Gascón (2011a, 2011b).

ACKNOWLEDGEMENT

I thank Josep Gascón for his reading of a previous version of this article and his helpful comments.

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ENDNOTES

1. This is the short form for “Action, Process, Object and Scheme”.

2. For the sake of simplicity, we do not distinguish between *task* and *type of task*, and between *technological* and *theoretical* elements among the logos elements, even if they are important distinctions in the ATD.

3. Actually, the last two components should be interpreted as evaluations of a technique in comparison with another technique. Indeed, we typically speak of a technique as being more or less economic or reliable than another technique. However, for the sake of simplicity, we do not take into account this aspect here.

4. It is a remarkable fact that Suppes was the main promoter of the semantic conception, direct precedent of the structuralist conception.